CHAPTER IV

STABILITY

4.1 THE STABILITY PROBLEM

The discussion of feedback systems presented up to this point has tacitly assumed that the systems under study were *stable*. A stable system is defined in general as one which produces a bounded output in response to any bounded input. Thus stability implies that

$$\int_{-\infty}^{\infty} |v_O(t)|dt \leq M < \infty \tag{4.1}$$

for *any* input such that

$$\int_{-\infty}^{\infty} |v_I(t)|dt \leq N < \infty \tag{4.2}$$

If we limit our consideration to linear systems, stability is independent of the input signal, and the sufficient and necessary condition for stability is that all poles of system transfer function lie in the left half of the s plane. This condition follows directly from Eqn. 4.1, since any right-half-plane poles contribute terms to the output that grow exponentially with time and thus are unbounded. Note that this definition implies that a system with poles on the imaginary axis is unstable, since its output is not bounded unless its input is rather carefully chosen.

The origin of the stability problem can be described in intuitively appealing through nonrigorous terms as follows. If a feedback system detects an error between the actual and desired outputs, it attempts to reduce this error to zero. However, changes in the error signal that result from corrective action do not occur instantaneously because of time delays around the loop. In a high-gain system, these delays can cause a tendency to overcorrect. If the magnitude of the overcorrection exceeds the magnitude of the initial error, instability results. Signal amplitudes grow exponentially until some nonlinearity limits further growth, at which time the system either saturates or oscillates in a constant-amplitude fashion called a *limit cycle*.[1] The feedback system designer must always temper his desire to

[1] The effect of nonlinearities on the steady-state amplitude reached by an unstable system is investigated in Chapter 6.

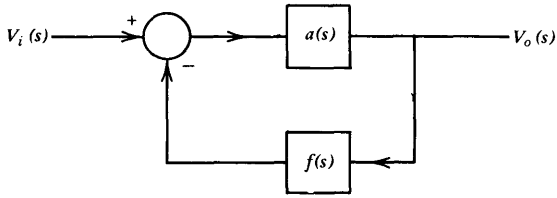


Figure 4.1 Block diagram of single-loop amplifier.

provide a large magnitude and a high unity-gain frequency for the loop transmission with the certain knowledge that sufficiently high values for these quantities invariably lead to instability.

As a specific example of a system with potentially unstable behavior, consider a simple single-loop system of the type shown in Fig. 4.1, with

$$a(s) = \frac{a_0}{(s+1)^3} \tag{4.3}$$

and

$$f(s) = 1 \tag{4.4}$$

The loop transmission for this system is

$$-a(s)f(s) = \frac{-a_0}{(s+1)^3} \tag{4.5}$$

or for sinusoidal excitation,

$$-a(j\omega)f(j\omega) = \frac{-a_0}{(j\omega+1)^3} = \frac{-a_0}{-j\omega^3 - 3\omega^2 + 3j\omega + 1} \tag{4.6}$$

If we evaluate Eqn. 4.6 at $\omega = \sqrt{3}$, we find that

$$-a(j\sqrt{3})f(j\sqrt{3}) = \frac{a_0}{8} \tag{4.7}$$

If the quantity a_0 is chosen equal to 8, the system has a real, *positive* loop transmission with a magnitude of one for sinusoidal excitation at three radians per second.

We might suspect that a system with a loop transmission of +1 is capable of oscillation, and this suspician can be confirmed by examining the closed-loop transfer function of the system with $a_0 = 8$. In this case,

$$A(s) = \frac{a(s)}{1 + a(s)f(s)} = \frac{8}{s^3 + 3s^2 + 3s + 9}$$

$$= \frac{8}{(s+3)(s+j\sqrt{3})(s-j\sqrt{3})} \tag{4.8}$$

This transfer function has a negative, real-axis pole and a pair of poles located on the imaginary axis at $s = \pm j\sqrt{3}$. An argument based on the properties of partial-fraction expansions (see Section 3.2.2) shows that the response of this system to many common (bounded) transient signals includes a constant-amplitude sinusoidal component.

Further increases in low-frequency loop-transmission magnitude move the pole pair into the right-half plane. For example, if we combine the forward-path transfer function

$$a(s) = \frac{64}{(s + 1)^3} \tag{4.9}$$

with unity feedback, the resultant closed-loop transfer function is

$$\begin{aligned} A(s) &= \frac{64}{s^3 + 3s^2 + 3s + 65} \\ &= \frac{64}{(s + 5)(s - 1 + j2\sqrt{3})(s - 1 - j2\sqrt{3})} \end{aligned} \tag{4.10}$$

With this value for a_0, the system transient response will include a sinusoidal component with an exponentially growing envelope.

If the dynamics associated with the loop transmission remain fixed, the system will be stable only for values of a_0 less than 8. This stability is achieved at the expense of desensitivity. If a value of $a_0 = 1$ is used so that

$$a(s)f(s) = \frac{1}{(s + 1)^3} \tag{4.11}$$

we find all closed-loop poles are in the left-half plane, since

$$\begin{aligned} A(s) &= \frac{1}{s^3 + 3s^2 + 3s + 2} \\ &= \frac{1}{(s + 2)(s + 0.5 + j\sqrt{3}/2)(s + 0.5 - j\sqrt{3}/2)} \end{aligned} \tag{4.12}$$

in this case.

In certain limited cases, a binary answer to the stability question is sufficient. Normally, however, we shall be interested in more quantitative information concerning the "degree" of stability of a feedback system. Frequently used measures of relative stability include the peak magnitude of the frequency response, the fractional overshoot in response to a step input, the damping ratio associated with the dominant pole pair, or the variation of a certain parameter that can be tolerated without causing absolute instability. Any of the measures of relative stability mentioned above can be found by direct calculations involving the system transfer function. While such determinations are practical with the aid of machine computation, insight into system operation is frequently obscured if this process is used. The techniques described in this chapter are intended not only to provide answers to questions concerning stability, but also (and more important) to indicate how to improve the performance of unsatisfactory systems.

4.2 THE ROUTH CRITERION

The Routh test is a mathematical method that can be used to determine the number of zeros of a polynomial with positive real parts. If the test is applied to the denominator polynomial of a transfer function (also called the *characteristic equation*) the absence of any right-half-plane zeros of the characteristic equation guarantees system stability. One computational advantage of the Routh test is that it is not necessary to factor the polynomial to apply the test.

4.2.1 Evaluation of Stability

The test is described for a polynomial of the form

$$P(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n \tag{4.13}$$

A necessary but not sufficient condition for all the zeros of Eqn. 4.13 to have negative real parts is that all the a's be present and that they all have the same sign. If this necessary condition is satisfied, an array of numbers is generated from the a's as follows. (This example is for n even. For n odd, a_n terminates the second row.)

$$\begin{array}{cccccc}
a_0 & a_2 & a_4 & \cdot\ \cdot & a_{n-2} & a_n \\
a_1 & a_3 & a_5 & \cdot\ \cdot & a_{n-1} & 0 \\
\dfrac{a_1a_2 - a_0a_3}{a_1} = b_1 & \dfrac{a_1a_4 - a_0a_5}{a_1} = b_2 & \cdot & \cdot\ \cdot & \dfrac{a_1a_n - a_0\cdot 0}{a_1} = b_{n/2} & 0 \\
\dfrac{b_1a_3 - a_1b_2}{b_1} = c_1 & \dfrac{b_1a_5 - a_1b_3}{b_1} = c_2 & \cdot & \cdot\ \cdot & 0 & 0 \\
\dfrac{c_1b_2 - b_1c_2}{c_1} = d_1 & \cdot & \cdot & \cdot\ \cdot & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot \\
0 & 0 & \cdot & \cdot\ \cdot & 0 & 0
\end{array} \tag{4.14}$$

As the array develops, progressively more elements of each row become zero, until only the first element of the $n + 1$ row is nonzero. The total number of sign changes in the first column is then equal to the number of zeros of the original polynomial that lie in the right-half plane.

The use of the Routh criterion is illustrated using the polynomial

$$P(s) = s^4 + 9s^3 + 14s^2 + 266s + 260 \tag{4.15}$$

Since all coefficients are real and positive, the necessary condition for all roots of Eqn. 4.15 to have negative real parts is satisfied. The array is

$$\begin{array}{lll}
1 & 14 & 260 \\
9 & 266 & 0 \\
\dfrac{9 \times 14 - 1 \times 266}{9} = -\dfrac{140}{9} & \dfrac{9 \times 260 - 1 \times 0}{9} = 260 & 0 \\
\text{(sign change)} & & \\
\dfrac{-(140/9) \times 266 - 9 \times 260}{-(140/9)} = +\dfrac{2915}{7} & 0 & 0 \\
\text{(sign change)} & & \\
\dfrac{(2915/7) \times 260 - [-(140/9) \times 0]}{2915/7} = 260 & 0 & 0
\end{array} \tag{4.16}$$

The two sign changes in the first column indicate two right-half-plane zeros. This result can be verified by factoring the original polynomial, showing that

$$s^4 + 9s^3 + 14s^2 + 266s + 260 = (s - 1 + j5)(s - 1 - j5)(s + 1)(s + 10) \tag{4.17}$$

A second example is provided by the polynomial

$$P(s) = s^4 + 13s^3 + 58s^2 + 306s + 260 \tag{4.18}$$

The corresponding array is

$$\begin{array}{lll}
1 & 58 & 260 \\
13 & 306 & 0 \\
\dfrac{13 \times 58 - 1 \times 306}{13} = \dfrac{448}{13} & \dfrac{13 \times 260 - 1 \times 0}{13} = 260 & 0 \\
\dfrac{(448/13) \times 306 - 13 \times 260}{448/13} = \dfrac{23287}{112} & 0 & 0 \\
\dfrac{(23287/112) \times 260 - (448/13) \times 0}{23287/112} = 260 & 0 & 0
\end{array} \tag{4.19}$$

Factoring verifies the result that there are no right-half-plane zeros for this polynomial, since

$$s^4 + 13s^3 + 58s^2 + 306s + 260 = (s + 1 + j5)(s + 1 - j5)(s + 1)(s + 10) \quad (4.20)$$

Two kinds of difficulties can occur when applying the Routh test. It is possible that the first element in one row of the array is zero. In this case, the original polynomial is multiplied by $s + \alpha$, where α is any positive real number, and the test is repeated. This procedure is illustrated using the polynomial

$$P(s) = s^5 + s^4 + 10s^3 + 10s^2 + 20s + 5 \quad (4.21)$$

The first element of the third row of the array is zero.

$$\begin{array}{ccc} 1 & 10 & 20 \\ 1 & 10 & 5 \\ 0 & 15 & 0 \end{array} \quad (4.22)$$

The difficulty is resolved by multiplying Eqn. 4.21 by $s + 1$, yielding

$$P'(s) = s^6 + 2s^5 + 11s^4 + 20s^3 + 30s^2 + 25s + 5 \quad (4.23)$$

The array for Eqn. 4.23 is

$$\begin{array}{cccc} 1 & 11 & 30 & 5 \\ 2 & 20 & 25 & 0 \\ 1 & 17.5 & 5 & 0 \\ -15 & 15 & 0 & 0 \\ -18.5 & 5 & 0 & 0 \\ 10.95 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{array} \quad (4.24)$$

Since multiplication by $s + 1$ did not add any right-half-plane zeros to Eqn. 4.21, we conclude that the two right-half-plane zeros indicated by the array of Eqn. 4.24 must be contained in the original polynomial.

The second possibility is that an entire row becomes zero. This condition indicates that there is a pair of roots on the imaginary axis, a pair of real roots located symmetrically with respect to the origin, or both kinds of pairs in the original polynomial. The terms in the row above the all-zero row are used as coefficients of an equation in even powers of s called the *auxiliary equation.* The zeros of this equation are the pairs mentioned above. The auxiliary equation can be differentiated with respect to s, and the resultant coefficients are used in place of the all-zero row to continue the array. This type of difficulty is illustrated with the polynomial

$$P(s) = s^4 + 11s^3 + 11s^2 + 11s + 10 = (s + j)(s - j)(s + 1)(s + 10) \tag{4.25}$$

The array is

$$\begin{array}{ccc} 1 & 11 & 10 \\ 11 & 11 & 0 \\ 10 & 10 & 0 \\ 0 & 0 & 0 \end{array} \tag{4.26}$$

The auxiliary equation is

$$Q(s) = 10s^2 + 10 \tag{4.27}$$

The roots of the equation are the two imaginary zeros of Eqn. 4.25. Differentiating Eqn. 4.27 and using the nonzero coefficient to replace the first element of row 4 of Eqn. 4.26 yields a new array.

$$\begin{array}{ccc} 1 & 11 & 10 \\ 11 & 11 & 0 \\ 10 & 10 & 0 \\ 20 & 0 & 0 \\ 10 & 0 & 0 \end{array} \tag{4.28}$$

The absence of sign changes in the array verifies that the original polynomial has no zeros in the right-half plane.

Note that, while there are no closed-loop poles in the right-half plane, a system with a characteristic equation given by Eqn. 4.25 is unstable by our definition since it has a pair of poles on the imaginary axis. Examining only the left-hand column of the Routh array only identifies the number of right-half-plane zeros of the tested polynomial. Imaginary-axis zeros can be found by the manipulations involving the auxiliary equation.

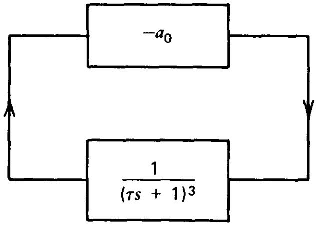


Figure 4.2 Block diagram of phase-shift oscillator.

4.2.2 Use as a Design Aid

The Routh criterion is most frequently used to determine the stability of a feedback system. In certain cases, however, more quantitative design information is obtainable, as illustrated by the following examples.

A phase-shift oscillator can be constructed by applying sufficient negative feedback around a network that has three or more poles. If an amplifier with frequency-independent gain is combined with a network with three coincident poles, the block diagram for the resultant system is as shown in Fig. 4.2. The value of a_0 necessary to sustain oscillations can be determined by Routh analysis.[2]

Stability investigations for Fig. 4.2 are complicated by the fact that the oscillator has no input; thus we cannot use the poles of an input-to-output transfer function to determine stability. We should note that the stability of a linear system is a property of the system itself and is thus independent of input signals that may be applied to it. Any unstable physical system will demonstrate its instability with no input, since runaway behavior will be stimulated by always present noise. Even in a purely mathematical linear system, stability is determined by the location of the closed-loop poles, and these locations are clearly input independent.

The analysis of the oscillator is initiated by recalling that the characteristic equation of any feedback system is one minus its loop transmission. Therefore

$$P(s) = 1 + \frac{a_0}{(\tau s + 1)^3} \tag{4.29}$$

In this and other calculations involving the characteristic equation, it is possible to clear fractions since the location of the zeros are not altered

[2] The Routh test applied to this example offers computational advantages compared to the direct factoring used for a similar transfer function in the example of Section 4.1.

by this operation. After clearing fractions and identifying coefficients, the Routh array is

$$\begin{array}{ll} \tau^3 & 3\tau \\ 3\tau^2 & 1 + a_0 \\ \dfrac{(8 - a_0)\tau}{3} & 0 \\ 1 + a_0 & 0 \end{array} \tag{4.30}$$

Assuming τ is positive, roots with positive real parts occur for $a_0 < -1$ (one right-half-plane zero) and for $a_0 > +8$ (two right-half-plane zeros). Laplace analysis indicates that generation of a constant-amplitude sinusoidal oscillation requires a pole pair on the imaginary axis. In practice, a complex pole pair is located slightly to the right of the imaginary axis. An intentionally introduced nonlinearity can then be used to limit the amplitude of the oscillation (see Section 6.3.3). Thus, a practical oscillator circuit is obtained with $a_0 > 8$.

The frequency of oscillation with $a_0 = 8$ can be determined by examining the array with this value for a_0. Under these conditions the third row becomes all zero. The auxiliary equation is

$$Q(s) = 3\tau^2 s^2 + 9 \tag{4.31}$$

and the equation has zeros at $s = \pm j\sqrt{3}/\tau$, indicating oscillation at $\sqrt{3}/\tau$ radians per second for $a_0 = 8$.

