PROFESSOR: Welcome back. Today, we will consider solutions to systems of an infinite number of degrees of freedom. What we'll consider is when you have identical oscillators, which are coupled only to the neighbors, and there is an infinite number of them. So each one is an infinitely small oscillator, as I say, coupled to each neighbor.

The simplest example is that of a taut string. Each piece of that string is a harmonic oscillator. It oscillates transversely. And each piece is identical to every other piece. And it's coupled to its neighbors. So let's consider the following problem.

The problem we are going to talk about-- the solution of the following problem. Suppose you have a string. For practical purposes, an infinite string. It has a mass per unit length of new mu, a constant tension T, and at one end it's connected to a massless ring. This is an idealized situation. You could imagine, suppose that ring was on your finger. You held the string taut and you moved your finger in some ways, which caused a distortion, which propagated down this string. You got a progressive pulse going down this string. So this is an idealized diagram of that situation.

I've indicated the finger holding the string by, you could imagine, a massless ring sliding on a frictionless rod, for example. And somehow or other, a force is applied to this ring. But you can think of the actual problem where you're holding the ring with your finger and moving this up and down.

Now, somehow or other, you're moving this ring such that as a result of your motion, there is a progressive pulse. And the idealized pulse we'll consider is a triangular shape like this, where this height is H and the length from here and here is L. It is idealized. In reality, you can't have sharp corners like that. You can imagine there is a very tiny curvature at those locations.

Now, what we are told is that at some instant t given by 4L over-- and this is the phase velocity of propagation down the pulse over square root of t over mu. At this instant of time, you look at the string and you see that this is the shape of that pulse. And as a function of time, that propagates to the right. OK?

The question is, what is the motion of this ring as a function of time? That's one. Two, what is the power that the force which drives this string delivers to the string? So what's the power as a function of time?

All right, next. What is at any instant of time the potential energy stored in this pulse? Four, what is the kinetic energy stored in this pulse as it's propagating? And finally the question, are your results to the power that's delivered, the potential energy stored in the distortion, the kinetic energy in the distortion consistent with each other?

These are the things we are told to assume. And I repeat some of them. We assume the ring is massless. The distortion is always sufficiently small so that this height is much smaller than this. So that this angle here is sufficiently small that we can approximate the sine angle the angle or the tangent of the angle, which of course is this height divided by that. So it's h over L over T.

We are told that the mass per unit length is mu and that the tension is uniform. In other words, because this distortion is small, we are making the assumption that the tension in this string is constant and the same everywhere. There are no losses, no frictional losses.

And finally, all the motion is-- because this is so small, is in the transverse direction. So any piece of the string moves in the transverse direction.

This is a reasonable approximation to reality. Or put it the other way around, the reality would be a reasonable approximation to this idealized situation, which we will try to understand and solve. OK, so how do we go about it?

The usual way. We first have to represent the problem in terms of mathematics. So what do we know?

The system is a continuous infinite row of oscillators. Each one connected only to its neighbor. And we know that such a system can be represented by a wave equation. The equation of motion for this system is a wave equation.

In essence, if you remember when we had coupled oscillators, three oscillators, four, et cetera. Every time you add one more oscillator, you have one more equation of motion. So if you have an infinite number of oscillators, you expect to have an infinite number of coupled equations of motions. That's what a wave equation is.

For every position x, you have an equation of motion. Why the distortion is a function of x?

So essentially for every value of x, you have one equation. Since x is a continuum, this, in essence, is an infinite number of coupled differential equations. And we call this the wave equation.

So the equation of motion of the string is that. What else do we know? How can we use the laws of nature to describe this situation?

Well, let's consider the ring. On the ring, there will be forces. One is the string is attached, so there will be the tension in the string pulling on the ring. We said that the ring cannot-- it moves only up and down, so there must be some constraining force on it. If you're imagining it to be on this frictionless rod, then that constraining force is the reaction of the rod on the ring.

If you're thinking of it in terms of a finger holding onto that ring, then your finger is preventing the ring moving backwards or forwards. So that's this reaction force.

Now, on top of that, there is the vertical force, the one we are interested in, which must be the cause of the distortion. So those are the three forces. What else do we know? We said that we are going to idealize this situation. We're going to assume that this ring has no mass. It's only there so you can essentially hold the tip of the string.

Now if mass is 0, then in Newton's laws of motion there can be no net force on that mass. Because if you applied any force to a 0 mass, it will have an infinite acceleration. Your ring would disappear.

