Stability Analysis

Linearity Series

Instructor's Guide

Contents

Intro

Coursework

Resources

Table of Contents

Developed by the Teaching and Learning Laboratory at MIT
for the Singapore University of Technology and Design

Introduction

When to Use this Video

- In SOPH 301, at home, after Lecture 6: Eigenvalues and the phase plane, part I.
- In EPDM 301, in lecture, before Lecture 19: Single DOF system: equilibrium, linearization, stability analysis.
- Prior knowledge: elementary mechanics, ordinary differential equations, and eigenproblems.

Key Information

Duration: 14:32
Narrator: Prof. Tony Patera
Materials Needed:
• Paper
• Pencil

Learning Objectives

After watching this video students will be familiear with the framework of equilibrium and stability analysis.

Motivation

- This video aims to motivate students to study this topic further.
- Students have difficulty connecting the linear stability analysis framework to problems. This video aims to distinguish between the physical pendulum, the nonlinear pendulum model, and the linear pendulum model, disambiguating the various models obtained at various stages in this framework.

Student Experience

It is highly recommended that the video is paused when prompted so that students are able to attempt the activities on their own and then check their solutions against the video.

During the video, students will:

- Identify whether two equilibria for a physical model are stable or unstable based on phyiscal intuition.
- Find a linear approximation for the sine function near 0.
- Determine what happens to the solution when both eigenvalues for the eigenproblem near the equilibrium have negative real part.
- Determine what happens to the solution when at least one eigenvalue has positive real part.
- Determine what happens to the solution when the eigenvalues have zero real part.
- Work through the linear stability analysis framework for the top equilibrium of the pendulum.

Video Highlights

This table outlines a collection of activities and important ideas from the video.

Time	Feature	Comments
0:00	Heating and Cooling a small region	IR footage of the long term behavior after heating and cooling a small region of a countertop motivates the notion of stable equilibrium.
1:10	Prerequisites and Learning Objectives	What students need to know before watching the video, and what they will learn after watching the video.
1:28	Chatper 1: Equilibrium and Stability Analysis	
2:15	Stability is defined	
2:50	Energy argument for heat example	
3:35	Chapter 2: Pendulum	
3:55	Framework for linear stability analysis	A step-by-step procedure for linear stability analysis is given, and worked through for the example of the pendulum throughout the chapter.
4:40	Model of pendulum developed	
6:15	Model tested numerically	
6:49	Equilibrium identified	
7:27	Activity	Students are asked to use physical intuition to decide whether the equilibria identified are stable or unstable.
8:09	Linearize equation at bottom equilibrium	
8:45	Activity	Students are asked to linearize the sine function near 0.
9:40	Eigenproblem formulated	
10:15	Activity: Inspect eigenvalues	Students determine how the signs of the real part of the eigenvalues effect the long term behavior of solutions to the differential equation.
12:34	Compare numerical and linear models	
13:00	To Review	Video content is summarized.

Video Summary

This video introduces students to the notion of stability of equilibria. A temperature example is explored using an energy argument, and then the typical linear stability analysis framework is introduced and worked through in detail through the example of the pendulum.

Course Materials

Pre-Video Materials

When appropriate, this guide is accompanied by additional materials to aid in the delivery of some of the following activities and discussions.

The following problems are intended to refenforce student understanding of the behavior of linear 2nd order differential equations, the use of linear algebra to solve such equations, and some basic knowledge about eigenvalues and eigenvectors of matrices and their use in solving differential equations.

1. Find the general solution of this first order, linear ODE.

$$\frac{dy}{dt} = (t-3)(y-2)$$

(a) What is the steady state behavior of this solution? (What happens as time approaches infinity?)

(b) What is an intial condition for the equilibrium solution to this equation?

(c) Is the equilibrium solution stable or unstable?