As a second example of the type of design information that can be obtained via Routh analysis, consider an operational amplifier with an open-loop transfer function

$$a(s) = \frac{a_0}{(s + 1)(10^{-6}s + 1)(10^{-7}s + 1)} \tag{4.32}$$

It is assumed that this amplifier is connected as a unity-gain noninverting amplifier, and we wish to determine the range of values of a_0 for which all closed-loop poles have real parts more negative than -2×10^5 sec^{-1}. This condition on closed-loop pole location implies that any pulse response of the system will decay at least as fast as $Ke^{-2 \times 10^5 t}$ after the exciting pulse returns to zero. The constant K is dependent on conditions at the time the input becomes zero.

The characteristic equation for the amplifier is (after dropping insignificant terms)

$$P(s) = 10^{-13}s^3 + 1.1 \times 10^{-6}s^2 + s + 1 + a_0 \tag{4.33}$$

In order to determine the range of a_0 for which all zeros of this characteristic equation have real parts more negative than $-2 \times 10^5 \text{ sec}^{-1}$, it is only necessary to make a change of variable in Eqn. 4.33 and apply Routh's criterion to the modified equation. In particular, application of the Routh test to a polynomial obtained by substituting

$$\lambda = s + c \tag{4.34}$$

will determine the number of zeros of the original polynomial with real parts more positive than $-c$, since this substitution shifts singularities in the s plane to the right by an amount c as they are mapped into the λ plane. If the indicated substitution is made with $c = 2 \times 10^5 \text{ sec}^{-1}$, Eqn. 4.33 becomes

$$P(\lambda) = 10^{-13}\lambda^3 + 10^{-6}\lambda^2 + 0.57\lambda - 1.57 \times 10^5 + a_0 \tag{4.35}$$

The Routh array is

$$\begin{array}{ll} 10^{-13} & 0.57 \\ 10^{-6} & -1.57 \times 10^5 + a_0 \\ 0.59 - 10^{-7} a_0 & 0 \\ -1.57 \times 10^5 + a_0 & 0 \end{array} \tag{4.36}$$

This array shows that Eqn. 4.33 has one zero with a real part more positive than $-2 \times 10^5 \text{ sec}^{-1}$ for $a_0 < 1.57 \times 10^5$, and has two zeros to the right of the dividing line for $a_0 > 5.9 \times 10^6$. Accordingly, all zeros have real parts more negative than $-2 \times 10^5 \text{ sec}^{-1}$ only for

$$1.57 \times 10^5 < a_0 < 5.9 \times 10^6 \tag{4.37}$$

4.3 ROOT-LOCUS TECHNIQUES

A single-loop feedback amplifier is shown in the block diagram of Fig. 4.1. The closed-loop transfer function for this amplifier is

$$\frac{V_o(s)}{V_i(s)} = A(s) = \frac{a(s)}{1 + a(s)f(s)} \tag{4.38}$$

Root-locus techniques provide a method for finding the poles of the closed-loop transfer function $A(s)$ [or equivalently the zeros of $1 + a(s)f(s)$] given the poles and zeros of $a(s)f(s)$ and the d-c loop-transmission magnitude a_0f_0.[3] Notice that since the quantity a_0f_0 must appear in one or more terms

[3] If the loop transmission has one or more zeros at the origin so that its d-c magnitude is zero, the closed-loop poles are found from the midband value of af.

of the characteristic equation, the locations of the poles of $A(s)$ must depend on $a_0 f_0$. A *root-locus diagram* consists of a collection of branches or loci in the s plane that indicate how the locations of the poles of $A(s)$ change as $a_0 f_0$ varies.

The root-locus diagram provides useful information concerning the performance of a feedback system since the relative stability of any linear system is uniquely determined by its close-loop pole locations. We shall find that approximate root-locus diagrams are easily and rapidly sketched, and that they provide readily interpreted insight into how the closed-loop performance of a system responds to changes in its loop transmission. We shall also see that root-locus techniques can be combined with simple algebraic methods to yield exact answers in certain cases.

4.3.1 Forming the Diagram

A simple example that illustrates several important features of root-locus techniques is provided by the system shown in Fig. 4.1 with a feedback transfer function f of unity and a forward transfer function

$$a(s) = \frac{a_0}{(\tau_a s + 1)(\tau_b s + 1)} \tag{4.39}$$

The corresponding closed-loop transfer function is

$$A(s) = \frac{a(s)}{1 + a(s)f(s)} = \frac{a_0}{\tau_a \tau_b s^2 + (\tau_a + \tau_b)s + (1 + a_0)} \tag{4.40}$$

The closed-loop poles can be determined by factoring the characteristic equation of $A(s)$, yielding

$$s_1 = \frac{-(\tau_a + \tau_b) + \sqrt{(\tau_a + \tau_b)^2 - 4(1 + a_0)\tau_a \tau_b}}{2\tau_a \tau_b} \tag{4.41a}$$

$$s_2 = \frac{-(\tau_a + \tau_b) - \sqrt{(\tau_a + \tau_b)^2 - 4(1 + a_0)\tau_a \tau_b}}{2\tau_a \tau_b} \tag{4.41b}$$

The root-locus diagram in Fig. 4.3 is drawn with the aid of Eqn. 4.41. The important features of this diagram include the following.

(a) The loop-transmission pole locations are shown. (Loop-transmission zeros are also indicated if they are present.)

(b) The poles of $A(s)$ coincide with loop-transmission poles for $a_0 = 0$.

(c) As a_0 increases, the locations of the poles of $A(s)$ change along the loci as shown. Arrows indicate the direction of changes that result for increasing a_0.

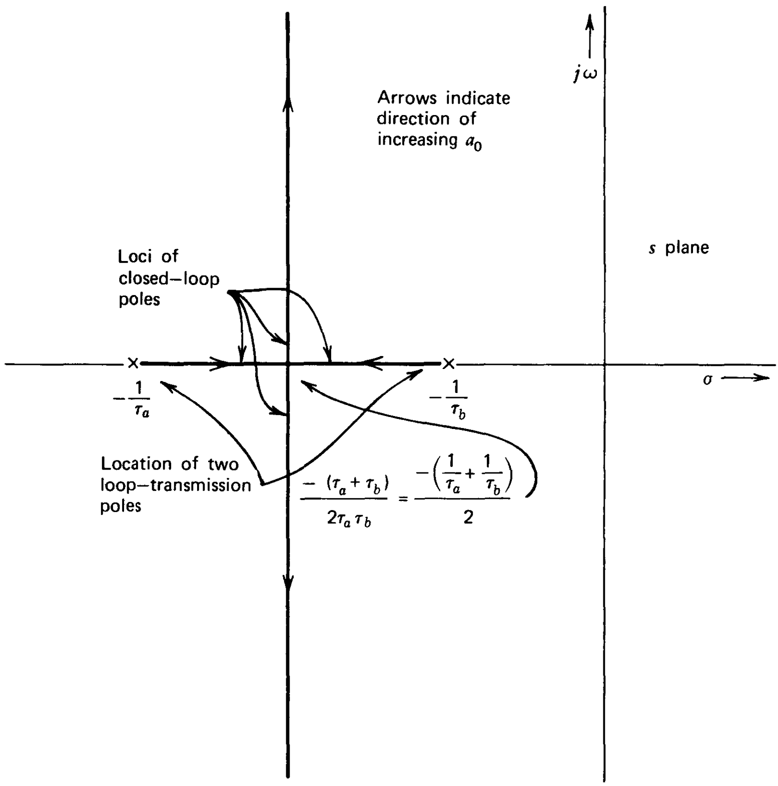


Figure 4.3 Root-locus diagram for second-order system.

(d) The two poles coincide at the arithmetic mean of the loop-transmission pole locations for zero radicand in Eqn. 4.41, or for

$$a_0 = \frac{(\tau_a + \tau_b)^2}{4\tau_a\tau_b} - 1 \tag{4.42}$$

(e) For increases in a_0 beyond the value of Eqn. 4.42, the closed-loop pole pair is complex with constant real part and a damping ratio that is a monotonic decreasing function of a_0. Consequently, ω_n increases with increasing a_0 in this range.

Certain important features of system behavior are evident from the diagram. For example, the system does not become unstable for any positive value of a_0. However, the relative stability decreases as a_0 increases beyond the value indicated in Eqn. 4.42.

It is always possible to draw a root-locus diagram by directly factoring the characteristic equation of the system under study as in the preceding example. Unfortunately, the effort involved in factoring higher-order polynomials makes machine computation mandatory for all but the simplest systems. We shall see that it is possible to approximate the root-locus diagrams and thus retain the insight often lost with machine computation when absolute accuracy is not required.

The key to developing the rules used to approximate the loci is to realize that closed-loop poles occur only at zeros of the characteristic equation or at frequencies s_1 such that[4]

$$1 + a(s_1)f(s_1) = 0 \tag{4.43a}$$

or

$$a(s_1)f(s_1) = -1 \tag{4.43b}$$

Thus, if the point s_1 is a point on a branch of the root-locus diagram, the two conditions

$$|a(s_1)f(s_1)| = 1 \tag{4.44a}$$

and

$$\measuredangle\, a(s_1)f(s_1) = (2n + 1)\, 180^\circ \tag{4.44b}$$

where n is any integer, must be satisfied. The angle condition is the more important of these two constraints for purposes of forming a root-locus diagram. The reason is that since we plot the loci as a_0f_0 is varied, it is possible to find a value for a a_0f_0 that satisfies the magnitude condition at any point in the s plane where the angle condition is satisfied.

By concentrating primarily on the angle condition, we are able to formulate a set of rules that greatly simplify root-locus-diagram construction compared with brute-force factoring of the characteristic equation. Here are some of the rules we shall use.

1. The number of branches of the diagram is equal to the number of poles of $a(s)f(s)$. Each branch starts at a pole of $a(s)f(s)$ for small values of a_0f_0 and approaches a zero of $a(s)f(s)$ either in the finite s plane or at infinity for large values of a_0f_0. The starting and ending points are demonstrated by considering

$$a(s)f(s) = a_0f_0g(s) \tag{4.45}$$

where $g(s)$ contains the frequency-dependent portion of the loop trans-

[4] It is assumed throughout that the system under study is a negative feedback system with the topology shown in Fig. 4.1.

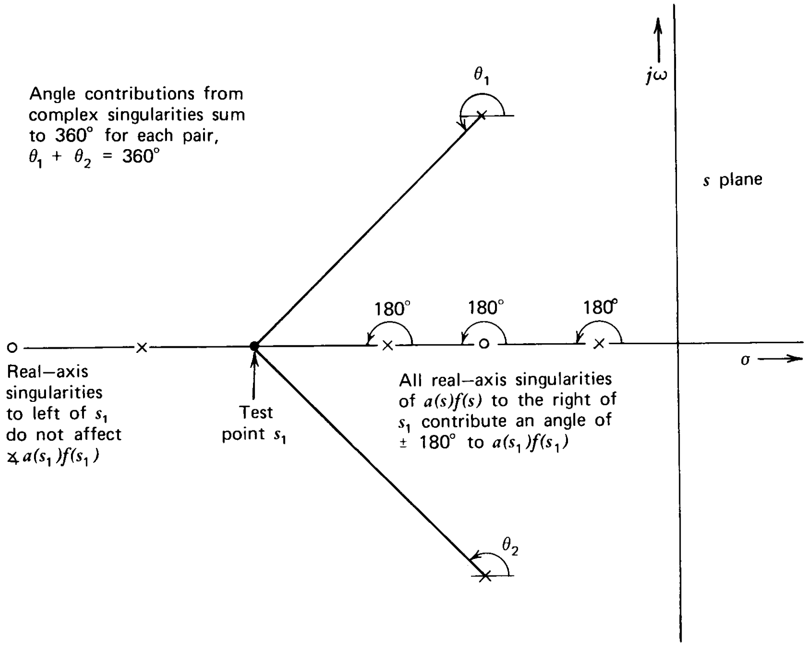


Figure 4.4 Loci on real axis.

mission and the value of $g(0) \triangleq g_0$ is unity. Rearranging Eqn. 4.44 and using this notation yields

$$|g(s_1)| = \frac{1}{a_0 f_0} \tag{4.46}$$

at any point s_1 on a branch of the root-locus diagram. Thus for small values of $a_0 f_0$, $|g(s_1)|$ must be large, implying that the point s_1 is close to a pole of $g(s)$. Conversely, a large value of $a_0 f_0$ requires proximity to a zero of $g(s)$.

2. Branches of the diagram lie on the real axis to the left of an odd number of real-axis poles and zeros of $a(s)f(s)$.[5] This rule follows directly from Eqn. 4.44b as illustrated in Fig. 4.4. Each real-axis zero of $a(s)f(s)$ to the right of s_1 adds 180° to the angle of $a(s_1)f(s_1)$ while each real-axis pole to the right of s_1 subtracts 180° from the angle. Real-axis singularities to the left of point s_1 do not influence the angle of $a(s_1)f(s_1)$. Similarly, since complex singularities must always occur in conjugate pairs, the net angle con-

[5] Special care is necessary for systems with right-half-plane open-loop singularities. See Section 4.3.3.

tribution from these singularities is zero. This rule is thus sufficient to satisfy Eqn. 4.44b. We are further guaranteed that branches *must* exist on all segments of the real axis to the left of an odd number of singularities of $a(s)f(s)$, since there is some value of $a_0 f_0$ that will exactly satisfy Eqn. 4.44a at every point on these segments, and the satisfaction of Eqns. 4.44a and 4.44b is both necessary and sufficient for the existence of a pole of $A(s)$.

3. The two separate branches of the diagram that must exist between pairs of poles or pairs of zeros on segments of the real axis that satisfy rule 2 must at some point depart from or enter the real axis at right angles to it. Frequently the precise break-away point is not required in order to sketch the loci to acceptable accuracy. If it is necessary to have an exact location, it can be shown that the break-away points are the solutions of the equation

$$\frac{d[g(s)]}{ds} = 0 \tag{4.47}$$

for systems without coincident singularities.

4. If the number of poles of $a(s)f(s)$ exceeds the number of zeros of this function by two or more, the average distance of the poles of $A(s)$ from the imaginary axis is independent of $a_0 f_0$. This rule evolves from a property of algebraic polynomials. Consider a polynomial

$$\begin{aligned} P(s) &= (a_1 s + a_1 s_1)(a_2 s + a_2 s_2)(a_3 s + a_3 s_3) \cdots (a_n s + a_n s_n) \\ &= (a_1 a_2 \cdots a_n)(s + s_1)(s + s_2)(s + s_3) \cdots (s + s_n) \\ &= (a_1 a_2 \cdots a_n)[s^n + (s_1 + s_2 + s_3 + \cdots + s_n)s^{n-1} \\ &\qquad + \cdots + s_1 s_2 s_3 \cdots s_n] \end{aligned} \tag{4.48}$$

From the final expression of Eqn. 4.48, we see that the ratio of the coefficients of the s^{n-1} term and the s^n term (denoted as $-n\bar{s}$) is

$$-n\bar{s} = s_1 + s_2 + s_3 + \cdots + s_n \tag{4.49}$$

Since imaginary components of terms on the right-hand side of Eqn. 4.49 must occur in conjugate pairs and thus cancel, the quantity

$$s = -\frac{(s_1 + s_2 + s_3 + \cdots + s_n)}{n} \tag{4.50}$$

is the average distance of the roots of $P(s)$ from the imaginary axis. In order to apply Eqn. 4.50 to the characteristic equation of a feedback system, assume that

$$a(s)f(s) = a_0 f_0 \frac{p(s)}{q(s)} \tag{4.51}$$

Then

$$A(s) = \frac{a(s)}{1 + a(s)f(s)} = \frac{a(s)}{1 + a_0 f_0[p(s)/q(s)]} = \frac{a(s)q(s)}{q(s) + a_0 f_0 p(s)} \tag{4.52}$$

If the order of $q(s)$ exceeds that of $p(s)$ by two or more, the ratio of the coefficients of the two highest-order terms of the characteristic equation of $A(s)$ is independent of $a_0 f_0$, and thus the average distance of the poles of $A(s)$ from the imaginary axis is a constant.

5. For large values of $a_0 f_0$, $P - Z$ branches approach infinity, where P and Z are the number of poles and finite-plane zeros of $a(s)f(s)$, respectively. These branches approach asymptotes that make angles with the real axis given by

$$\theta_n = \frac{(2n + 1)\ 180°}{P - Z} \tag{4.53}$$

In Eqn. 4.53, n assumes all integer values from 0 to $P - Z - 1$. The asymptotes all intersect the real axis at a point

$$\frac{\Sigma \text{ real parts of poles of } a(s)f(s) - \Sigma \text{ real parts of zeros of } a(s)f(s)}{P - Z}$$

The proof of this rule is left to Problem P4.4.