So the fact that the ring is massless-- in other words, you're just holding the very tip of the string. That's what it, in essence, means. It means there is no net force on the ring.

The ring does not move backwards and forwards. There's no net force in that direction. So the reaction of the rod on the string must be exactly equal to the horizontal component of the tension. Actually, we don't need this for the solving of the problem.

On the other hand, in the vertical direction we also know that there is no net force vertically. And so the magnitude of the force at any instant of time will, of course, be equal and opposite to the magnitude of the force due to the tension in this string, the component of that in the vertical direction. So the F of t at any instant of time will be equal to minus the vertical component of the tension on the string.

OK, so that's everything we know about the dynamics. This plus the knowledge that at the time we were given this was the shape of the distortion, and it was progressing to the right, must be enough to be able to predict the motion of the ring, the power delivered. Now, why do I say that?

Well, let's go back for a second. This will tell us what the ring is doing and what the forces are acting on it.

Now, if you have a force acting on something, and that something is moving, the power that you will deliver that to that object is the dot product between the force and the velocity of the object. These we know because power is work per unit time. Work done by a force is force times the distance moved. So therefore, work times distance moved per second is the force times the velocity. So this is the power generated by a force if it moves an object with a velocity v of t.

From what I told you earlier, I'm just thinking through whether we can solve this problem. From the part I discussed earlier, we will be able to figure out the force. We will be able to figure out the motion. Therefore, we will be able to calculate this and answer the second part of the problem. Let's continue.

We want to find the energy stored in the distortion of this ring. How would we do that to make sure that we have everything in place?

Well, if you have some system which you distort from equilibrium, the potential energy stored by conservation of energy is equal to the work you do to distort that system.

In our case, the pulse is a distorted string from equilibrium. So if I calculate the work that I will have to do to produce that shape of the string, that will be equal to the potential energy stored in the string. So potential energy, in other words, is work to distort the string. And that is simply the integral of the force times the distance that will move that force to distort the string. So that we will be able to do since we-- as I'll show in a second.

The next part was, what is the kinetic energy in the pulse? Well kinetic energy is, of course, 1/2 mass times velocity squared of a system. The string is-- it's a continuous distribution of masses. If I take every piece of that string and calculate its transverse velocity-- that's the only velocity that it has. We said that this string has only transverse motion. So if I calculate the transverse velocity of every piece of the string and add up all the pieces, I'll have the total kinetic energy stored.

So at any instant of time, I have to just find out what every piece of the string is doing, calculate its velocity squared multiplied by 1/2 by the mass of that piece, add them all up, and that's equal to-- therefore, we will have to integrate across the whole string 1/2 times the transverse velocity squared times the density of the-- the string has certain mass per unit length, mu. So I have to take this and integrate a mu dx times that. That'll give us the total kinetic energy. OK.

So now, here we have the problem in terms of mathematics. And we'll switch over and try to solve it. OK.

This wave equation has, as I told you, is equivalent to an infinite coupled differential equations. It has infinite number of solutions. Trying like we did for simpler system, like system with one degree of freedom or two degree of freedom is no longer practical. I can't go through all the possibilities and then guess which one applies. So I have to use some more knowledge or experience.

In general, as I say, there's an infinite number of solutions. But we know that we are looking for a very specific kind of solution. We know that this pulse we're told was generated by this mass-- this ring. Sorry, not the mass. By this ring being moved by a finger. Which gave rise to a progressive wave. Meaning a shape like this which is moving to the right. That immediately gives us a clue.

We know that there are some general classes of solutions of this wave equation. There are the normal modes, which are every piece of this string moving with the same frequency and phase, et cetera. There is another one which isn't obvious, but which Professor Walter Lewin discussed in class, that is actually quite-- I consider it the miraculous one. It's so not obvious.

There is a solution of this wave equation, which consists of any distortion. You can take any shape. And if somehow or other you manage to make that distortion move with a uniform velocity corresponding to this velocity in the wave equation-- so for our string, the square root of this number-- that will progress forever. It's a progressive wave solution. And it's for any shape.

So here, clearly we are talking of that kind of a solution of this wave equation. And as I say, from what you've learned from Professor Walter Lewin, we know that the wave equation does have that class of solutions.

So the solution to this problem must be that of a progressive wave. Specifically, a shape like this which moves to the right with a velocity square root of T over mu. And I cannot resist emphasizing that no part of the string is moving here to the right.

