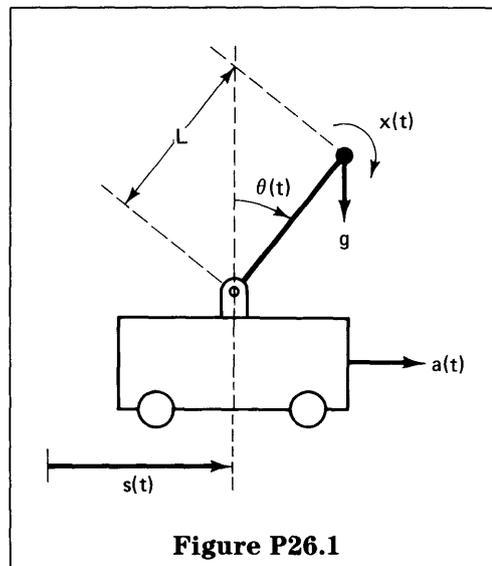


26 Feedback Example: The Inverted Pendulum

Recommended Problems

P26.1

Consider an inverted pendulum mounted on a movable cart as depicted in Figure P26.1. We have modeled the pendulum as consisting of a massless rod of length L , with a mass m attached at the end. The variable $\theta(t)$ denotes the pendulum's angular deflection from the vertical, g is gravitational acceleration, $s(t)$ is the position of the cart with respect to some reference point, $a(t)$ is the acceleration of the cart, and $x(t)$ represents the angular acceleration resulting from any disturbances, such as gusts of wind.



Our goal in this problem is to analyze the dynamics of the inverted pendulum and more specifically to investigate the problem of balancing the pendulum by judicious choice of the cart acceleration $a(t)$. The differential equation relating $\theta(t)$, $a(t)$, and $x(t)$ is

$$L \frac{d^2\theta(t)}{dt^2} = g \sin [\theta(t)] - a(t) \cos [\theta(t)] + Lx(t) \quad (\text{P26.1-1})$$

This relation merely equates the actual acceleration of the mass along a direction perpendicular to the rod and the applied accelerations (gravity, the disturbance acceleration due to $x(t)$, and the cart acceleration) along this direction.

Note that eq. (P26.1-1) is a nonlinear differential equation relating $\theta(t)$, $a(t)$, and $x(t)$. The detailed, exact analysis of the behavior of the pendulum therefore requires that we examine this nonlinear equation; however, we can obtain a great deal of insight into the dynamics of the inverted pendulum by performing a linear analysis. Specifically, we will examine the dynamics of the pendulum when it is nearly vertical (i.e., when $\theta(t)$ is small). In this case, we can make the approximations

$$\sin [\theta(t)] \approx \theta(t), \quad \cos [\theta(t)] \approx 1 \quad (\text{P26.1-2})$$

- (a) Suppose that the cart is stationary (i.e., $a(t) = 0$) and consider the causal LTI system with input $x(t)$ and output $\theta(t)$ described by eq. (P26.1-1) together with the approximations given in eq. (P26.1-2). Find the system function for this system and show that it has a pole in the right half-plane, implying that the system is unstable.

The result of part (a) indicates that if the cart is stationary, any minor angular disturbance caused by $x(t)$ will lead to growing angular deviations from the vertical. Clearly, at some point these deviations will become sufficiently large that the approximations of eq. (P26.1-2) will no longer be valid. At this point the linear analysis is no longer accurate, but the fact that it is accurate for small angular displacements allows us to conclude that the vertical equilibrium position is unstable, as small angular displacements will grow rather than diminish.

- (b) We now wish to consider the problem of stabilizing the vertical position of the pendulum by moving the cart in an appropriate fashion. Suppose that we try proportional feedback, with $a(t) = K\theta(t)$. Assume that $\theta(t)$ is small so that the approximations in eq. (P26.1-2) are valid. Draw a block diagram of the linear system with $\theta(t)$ as output, $x(t)$ as the external input, and $a(t)$ as the signal that is fed back. Show that the resulting closed-loop system is unstable. Find a value of K such that if $x(t) = \delta(t)$, the pendulum will sway back and forth in an undamped oscillatory fashion.

- (c) Consider using proportional plus derivative (PD) feedback:

$$a(t) = K_1\theta(t) + K_2 \frac{d\theta(t)}{dt}$$

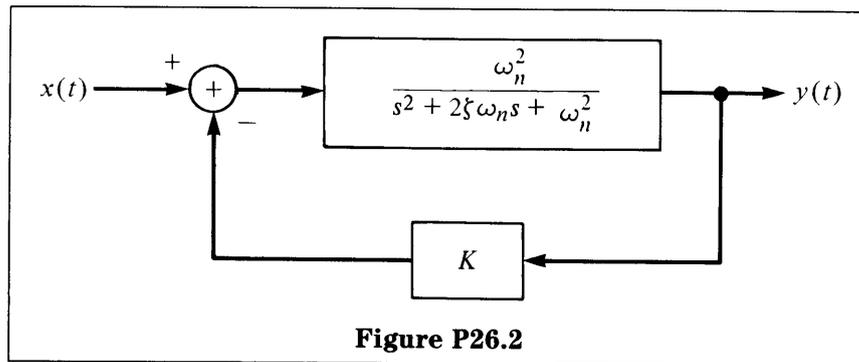
Show that we can find values of K_1 and K_2 that do stabilize the pendulum. In fact, using

$$g = 9.8 \text{ m/s}^2, \quad L = 0.5 \text{ m}, \quad (\text{P26.1-3})$$

choose values of K_1 and K_2 so that the damping ratio of the closed-loop system is 1 and the natural frequency is 3 rad/s.

