

# Compensation Example 10

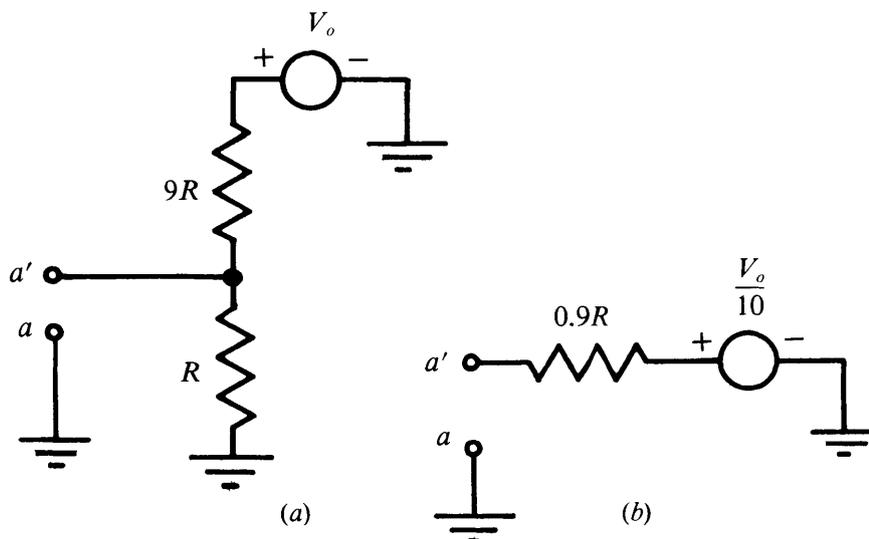
**Note:** All references to Figures and Equations whose numbers are *not* preceded by an “S” refer to the textbook.

- (a) The solution of this problem is outlined in the discussion on p. 183 of the textbook. Associated with this discussion are the circuit and block diagrams of Figure 5.13, which are applicable to this problem. In the textbook, the block diagram of Figure 5.13*b* is presented without derivation. Here, we fill in the details of this derivation.

## Solution 10.1 (P5.8)

When faced with deriving a block diagram for the circuit of Figure 5.13*a*, one may proceed by writing network equations in terms of  $V_i$  and  $V_o$ . Then, after some algebraic manipulation, these equations are used to draw the block diagram. The disadvantage of this approach is that it is algebra intensive and tends to obscure physical insight. What is perhaps a more illuminating approach is detailed below.

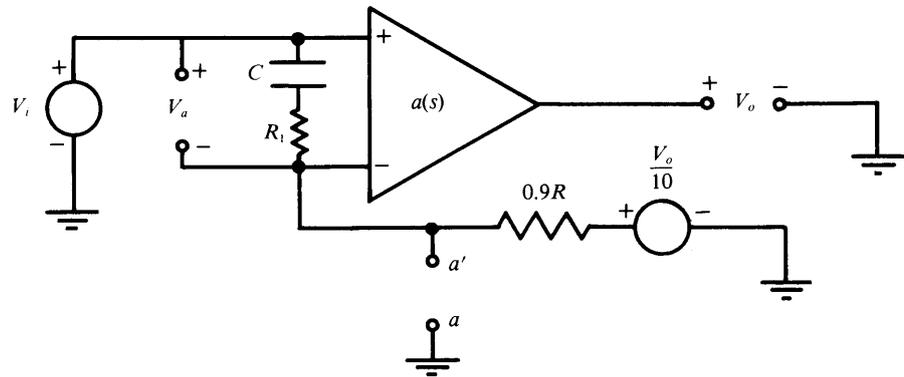
We start by constructing the Thevenin equivalent circuit for the  $R$ - $9R$  feedback network as shown in Figure S10.1*a* and *b*.



**Figure S10.1** Analysis of Problem 10.1 (P5.8) through the use of a Thevenin equivalent circuit. (a)  $R$ - $9R$  feedback network. (b) Thevenin equivalent as seen at terminal pair  $aa'$ .

