Stability 4

Note: All references to Figures and Equations whose numbers are *not* preceded by an "S" refer to the textbook.

Solution 4.1 (P4.1)

Because the coefficients of the polynomial

$$s^5 + s^4 + 3s^3 + 4s^2 + s + 2 \qquad \textbf{(S4.1)}$$

are all present and of the same sign, the necessary condition for all roots to have negative real parts is satisfied. The Routh array is:

$$\begin{array}{ccc}
1 & 3 & 1 \\
1 & 4 & 2 \\
\dfrac{(1\times 3)-(1\times 4)}{1} = -1 & \dfrac{(1\times 1)-(1\times 2)}{1} = -1 & 0 \\
\dfrac{(-1\times 4)-(1\times -1)}{-1} = 3 & \dfrac{(-1\times 2)-(1\times 0)}{-1} = 2 & 0 \\
\dfrac{(3\times -1)-(-1\times 2)}{3} = -\dfrac{1}{3} & 0 & 0 \\
\dfrac{(-\frac{1}{3}\times 2)-(3\times 0)}{-\frac{1}{3}} = 2 & 0 & 0
\end{array} \qquad \textbf{(S4.2)}$$

Redrawing the array for clarity, we have

$$\begin{array}{ccc}
1 & 3 & 1 \\
1 & 4 & 2 \\
-1 & -1 & 0 \\
3 & 2 & 0 \\
-\frac{1}{3} & 0 & 0 \\
2 & 0 & 0
\end{array} \qquad \textbf{(S4.3)}$$

There are four sign changes in the first column, and thus four right-half-plane zeros of the polynomial.

Solution 4.2 (P4.2)

Following the development on pp. 116 and 117 of the textbook, we write the characteristic equation as 1 minus the loop transmission. That is,

$$\text{Characteristic equation} = 1 - L(s) = 1 + \frac{a_o}{(\tau s + 1)^4} \tag{S4.4}$$

After clearing fractions, the characteristic polynomial is

$$\begin{aligned} P(s) &= (\tau s + 1)^4 + a_o \\ &= \tau^4 s^4 + 4\tau^3 s^3 + 6\tau^2 s^2 + 4\tau s + 1 + a_o \end{aligned} \tag{S4.5}$$

The Routh array associated with this polynomial is

$$\begin{array}{ccc} \tau^4 & 6\tau^2 & 1 + a_o \\ 4\tau^3 & 4\tau & 0 \\ 5\tau^2 & 1 + a_o & 0 \\ 4\tau\left(\dfrac{4 - a_o}{5}\right) & 0 & 0 \\ 1 + a_o & 0 & 0 \end{array} \tag{S4.6}$$

The reader should check the algebra used to derive this array.

Assuming τ is positive, roots with positive real parts occur for $a_o < -1$ (one right-half-plane zero), and for $a_o > 4$ (two right-half-plane zeros). Recall that the problem asks for the a_o that results in a pair of complex roots on the imaginary axis. Only the value $a_o = 4$ satisfies this condition. With $a_o = 4$, the entire fourth row is zero, and we can solve for the pole locations by using the auxiliary equation. Using the coefficients of the third row, the auxiliary equation is

$$5\tau^2 s^2 + 5 = 0 \tag{S4.7}$$

The equation has solutions at

$$s = \pm \frac{j}{\tau} \tag{S4.8}$$

indicating that with $a_o = 4$, the system will oscillate at $\frac{1}{\tau}$ rad/sec.

Now that we have found two of the poles for $a_o = 4$, they can be factored out to find the two other roots of the characteristic equation. That is, we can factor $P(s)$ as the term $\tau^2 s^2 + 1$ multiplied by a quadratic. This quadratic can be found by applying synthetic division to $P(s)$ as shown below.

$$
\begin{array}{r}
\tau^2 s^2 + 4\tau s + 5 \\
\tau^2 s^2 + 1 \,\overline{)\, \tau^4 s^4 + 4\tau^3 s^3 + 6\tau^2 s^2 + 4\tau s + 5} \\
\underline{\tau^4 s^4 + \tau^2 s^2 \qquad\qquad\qquad\qquad} \\
4\tau^3 s^3 + 5\tau^2 s^2 + 4\tau s + 5 \\
\underline{4\tau^3 s^3 + 4\tau s \qquad\qquad} \\
5\tau^2 s^2 + 5 \\
\underline{5\tau^2 s^2 + 5} \\
0
\end{array}
\tag{S4.9}
$$

Then the two remaining poles are solutions of

$$\tau^2 s^2 + 4\tau s + 5 = 0 \tag{S4.10}$$

which is solved by the quadratic formula to give

$$s = \frac{-2 + j}{\tau} \tag{S4.11a}$$

and

$$s = \frac{-2 - j}{\tau} \tag{S4.11b}$$

as the two other closed-loop pole locations when $a_o = 4$.

RES.6-010 Electronic Feedback Systems
Spring 2013