The string is only moving in the transverse direction. But it is the distortion which is moving with that velocity v . It's called the phase velocity. All right. So our knowledge of the solutions of wave equations and these descriptions make sense. And so we know the class of solutions that describe this. In fact, it is the progressive wave solution of this problem. Fantastic.

So we know what this string is doing at all times, a earlier times and later times. From now on, this triangle will just move forever like that. That will be the distortion. I can turn the clock around. I know that earlier it was here, here, et cetera. So I know the shape of this string at all times.

Knowing that, I can-- it's a funny way of saying it, predict because it happened earlier. But I can tell you what the distortion must have been at earlier times. Going back with velocity v until-- up to the times when this distortion was here. Of course, the string was doing nothing. But at some instant of-- so the total picture is the following.

We start off with the taut string. At some instant of time, the person must have moved the string with uniform velocity up, which I call the transverse velocity the transverse. That'll start this wave front. You will have come to a stop and start moving downwards again, formed this triangle, and then I do no more. This triangle will then progress forward. It will progress with a phase velocity v with this velocity, which we know. It's not something I did in my head. I can derive this by studying how a string can be described by the wave equation.

But for this instant, you can say I took it from a book or somewhere, or from the lectures of Professor Walter Lewin. The phase velocity of a pulse on this string is-- of an ideal string is the square root of the tension divided in mass per unit length. OK.

And I was telling you a second ago, when I go back in time at any instant of time when they ring was moving, this string has that shape. And as I move the ring up and generate this wave front, every piece of the string is moving up with a transverse velocity v tr, which I don't know. But the result is that this part of the pulse moves to the right with velocity [INAUDIBLE], phase velocity of the pulse.

Don't get confused. This piece of the string-- and I'm repeating myself-- is not moving to the right. This piece is moving up with this transverse velocity. This piece of the string is moving up. This piece isn't moving yet. But shortly afterwards, it will. And so the result of the movement of every piece of this string up is to produce a pulse which is moving to the right. OK.

So now we want to calculate what is this transverse velocity. Well, from this picture it will be-- this velocity is related to that one by that angle. And so the motion of the ring is the transverse velocity at time t of the ring-- this part of it. OK, we're talking about this-- is equal to v times H over L over 2. That's the tangent of this angle. That's the relation between this velocity and this transverse velocity. Which is equal to 2v H over L.

And this is the motion of this string from the time when this is back here until the time when this is moved for the ring to be at the maximum height H. If we remember that we were told that the shape of the pulse-- and I have to refer you now back to do original picture.

If you look at that original picture, we were told that at time 4L over the square root of T over mu, that pulse was-- well, the back end of it was 2L away from the ring and the forward part was 3L. That pulse is moving with velocity v. And therefore, we can calculate the time when the different parts of that pulse were generated by the ring.

From the time L over v until 3L over 2v, this part of the pulse was generated. And so the transverse velocity during that period of the ring is 2v H over L.

Later on, now the ring is at the top and just the front edge of the pulse has been generated. We now suddenly-- this is an idealized situation. So this ring was going up with uniform velocity. And it suddenly reverses direction. Obviously, that's not the physical situation. That's infinite acceleration. But we can imagine it happens very fast that it's almost infinite acceleration.

And so suddenly, we change from the ring, pulling the string upwards to the ring pulling the string downwards. We are now creating the back edge of that pulse. The situation is symmetric to the previous case.

The only difference is that it's in the opposite direction. And we know that the magnitude of the two are the same because the pulse is symmetric. In one case, we are producing a wave front which is at some angle like that. And then the other like this. But they're symmetric. The angles are the same, but in the opposite direction.

So the transverse velocity here now is therefore minus 2v H over L. And this happens this time interval. By the time we've reached time 2L over V, the complete pulse has left the ring. From then on, the ring must be stationary. Because if it moved, it would produce new distortions in the string.

OK, so we have completely described the motion of the ring. That was the first part of the problem. And I repeat quickly, you have a string which is straight. I'm holding the ring. I'm waiting, waiting. At some time when I feel like it, I start moving it with uniform velocity up. I then reverse instantaneously and move with uniform velocity down.

When I'm moving up, I generate the front part of that pulse. When I'm moving down, I generate the back part. Then, I stop. I'm doing nothing. The ring is doing nothing more. The string is attached to it horizontally. But as a result of that motion of the ring, I've introduced a triangular distortion on the string. That distortion will now propagate forever with the phase velocity square root T over mu.