6. Near a complex pole of $a(s)f(s)$, the angle of a branch with respect to the pole is

$$\theta_p = 180° + \Sigma \measuredangle z - \Sigma \measuredangle p \tag{4.54}$$

where $\Sigma \measuredangle z$ is the sum of the angles of vectors drawn from all the zeros of $a(s)f(s)$ to the complex pole in question and $\Sigma \measuredangle p$ is the sum of the angles of vectors drawn from all other poles of $a(s)f(s)$ to the complex pole. Similarly, the angle a branch makes with a loop-transmission zero in the vicinity of the zero is

$$\theta_z = 180° - \Sigma \measuredangle z + \Sigma \measuredangle p \tag{4.55}$$

These conditions follow directly from Eqn. 4.44b.

7. If the singularities of $a(s)f(s)$ include a group much nearer the origin than all other singularities of $a(s)f(s)$, the higher-frequency singularities can be ignored when determining loci in the vicinity of the origin. Figure 4.5 illustrates this situation. It is assumed that the point s_1 is on a branch if the high-frequency singularities are ignored, and thus the angle of the low-frequency portion of $a(s)f(s)$ evaluated at $s = s_1$ must be $(2n + 1)\ 180°$. The geometry shows that the angular contribution attributable to remote singularities such as that indicated as θ_1 is small. (The two angles from a remote complex-conjugate pair also sum to a small angle.) Small changes in the

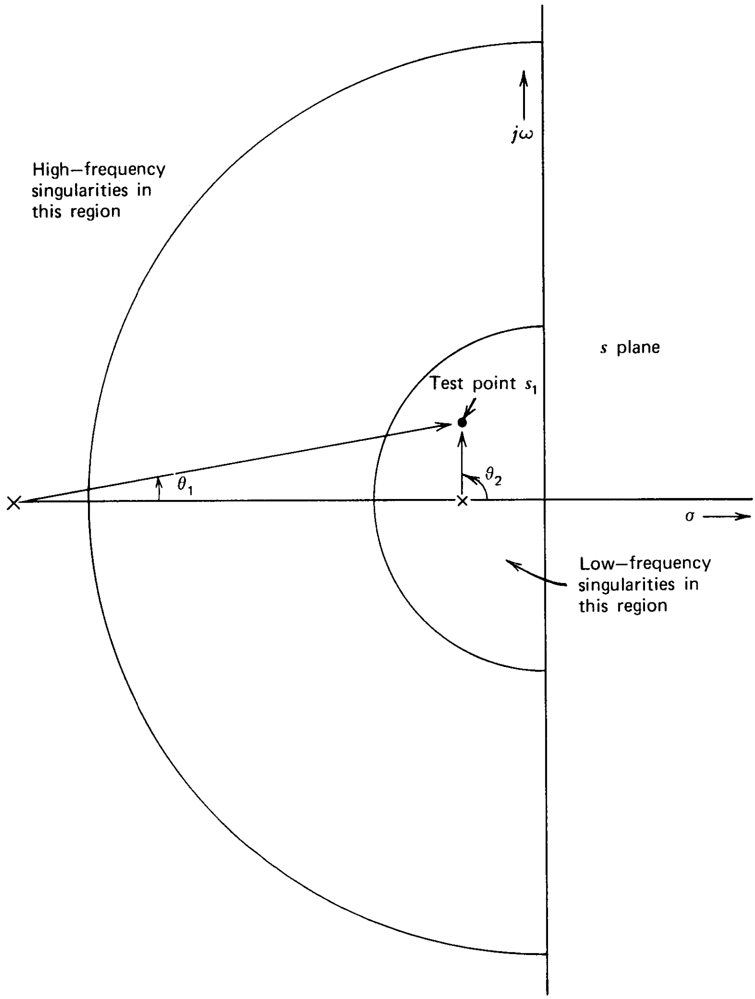


Figure 4.5 Loci in vicinity of low-frequency singularities.

location of s_1 that can cause relatively large changes in the angle (e.g., θ_2) from low-frequency singularities offset the contribution from remote singularities, implying that ignoring the remote singularities results in insignificant changes in the root-locus diagram in the vicinity of the low-frequency singularities. Furthermore, all closed-loop pole locations will lie relatively close to their starting points for low and moderate values of $a_0 f_0$. Since the discussion of Section 3.3.2 shows that $A(s)$ will be dominated by its lowest-frequency poles, the higher-frequency singularities of $a(s)f(s)$ can be ignored when we are interested in the performance of the system for low and moderate values of a_0f_0.

8. The value of a_0f_0 required to make a closed-loop pole lie at the point s_1 on a branch of the root-locus diagram is

$$a_0f_0 = \frac{1}{|g(s_1)|} \tag{4.56}$$

where $g(s)$ is defined in rule 1. This rule is required to satisfy Eqn. 4.44a.

4.3.2 Examples

The root-locus diagram shown in Fig. 4.3 can be developed using the rules given above rather than by factoring the denominator of the closed-loop transfer function. The general behavior of the two branches on the real axis is determined using rules 2 and 3. While the break-away point can be found from Eqn. 4.47, it is easier to use either rule 4 or rule 5 to establish off-axis behavior. Since the average distance of the closed-loop poles from the imaginary axis must remain constant for this system [the number of poles of $a(s)f(s)$ is two greater than the number of its zeros], the branches must move parallel to the imaginary axis after they leave the real axis. Furthermore, the average distance must be identical to that for $a_0f_0 = 0$, and thus the segment parallel to the imaginary axis must be located at $-\frac{1}{2}[(1/\tau_a) + (1/\tau_b)]$. Rule 5 gives the same result, since it shows that the two branches must approach vertical asymptotes that intersect the real axis at $-\frac{1}{2}[(1/\tau_a) + (1/\tau_b)]$.

More interesting root-locus diagrams result for systems with more loop-transmission singularities. For example, the transfer function of an amplifier with three common-emitter stages normally has three poles at moderate frequencies and three additional poles at considerably higher frequencies. Rule 7 indicates that the three high-frequency poles can be ignored if this type of amplifier is used in a feedback connection with moderate values of d-c loop transmission. If it is assumed that frequency-independent negative feedback is applied around the three-stage amplifier, a representative *af* product could be[6]

$$a(s)f(s) = \frac{a_0f_0}{(s+1)(0.5s+1)(0.1s+1)} \tag{4.57}$$

[6] The corresponding pole locations at -1, -2, and -10 sec^{-1} are unrealistically low for most amplifiers. These values result, however, if the transfer function for an amplifier with poles at -10^6, -2×10^6, and -10^7 sec^{-1} is normalized using the microsecond rather than the second as the basic time unit. Such frequency scaling will often be used since it eliminates some of the unwieldy powers of 10 from our calculations.

The root-locus diagram for this system is shown in Fig. 4.6. Rule 2 determines the diagram on the real axis, while rule 5 establishes the asymptotes. Rule 4 can be used to estimate the branches off the real axis, since the branches corresponding to the two lower-frequency poles must move to the right to balance the branch going left from the high-frequency pole. The break-away point can be determined from Eqn. 4.47, with

$$\frac{d[g(s)]}{ds} = \frac{-[0.15s^2 + 1.3s + 1.6]}{[(s + 1)(0.5s + 1)(0.1s + 1)]^2} \tag{4.58}$$

Zeros of Eqn. 4.58 are at $-7.2\ \text{sec}^{-1}$ and $-1.47\ \text{sec}^{-1}$. The higher-frequency location is meaningless for this problem, and in fact corresponds to a break-away point which results if positive feedback is applied around the amplifier. Note that the break-away point can be accurately estimated using rule 7. If the relatively higher-frequency pole at $10\ \text{sec}^{-1}$ is ignored, a break-away point at $-1.5\ \text{sec}^{-1}$ results for the remaining two-pole transfer function.

Algebraic manipulations can be used to obtain more quantitative information about the system. Figure 4.6 shows that the system becomes unstable as two poles move into the right-half plane for sufficiently large values of a_0f_0. The value of a_0f_0 that moves the pair of closed-loop poles onto the imaginary axis is found by applying Routh's criterion to the characteristic equation of the system, which is (after clearing fractions)

$$\begin{aligned} P(s) &= (s + 1)(0.5s + 1)(0.1s + 1) + a_0f_0 \\ &= 0.05s^3 + 0.65s^2 + 1.6s + 1 + a_0f_0 \end{aligned} \tag{4.59}$$

The Routh array is

$$\begin{array}{cc} 0.05 & 1.6 \\ 0.65 & 1 + a_0f_0 \\ \frac{1}{0.65}(0.99 - 0.05a_0f_0) & 0 \\ 1 + a_0f_0 & 0 \end{array} \tag{4.60}$$

Two sign reversals indicating instability occur for $a_0f_0 > 19.8$. With this value of a_0f_0, the auxiliary equation is

$$Q(s) = 0.65s^2 + 20.8 \tag{4.61}$$

The roots of this equation indicate that the poles cross the imaginary axis at $s = \pm j(5.65)$.

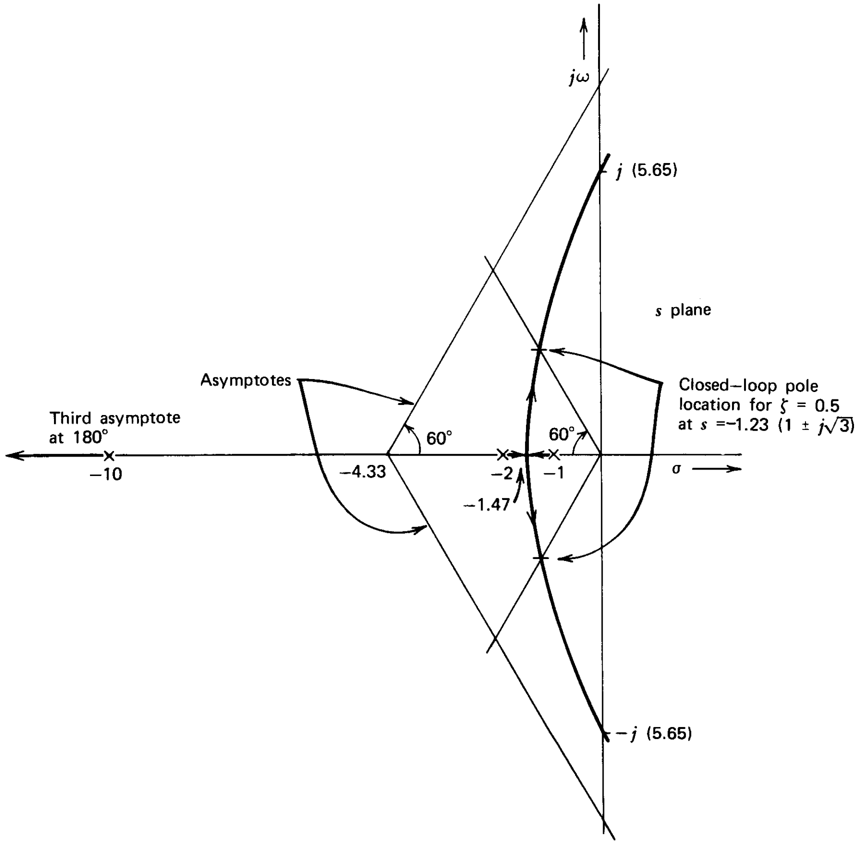


Figure 4.6 Root-locus diagram for third-order system.

It is also possible to determine values for $a_0 f_0$ that result in specified closed-loop pole configurations. This type of calculation is illustrated by finding the value of $a_0 f_0$ required to provide a damping ratio of 0.5, corresponding to complex-pair poles located 60° from the real axis. The magnitude of the ratio of the imaginary part to the real part of the pole location for a pole pair with $\zeta = 0.5$ is $\sqrt{3}$. Thus the characteristic equation for this system, when the damping ratio of the complex pole pair is 0.5, is

$$\begin{aligned} P'(s) &= (s+\gamma)(s+\beta+j\sqrt{3}\beta)(s+\beta-j\sqrt{3}\beta) \\ &= s^3 + (\gamma+2\beta)s^2 + 2\beta(\gamma+2\beta)s + 4\gamma\beta^2 \end{aligned} \tag{4.62}$$

where $-\gamma$ is the location of the real-axis pole.

The parameters are determined by multiplying Eqn. 4.59 by 20 (to make the coefficient of the s^3 term unity) and equating the new equation to $P'(s)$.

$$s^3 + 13s^2 + 32s + 20(1 + a_0f_0)$$
$$= s^3 + (\gamma + 2\beta)s^2 + 2\beta(\gamma + 2\beta)s + 4\gamma\beta^2 \qquad (4.63)$$

Equation 4.63 is easily solved for γ, β, and a_0f_0, with the results

$$\gamma = 10.54$$
$$\beta = 1.23$$
$$a_0f_0 = 2.2 \qquad (4.64)$$

Several features of the system are evident from this analysis. Since the complex pair is located at $s = -1.23\,(1 \pm j\sqrt{3})$ when the real-axis pole is located at $s = -10.54$, a two-pole approximation based on the pair should accurately model the transient or frequency response of the system. The relatively low desensitivity $1 + a_0f_0 = 3.2$ results if the damping ratio of the complex pair is made 0.5, and any increase in desensitivity will result in poorer damping. The earlier analysis shows that attempts to increase desensitivity beyond 20.8 result in instability.

Note that since there was only one degree of freedom (the value of a_0f_0) existed in our calculations, only one feature of the closed-loop pole pattern could be controlled. It is not possible to force arbitrary values for more than one of the three quantities defining the closed-loop pole locations (ζ and ω_n for the pair and the location of the real pole) unless more degrees of design freedom are allowed.

Another example of root-locus diagram construction is shown in Fig. 4.7, the diagram for

$$a(s)f(s) = \frac{a_0f_0}{(s + 1)\,(s^2/8 + s/2 + 1)} \qquad (4.65)$$

Rule 5 establishes the asymptotes, while rule 6 is used to determine the loci near the complex poles. The value of a_0f_0 for which the complex pair of poles enters the right-half plane and the frequency at which they cross the imaginary axis are found by Routh's criterion. The reader should verify that these poles cross the imaginary axis at $s = \pm j2\sqrt{3}$ for $a_0f_0 = 6.5$.

The root-locus diagram for a system with

$$a(s)f(s) = \frac{a_0f_0(0.5s + 1)}{s(s + 1)} \qquad (4.66)$$

is shown in Fig. 4.8. Rule 2 indicates that branches are on the real axis between the two loop-transmission poles and to the left of the zero. The

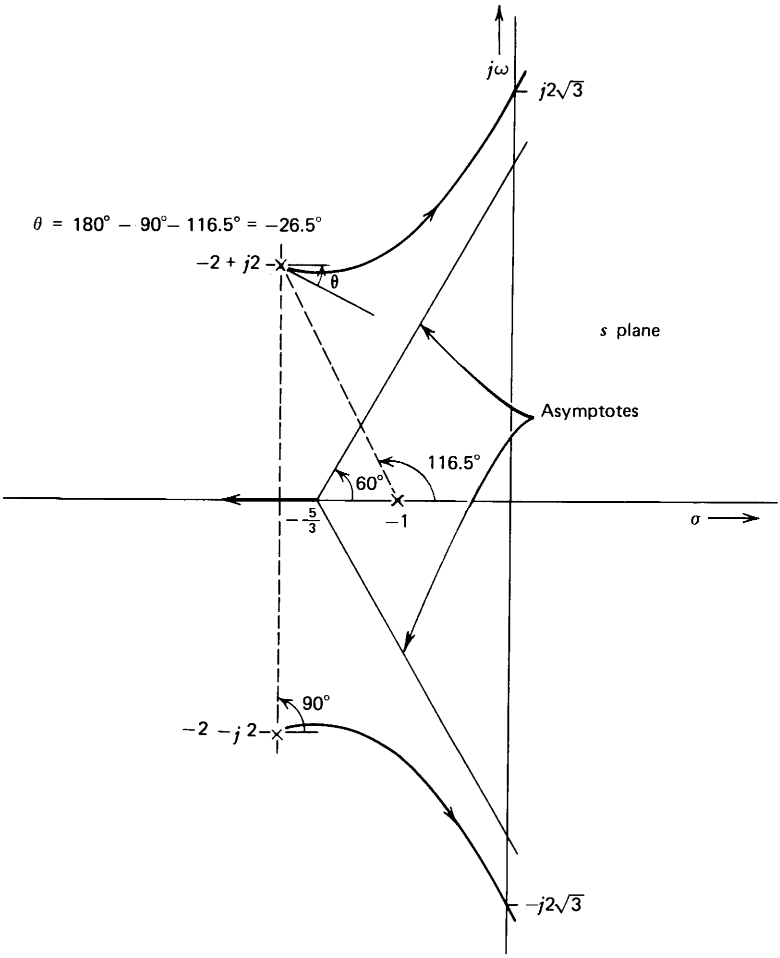


Figure 4.7 Root-locus diagram for $a(s)f(s) = a_0f_0/[(s+1)(s^2/8 + s/2 + 1)]$.

points of departure from and reentry to the real axis are obtained by solving

$$\frac{d}{ds}\left[\frac{(0.5s+1)}{s(s+1)}\right] = 0 \tag{4.67}$$

yielding $s = -2 \pm \sqrt{2}$.