With this manipulation, the circuit diagram is as shown in Figure S10.2,

**Figure S10.2** Modified circuit diagram for Problem 10.1 (P5.8).



where  $V_a$  is the differential input voltage. That is,  $V_o(s) = a(s)V_a(s)$ . Then, by superposition and the voltage divider relationship,

$$\begin{aligned} V_a(s) &= \frac{R_1 + \frac{1}{Cs}}{R_1 + \frac{1}{Cs} + 0.9R} \left[ V_i(s) - \frac{V_o(s)}{10} \right] \\ &= \frac{\tau s + 1}{\alpha \tau s + 1} \left[ V_i(s) - \frac{V_o(s)}{10} \right] \end{aligned} \quad (\text{S10.1})$$

where  $\alpha = \frac{R_1 + 0.9R}{R_1}$  and  $\tau = R_1 C$  as defined on p. 181 of the textbook. The block diagram of Figure 5.13c follows directly from Equation S10.1 and our definition of  $V_a$ .

The negative of the loop transmission for this system is then as given by Equation 5.14.

$$a''(s)f''(s) = 0.1 \frac{\tau s + 1}{\alpha \tau s + 1} a(s) \quad (\text{S10.2})$$

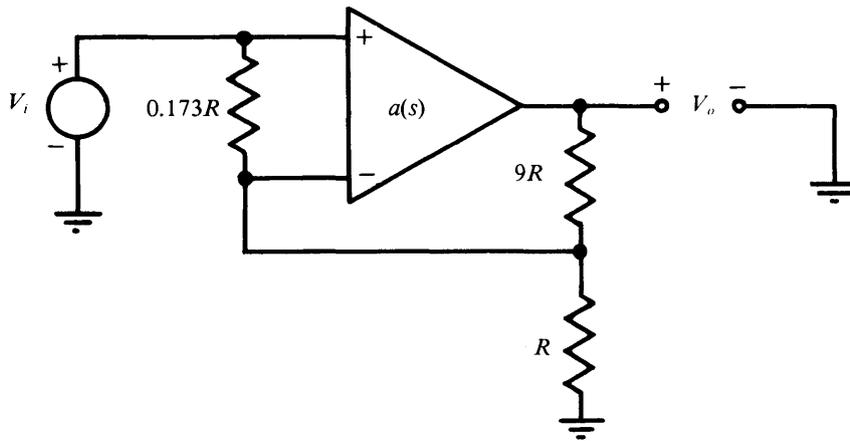
If we short out the capacitor (i.e., let  $C \rightarrow \infty$ , and thus  $\tau \rightarrow \infty$ ),

$$a'''(s)f'''(s) = \frac{0.1}{\alpha} a(s) \quad (\text{S10.3})$$

To lower the loop transmission by 6.2 at all frequencies then requires that

$$\alpha = 6.2 = \frac{R_1 + 0.9R}{R_1} \quad (\text{S10.4})$$

This is solved by  $R_1 = 0.173R$ , as suggested in the textbook. Thus the circuit with  $a_{\omega}f_{\omega}$  reduced by a factor of 6.2 is as shown in Figure S10.3.



**Figure S10.3** Circuit with  $a_{\omega}f_{\omega}$  reduced by a factor of 6.2.

To lower the lowest-frequency loop-transmission pole by a factor of 6.2, we use the lag-network zero to cancel the pole of  $a(s)$  at  $s = -1$ . Then, the lag network  $\alpha$  is set at 6.2, so that the lag-network pole is at  $s = \frac{-1}{6.2}$ . That is, given that the negative of the loop transmission for Figure 5.13c is