I don't have to discuss how this pulse is progressing forward. That we've done in general, or Professor Walter Lewin has done in general. He has shown that there is this progressive wave solution to the wave equation, which describes this string.

OK, so we've finally understood the motion of the ring and the subsequent motion of the pulse progressing down the string. End of that.

The next question was the power delivered by the string. Well, clearly, during the time when the ring is stationary, isn't doing no work on anything. My finger doesn't have to do any work on the ring or the string. There's no power generating. But during the time when I was moving the ring up, I had to exert a force. So I was pulling with a force. Therefore, I was doing work. And the rate of doing work is the power that I'm generating.

Similarly when I'm going down, I will have to do work against the force, the tension in the string, and I will be exerting power during that time. How do we calculate that?

Well, I repeat what I said earlier, so I don't have to look at the board. The instantaneous power generated or exerted by a force is force dot the velocity. OK.

Now, we did calculate the force. Sorry, we can calculate the force that I have to generate to move this ring. I'm just redrawing the sketch for you. Here is the ring. What are the forces?

I am repeating, there is the reaction force here. That's this force F which I am exerting. And I've chosen this picture at the time when the string is like this. At other times, the string will be in that direction. But I've sketched this at a time when the string is in this direction. That is the tension in it. It exerts a force. So those are the three forces.

And as I've discussed earlier, this is a massless ring. Therefore, there's no net force. At this instant, this force must be equal and in the opposite direction than the vertical component of this tension T. So it is equal to T times the tangent of this angle. So at that instant of time, the magnitude of this force-- and it's in the up direction-- is plus T H over L over 2, which is 2H T over L.

This corresponds to the period when the string is like this at the ring. And earlier on, I told you that corresponds to this time interval. OK.

Later, I've reached the top. I'm starting to move down. The string now is upwards, but the angle is the same. And so the force that I have to exert is the same magnitude as this but in the opposite direction. Hence, the minus sign. So it's minus T over that, which is minus 2H T over L. The two are, of course, equal.

So I first have to-- it actually makes sense if you imagine a string. If I want to distort it up, I have to pull on that. Later on, I want to pull it in the opposite. I pull it down. Those are the two types. So we know at every instant of time what is the force that I exert and the direction of it.

Now, the power that I generate is the force dot velocity. So I have to take the dot product between this force and the corresponding velocity. All the motion is up and down, so I don't have to-- I'm doing just the components. And so the dot product is just the multiplication of the force times the velocity.

And the only thing I have to watch out for is whether they're in the same direction, in the opposite direction. Notice that when the transverse velocity is positive-- in other words, upwards-- that's the time when I'm pulling upwards. And so the force and the velocity have the same sign. So the power I exert is this 2H T over L times the transverse velocity 2v H over L. OK?

Now, afterwards when we change to the other-- the back edge, the force reverses. It's minus. So that's the transverse velocity. And so that product, once again, is positive. So both on the leading edge of that pulse and the back edge, I do positive amount of work. I do work on the ring and not the ring on me.

And of course, immediately that power is transmitted to the string. So for both the front edge and the back edge of the piles, I have to do work on the string. It's positive. And at every instant of time during my motion, this is the power. It's the product of the force times the transverse velocity. So this now is true for the complete time period during which I am generating the pulse. OK.

I can now plot what is the power that I have to exert as a function of time. So here is the power and in this direction is time. And what we've shown from a time of L over v to a time 2L over v. And by the way, just I'm reminding you, halfway through-- this is when I'm moving it in one direction. This is in the other. But the power is positive in both cases. So this is the power as a function of time. And so what is the total energy that I have to give to the string to produce that pulse?

Well, total energy given is the integral of the power over the time during which I exert that power. So it's the integral of P of t dt over that interval. Well, that's just the area of that pulse. So it's this height times this time. And if I do that, I get-- if I take this and multiply it by that length, I get 4H squared T over L. All right, so we've now understood what the ring does.

We understand how much energy I had to apply to it in order to generate that pulse. OK, finally we were asked about something about the pulse. I've put some energy into it, where did that energy go?

That energy went into the potential energy stored because of the shape of the pulse and the kinetic energy in it. So the essence of the next parts of the question is, let's now, from first principles-- knowing what pulse we have, let's try to calculate how much energy we've stored in the pulse in the form of potential energy. How much energy we've stored in the pulse in the form of kinetic energy. And just to check that we didn't make a mistake, let's see that energy is conserved. We know the total energy we've put into the system. Let's see whether that's equal to the potential energy plus the kinetic energy. Of course, it's going to work out right, so it's more a test I haven't made a mistake. And we are not going to discover non-conservation of energy here.