2. Harmonic Oscillator: Appendix A1

Suppose we have 4 ideal spring systems, each with the same mass, but different springs modeled by the 4 equations below.

$$\begin{array}{rl} i. & \ddot{x} + 8x = 0 \\ ii. & \ddot{x} + 2x = 0 \\ iii. & 2\ddot{x} + x = 0 \\ iv. & 8\ddot{x} + x = 0 \end{array}$$

(a) Which spring is stiffer?

(b) Which spring will have the highest frequency of motion?

3. We can write the 2nd order differential equation modeling the spring as a system of first order equations in matrix form.

$$\begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \omega^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$$

(a) What are the eigenvalues and eigenvectors of this matrix?

(b) Express the solutions to this differential equation in terms of these eigenvalues and vectors.

Post-Video Materials

1. Solve the eigenproblem for the top equilibrium of the pendulum to determine the stability by doing the following:

(a) Linearize the system of ODEs near the equilibrium $(\pi, 0)$.

(b) Write as a matrix equation and find the eigenvalues.

(c) Idenfity the sign of the real part of the eigenvalues. What does this mean about stability?

(d) Discuss with a classmate what further analysis of this system, the solution, and the eigenvalues leads you to conclude.

2. Book Problem: Appendix A2

(The following problem is from MIT course 2.086/2.088 Fall 2012, recitation 7 Matlab Exercise.)

Consider the stability of a parallelpiped, which roughly has the dimensions and shape as a book. The dimensions, principle directions, and moments of inertia are labeled in the diagram in Appendix A2.

The equations of for the moments of inertia are given by the following set of equations.

$$I_1 = \frac{1}{3}(a^2 + b^2)m$$
$$I_2 = \frac{1}{3}(a^2 + c^2)m$$
$$I_3 = \frac{1}{3}(b^2 + c^2)m$$

The system of differential equations that models this is the following.

$$I_1 \frac{d\omega_1}{dt} = -\omega_2\omega_3(I_3 - I_2)$$
$$I_2 \frac{d\omega_2}{dt} = -\omega_3\omega_1(I_1 - I_3)$$
$$I_3 \frac{d\omega_3}{dt} = -\omega_1\omega_2(I_2 - I_1)$$

Work through the stability analysis framework.

(a) Identify the equilibria for this system.

(b) Linearize the system of equations at each equilibrium, and determine the stability of each.

Additional Resources

Going Further

Engineering involves many modeling problems, which are typically nonlinear. A qualitative analysis of the behavior typically involves a linear stability analysis. Linear differential equations are solvable analytically, and this behavior makes them ideal starting point for gaining intuitive understanding of system behavior.

Once students are comfortable with linear systems, they can embark into solving nonlinear systems numerically. In numerical methods, it is important for students to have a strong understanding of stable and unstable steady state solutions in order to create numerical schemes that behave appropriately.

References

The following article models the simple and non-linear pendulum, and suggests a student activity that allows students to explore the effects of simplifying assumptions on the model, and compares the model to experimental data collected by the student.

- Reid, T. S. & King, S. C., (2009). Pendulum motion and differential equations. *PRIMUS, 19*(2), 205–217.

These vides discuss pendulum behavior, the first from a physics perspective, the second from a mathematical perspective.

- Lewin, W., *8.01 Classical Mechanics*, Fall 1999. (Massachusetts Institute of Technology: MIT OpenCouseWare), http://ocw.mit.edu (Accessed November 27, 2012). License: Creative Commons BY-NC-SA
  -Lecture 10: Hook's Law and Simple Harmonic Motion
- Mattuck, A., *18.03 Differential Equationss*, Spring 2003. (Massachusetts Institute of Technology: MIT OpenCouseWare), http://ocw.mit.edu (Accessed November 27, 2012). License: Creative Commons BY-NC-SA
  -Lectures 31: Non-linear Autonomous Systems

For a review of eigenvalues and eigenvectors, consultation of the following linear algebra text may be useful.

- Strang, G. (1998). *Introduction to Linear Algebra.* Wellesley, MA: Wellesley-Cambridge Press.

For an introduction to linear stability analysis, we recommend the following introductory text.