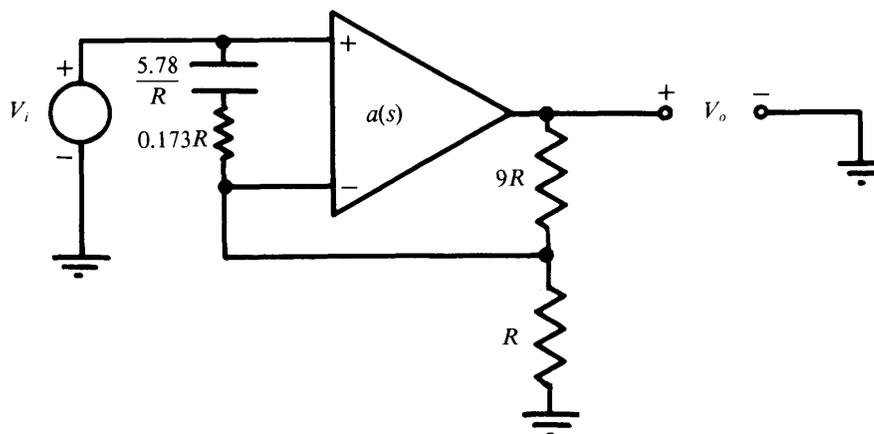
$$a''(s)f''(s) = 0.1 \frac{\tau s + 1}{\alpha \tau s + 1} \frac{5 \times 10^5}{(s + 1)(10^{-4}s + 1)(10^{-5}s + 1)} \quad (\text{S10.5a})$$

if we let  $\tau = 1$  and  $\alpha = 6.2$  this becomes

$$a''(s)f''(s) = 0.1 \frac{5 \times 10^5}{(6.2s + 1)(10^{-4}s + 1)(10^{-5}s + 1)} \quad (\text{S10.5b})$$

and the lowest-frequency pole has been effectively moved down by a factor of 6.2. From earlier results, for  $\alpha = 6.2$ ,  $R_1 = 0.173R$ . Then because  $\tau = R_1C$ , for  $\tau = 1$  we have  $R_1C = 1$ , which is solved by  $C = \frac{5.78}{R}$ . Thus, the circuit that implements this pole lowering is as shown in Figure S10.4.

**Figure S10.4** Circuit with lowest-frequency pole moved down by a factor of 6.2.



- (b) The loop-transmission magnitude Bode plots for the two compensation schemes are shown in the textbook in Figure 5.16. The corresponding angle curves are not difficult to sketch, and thus are not included here.
- (c) The phase curves for reduced  $a_o f_o$ , and the lowered first pole differ only for frequencies below about 10 rad/sec, due to the difference in the low-frequency pole location. They are identical in the vicinity of crossover. At the crossover frequency of  $6.7 \times 10^3$  rad/sec, the phase for both compensation schemes is  $-128^\circ$ . Thus, the phase margin is  $52^\circ$ , which is better than the lag compensation by about  $5^\circ$ . This improvement is due to the fact that the lag network has a residual phase of  $-5^\circ$  at the crossover frequency. By moving the lag network to a lower frequency, the phase margin may be slightly improved (up to  $5^\circ$ ) at the expense of midband desensitivity.

### Solution 10.2 (P5.12)

As a matter of cultural interest, the factor

$$\frac{(s^2/12) - (s/2) + 1}{(s^2/12) + (s/2) + 1} \quad (\text{S10.6})$$

is the second-order Pade approximation to a 1-second time delay. See p. 530 of the textbook for further discussion of this topic. This factor has a pole-zero pattern as shown in Figure 12.26 and a phase characteristic as shown in Figure 12.27.

To compensate this system, a lead network alone is not useful, because its transfer function magnitude increases with frequency, and the Pade approximation magnitude is constant. This is discussed in greater detail in Section 5.2.6 of the textbook. To force the loop to crossover, we must introduce a compensating element that provides attenuation with increasing frequency. At the same time, the compensating element should introduce a minimum of negative phase shift. A single pole satisfies these criteria, so we will compensate the loop by using a dominant pole.

