Solution 16.1

The squared magnitude function for a fifth order Butterworth filter with cutoff frequency $\Omega_c = 2\pi \times 10^3$ is given by

$$H(s)\,H(-s) = \frac{1}{1+\left(\frac{s}{j2\pi \times 10^3}\right)^{10}}$$

The poles of $H(s)H(-s)$ are the roots of $1 + \left(\frac{s}{j2\pi \times 10^3}\right)^{10} = 0$ or

$$s = (-1)^{\frac{1}{10}}\,(j2\pi \times 10^3)$$

as indicated in Figure S16.1-1

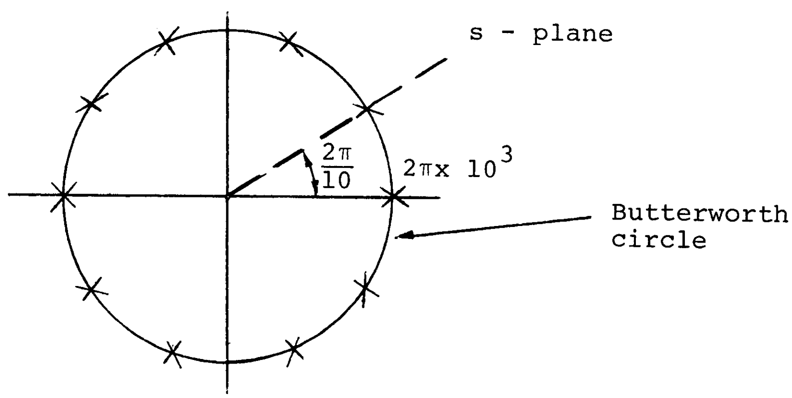

Figure S16.1-1

Since $H(s)$ corresponds to a stable, causal filter, we factor the squared magnitude function so that the left-half plane poles correspond to $H(s)$ and the right-half plane poles correspond to $H(-s)$. Thus the poles of $H(s)$ are as indicated in Figure S16.1-2.

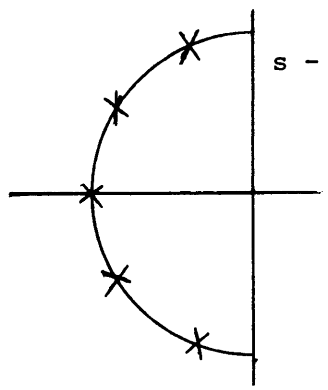

Figure S16.1-2

Solution 16.2

Since the Butterworth filter has a monotonic frequency response with unity magnitude at $\omega = 0$ the stated specifications will be met if we require that

$$|H(e^{j0})| = 1$$

$$-0.75 \le 20 \log_{10}|H(e^{j0.2613\pi})|$$

$$20 \log_{10}|H(e^{j0.4018\pi})| \le -20$$

or, equivalently,

$$|H(e^{j0.2613\pi})|^2 \ge 10^{-.075}$$

and

$$|H(e^{j0.4018\pi})|^2 \le 10^{-2}$$

Using impulse invariance with T = 1 and neglecting aliasing we require that the analog filter $H_a(j\Omega)$ meet the specifications

$$|H_a(j0.2613\pi)|^2 \geq 10^{-.075}$$

$$|H_a(j0.4018\pi)|^2 \leq 10^{-2}$$

We will first consider meeting these specifications with equality. Thus,

$$1 + \left(\frac{j0.2613\pi}{j\Omega_c}\right)^{2N} = 10^{.075}$$

$$1 + \left(\frac{j0.4018\pi}{j\Omega_c}\right)^{2N} = 10^{2}$$

or

$$2N\ \log(0.4018\pi) - 2N\ \log\Omega_c = \log(10^2-1)$$

$$2N\ \log(0.2613\pi) - 2N\ \log\Omega_c = \log(10^{.075}-1)$$

Subtracting we obtain

$$2N\ \log\left(\frac{0.4018\pi}{0.2613\pi}\right) = \log\left(\frac{10^2-1}{10^{.075}-1}\right)$$

or

$$N = 7.278.$$

Since N must be an integer, we choose $N = 8$. Then, to compensate for the effect of aliasing we can choose Ω_c to meet the passband edge specifications in which case the stopband specifications will be exceeded. Determining Ω_c on this basis we have

$$\log\Omega_c = \log(0.2613\pi) - \frac{1}{2N}\log(10^{.075}-1)$$

or

$$\Omega_c = 0.911.$$

Thus the analog filter squared magnitude function is

$$H_a(s)\,H_a(-s) = \frac{1}{1 + \left(\frac{s}{j0.911}\right)^{16}}\,.$$

The poles of the squared magnitude function are indicated in Figure S16.2-1.