OK, so now let's try to calculate how much potential energy is stored in the pulse. OK, you can do it many ways. But normally, I like going to first principles. This string normally in equilibrium is straight.

In the moving pulse, it is distorted. It's this shape. In order to distort it to this shape-- and ti doesn't matter, I'm talking at some instant of time. I don't care whether anything is moving here or not. At some instant of time, this has a certain amount of potential energy. It is the energy stored in the system because it is distorted from its equilibrium straight path. I don't care whether any piece here is moving or not. That's nothing to the potential energy. How much energy is there if I have a piece of string shaped like this whether it's part of the pulse or not?

Irrelevant. OK, how do we do it?

I will do a thought experiment, a Gedanken experiment. I will imagine I took a string, nailed these points. They don't move separated by a distance L. In the middle, I get hold of this and from it being straight, I start pulling up. All right? Until I reach a height H.

At that stage, I've generated this shape. The potential energy in that shape must come from the work I did in distorting it. How much work do I do?

Well, I have to exert a force pulling it up. And the integral of that force times the total distance, that will be the total work I have done. So that's what I have to do. All right? So what I'm doing?

I'm going to integrate from this height being 0 to the height being H. I mean, integral is just the addition of bits of work as I pull this up. So integral of the force, which I call little f of y dy. Now, what is the force?

I'm not [INAUDIBLE] where I'm doing this. I'm doing it, pulling it up very slowly at uniform velocity. So I'm not accelerating anything. So the net force on this must be equal and opposite to the force exacted on my fingers by this string.

Well, there is a tension in this string pulling in these directions. The horizontal components of the tensions cancel, but the vertical ones add up. The two vertical components of the string on this side and that are equal and opposite to my force f.

Now, what we said, the assumption here is that this distortion is so small that in the process of me pulling this and changing it, this tension does not change. In reality, it would change slightly. But if this is small enough, it's negligible. So I'm going to make the assumption that this tension does not change as I'm moving this up. So the force I'll be applying will be constant, actually. And so what that is?

Well, it's twice the tension times the component of it, this tangent of the angle in the vertical direction from 0 to H times dy. But as I say, T will be constant throughout this period. But y-- there is a y here, which is not a constant. This goes from 0 to H. And so if I integrate that with respect to y, I'll get y squared over 2 from 0 to H. And so this integral is equal 4T H squared over L over 2. This 2 is from the integral of y dy. It's y squared over 2 taken from those limits gives me that. OK, which is 2H squared T over L.

This is the work I did distorting the string. It must be equal to the energy stored in the string in that process. OK, so we've answered that part. How about the kinetic energy?

There is a pulse which is moving to the right. Meaning that every piece of this string from here to here is actually in motion. Over here, it's stationary. Over here, it's stationary. But to the left of center, every piece in here it's actually moving down. We discussed that earlier. Every piece along here is moving up.

The result is you get the impression of the pulse moving to the right. But the actual motion of the mass is down here and up here. So at any instant of time when you have this pulse, there is kinetic energy of the string in this piece and in that piece. And we have to calculate that.

They are both positive kinetic energy. There is no such thing as negative kinetic energy. So those are moving. And the transverse velocity is down and here up. Kinetic energy is 1/2 mass times velocity squared. They're both positive, OK? So let's now just calculate.

Again, it will be the integral, the sum of the kinetic energy of every piece along here. So if I take a piece of mass dm multiply it by 1/2 times its transverse-- the motion of that mass squared. That's the total velocity of that mass. It's not moving horizontally. We've made that assumption in this idealized case. So this is the kinetic energy of a piece of the mass. And I have to integrate that over all the pieces of m from here to here. There's no kinetic energy there and there.

OK, this is a constant, actually. It's only true because this is a straight line that this pulse consists of two straight lines. If it wasn't, it would not be constant. So this integral is that 1/2 times this transverse velocity, which we calculated earlier the transverse. We've calculated over here the transverse velocity. And you can see it's independent of position of the string. So back to here.

So we know that transverse velocity. And what is the piece of mass?

Well, that will be the density of the string, mass per unit length mu times the little piece of string of length l. This is a trivial integral, so they're all constants here.