4.3.3 Systems With Right-Half-Plane Loop-Transmission Singularities

It is necessary to be particularly careful about the sign of the loop transmission when root-locus diagrams are drawn for systems with right-half-

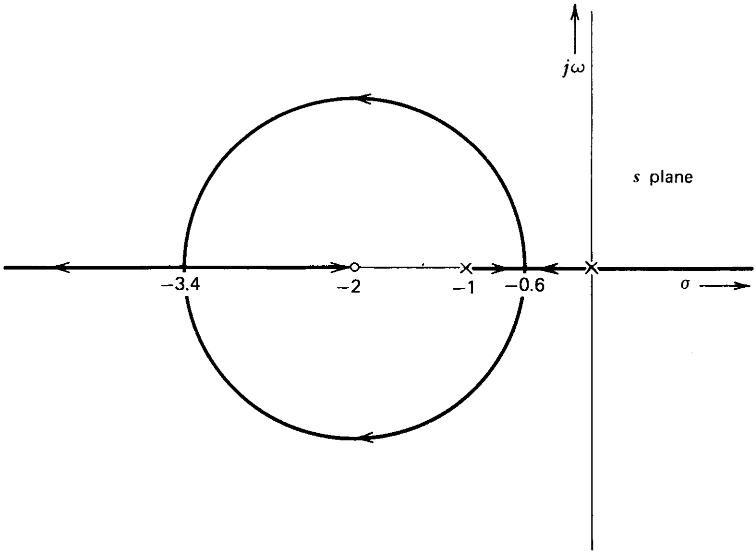


Figure 4.8 Root-locus for diagram $a(s)f(s) = a_0f_0(0.5s + 1)/[s(s + 1)]$.

plane loop-transmission singularities. Some systems that are unstable without feedback have one or more loop-transmission poles in the right-half plane. For example, a large rocket does not become aerodynamically stable until it reaches a certain critical speed, and would tip over shortly after lift off if the thrust were not vectored by means of a feedback system. It can be shown that the transfer function of the rocket alone includes a real-axis right-half-plane pole.

A more familiar example arises from a single-stage common-emitter amplifier. The transfer function of this type of amplifier includes a pole at moderate frequency, a second pole at high frequency, and a high-frequency right-half-plane zero that reflects the signal fed forward from input to output through the collector-to-base capacitance of the transistor. A representative *af* product for this type of amplifier with frequency-independent feedback applied around it is

$$a(s)f(s) = \frac{a_0f_0(-10^{-3}s + 1)}{(10^{-3}s + 1)(s + 1)} \tag{4.68}$$

The singularities for this amplifier are shown in Fig. 4.9. If the root-locus rules are applied blindly, we conclude that the low-frequency pole moves to the right, and enters the right-half plane for d-c loop-transmission magnitudes in excess of one. Fortunately, experimental evidence refutes

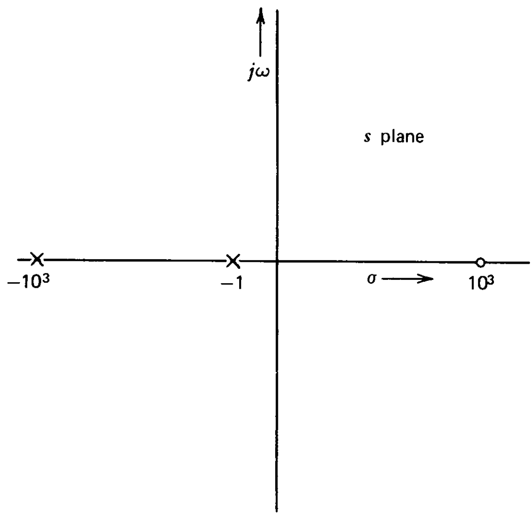


Figure 4.9 Singularities for common-emitter amplifier.

this result. The difficulty stems from the sign of the low-frequency gain. It has been assumed throughout this discussion that loop transmission is negative at low frequency so that the system has negative feedback. The rules were developed assuming the topology shown in Fig. 4.1 where negative feedback results when a_0 and f_0 have the same sign. If we consider positive feedback systems, Eqn. 4.44b must be changed to

$$\measuredangle\, a(s_1)f(s_1) = n\, 360° \tag{4.69}$$

where n is any integer, and rules evolved from the angle condition must be appropriately modified. For example, rule 2 is changed to "branches lie on the real axis to the left of an even number of real-axis singularities for positive feedback systems."

The singularity pattern shown in Fig. 4.9 corresponds to a transfer function

$$a'(s)f'(s) = \frac{a_0 f_0(10^{-3}s - 1)}{(10^{-3}s + 1)(s + 1)} = \frac{-a_0 f_0(-10^{-3}s + 1)}{(10^{-3}s + 1)(s + 1)} \tag{4.70}$$

because the vector from the zero to $s = 0$ has an angle of 180°. The sign reversal associated with the zero when plotted in the s plane diagram has changed the sign of the d-c loop transmission compared with that of Eqn. 4.68. One way to reverse the effects of this sign change is to substitute Eqn. 4.69 for Eqn. 4.44b and modify all angle-dependent rules accordingly.

A far simpler technique that works equally well for amplifiers with the right-half plane zeros located at high frequencies is to ignore these zeros when forming the root-locus diagram. Since elimination of these zeros eliminates associated sign reversals, no modification of the rules is necessary. Rule 7 insures that the diagram is not changed for moderate magnitudes of loop transmission by ignoring the high-frequency zeros.

4.3.4 Location of Closed-Loop Zeros

A root-locus diagram indicates the location of the closed-loop poles of a feedback system. In addition to the stability information provided by the pole locations, we may need the locations of the closed-loop zeros to determine some aspects of system performance.

The method used to determine the closed-loop zeros is developed with the aid of Fig. 4.10. Part *a* of this figure shows the block diagram for a single-loop feedback system. The diagram of Fig. 4.10*b* has the same input-output transfer function as that of Fig. 4.10*a*, but has been modified so that

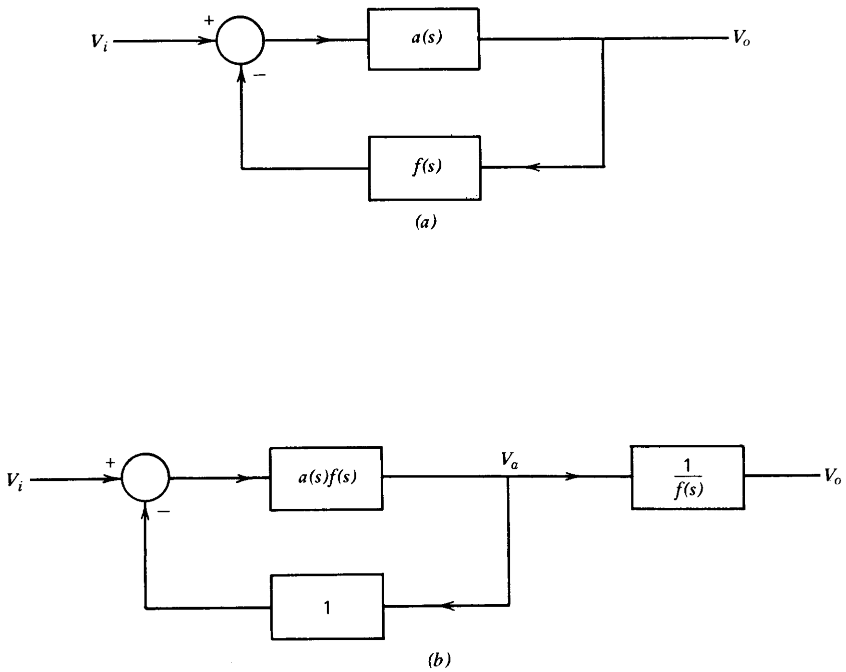


Figure 4.10 System used to determine closed-loop zeros. (*a*) Single-loop feedback system. (*b*) Modified block diagram.

the feedback path inside the loop has unity gain. We first consider the closed-loop transfer function

$$\frac{V_a(s)}{V_i(s)} = \frac{a(s)f(s)}{1 + a(s)f(s)} \tag{4.71}$$

A root-locus diagram gives the pole locations for this closed-loop transfer function directly, since the diagram indicates the frequencies at which the denominator of Eqn. 4.71 is zero. The zeros of Eqn. 4.71 coincide with the zeros of the transfer function $a(s)f(s)$. However, from Fig. 4.10*b*,

$$A(s) = \left[\frac{V_o(s)}{V_i(s)}\right] = \left[\frac{V_a(s)}{V_i(s)}\right]\left[\frac{V_o(s)}{V_a(s)}\right] = \left[\frac{V_a(s)}{V_i(s)}\right]\left[\frac{1}{f(s)}\right] \tag{4.72}$$

Thus in addition to the singularities associated with Eqn. 4.71, $A(s)$ has poles at poles of $1/f(s)$, or equivalently at zeros of $f(s)$, and has zeros at poles of $f(s)$. The additional poles of Eqn. 4.72 cancel the zeros of $f(s)$ in Eqn. 4.71, with the net result that $A(s)$ has zeros at zeros of $a(s)$ and at poles of $f(s)$. It is important to recognize that the zeros of $A(s)$ are independent of $a_0 f_0$.

An alternative approach is to recognize that zeros of $A(s)$ occur at zeros of the numerator of this function *and* at frequencies where the denominator becomes infinite while the numerator remains finite. The later condition is satisfied at poles of $f(s)$, since this term is included in the denominator of $A(s)$ but not in its numerator.

Note that the singularities of $A(s)$ are particularly easy to determine if the feedback path is frequency independent. In this case, (as always) closed-loop poles are obtained directly from the root-locus diagram. The zeros of $a(s)$, which are the only zeros plotted in the diagram when $f(s) = f_0$, are also the zeros of $A(s)$.

These concepts are illustrated by means of two examples of frequency-selective feedback amplifiers. Amplifiers of this type can be constructed by combining twin-T networks with operational amplifiers. A twin-T network can have a voltage transfer function that includes complex zeros with positive, negative, or zero real parts. It is assumed that a twin-T with a voltage-transfer ratio[7]

$$T(s) = \frac{s^2 + 1}{s^2 + 2s + 1} \tag{4.73}$$

is available.

[7] The transfer function of a twin-T network includes a third real-axis zero, as well as a third pole. Furthermore, none of the poles coincide. The departure from reality represented by Eqn. 4.73 simplifies the following development without significantly changing the conclusions. The reader who is interested in the transfer function of this type of network is referred to J. E. Gibson and F. B. Tuteur, *Control System Components*, McGraw-Hill, New York, 1958, Section 1.26.

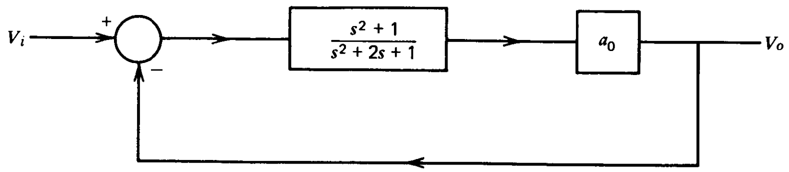


Figure 4.11 Rejection amplifier.

Figures 4.11 and 4.12 show two ways of combining this network with an amplifier that is assumed to have constant gain a_0 at frequencies of interest. Since both of these systems have the same loop transmission, they have identical root-locus diagrams as shown in Fig. 4.13. The closed-loop poles leave the real axis for any finite value of a_0 and approach the j-axis zeros along circular arcs. The closed-loop pole location for one particular value of a_0 is also indicated in this figure.

The rejection amplifier (Fig. 4.11) is considered first. Since the connection has a frequency-independent feedback path, its closed-loop zeros are the two shown in the root-locus diagram. If the signal V_i is a constant-amplitude sinusoid, the effects of the closed-loop poles and zeros very nearly cancel except at frequencies close to one radian per second. The closed-loop frequency response is indicated in Fig. 4.14*a*. As a_0 is increased, the distance between the closed-loop poles and zeros becomes smaller. Thus the band of frequencies over which the poles and zeros do not cancel becomes narrower, implying a sharper notch, as a_0 is increased.

The bandpass amplifier combines the poles from the root-locus diagram with a second-order closed-loop zero at $s = -1$, corresponding to the pole pair of $f(s)$. The closed-loop transfer function has no other zeros, since $a(s)$ has no zeros in this connection. The frequency response for this amplifier is shown in Fig. 4.14*b*. In this case the amplifier becomes more selective and provides higher gain at one radian per second as a_0 increases, since the damping ratio of the complex pole pair decreases.

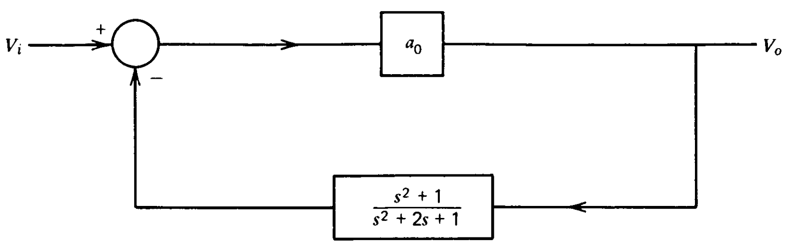


Figure 4.12 Bandpass amplifier.

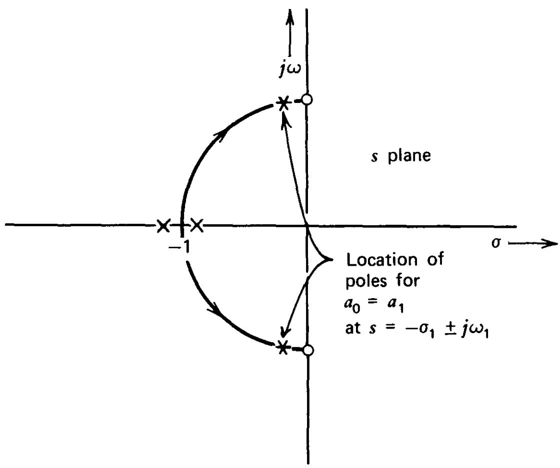


Figure 4.13 Root-locus diagram for systems of Figs. 4.11 and 4.12.

4.3.5 Root Contours

The root-locus method allows us to determine how the locations of the closed-loop poles of a feedback system change as the magnitude of the low-frequency loop transmission is varied. There are many systems where relative stability as a function of some parameter other than gain is required. We shall see, for example, that the location of an open-loop singularity in the transfer function of an operational amplifier is frequently varied to compensate the amplifier and thus improve its performance in a given application. Root-locus techniques could be used to plot a family of root-locus diagrams corresponding to various values for a system parameter other than gain. It is also possible to extend root-locus concepts so that the variation in closed-loop pole location as a function of some single parameter other than gain is determined for a fixed value of $a_0 f_0$. The generalized root-locus diagram that results from this extension is called a *root-contour* diagram.

In order to see how the root contours are constructed, we recall that the characteristic equation for a negative feedback system can be written in the form

$$P(s) = q(s) + a_0 f_0 p(s) \tag{4.74}$$

where it is assumed that

$$a(s)f(s) = a_0 f_0 \frac{p(s)}{q(s)}$$

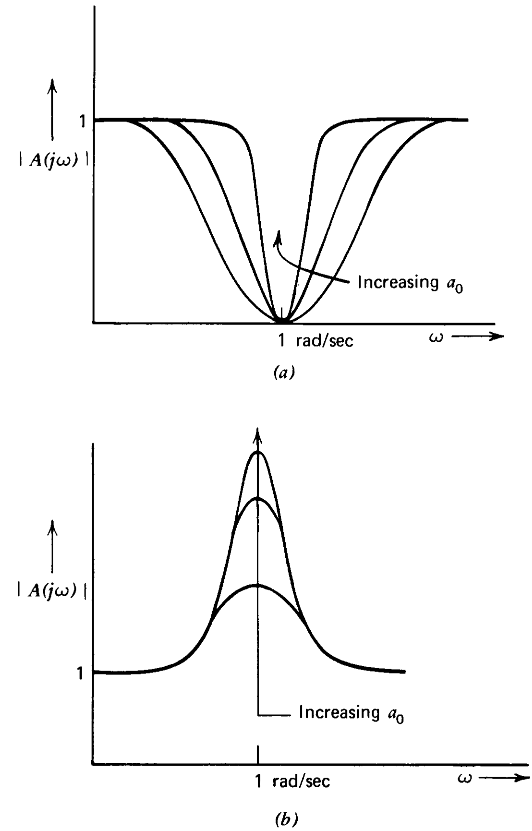


Figure 4.14. Frequency responses for selective amplifiers. (*a*) Rejection amplifier. (*b*) Bandpass amplifier.