Let's design the loop compensation for  $45^\circ$  of phase margin. Near crossover, the dominant pole will contribute a phase of  $-90^\circ$  because it is to be located well below crossover. Thus, to have  $45^\circ$  of phase margin, crossover should be set at the frequency where the Pade approximation has a phase of  $-45^\circ$ . From Figure 12.27, then, crossover must occur at a frequency somewhat below 1 rad/sec. The exact expression for the phase of the Pade approximation is given in Equation 12.65. However, as explained on p. 531 of the textbook, for  $\omega$  less than 2 rad/sec, the phase is well approximated by an angle of  $-57.3^\circ\omega$ . Using this approximation, the frequency at which the phase is  $-45^\circ$  is  $\omega_c = \frac{45}{57.3} = 0.79$  rad/sec. Because the Pade approximation has unity magnitude, then, the compensating transfer function must pass through unity magnitude at  $\omega_c = 0.79$  rad/sec. For a dominant pole located at  $\omega_p$ , where  $\omega_p \ll \omega_c$ , and with a d-c gain of  $a_o$ , the compensating transfer function is

$$H(j\omega) = \frac{a_o}{j\frac{\omega}{\omega_p} + 1} \quad (\text{S10.7})$$

To have unity magnitude at  $\omega_c$  then requires that  $a_o = \frac{\omega_c}{\omega_p} = \frac{0.79}{\omega_p}$ . Any compensation satisfying the above conditions will yield a phase margin of  $45^\circ$ .

To achieve maximum desensitivity,  $a_o$  should be made as large as possible. In the limit, let  $\omega_p \rightarrow 0$  and  $a_o \rightarrow \infty$ , while maintaining the relationship  $a_o = \frac{0.79}{\omega_p}$ . The result is

$$H(j\omega) = \frac{0.79}{j\omega} \quad (\text{S10.8})$$

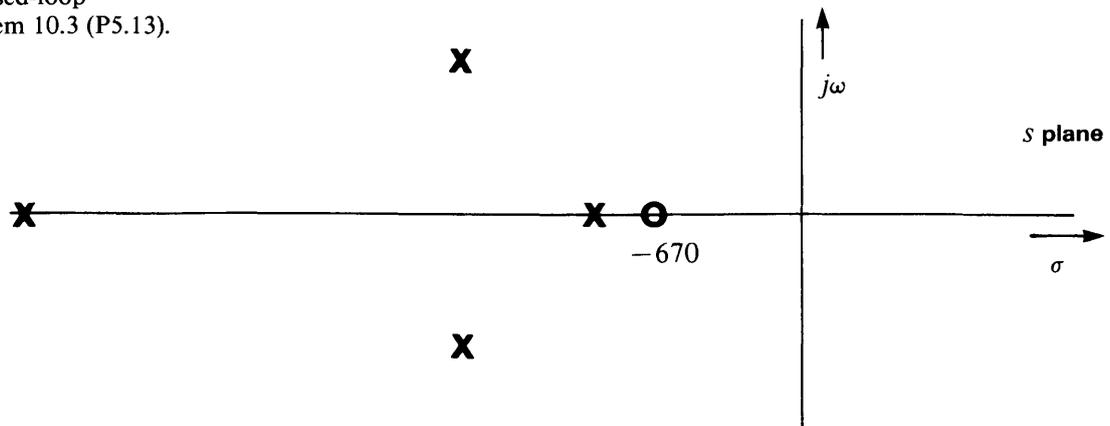
That is, the compensation is an integrator, which has infinite gain at d-c, and thus infinite desensitivity at d-c. As expected, this has unity magnitude at  $\omega_c = 0.79$  rad/sec, and an angle of  $-90^\circ$  at all frequencies, thus the loop has  $45^\circ$  of phase margin.

More complex schemes, such as using a double integration ( $1/s^2$ ) with lead compensation, are also possible. This would offer higher desensitivity at midband frequencies. However, when implemented with real hardware, such a scheme is more sensitive to component variations than the single-pole compensation, which is quite robust. Therefore, except under special circumstances, the single-pole compensation is the most reasonable solution.