By the way, this integral-- maybe I should've said it-- goes over a length from 0 to L. That's the length of the string we're considering. So we get up with that equal to that, which is equal to-- because I could replace v. We know v. v is the square root of T over mu. And so if I replace v by v squared by T over mu, it cancels the mu, but we get a T here. And so the total energy, kinetic energy, is this. And we are essentially whole. Because now we've calculated the total potential energy stored in the pulse. We've calculated the total kinetic energy. So the total energy stored in that policy is the sum of those two, which is of course, 4H squared T over L. So in other words, as this pulse moves along, it has a total energy of this plus that 4H squared T over L. And it will continue forever. Energy is conserved, it will go forever. Where did it come from?

We said it came from the motion of the ring. And I calculated how much energy I did initially. It was 4H squared T over L. And lo and behold, big surprise, it's exactly what's in the pulse. So the work I do is right at the beginning. The work I do is at the time when I generate the pulse. From then on, the energy just progresses forever. That's the end to that problem. We'll now deal with the next problem.

So we now come to the second problem to do with progressive waves. And the problem I've taken is the following. You have two coaxial cables. They're connected at some place. And we are told that on the left cable-- this one here, left cable-- there is a voltage at every location given by v1 cosine omega 1t minus 2 pi over lambda 1t. And there is some kind of reflection to it. They don't tell us much about it.

On the right-hand side, they tell us the voltage wave v2 cosine omega 2t minus 2 pi lambda 2 over t. And furthermore, they tell us that we have to assume that these cables are ideal and lossless. That's one. That the phase velocity on the left one is v1 and that whenever there is a voltage on it at any location, there is a current, which is v1 divided by some constant z1.

So on the left one, the phase velocity v1 and I1 is v1 over z1. And on the right cable, the phase velocity is v2 and the current I2 is v2 over z2. The question is, what is the ratio of omega 2 to omega 1? What's the ratio of lambda 2 to lambda 1? And what's the ratio of v2 to v1?

Now, you may have never seen a coaxial cable. You may have seen a coaxial cable, but you've never discussed it or learned about it. Certainly, Professor Walter Lewin did not cover this. Why did I do this problem?

Precisely to show that by learning how to solve problems or understanding progressive waves on one system like this string, we can do transfer the whole knowledge we have to understanding how to analyze, in fact, a much more interesting situation. A situation you'll come across very often in life, progressive waves down coaxial cables.

Now, from the way the problem is worded, we can in fact conclude that we should be able to solve this problem from our knowledge of what happens on taut string.

We are told that on this one, there is a voltage, which has this form. If you look at this form, this is a function of some constants times time minus-- I'm sorry, this should be an x and an x. Let me immediately correct this, x here and x here. So it's a function of some constant times t minus a constant times x. And we know that such a function represents a progressive wave.

If you plot this at different times and different locations, what you will see is a sinusoidal function which is moving to the right with some, what we call, phase velocity. Which in the problem, they tell us what it is. It's v1. So from this form, we immediately realize that the voltage on this cable is a progressive wave, which is going to the right. Therefore, this system, the cable, must have a wave equation for it to have a wave as a solution-- a progressive wave as a solution.

OK, next. If we know omega 1, what is omega 2? If we know omega 1, what is lambda 1? What is lambda 2? And if we know v1, what is v2?

In other words, we immediately realize that these two-- the angular frequencies here or the wavelengths of the progressive waves-- do not necessarily have the same phase velocity. And in fact, they tell us that it is different. So the propagation down these two cables will be different. And so these cables must be made structurally in a different way. They are different cables, but they're just connected together. OK.

And of course, electrically-- that's the other thing I should-- the outside has to be connected. All right, how do we proceed?

So as I started saying, from the wording of the problem we immediately conclude that a coaxial cable, in fact, is the continuum limit of some row of oscillators. Is that surprising?

Well, if you stop and think and magnified the coaxial cable, in essence what you see is two parallel wires. They have a capacitance with each other. The two conductors have the central wire and the sheathing around that. Each have self-inductance. And so schematically, it looks something like this. You have a row of inductances, a row of capacitances connected like this. So this is a schematic diagram of a coaxial cable. What is this?

Nothing other than each piece here is like a simple harmonic oscillator we've solved many times. It's simply an inductance in the capacitance. And they are furthermore, identical harmonic oscillators only connected to their neighbors. And so this, from a point of view of the response of this system, is identical to a taut string, where at each point in the s we had a harmonic oscillator which was connected only to its neighbors.