If the $a_0 f_0$ product is constant, but some other system parameter τ varies, the characteristic equation can be rewritten

$$P(s) = q'(s) + \tau p'(s) \tag{4.75}$$

All of the terms that multiply τ are included in $p'(s)$ in Eqn. 4.75, so that $q'(s)$ and $p'(s)$ are both independent of τ. The root-contour diagram as a function of τ can then be drawn by applying the construction rules to a singularity pattern that has poles at zeros of $q'(s)$ and zeros at zeros of $p'(s)$.

An operational amplifier connected as a unity-gain follower is used to illustrate the construction of a root-contour diagram. This connection has unity feedback, and it is assumed that the amplifier open-loop transfer function is

$$a(s) = \frac{10^6(\tau s + 1)}{(s + 1)^2} \tag{4.76}$$

The characteristic equation after clearing fractions is

$$P(s) = s^2 + 2s + (10^6 + 1) + \tau 10^6 s \tag{4.77}$$

Identifying terms in accordance with Eqn. 4.75 results in

$$p'(s) = 10^6 s \tag{4.78a}$$

$$q'(s) = s^2 + 2s + 10^6 + 1 \simeq s^2 + 2s + 10^6 \tag{4.78b}$$

Thus the singularity pattern used to form the root contours has a zero at the origin and complex poles at $s = -1 \pm j10^3$. The root-contour diagram is shown in Fig. 4.15. Rule 8 is used to find the value of τ necessary to locate

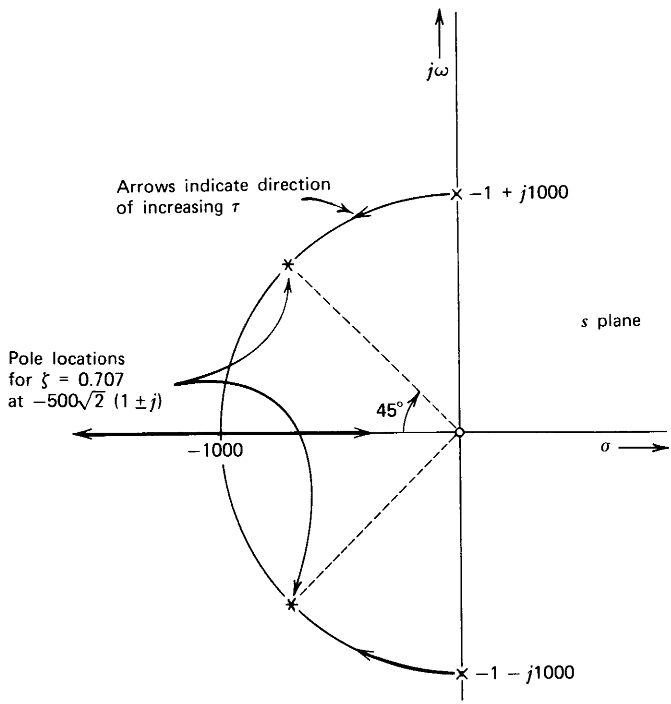


Figure 4.15 Root-contour diagram for $p'(s)/q'(s) = 10^6 s/(s^2 + 2s + 10^6)$.

the complex pole pair 45° from the negative real axis corresponding to a damping ratio of 0.707. From Eqn. 4.56, the required value is

$$\tau = \left|\frac{q'(s)}{p'(s)}\right|_{s\,=\,-500\sqrt{2}\,(1+j)}$$

$$= \left|\frac{s^2 + 2s + 10^6}{10^6 s}\right|_{s\,=\,-500\sqrt{2}\,(1+j)} = \sqrt{2} \times 10^{-3} \tag{4.79}$$

4.4 STABILITY BASED ON FREQUENCY RESPONSE

The Routh criterion and root-locus methods provide information concerning the stability of a feedback system starting with either the characteristic equation or the loop-transmission singularities of the system. Thus both of these techniques require that the system loop transmission be expressible as a ratio of polynomials in s. There are two possible difficulties. The system may include elements with transfer functions that cannot be expressed as a ratio of finite polynomials. A familiar example of this type of element is the pure time delay of τ seconds with a transfer function $e^{-s\tau}$. A second possibility is that the available information about the system consists of an experimentally determined frequency response. Approximating the measured data in a form suitable for Routh or root-locus analysis may not be practical.

The methods described in this section evaluate the stability of a feedback system starting from its loop transmission as a function of frequency. The only required data are the magnitude and angle of this transmission, and it is not necessary that these data be presented as analytic expressions. As a result, stability can be determined directly from experimental results.

4.4.1 The Nyquist Criterion

It is necessary to develop a method for determining absolute and relative stability information for feedback systems based on the variation of their loop transmissions with frequency. The topology of Fig. 4.1 is assumed. If there is some frequency ω at which

$$a(j\omega)f(j\omega) = -1 \tag{4.80}$$

the loop transmission is $+1$ at this frequency. It is evident that the system can then oscillate at the frequency ω, since it can in effect supply its own driving signal without an externally applied input. This kind of intuitive argument fails in many cases of practical interest. For example, a system with a loop transmission of $+10$ at some frequency may or may not be stable depending on the loop-transmission values at other frequencies. The Nyquist criterion can be used to resolve this and other stability questions.

The test determines if there are any values of s with positive real parts for which $a(s)f(s) = -1$. If this condition is satisfied, the characteristic equation of the system has a right-half-plane zero implying instability. In order to use the Nyquist criterion, the function $a(s)f(s)$ is evaluated as s takes on values along the contour shown in the s-plane plot of Fig. 4.16. The contour includes a segment of the imaginary axis and is closed with a large semicircle of radius R that lies in the right half of the s plane. The values of $a(s)f(s)$ as s varies along the indicated contour are plotted in gain-phase form in an af plane. A possible af-plane plot is shown in Fig. 4.17. The symmetry about the 0° line in the af plane is characteristic of all such plots since $\text{Im}[a(j\omega)f(j\omega)] = -\text{Im}[a(-j\omega)f(-j\omega)]$.

Our objective is to determine if there are any values of s that lie in the shaded region of Fig. 4.16 for which $a(s)f(s) = -1$. This determination is simplified by recognizing that the transformation involved maps closed contours in the s plane into closed contours in the af plane. Furthermore,

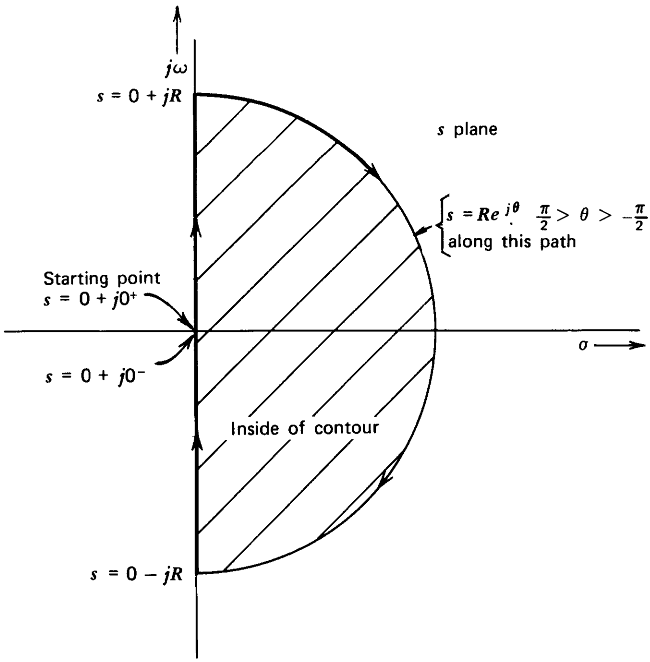


Figure 4.16 Contour Used to evaluate $a(s)f(s)$.

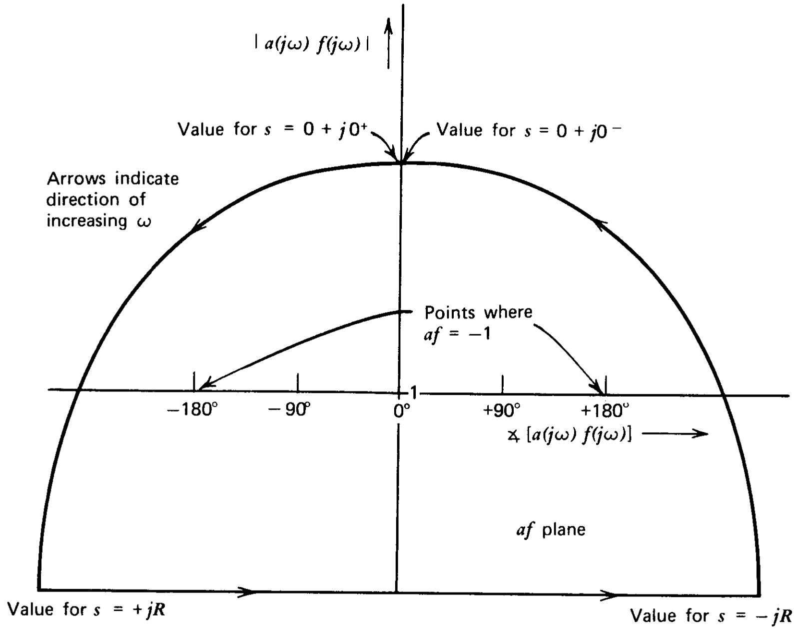


Figure 4.17 Plot of $a(s)f(s)$ as s varies along contour of Fig. 4.16.

all values of s that lie on one side of a contour in the s plane must map to values of af that lie on one side of the corresponding contour in the af plane. The -1 points are clearly indicated in the af-plane plot. Thus the only remaining task is to determine if the shaded region in Fig. 4.16 maps to the inside or to the outside of the contour in Fig. 4.17. If it maps to the inside, there are two values of s in the right-half plane for which $a(s)f(s) = -1$, and the system is unstable.

The form of the af-plane plot and corresponding regions of the two plots are easily determined from $a(s)f(s)$ as illustrated in the following examples. Figure 4.18 indicates the general shape of the s-plane and af-plane plots for

$$a(s)f(s) = \frac{10^3}{(s + 1)(0.1s + 1)(0.01s + 1)} \tag{4.81}$$

Note that the magnitude of af in this example is 10^3 and its angle is zero at $s = 0$. As s takes on values approaching $+jR$, the angle of af changes from 0° toward $-270°$, and its magnitude decreases. These relationships are readily obtained from the usual vector manipulations in the s plane. For a sufficiently large value of R, the magnitude of af is arbitrarily small,

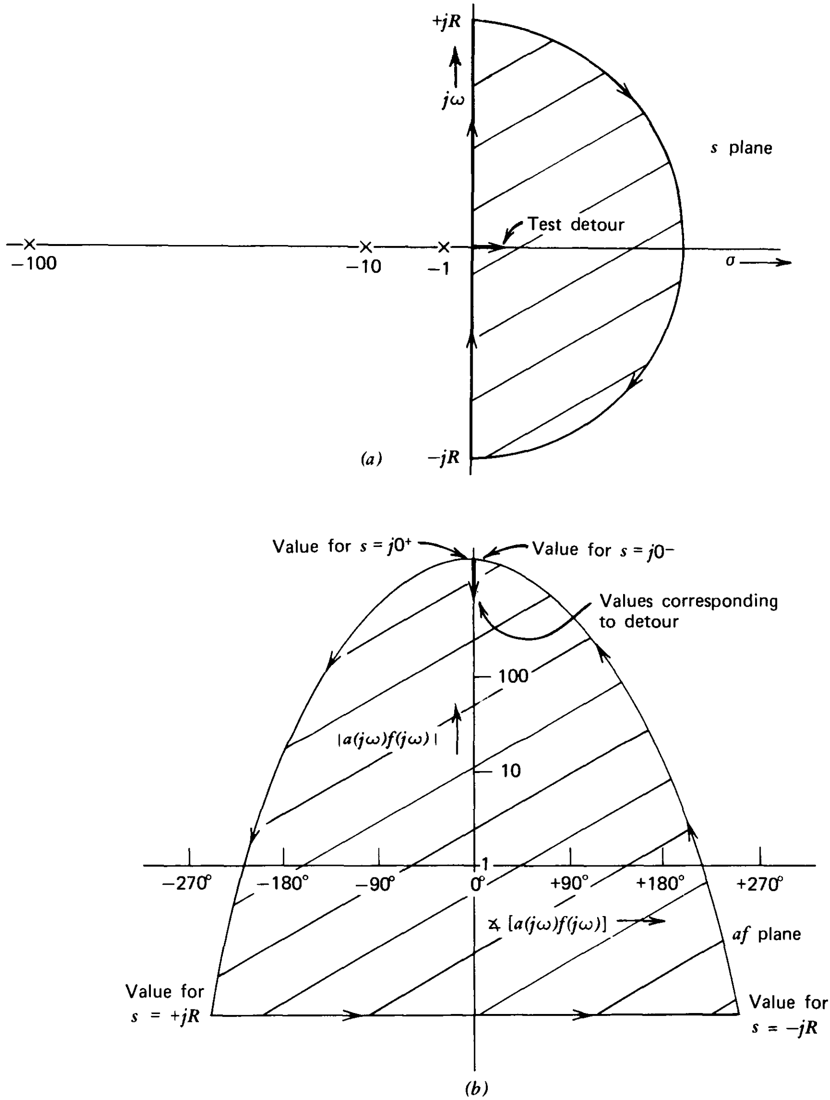


Figure 4.18 Nyquist test for $a(s)f(s) = 10^3/[(s + 1)(0.1s + 1)(0.01s + 1)]$. (*a*) s-plane plot. (*b*) af-plane plot.

and its angle is nearly $-270°$. As s assumes values in the right-half plane along a semicircle of radius R, the magnitude of af remains constant (for R much greater than the distance of any singularities of af from the origin), and its angle changes from $-270°$ to $0°$ as s goes from $+jR$ to $+R$. The remainder of the af-plane plot must be symmetric about the $0°$ line.

In order to show that the two shaded regions correspond to each other, a small detour from the contour in the s plane is made at $s = 0$ as indicated in Fig. 4.18a. As s assumes real positive values, the magnitude of $a(s)f(s)$ decreases, since the distance from the point on the test detour to each of the poles increases. Thus the detour produces values in the af plane that lie in the shaded region. While we shall normally use a test detour to determine corresponding regions in the two planes, the angular relationships indicated in this example are general ones. Because of the way axes are chosen in the two planes, right-hand turns in one plane map to left-hand turns in the other. A consequence of this reversal is illustrated in Fig. 4.18. Note that if we follow the contour in the s plane in the direction of the arrows, the shaded region is to our right. The angle reversal places the corresponding region in the af plane to the left when its boundary is followed in the direction of the arrows.

Since the two -1 points lie in the shaded region of the af plane, there are two values of s in the right-half plane for which $a(s)f(s) = -1$ and the system is unstable. Note that if a_0f_0 is reduced, the contour in the af plane slides downward and for sufficiently small values of a_0f_0 the system is stable. A geometric development or the Routh criterion shows that the system is stable for positive values of a_0f_0 smaller than 122.21.

Contours with the general shape shown in Fig. 4.19 result if a zero is added at the origin changing $a(s)f(s)$ to

$$a(s)f(s) = \frac{10^3 s}{(s + 1)(0.1s + 1)(0.01s + 1)} \tag{4.82}$$

In order to avoid angle and magnitude uncertainties that result if the s-plane contour passes through a singularity, a small-radius circular arc is used to avoid the zero. Two test detours on the s-plane contour are shown. As the first is followed, the magnitude of af increases since the dominant effect is that of leaving the zero. As the second test detour is followed, the magnitude of af increases since this detour approaches three poles and only one zero. The location of the shaded region in the af plane indicates that the -1 points remain outside this region for all positive values of a_0 and, therefore, the system is stable for any amount of negative feedback.