### Solution 10.3 (P5.13)

For the lag-compensated system described by Equation 5.15, the root locus is as shown in Figure 5.15c. For the given compensation, and value of  $a_o f_o$ , the closed-loop singularities are located approximately as shown in Figure S10.5.

**Figure S10.5** Closed-loop singularities for Problem 10.3 (P5.13).

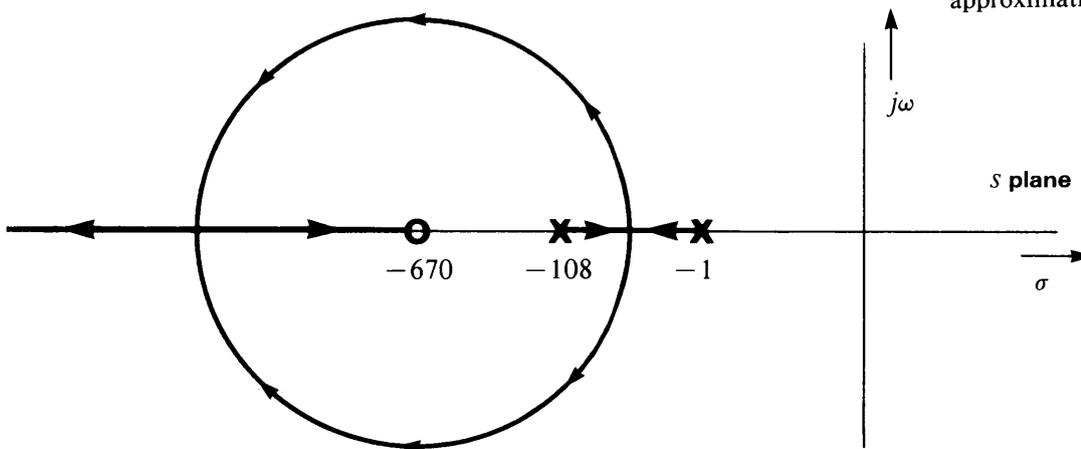


The pole-zero doublet near  $s = -670 \text{ sec}^{-1}$  is responsible for the long-time constant tail associated with the lag-compensated system.

As the lag-compensated system is fourth order, an exact solution of the closed-loop pole locations will require solving for the roots of the fourth-order equation  $1 + a''(s)f''(s) = 0$ . This is feasible with machine computation. However, as suggested in the problem assignment, a simplifying approximation is possible. That is, we ignore the poles at  $s = -10^4$  and  $s = -10^5$ , and assume that  $a(s)$  is given by

$$a(s) = \frac{5 \times 10^5(1.5 \times 10^{-3}s + 1)}{(s + 1)(9.3 \times 10^{-3}s + 1)} \quad (\text{S10.9})$$

The root locus for this simplified system is sketched in Figure S10.6.



**Figure S10.6** Root locus of approximating second-order system.

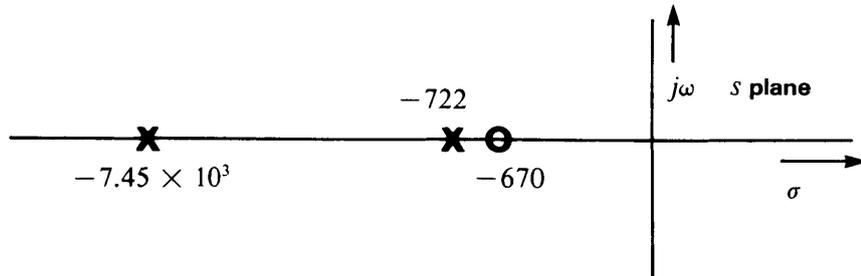
Note that this locus is very similar to a section of the root locus of Figure 5.15c in the textbook. Further, since the closed-loop complex pair of the full fourth-order representation is located a decade up in frequency from the zero at  $s = -670$ , it is reasonable to expect that the complex pair has only a slight influence on the locus in the vicinity of the zero.