So the equation of motion for this system-- well, the variable was the voltage, the potential difference between the central conductors and the outside conductor, must satisfy a wave equation like this where v is the phase velocity of the propagation of v down the cable. And we are told on the left-hand side is v1, on the right-hand side is v2.

Furthermore, in the problem they told us that if there is a voltage propagating down this cable, there will also be a current propagating. And they tell us that the ratio of the voltage to the current is a constant. And it's different for the two cables. In fact, let me sort of digress and tell you it is the two quantities, the phase velocity and this constant z, which is called the characteristic impedance, which characterizes any coaxial cable. Or in fact, any system consisting of two parallel conductors. So a wire, typical two-conductor wire, would be a coaxial cable in that sense.

Since I is proportional to the voltage, if this satisfies the wave equation so must the current. OK. Furthermore, what do they tell us?

They tell us that if you look at the first cable, the one on the left, we see the progressive wave going to the right. But also, there may be a reflected wave they tell us. On the right-hand side, they tell us there is only a propagating wave to the right. From that, I can immediately conclude something about the boundary conditions of this system.

Take the right-hand side. There's a wave going to the right, but nothing to the left. Therefore, there must be no reflection of the progressive wave at the far end of the right cable. How about the left cable?

Well, the pulse is coming in. There is a junction between two cables which are not the same. And whenever you have such a situation where you have a continuous system of oscillators coming to another continuous system, like two wires of different density for example, or different tensions, or something of that kind, at the junction you are going to get a reflection.

So where you're going to get a reflection at that junction, that reflected wave will go all the way to the left, come to the far end of that cable. And normally it would be reflected there unless one works hard to prevent that from happening.

And here, they're telling us nothing is-- there's no reflection. There isn't any further reflection from that side. There is just the original pulse coming from left to the right.

OK, so now, how do we answer the questions they do? And so now, I will go from what we've learned about the physical system to a mathematical description of it. And at this stage, you probably won't be able to almost tell the difference with I'm talking about-- two strings connected together or I'm talking a coaxial cable. The behavior, mathematical behavior, will be the same. The mathematical description will be the same.

I should digress for a second. If you're curious what happens here, I was talking about two pulses going down, the voltage pulse and the current. What happens on the string?

Well, in a string there are also many pulses. We normally don't talk about all of them. For example, if you have a propagating pulse of displacement on a string, then there will be also a pulse going down corresponding to the transverse velocity of the string at every point. So there's already two. There's another one.

As the pulse on a string goes along, the left part of the string acts a force on the right part of the string to generate that pulse. And that also propagates. So even on a simple thing like a string, there will be simultaneously three progressive waves going down the string-- the displacement, the transverse velocity, and the force.

Here, we are talking about two-- the current and the voltage. All right, so I've translated what I've discussed before. I'll call x equals 0 the junction between the two cables.

On the left and on the vertical axis here, I'm going to either be plotting the voltage across the cable or the current flowing down the cable. I'm just too lazy to draw two separate plots. So what we are told, there is the voltage, which is given by this expression. That's a progressive wave, a sinusoidal progressive wave moving to the right.

There is a reflected one, which is given by some amplitude, v reflected times cosine. And this will now have a plus sign because it's now propagating to the left.

Here, the minus sign tells you it's going to the right. The plus sign going to the left.

Now, we know from-- for example, our studies of coupled oscillators, that in a steady state situation, every oscillator is moving with the same frequency and phase. So if I extend that to the infinite case, I know that the frequency of this wave, which was only one frequency. So the progressive wave going this way has only one frequency, omega 1. If it drives the oscillators as it goes along, it will drive them to oscillate at the same frequency as its frequency. Another way to say it, in this problem there's only one frequency. And so the reflected wave will have the same frequency as the incoming one.

Similarly, the transmitted wave will have the same frequency. That's why I wrote this to be equal to that. We can actually formally show that that has to be the case by considering what happens at the boundary. You could never have the two separate halves moving with different frequencies and the string not be broken. Or in this case, the voltage being not the same on both sides of the string, on both sides of the boundary of the string.

OK, so everything we learned over there we can write as this sinusoidal progressive wave to the right with amplitude v1, the reflected wave with amplitude v reflected. And there will be a transmitted wave of some magnitude v2. And a cosine omega 2t minus 2 pi lambda to the x.

I went ahead of myself to explain that this omega 1 has to be the same as that. In a second using the same argument, I'll show that this has to be equal to that. What else do we know about the situation?