The Nyquist test can also be used for systems that have one or more loop-transmission poles in the right-half plane and thus are unstable without

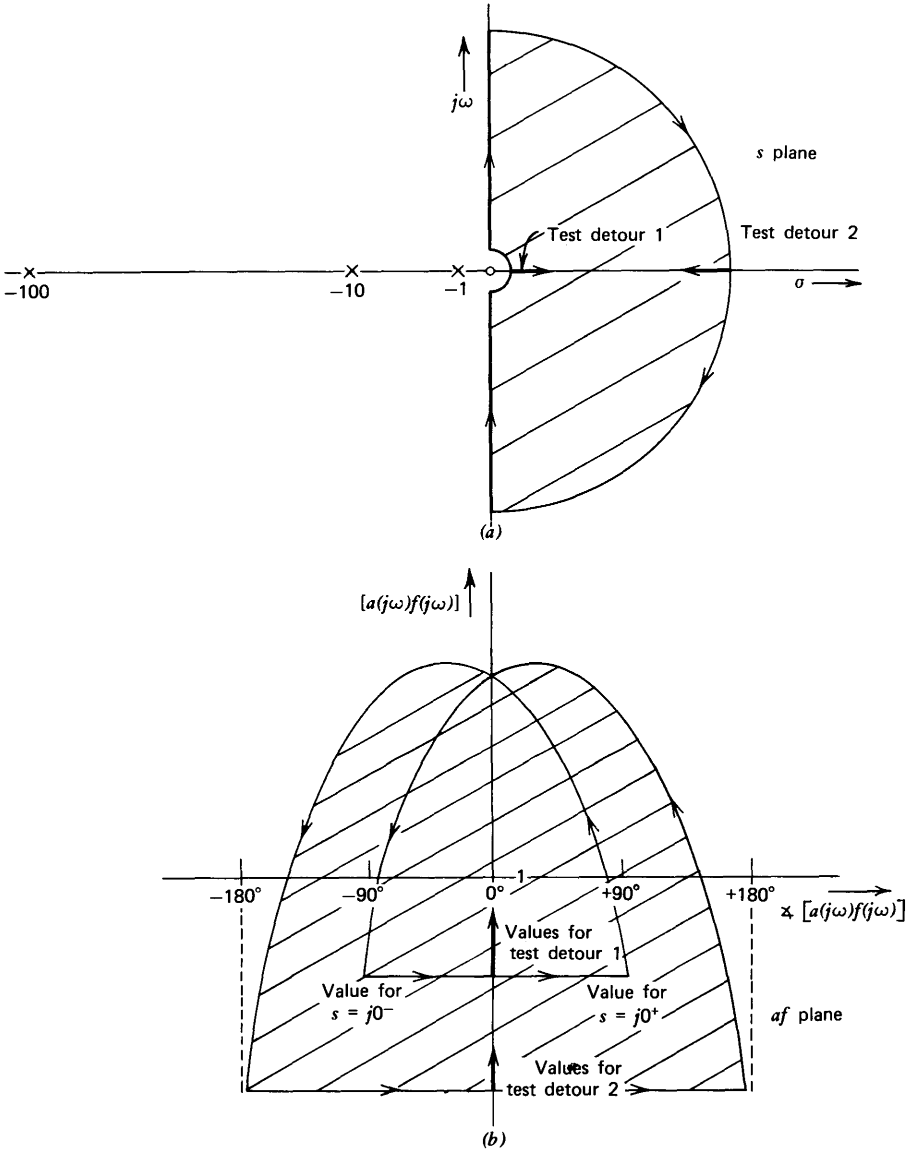


Figure 4.19 Nyquist test for $a(s)f(s) = 10^3 s/[(s + 1)(0.1s + 1)(0.01s + 1)]$. (*a*) *s*-plane plot. (*b*) *af*-plane plot.

feedback. An example of this type of system results for

$$a(s)f(s) = \frac{a_0}{s - 1} \tag{4.83}$$

with s-plane and af-plane plots shown in Figs. 4.20*a* and 4.20*b*. The line indicated by + marks in the af-plane plot is an attempt to show that for this system the angle must be continuous as s changes from $j0^-$ to $j0^+$. In order to preserve this necessary continuity, we must realize that $+180°$ and $-180°$ are identical angles, and conceive of the af plane as a cylinder joined at the $\pm 180°$ lines. This concept is made somewhat less disturbing by using polar coordinates for the af-plane plot as shown in Fig. 4.20*c*. Here the -1 point appears only once. The use of the test detour shows that values of s in the right-half plane map outside of a circle that extends from 0 to $-a_0$ as shown in Fig. 4.20*c*. The location of the -1 point in either af-plane plot shows that the system is stable only for $a_0 > 1$.

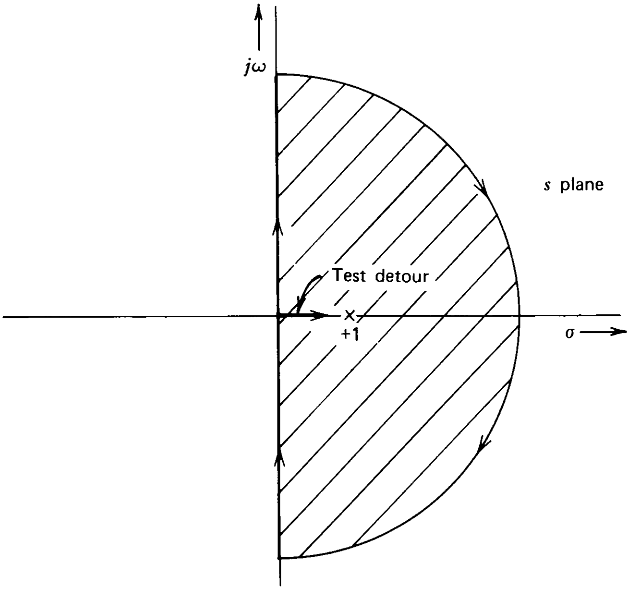
Note that the -1 points in the af plane corresponding to angles of $\pm 180°$ collapse to one point when the af cylinder necessary for the Nyquist construction for this example is formed. This feature and the nature of the af contour show that when a_0 is less than one, there is only one value of s for which $a(s)f(s) = -1$. Thus this system has a single closed-loop pole on the positive real axis for values of a_0 that result in instability.

This system indicates another type of difficulty that can be encountered with systems that have right-half-plane loop-transmission singularities. The angle of $a(j\omega)f(j\omega)$ is 180° at low frequencies, implying that the system actually has positive feedback at these frequencies. (Recall the additional inversion included at the summation point in our standard representation.) The s-plane representation (Fig. 4.20*a*) is consistent since it indicates an angle of 180° for $s = 0$. Thus no procedural modification of the type described in Section 4.3.3 is necessary in this case.

4.4.2 Interpretation of Bode Plots

A Bode plot does not contain the information concerning values of af as the contour in the s plane is closed, which is necessary to apply the Nyquist test. Experience shows that the easiest way to determine stability from a Bode plot of an arbitrary loop transmission is to roughly sketch a complete af-plane plot and apply the Nyquist test as described in Section 4.4.1. For many systems of practical interest, however, it is possible to circumvent this step and use the Bode information directly.

The following two rules evolve from the Nyquist test for systems that have negative feedback at low or mid frequencies and that have no right-half-plane singularities in their loop transmission.



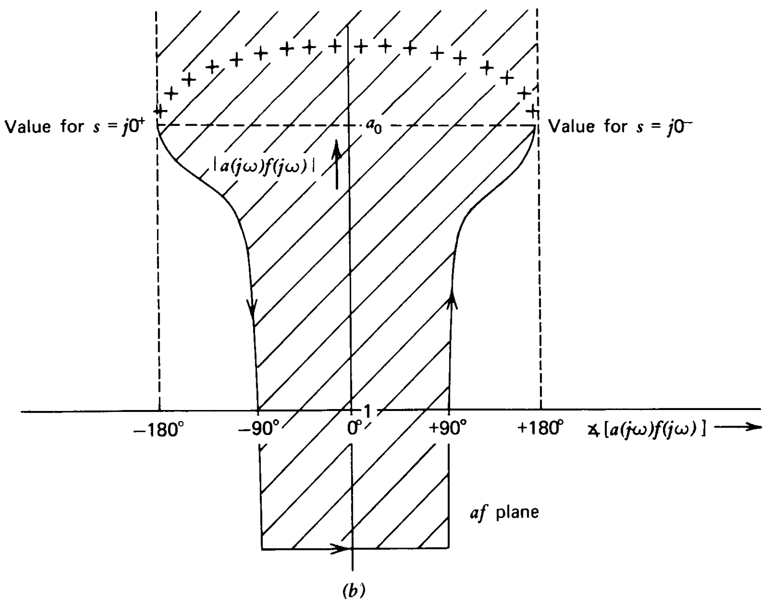


Figure 4.20 Nyquist test for $a(s)f(s) = a_0/(s - 1)$. (*a*) *s*-plane plot. (*b*) *af*-plane plot. (*c*) *af*-plane plot (polar coordinates).

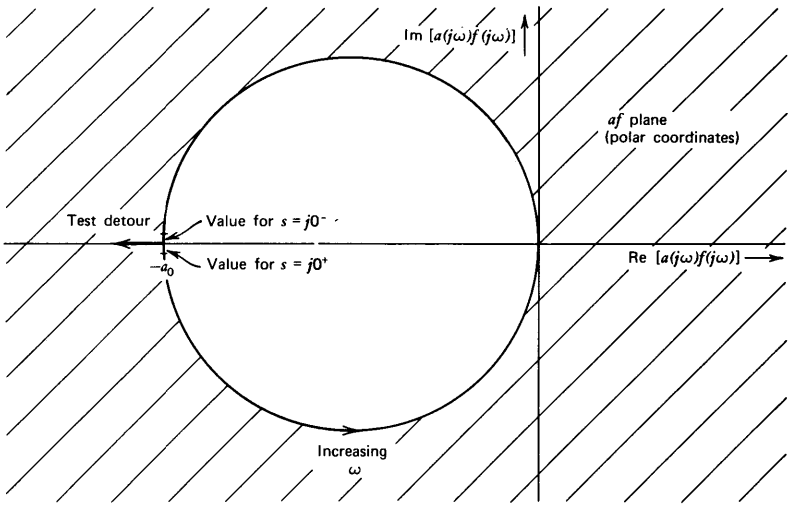


Figure 4.20—Continued

1. If the magnitude of *af* is 1 at only one frequency, the system is stable if the angle of *af* is between +180° and −180° at the unity-gain frequency.
2. If the angle of *af* passes through +180° or −180° at only one frequency, the system is stable if the magnitude of *af* is less than 1 at this frequency.

Information concerning the relative stability of a feedback system can also be determined from a Bode plot for the following reason. The values of s for which $af = -1$ are the closed-loop pole locations of a feedback system. The Nyquist test exploits this relationship in order to determine the absolute stability of a system. If the system is stable, but a pair of −1's of *af* occur for values of s close to the imaginary axis, the system must have a pair of closed-loop poles with a small damping ratio.

The quantities shown in Fig. 4.21 provide a useful estimation of the proximity of −1's of *af* to the imaginary axis and thus indicate relative stability. The *phase margin* is the difference between the angle of *af* and −180° at the frequency where the magnitude of *af* is 1. A phase margin of 0° indicates closed-loop poles on the imaignary axis, and therefore the phase margin is a measure of the additional negative phase shift at the unity-magnitude frequency that will cause instability. Similarly, the *gain margin* is the amount of gain increase required to make the magnitude of *af* unity at the frequency where the angle of *af* is −180°, and represents the

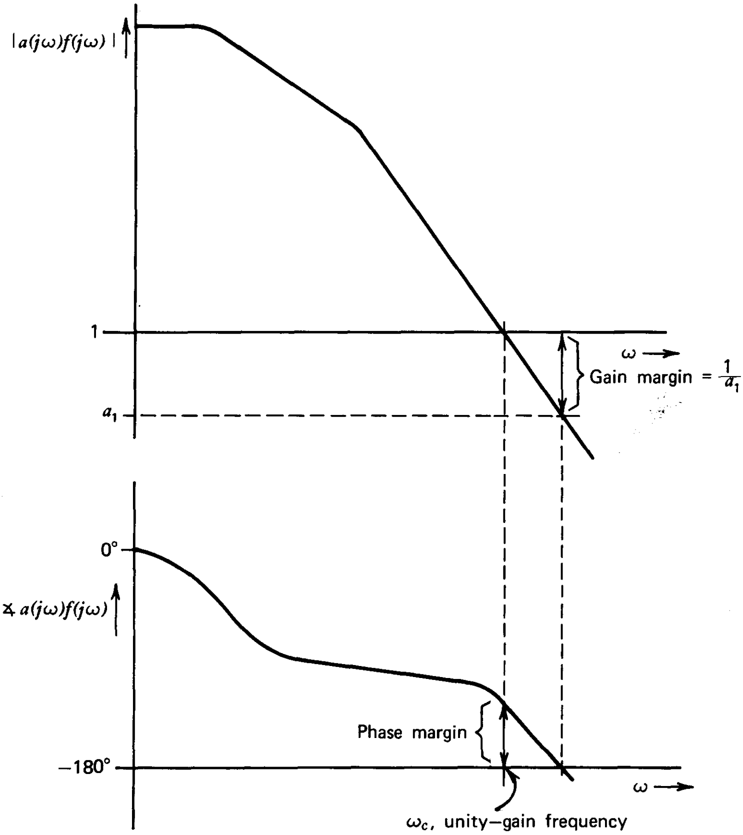


Figure 4.21 Loop-transmission quantities.

amount of increase in $a_0 f_0$ required to cause instability. The frequency at which the magnitude of af is unity is called the *unity-gain frequency* or the *crossover frequency*. This parameter characterizes the relative frequency response or speed of the time response of the system.

A particularly valuable feature of analysis based on the loop-transmission characteristics of a system is that the gain margin and the phase margin, quantities that are quickly and easily determined using Bode techniques, give surprisingly good indications of the relative stability of a feedback system. It is generally found that gain margins of three or more combined with phase margins between 30 and 60° result in desirable trade-offs between bandwidth or rise time and relative stability. The smaller values for gain and phase margin correspond to lower relative stability and are avoided if small overshoot in response to a step or small frequency-response peaking is necessary or if there is the possibility of severe changes in parameter values.

The closed-loop bandwidth and rise time are almost directly related to the unity-gain frequency for systems with equal gain and phase margins. Thus any changes that increase the unity-gain frequency while maintaining constant values for gain and phase margins tend to increase closed-loop bandwidth and decrease closed-loop rise time.

Certain relationships between these three quantities and the corresponding closed-loop performance are given in the following section. Prior to presenting these relationships, it is emphasized that the simplicity and excellence of results associated with frequency-response analysis makes this method a frequently used one, particularly during the initial design phase. Once a tentative design based on these concepts is determined, more detailed information, such as the exact location of closed-loop singularities or the transient response of the system may be investigated, frequently with the aid of machine computation.

4.4.3 Closed-Loop Performance in Terms of Loop-Transmission Parameters

The quantity $a(j\omega)f(j\omega)$ can generally be quickly and accurately obtained in Bode-plot form. The effects of system-parameter changes on the loop transmission are also easily determined. Thus approximate relationships between the loop transmission and closed-loop performance provide a useful and powerful basis for feedback-system design.

The input-output relationship for a system of the type illustrated in Fig. 4.10*a* is

$$A(s) = \frac{V_o(s)}{V_i(s)} = \frac{a(s)}{1 + a(s)f(s)} \tag{4.84}$$

If the system is stable, the closed-loop transfer function of the system can be approximated for limiting values of loop transmission as

$$A(j\omega) \simeq \frac{1}{f(j\omega)} \qquad |a(j\omega)f(j\omega)| \gg 1 \tag{4.85a}$$

$$A(j\omega) \simeq a(j\omega) \qquad |a(j\omega)f(j\omega)| \ll 1 \tag{4.85b}$$

One objective in the design of feedback systems is to insure that the approximation of Eqn. 4.85a is valid at all frequencies of interest, so that the system closed-loop gain is controlled by the feedback element. The approximation of Eqn. 4.85b is relatively unimportant, since the system is effective operating without feedback in this case. While we normally do not expect to have the system provide precisely controlled closed-loop gain at frequencies where the magnitude of the loop transmission is close to one, the discussion of Section 4.4.2 shows that the relative stability of a system is largely determined by its performance in this frequency range.