For the above reasons, we expect that the second-order approximation of Equation S10.9 will be acceptably accurate. Using this approximation, the closed-loop transfer function is

$$\begin{aligned}
 A(s) &= \frac{a(s)}{1 + a(s)f(s)} \\
 &= \frac{5 \times 10^5(1.5 \times 10^{-3}s + 1)}{(s + 1)(9.3 \times 10^{-3}s + 1) + 5 \times 10^4(1.5 \times 10^{-3}s + 1)} \\
 &= 10 \frac{1.5 \times 10^{-3}s + 1}{1.86 \times 10^{-7}s^2 + 1.52 \times 10^{-3}s + 1} \quad (\text{S10.10})
 \end{aligned}$$

The poles of  $A(s)$  are at  $s = -722$  and  $s = -7.45 \times 10^3$ . That is, the closed-loop pole configuration is as sketched in Figure S10.7.

**Figure S10.7** Closed-loop singularities for the approximate system.



An exact analysis of the fourth-order case indicates that the pole of the pole-zero doublet is actually at  $s = -717$ . The close agreement with the approximate value of  $s = -722$  verifies the approximation. The settling time will be dominated by the pole-zero doublet. The step response of the doublet alone is given by

$$v(t) = 1 + 0.08e^{-722t} \quad (\text{S10.11})$$

This will settle to within 1% when

$$v(t_o) = 1.01 = 1 + 0.08e^{-722t_o} \quad (\text{S10.12})$$

or

$$0.01 = 0.08e^{-722t_o} \quad (\text{S10.13})$$

which is solved by  $t_o = 2.9$  msec. The solution is arrived at by recognizing that the initial value theorem requires that the step response at  $t \rightarrow 0+$  is  $\frac{722}{670} = 1.08$ . Further, as  $t \rightarrow \infty$  the final value theorem requires the step response to approach unity. The time constant of the exponential connecting these two values is  $\frac{1}{722} = 1.39$  msec.

For the first-order system, with a crossover at  $\omega = 6.7 \times 10^3$  rad/sec, the step response will be given by

$$v(t) = 1 - e^{-6.7 \times 10^3 t} \quad (\text{S10.14})$$

This will settle to 1% when  $e^{-6.7 \times 10^3 t_1} = 0.01$ . This is solved by  $t_1 = 0.69$  msec. That is, the first-order system is about a factor of 4 faster than the lag-compensated system. In many instances, such as analog-to-digital conversion, settling time is quite important. The lesson of this problem is that a pole-zero doublet can have a very significant effect on an amplifier settling time.

The principal objective of this problem is to provide the student with the experience of applying analytical results in the laboratory. As the laboratory portion of the problem is essential, and the hands-on experience more important than the actual answers, extensive solutions are not provided here. Furthermore, as with most design problems, there is no one correct answer, so it is expected that each student's solution will differ in some respects from other students' solutions.

### Solution 10.4 (P5.15)

Included here are some general guidelines and suggestions for approaching the problem, as well as answers to some of the questions posed in the problem statement. Appropriate topologies for each of the three compensation techniques are also given. If the analytical and experimental portions of the problem are properly solved, the student should gain confidence that the analytical approaches we have studied thus far are useful design techniques and can be applied with accuracy to real circuit problems.

The problem statement suggests the use of a resistive attenuation at the amplifier input. A possible topology is shown in Figure S10.8.