Well, we know that at the boundary, the two y's are connected. Therefore, there will be the same potential difference between the central conductor and the outside conductor just to the left of the boundary and to the right. So at x equals 0 for all times, the voltage VL will be equal to the potential difference on the right-hand side. So that's one boundary condition.

The other one, we know that charges are conserved. And therefore, if you have junction of two wires, the total current going into that junction or coming away from it must be 0 or you would be creating charges. So the sum of all the currents coming in or out from the junction for all times and x equals 0 must be 0. So this is now all in mathematics. You don't need to know this, but this is anything to do with voltages, currents, et cetera. This is the mathematical description of the situation. And let's try to now answer the questions that were posed.

OK, the first question was, what is the ratio of omega 2 to omega 1? And I've answered it already. The only frequency in this problem is omega 1. That's what comes in. It'll drive any oscillator anywhere at that frequency under the steady-state conditions, which we have here. So omega 2 must equal to omega 1 or that ratio is equal to 1. First problem.

The next question we were asked, what is the ratio of the wavelengths?

Now, we know that for any harmonic wave, the product of the wavelengths times the frequency is the phase velocity. You can trivially prove it for yourself. Just take that formula for the progressive wave, harmonic progressive wave, and calculate what the frequency is. And if you have difficulty, draw the wave as a function of position at 2 times and see how far that picture has moved. And you'll find that lambda times f is equal to v. OK. This is true to the left of the boundary and on the right-hand side.

So I've rewritten this lambda times omega over 2 pi equals v. Therefore, if you have two strings of different phase velocity, for the first one this equation looks like that. For the second one, it's the same but 1's replaced by 2.

From these, I immediately divide one by the other, that lambda 2 over lambda 1 is equal to v2 over v1. They are no longer the same. The frequencies are the same, but the wavelengths are different. It depends on the properties of the two cables. So we get that the wavelength on the left-hand side, as I say, will be a different wavelength to the one on the right. And the ratio between them will be that of the phase velocity on both sides.

All right, next. The third part of the thing was, what's the ratio of the voltages? In other words, what is the ratio of v2 to v1? v1, I'm reminding you, is the amplitude of the voltage wave that comes from the left cable to the junction, and then continues out with a different amplitude. What is that amplitude?

To answer that, we make use of the boundary conditions, which we discussed earlier. These boundary conditions immediately tell us that how?

Well, the first boundary condition tells us that the voltage just to the left of x equals 0 must equal to just to the right to it. Now on the left-hand side, you have two waves-- one coming in, one coming out, the reflected one.

If you look at the equation for VL at x equals 0, for all times you'll find it's v1 times cosine omega 1t plus v reflected cosine omega 1t. On the right-hand side, you see that it is v2 times-- again, with x equals 0 cosine omega 2t. But omega 2t is equal to omega 1. And so at all times, the cosines cancel and you end up that v1 plus v reflected has to be v2. At all times that will be true at x equals 0. So you have one equation.

The other boundary condition gives us the other one. We know that if there is a wave coming in with amplitude v1, the current that comes in, it told us in the formulation of the problem, is that divided by z1, this characteristic impedance.

Similarly, for the reflected wave. But we have to watch it. There is a sign difference. Because why?

If the wave is coming from the left to the right, if we call that a positive current. After reflection if it goes the other way, it will be a negative. We'll be subtracting from that junction. So that's why there is this minus. And then, this is the one-- this is the total current coming into the junction. And that's got to be equal to the total current coming out on the other side, v2 over z2.

From this, I just multiply by z1, so I get this equation. And these are two trivial algebraic equations. If I add them, I get 2 v1, et cetera. From which, I get that this is the case. And again, I notice I forget v1. OK.

And so we get that the v2 over v1 is that quantity which was given. And we've solved the three parts of the problem. That's the end.

And the thing that I would just like you to keep in mind, that if you come across some kind of a problem to do with progressive waves, et cetera. If you've never seen the system, it doesn't mean you don't know how to solve it. Stop, think, see by analogy what it corresponds to a system which you have understood.

The corollary is, suppose you are having trouble with progressive waves on strings. But you're an electrical engineer and you feel very comfortable about waves on cables. You can use the study of waves on cables, reflection of pulses, reflection coefficients, transmission coefficients, et cetera. That which you've understood in the electrical system, you can translate it to what happens if you have mechanical systems connected together. Thank you.