- Edwards, C. H. & Penney, D. E. (2008). *Elementary Differential Equations with Boundary Value Problems* (6th ed.). Upper Saddle River, NJ: Pearson Prentice Hall.

Suppose we have 4 ideal spring systems, each with the same mass, but different springs modeled by the 4 equations below.

$$i. \qquad \ddot{x} + 8x = 0$$

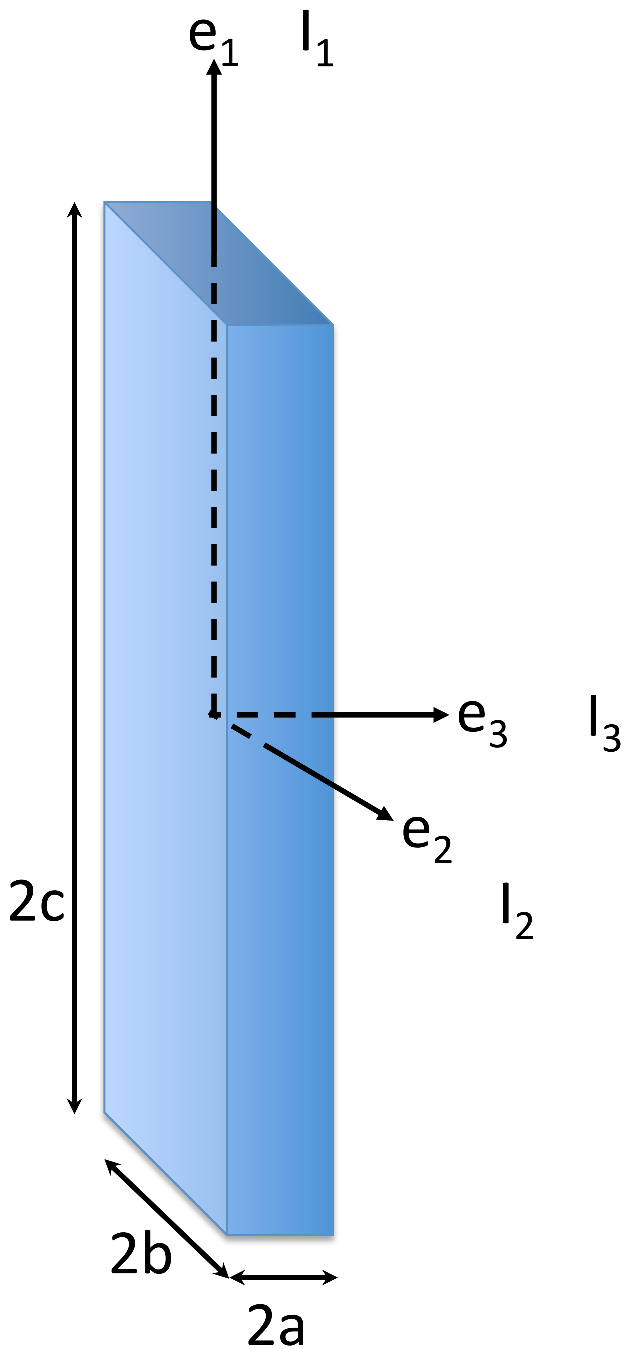
$$ii. \qquad \ddot{x} + 2x = 0$$

$$iii. \qquad 2\ddot{x} + x = 0$$

$$iv. \qquad 8\ddot{x} + x = 0$$

(a) Which spring is stiffer?

(b) Which spring will have the highest frequency of motion?



$$I_1 = \frac{1}{3}(a^2 + b^2)m$$

$$I_2 = \frac{1}{3}(a^2 + c^2)m$$

$$I_3 = \frac{1}{3}(b^2 + c^2)m$$

$$I_1 \frac{d\omega_1}{dt} = -\omega_2\omega_3(I_3 - I_2)$$

$$I_2 \frac{d\omega_2}{dt} = -\omega_3\omega_1(I_1 - I_3)$$

$$I_3 \frac{d\omega_3}{dt} = -\omega_1\omega_2(I_2 - I_1)$$