The *Nichols chart* shown in Fig. 4.22 provides a convenient method of evaluating the closed-loop gain of a feedback system from its loop trans-

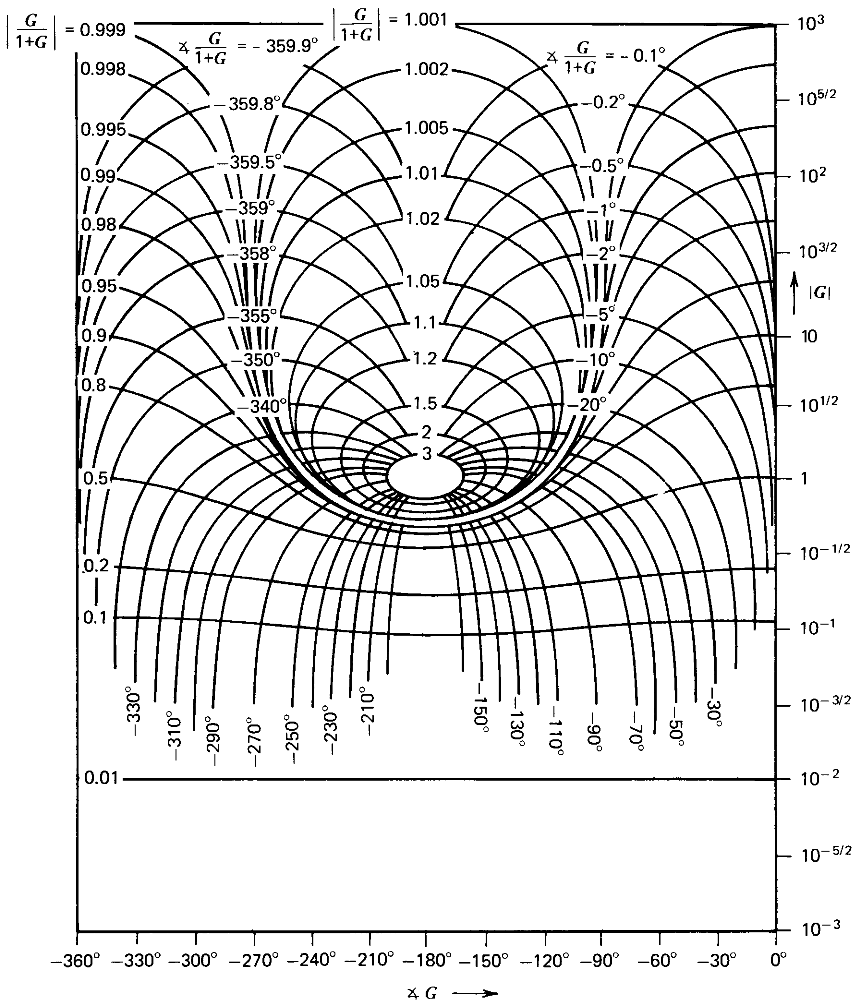


Figure 4.22 Nichols chart.

mission, and is particularly valuable when neither of the limiting approximations of Eqn. 4.85 is valid. This chart relates $G/(1 + G)$ to G where G is any complex number. In order to use the chart, the value of G is located on the rectangular gain-phase coordinates. The angle and magnitude of $G/(1 + G)$ are than read directly from the curved coordinates that intersect the value of G selected.

The gain-phase coordinates shown in Fig. 4.22 cover the complete 0° to −360° range in angle and a ratio of 10^6 in magnitude. This magnitude range is unnecessary, since the approximations of Eqn. 4.85 are usually valid when the loop-transmission magnitude exceeds 10 or is less than 0.1. Similarly, the range of angles of greatest interest is that which surrounds the −180° value and which includes anticipated phase margins. The Nichols chart shown in Fig. 4.23 is expanded to provide greater resolution in the region where it will normally be used.

One effective way to view the Nichols chart is as a three-dimensional surface, with the height of the surface proportional to the magnitude of the closed-loop transfer function corresponding to the loop-transmission parameters that define the point of interest. This visualization shows a "mountain" (with a peak of infinite height) where the loop transmission is +1.

The Nichols chart can be used directly for any unity-gain feedback system. The transformation indicated in Fig. 4.10*b* shows that the chart can be used for arbitrary single-loop systems by observing that

$$A(j\omega) = \frac{a(j\omega)}{1 + a(j\omega)f(j\omega)} = \left[\frac{a(j\omega)f(j\omega)}{1 + a(j\omega)f(j\omega)}\right]\left[\frac{1}{f(j\omega)}\right] \tag{4.86}$$

The closed-loop frequency response is determined by multiplying the factor $a(j\omega)f(j\omega)/[1 + a(j\omega)f(j\omega)]$ obtained via the Nichols chart by $1/f(j\omega)$ using Bode techniques.

One quantity of interest for feedback systems with frequency-independent feedback paths is the peak magnitude M_p, equal to the ratio of the maximum magnitude of $A(j\omega)$ to its low-frequency magnitude (see Section 3.5). A large value for M_p indicates a relatively less stable system, since it shows that there is some frequency for which the characteristic equation approaches zero and thus that there is a pair of closed-loop poles near the imaginary axis at approximately the peaking frequency. Feedback amplifiers are frequently designed to have M_p's between 1.1 and 1.5. Lower values for M_p imply greater relative stability, while higher values indicate that stability has been compromised in order to obtain a larger low-frequency loop transmission and a higher crossover frequency.

The value of M_p for a particular system can be easily determined from the Nichols chart. Furthermore, the chart can be used to evaluate the

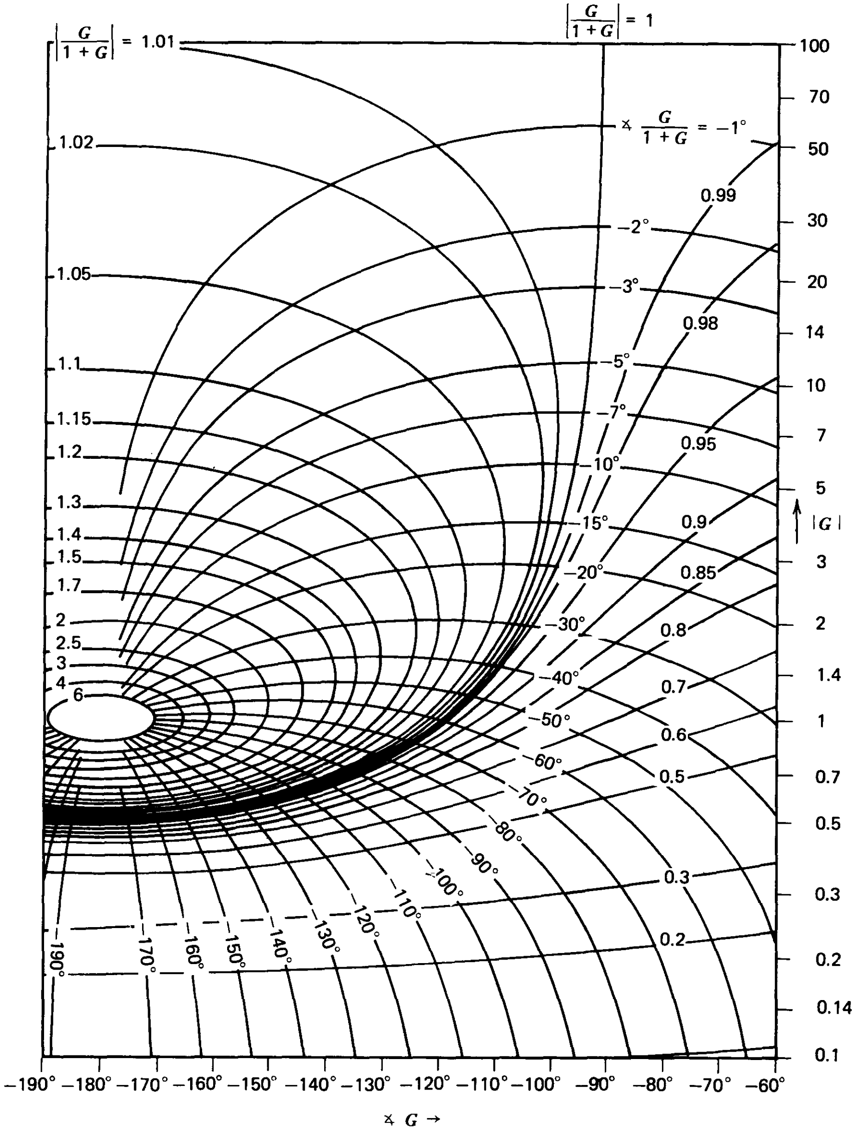


Figure 4.23 Expanded Nichols chart.

effects of variations in loop transmission on M_p. One frequently used manipulation determines the relationship between M_p and a_0f_0 for a system with fixed loop-transmission singularities. The quantity $a(j\omega)f(j\omega)/a_0f_0$ is first plotted on gain-phase coordinates using the same scale as the Nichols chart. If this plot is made on tracing paper, it can be aligned with the Nichols chart and slid up or down to illustrate the effects of different values of a_0f_0. The closed-loop transfer function is obtained directly from the Nichols chart by evaluating $A(j\omega)$ at various frequencies, while the highest magnitude curve of the Nichols chart touched by $a(j\omega)f(j\omega)$ for a particular value of a_0f_0 indicates the corresponding M_p.

Figure 4.24 shows this construction for a system with $f = 1$ and

$$a(s) = \frac{a_0}{(s + 1)(0.1s + 1)} \tag{4.87}$$

The values of a_0 for the three loop transmissions are 8.5, 22, and 50. The corresponding M_p's are 1, 1.4, and 2, respectively.

While the Nichols chart is normally used to determine the closed-loop function from the loop transmission, it is possible to use it to go the other way; that is, to determine $a(j\omega)f(j\omega)$ from $A(j\omega)$. This transformation is occasionally useful for the analysis of systems for which only closed-loop measurements are practical. The transformation yields good results when the magnitude of $a(j\omega)f(j\omega)$ is close to one. Furthermore, the approximation of Eqn. 4.85b shows tha $A(j\omega) \simeq a(j\omega)$ when the magnitude of the loop transmission is small. However, Eqn. 4.85a indicates that $A(j\omega)$ is essentially independent of the loop transmission when the loop-transmission magnitude is large. Examination of the Nichols chart confirms this result since it shows that very small changes in the closed-loop magnitude or angle translate to very large changes in the loop transmission for large loop-transmission magnitudes. Thus even small errors in the measurement of $A(j\omega)$ preclude estimation of large values for $a(j\omega)f(j\omega)$ with any accuracy.

The relative stability of a feedback system and many other important characteristics of its closed-loop response are largely determined by the behavior of its loop transmission at frequencies where the magnitude of this quantity is close to unity. The approximations presented below relate closed-loop quantities defined in Section 3.5 to the loop-transmission properties defined in Section 4.4.2. These approximations are useful for predicting closed-loop response, comparing the performance of various systems, and estimating the effects of changes in loop transmission on closed-loop performance.

The assumptions used in Section 3.5, in particular that f is one at all

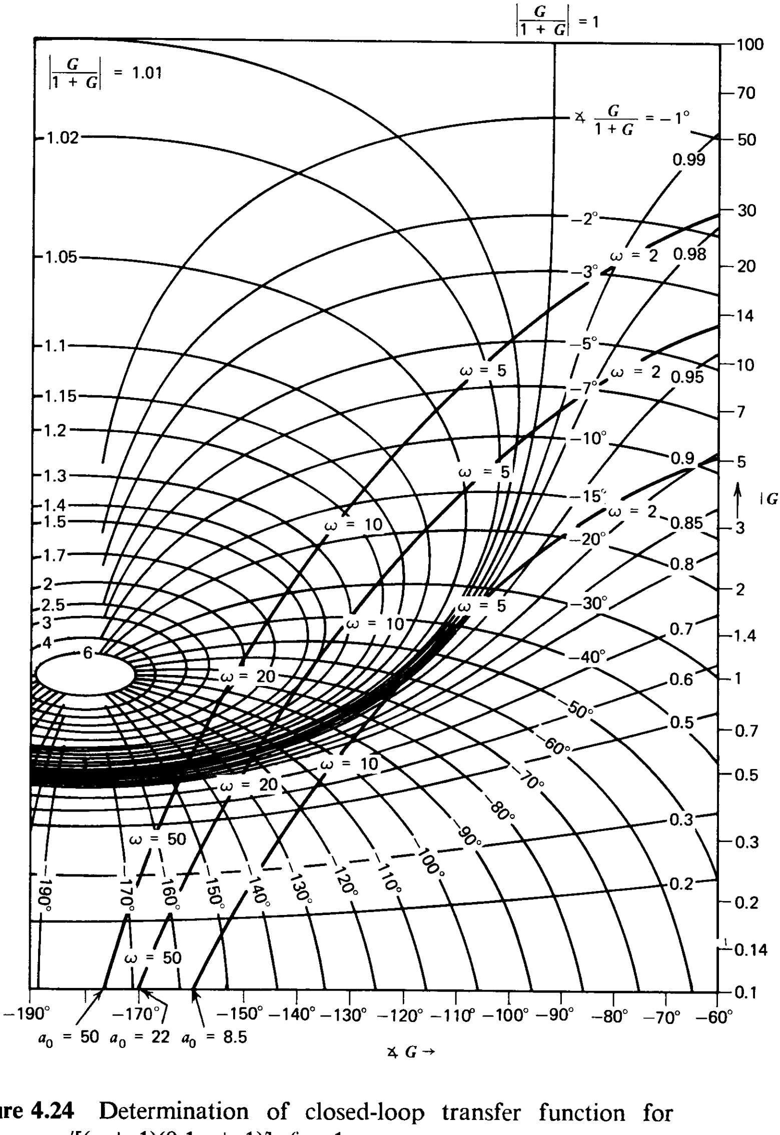


Figure 4.24 Determination of closed-loop transfer function for $a(s) = a_0/[(s+1)(0.1s+1)], f = 1$.

frequencies, that a_0 is large, and that the lowest frequency singularity of $a(s)$ is a pole, are assumed here. Under these conditions,

$$M_p \simeq \frac{1}{\sin \phi_m} \tag{4.88}$$

where ϕ_m is the phase margin. The considerations that lead to this approximation are illustrated in Fig. 4.25. This figure shows several closed-loop-magnitude curves in the vicinity of $M_p = 1.4$ and assumes that the system phase margin is 45°. Since the point $|G| = 1$, $\measuredangle G = -135°$ must exist

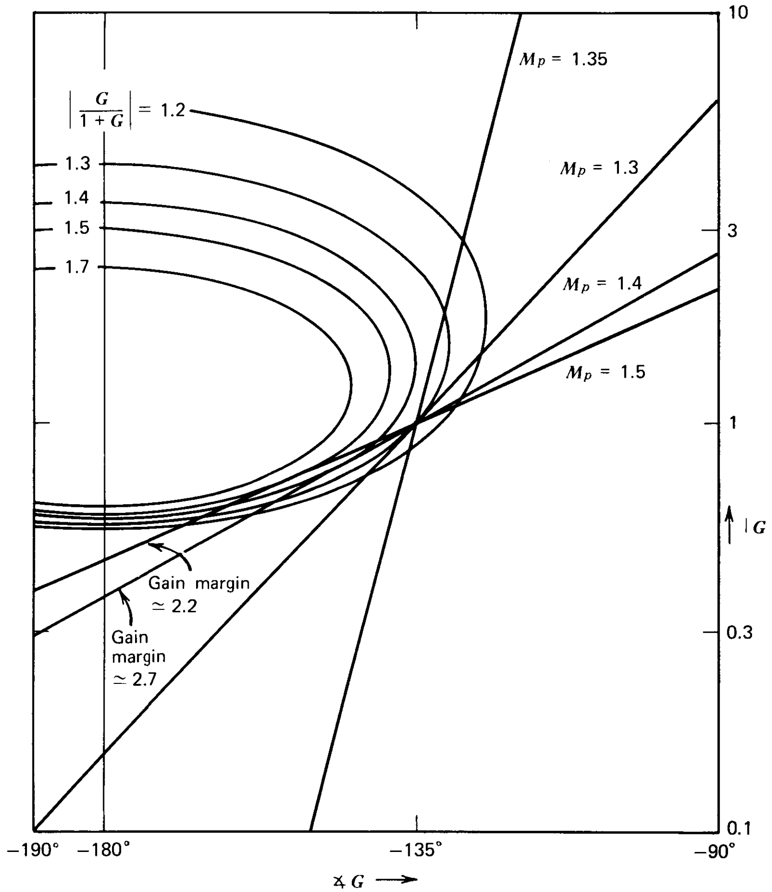


Figure 4.25 M_p for several systems with 45° of phase margin.

for a system with a 45° phase margin, there is no possible way that M_p can be less than approximately 1.3, and the loop-transmission gain-phase curve must be quite specifically constrained for M_p just to equal this value. If it is assumed that the magnitude and angle of G are linearly related, the linear constructions included in Fig. 4.25 show that M_p cannot exceed approximately 1.5 unless the gain margin is very small. Well-behaved systems are actually most likely to have a gain-phase curve that provides an extended region of approximate tangency to the $M_p = 1.4$ curve for a phase margin of 45°. Similar arguments hold for other values of phase margin, and the approximation of Eqn. 4.88 represents a good fit to the relationship between phase margin and corresponding M_p.