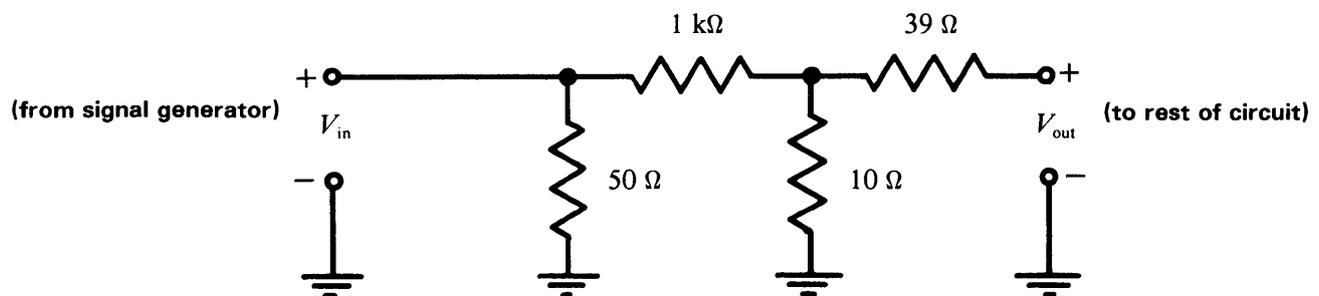


Figure S10.8 Attenuator.

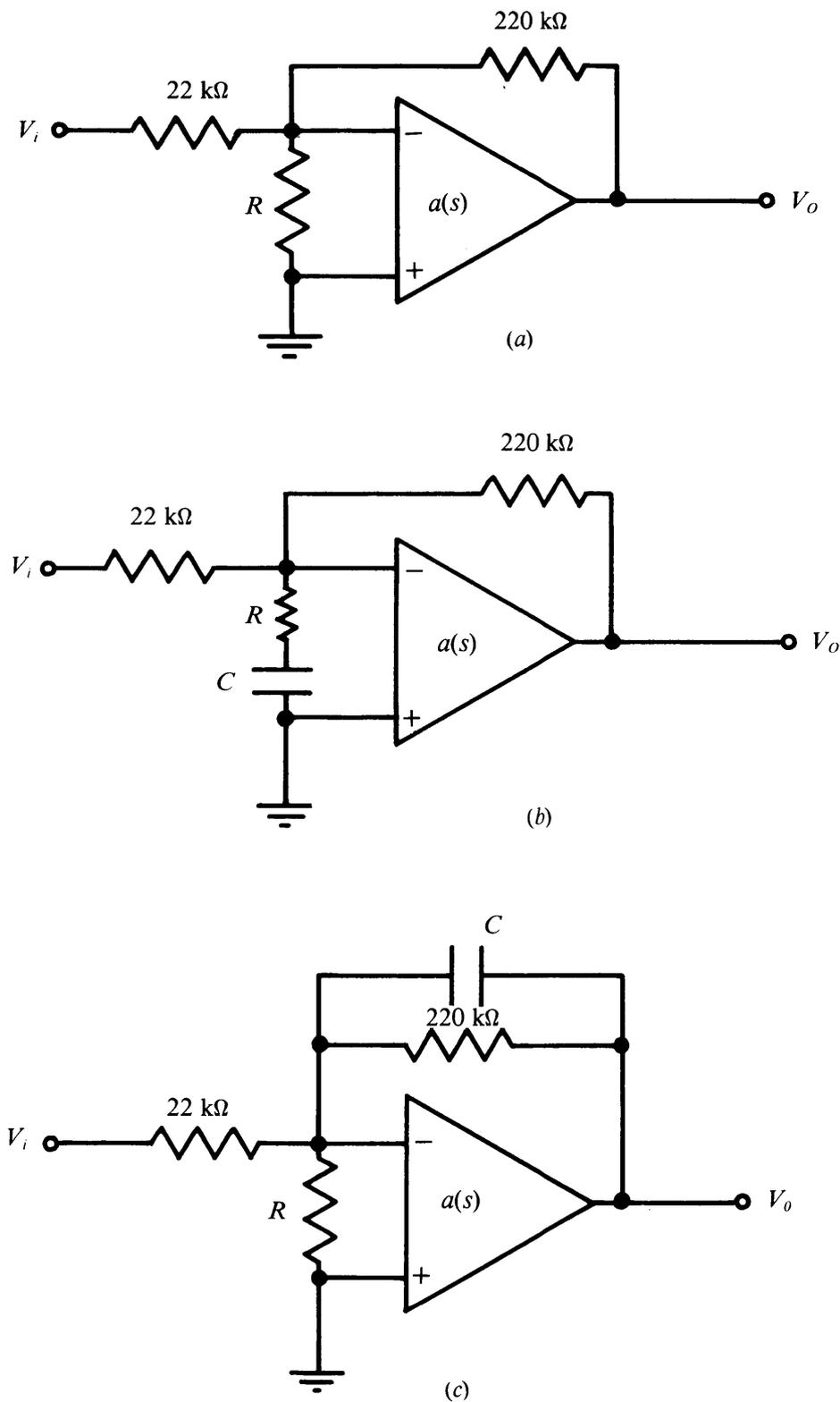
This attenuator provides a 100:1 attenuation ratio, with  $50\Omega$  input and output resistances. Thus, it is also useful for high-frequency applications, where a  $50\Omega$  impedance must be maintained. For the purposes of this lab, because only low frequencies are of interest, both the  $50\Omega$  and  $39\Omega$  resistor may be omitted.