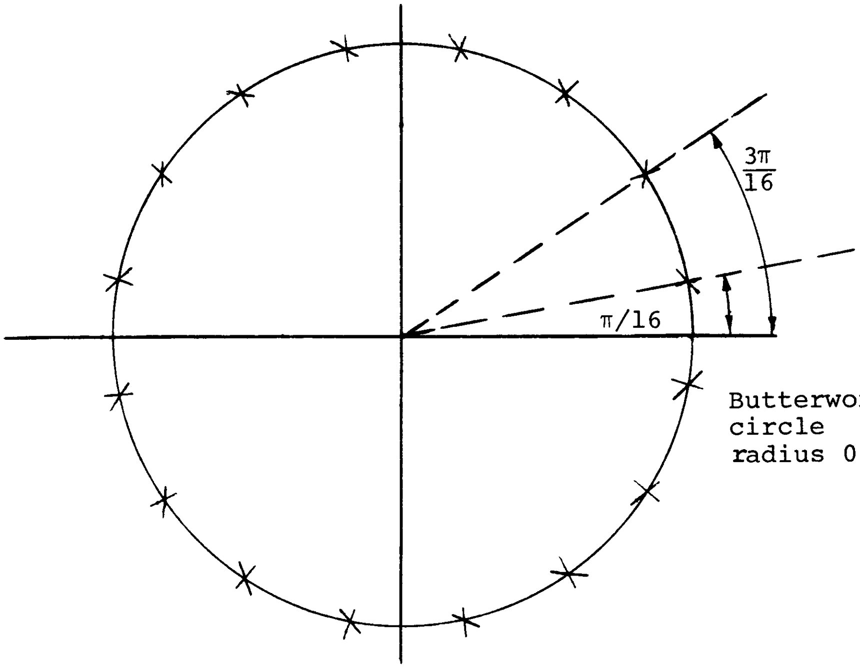

Figure S16.2-1

Therefore $H_a(s)$ has four complex conjugate pole pairs as indicated below:

pole-pair 1: $0.911\,e^{j\frac{9\pi}{16}}$, $0.911e^{-j\frac{9\pi}{16}}$

pole-pair 2: $0.911\,e^{j\frac{11\pi}{16}}$, $0.911e^{-j\frac{11\pi}{16}}$

pole-pair 3: $0.911\,e^{j\frac{13\pi}{16}}$, $0.911\,e^{-j\frac{13\pi}{16}}$

pole-pair 4: $0.911\,e^{j\frac{15\pi}{16}}$, $0.911\,e^{-j\frac{15\pi}{16}}$

From these pole-pairs it is straightforward to express $H_a(s)$ in factored form as

$$H_a(s) = \prod_{k=1}^{4} \frac{A}{(s-s_k)(s-s_k^*)}$$

where the factor A is determined so that $H_a(s)$ has unity gain at zero frequency, i.e.

$$H_a(0) = 1 \text{ or } A = \prod_{k=1}^{4} |s_k|^2 = (0.911)^8.$$

To transform this analog filter to the desired digital filter using impulse invariance, we would first expand $H_a(s)$ in a partial fraction expansion as

$$H_a(s) = \sum_{k=1}^{4} \left[\frac{a_k}{s-s_k} + \frac{a_k^*}{s-s_k^*} \right] .$$

The desired digital filter transfer function is then

$$H(z) = \sum_{k=1}^{4} \left[\frac{a_k}{1-e^{s_k} z^{-1}} + \frac{a_k^*}{1-e^{s_k^*} z^{-1}} \right] .$$

The residues a_k and a_k^* are evaluated as:

$$a_k = H_a(s)\ (s-s_k)\Big|_{s=s_k} .$$

Solution 16.3

Again the specifications on the digital filter are that

$$|H(e^{j0.2613\pi})|^2 \geq 10^{-.075}$$

and

$$|H(e^{j0.4018\pi})|^2 \leq 10^{-2} .$$

To obtain the specifications on the analog filter we must determine the analog frequencies Ω_p and Ω_s which will map to the digital frequencies of 0.2613π and 0.4018π respectively when the bilinear transformation is applied. With T = 1 in the bilinear transformation, these are given by

$$\Omega_p = 2 \tan\left(\frac{.2613\pi}{2}\right) = .8703$$

$$\Omega = 2 \tan\left(\frac{.4018\pi}{2}\right) = 1.4617$$

Thus, the specifications on the analog Butterworth filter are:

$$|H_a(j.8703)|^2 \geq 10^{-.075}$$

$$|H_a(j1.4617)|^2 \leq 10^{-2}$$

As in problem 16.2 we will consider meeting these specifications with equality. Thus:

$$1 + \left(\frac{.8703}{\Omega_c}\right)^{2N} = 10^{.075}$$

$$1 + \left(\frac{1.4617}{\Omega_c}\right)^{2N} = 10^{2}$$

Solving for N we obtain

$$N = 6.04.$$

This is so close to 6 that we might be willing to relax the specifications slightly and use a 6^{th} order filter. Alternatively we would use a 7^{th} order filter and exceed the specifications. Choosing the latter and picking Ω_c to exactly meet the pass band specifications,

$$14\ [\log(.8703) - \log\Omega_c] = \log[10^{.075}-1]$$

$$\Omega_c = .9805\ .$$

Thus the analog filter squared magnitude function is

$$H_a(s)\ H_a(-s) = \frac{1}{1 + \left(\frac{s}{.9805}\right)^{14}}$$

the poles of which are indicated in Figure S16.3-1.

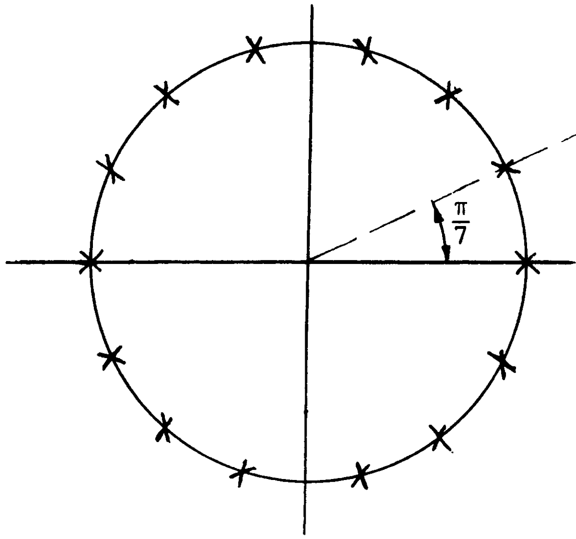

Figure S16.3-1

Therefore, $H_a(s)$ has three complex conjugate pole-pairs and one real pole as indicated below:

real pole: $-.9805$

pole-pair 1: $.9805\, e^{j\frac{8\pi}{14}}$, $.9805\, e^{-j\frac{8\pi}{14}}$

pole-pair 2: $.9805\, e^{j\frac{10\pi}{14}}$, $.9805\, e^{-j\frac{10\pi}{14}}$

pole-pair 3: $.9805\, e^{j\frac{12\pi}{14}}$, $.9805\, e^{-j\frac{12\pi}{14}}$

From these pole locations $H_a(s)$ can be easily expressed in factored form. The digital filter transfer function is then obtained as:

$$H(z) = H_a(s)\Big|_{s=2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}.$$

Solution 16.4

Let $H_\ell(z)$ denote the transfer function of the lowpass filter designed in Problem 16.3 and $H_h(z)$ the transfer function of the desired high-pass filter. To obtain $H_\ell(z)$ from $H_h(z)$ we apply the lowpass to high pass transformation (see table 7.1 page 434 of the text).

$$z^{-1} = -\frac{Z^{-1} + \alpha}{1 + \alpha Z^{-1}}$$

where

$$\alpha = -\frac{\cos\left(\frac{.4018\pi + .2613\pi}{2}\right)}{\cos\left(\frac{.4018\pi - .2613\pi}{2}\right)}$$

$$\alpha = -0.517$$

$$H_h(z) = H_\ell\left[-\frac{1+\alpha z^{-1}}{z^{-1}+\alpha}\right]$$

Next, let θ_s denote the stopband edge frequency for the lowpass filter and ω_s the stopband edge frequency for the highpass filter. Then, inverting the transformation

$$z^{-1} = -\frac{Z^{-1} + \alpha}{1+\alpha Z^{-1}}$$

we have

$$Z^{-1} = -\left(\frac{z^{-1} + \alpha}{\alpha z^{-1} + 1}\right)$$

Thus,

$$e^{-j\omega_s} = -\left(\frac{e^{-j\theta_s} + \alpha}{\alpha e^{-j\theta_s} + 1}\right)$$

then

$$\omega_s = \tan^{-1}\left(\frac{(1-\alpha^2)\sin\theta_s}{-2\alpha - (1+\alpha^2)\cos\theta_s}\right)$$

From this we obtain

$$\omega_s = .2616\,\pi\,.$$