Two other approximations relate the system transient response to its crossover frequency ω_c.

$$\frac{0.6}{\omega_c} < t_r < \frac{2.2}{\omega_c} \tag{4.89}$$

The shorter values of rise time correspond to lower values of phase margin.

$$t_s > \frac{4}{\omega_c} \tag{4.90}$$

The limit is approached only for systems with large phase margins.

We shall see that the open-loop transfer function of many operational amplifiers includes one pole at low frequencies and a second pole in the vicinity of the unity-gain frequency of the amplifier. If the system dynamics are dominated by these two poles, the damping ratio and natural frequency of a second-order system that approximates the actual closed-loop system can be obtained from Bode-plot parameters of a system with a frequency-independent feedback path using the curves shown in Fig. 4.26*a*. The curves shown in Fig. 4.26*b* relate peak overshoot and M_p for a second-order system to damping ratio and are derived using Eqns. 3.58 and 3.62. While the relationships of Fig. 4.26*a* are strictly valid only for a system with two widely spaced poles in its loop transmission, they provide an accurate approximation providing two conditions are satisfied.

1. The system loop-transmission magnitude falls off as $1/\omega$ at frequencies between one decade below crossover and the next higher frequency singularity.
2. Additional negative phase shift is provided in the vicinity of the crossover frequency by other components of the loop transmission.

The value of these curves is that they provide a way to determine an approximating second-order system from either phase margin, M_p, or peak

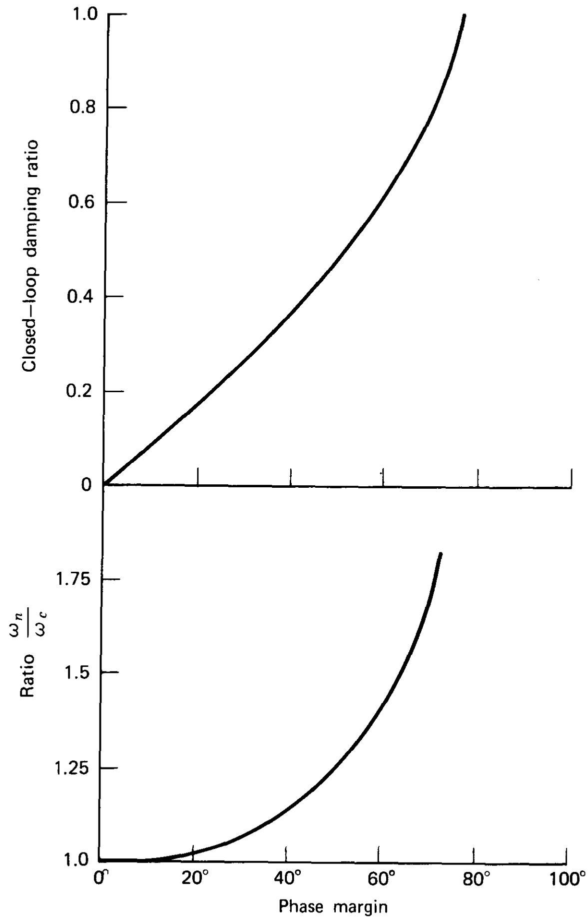


Figure 4.26*a* Closed-loop quantities from loop-transmission parameters for system with two widely spaced poles. Damping ratio and natural frequency as a function of phase margin and crossover frequency.

overshoot of a complex system. The validity of this approach stems from the fact that most systems must be dominated by one or two poles in the vicinity of the crossover frequency in order to yield acceptable performance. Examples illustrating the use of these approximations are included in later sections. We shall see that transient responses based on the approximation are virtually indistinguishable from those of the actual system in many cases of interest.

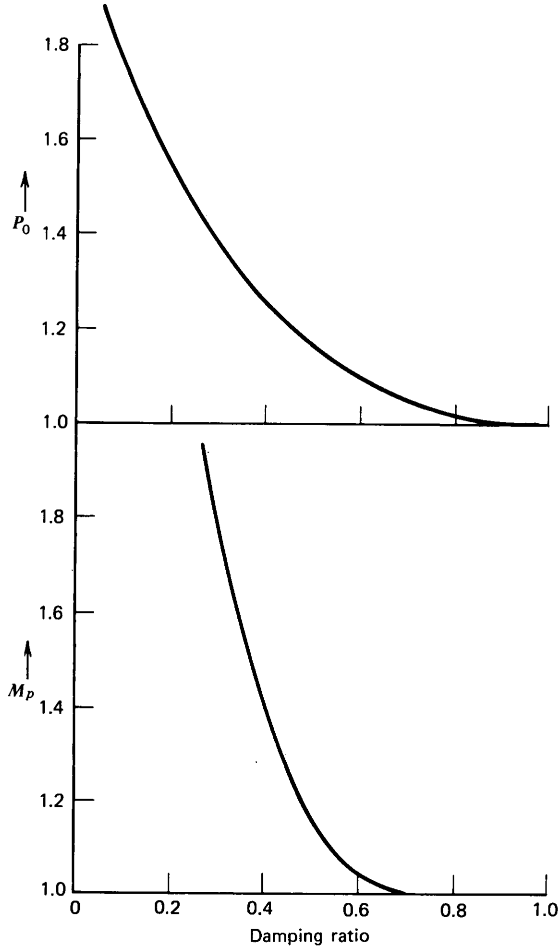


Figure 4.26*b* P_0 and M_p versus damping ratio for second-order system.

The first significant error coefficient for a system with unity feedback can also be determined directly from its Bode plot. If the loop transmission includes a wide range of frequencies below the crossover frequency where its magnitude is equal to k/ω^n, the error coefficients e_0 through e_{n-1} are negligible and e_n equals $1/k$.

PROBLEMS

P4.1

Find the number of right-half-plane zeros of the polynomial

$$P(s) = s^5 + s^4 + 3s^3 + 4s^2 + s + 2$$

P4.2

A phase-shift oscillator is constructed with a loop transmission

$$L(s) = -\frac{a_0}{(\tau s + 1)^4}$$

Use the Routh condition to determine the value of a_0 that places a pair of closed-loop poles on the imaginary axis. Also determine the location of the poles. Use this information to factor the characteristic equation of the system, thus finding the location of all four closed-loop poles for the critical value of a_0.

P4.3

Describe how the Routh test can be modified to determine the real parts of all singularities in a polynomial. Also explain why this modification is usually of little value as a computational aid to factoring the polynomial.

P4.4

Prove the root-locus construction rule that establishes the angle and intersection of branch asymptotes with the real axis.

P4.5

Sketch root-locus diagrams for the loop-transmission singularity pattern shown in Fig. 4.27. Evaluate part *c* for moderate values of $a_0 f_0$, and part *d* for both moderate and very large values of $a_0 f_0$.

P4.6

Consider two systems, both with $f = 1$. One of these systems has a forward-path transfer function

$$a(s) = \frac{a_0(0.5s + 1)}{(s + 1)(0.01s + 1)(0.51s + 1)}$$

while the second system has

$$a'(s) = \frac{a_0(0.51s + 1)}{(s + 1)(0.01s + 1)(0.5s + 1)}$$

Common sense dictates that the closed-loop transfer functions of these systems should be very nearly identical and, furthermore, that both should be similar to a system with

$$a''(s) = \frac{a_0}{(s + 1)(0.01s + 1)}$$

[The closely spaced pole-zero doublets in $a(s)$ and $a'(s)$ should effectively cancel out.] Use root-locus diagrams to show that the closed-loop responses are, in fact, similar.

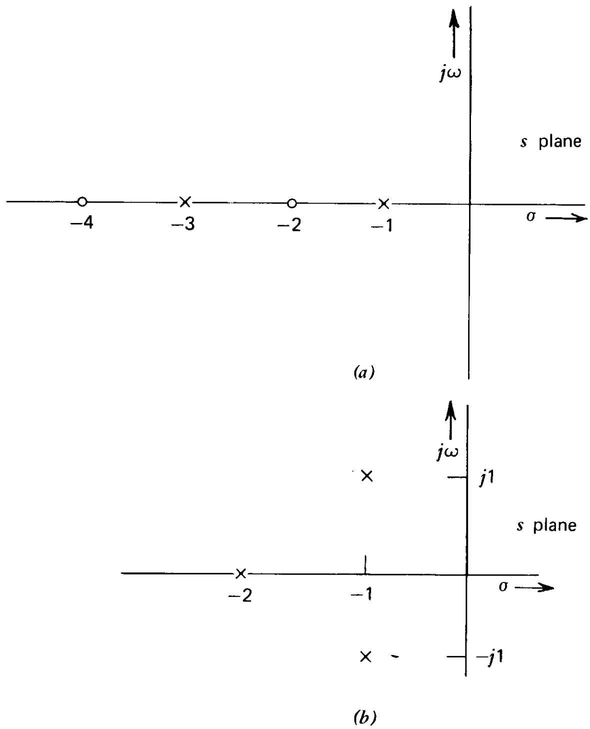


Figure 4.27 Loop-transmission singularity patterns.

P4.7

An operational amplifier has an open-loop transfer function

$$a(s) = \frac{10^6}{(0.1s + 1)(10^{-6}s + 1)^2}$$

This amplifier is combined with two resistors in a noninverting-amplifier configuration. Neglecting loading, determine the value of closed-loop gain that results when the damping ratio of the complex closed-loop pole pair is 0.5.

P4.8

An operational amplifier has an open-loop transfer function

$$a(s) = \frac{10^5}{(\tau s + 1)(10^{-6}s + 1)}$$

The quantity τ can be adjusted by changing the amplifier compensation.

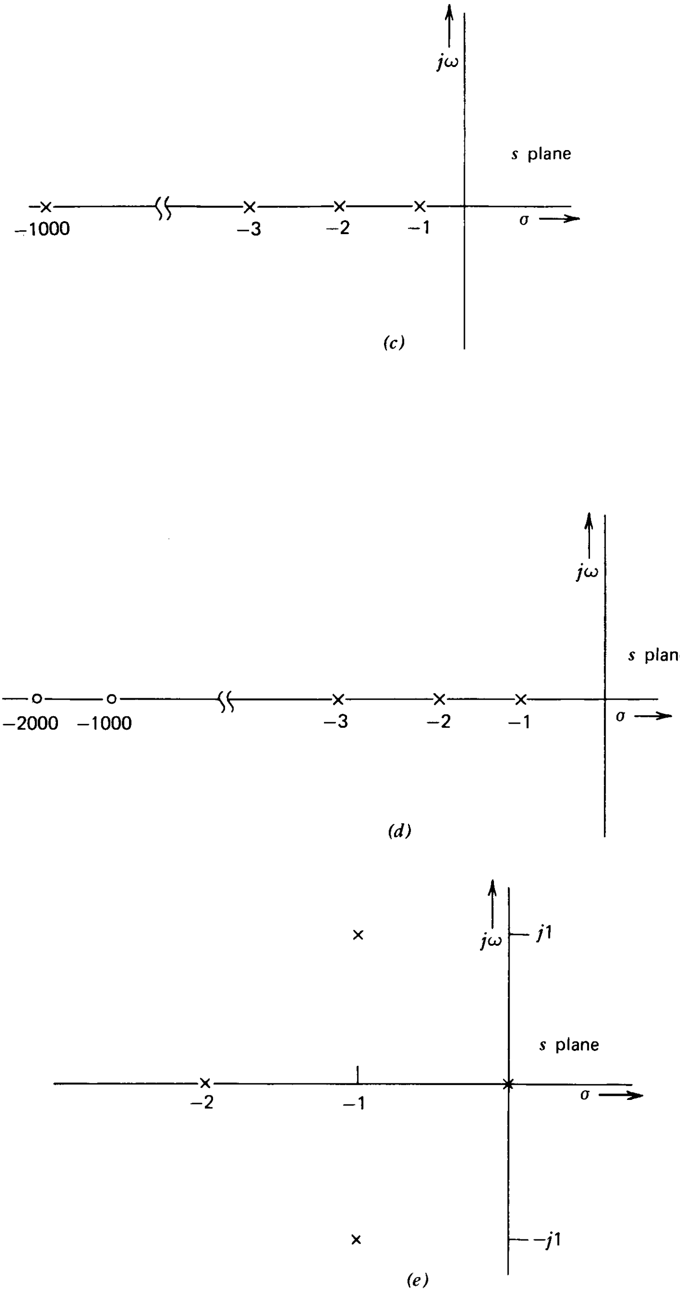


Figure 4.27—Continued

Use root-contour techniques to determine a value of τ that results in a closed-loop damping ratio of 0.707 when the amplifier is connected as a unity-gain inverter.

P4.9

A feedback system that includes a time delay has a loop transmission

$$L(s) = -\frac{a_0 e^{-0.01s}}{(s+1)}$$

Use the Nyquist test to determine the maximum value of a_0 for stable operation. What value of a_0 should be selected to limit M_p to a factor of 1.4? (You may assume that the feedback path of the system is frequency independent.)

P4.10

We have been investigating the stability of feedback systems that are generally low pass in nature, since the transfer functions of most operational-amplifier connections fall in this category. However, stability problems also arise in high-pass systems. For example, a-c coupled feedback amplifiers designed for use at audio frequencies sometimes display a low-frequency instability called "motor-boating." Use the Nyquist test to demonstrate the possibility of this type of instability for an amplifier with a loop transmission

$$L(s) = -\frac{a_0 s^3}{(s+1)(0.1s+1)^2}$$

Also show the potentially unstable behavior using root-locus methods. For what range of values of a_0 is the amplifier stable?

P4.11

Develop a modification of the Nyquist test that enables you to determine if a feedback system has any closed-loop poles with a damping ratio of less than 0.707. Illustrate your test by forming the modified Nyquist diagram for a system with $a(s) = a_0/(s+1)^2, f(s) = 1$. For what value of a_0 does the damping ratio of the closed-loop pole pair equal 0.707? Verify your answer by factoring the characteristic equation for this value of a_0.

P4.12

The open-loop transfer function of an operational amplifier is

$$a(s) = \frac{10^5}{(0.1s+1)(10^{-6}s+1)^2}$$

Determine the gain margin, phase margin, crossover frequency, and M_p for this amplifier when used in a feedback connection with $f = 1$. Also find the value of f that results in an M_p of 1.1. What are the values of phase and gain margin and crossover frequency with this value for f?

P4.13

A feedback system is constructed with

$$a(s) = \frac{10^6(0.01s + 1)^2}{(s + 1)^3}$$

and an adjustable, frequency-independent value for f. As f is increased from zero, it is observed that the system is stable for very small values of f, then becomes unstable, and eventually returns to stable behavior for sufficiently high values of f. Explain this performance using Nyquist and root-locus analysis. Use the Routh criterion to determine the two borderline values for f.

P4.14

An operational amplifier with a frequency-independent feedback path exhibits 40% overshoot and 10 to 90% rise time of 0.5 μs in response to a step input. Estimate the phase margin and crossover frequency of the feedback connection, assuming that its performance is dominated by two widely separated loop-transmission poles.

P4.15

Consider a feedback system with

$$a(s) = \frac{a_0}{s[(s^2/2) + s + 1]}$$

and $f(s) = 1$.

Show that by appropriate choice of a_0, the closed-loop poles of the system can be placed in a third-order Butterworth pattern. Find the crossover frequency and the phase margin of the loop transmission when a_0 is selected for the closed-loop Butterworth response. Use these quantities in conjunction with Fig. 4.26 to find the damping ratio and natural frequency of a second-order system that can be used to approximate the transient response of the third-order Butterworth filter. Compare the peak overshoot and rise time of the approximating system in response to a step with those of the Butterworth response (Fig. 3.10). Note that, even though this system is considerably different from that used to develop Fig. 4.26, the approximation predicts time-domain parameters with fair accuracy.