The capacitor  $C$  adjusts the location of the low-frequency pole associated with the LM301A. Therefore, because this pole is at a frequency much lower than  $10^3$  rad/sec,  $C$  may be used to adjust the loop-transmission crossover frequency without significantly affecting the phase at crossover. By adjusting  $C$  to bring the configuration to the verge of instability, the crossover frequency is set near the point where the negative phase shift of the loop transmission is slightly less than  $180^\circ$ . Stability can easily be ascertained by examining the amplifier step response. Adjust  $C$  to create a lightly damped step response. The longer the step response rings, the closer the poles are to the  $j\omega$  axis. A ring time of 200 to 500 msec is sufficient. Note that smaller values of  $C$  will result in longer ring times. Also make certain that the circuit is stable, that is, the ringing step response must decay with time.

In order to achieve good numerical agreement between theoretical and experimental results, use resistors that match the values indicated in Figures 5.28, 5.29, and 5.30 within  $\pm 1\%$ , and capacitors that match the indicated values within  $\pm 5\%$ . This capacitor tolerance does not apply to the two  $0.01\ \mu\text{F}$  decoupling capacitors. Do make sure to include these decoupling capacitors located close to the LM301A in order to avoid instabilities caused by power-supply lead inductance. In general, a bit of care in circuit construction will pay off in reliable circuit operation.

For the purposes of analysis, use the approximate transfer function for  $a(s)$  as given in the problem statement. Because the pole at  $s = -1$  is providing  $-90^\circ$  of phase shift in the vicinity of crossover, standardization by adjusting  $C$  places crossover near the point where the poles at  $10^3$  and  $10^4$  rad/sec are providing an additional  $-90^\circ$  of phase shift. This occurs at  $\omega = 3.16 \times 10^3$  rad/sec.

The Bode plot for the inverting gain-of-ten configuration of Figure 5.30 should indicate a crossover frequency of  $2 \times 10^3$  rad/sec, with a phase margin of  $15.3^\circ$ , and a gain margin of 2.4. With this phase margin, we expect  $M_p = \frac{1}{\sin 15.3^\circ} = 3.8$ .



**Figure S10.9** Suggested topologies for the three compensation schemes. (a) Reduced  $a_o f_o$ . (b) Lag. (c) Lead with reduced  $a_o f_o$ .

The three types of compensation have been covered in detail in this chapter, and thus are not solved for explicitly here. During the design and analysis of each compensation scheme, it is useful to have some form of computation that can provide results of transfer-function magnitude and phase versus frequency. A short program written on a computer or hand-held calculator will suffice. Appropriate topologies for the three forms of compensation are shown in Figure S10.9. Others are certainly possible.

MIT OpenCourseWare  
<http://ocw.mit.edu>

RES.6-010 Electronic Feedback Systems  
Spring 2013

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.