P26.2

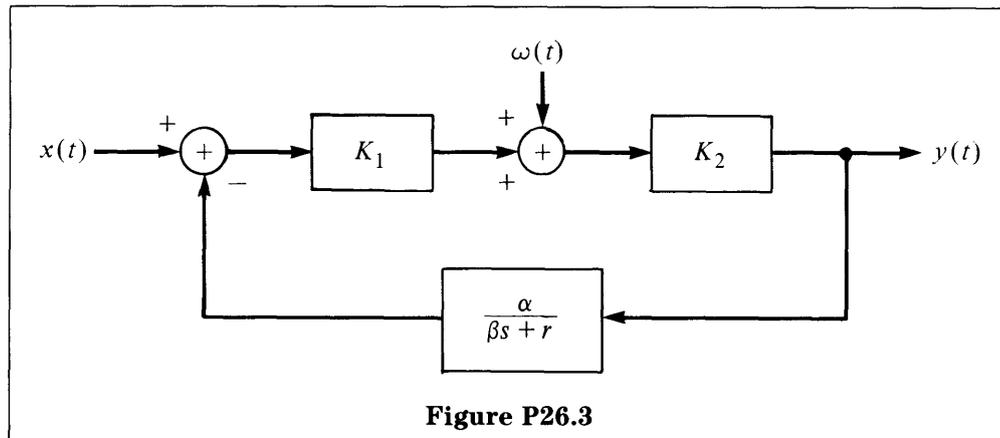
Consider the feedback system given in Figure P26.2.



- (a) The closed-loop system is a new second-order system. Find the new natural frequency $\hat{\omega}_n$ and the new damping factor $\hat{\zeta}$ for $K = 1$.
- (b) Find the poles of the closed-loop transfer function.
- (c) Sketch the trajectory of the poles as K goes from negative infinity to positive infinity.

P26.3

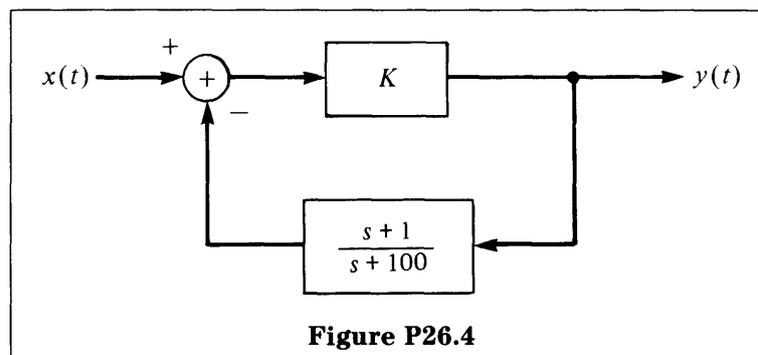
Consider the system in Figure P26.3.



- (a) Find $\frac{Y(s)}{X(s)} = H_1(s)$.
- (b) Find $\frac{Y(s)}{W(s)} = H_2(s)$.
- (c) For what range of values of α , β , and r will both $H_1(s)$ and $H_2(s)$ be stable?

P26.4

Consider the feedback system in Figure P26.4.

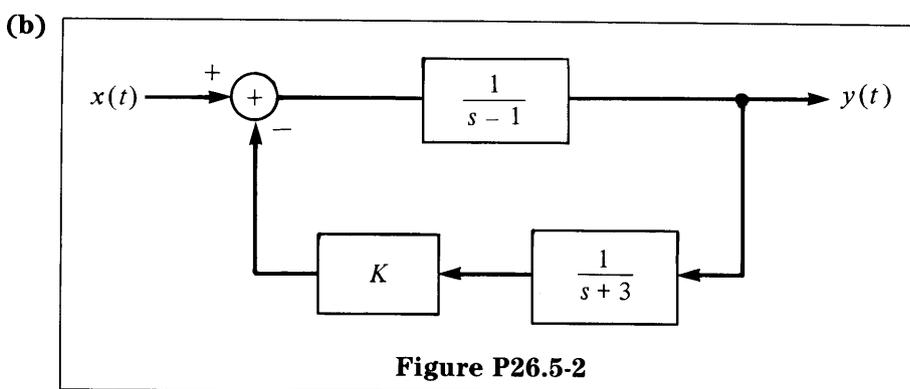
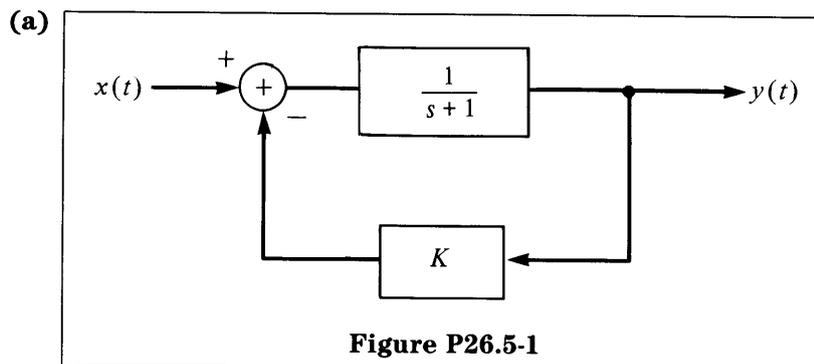


Find the closed-loop poles and zeros of this system for the following values of K :

- (a) $K = 0.01$
- (b) $K = 1$
- (c) $K = 10$
- (d) $K = 100$

P26.5

Sketch the root loci for $K > 0$ and $K < 0$ for each of the following systems.



Optional Problems

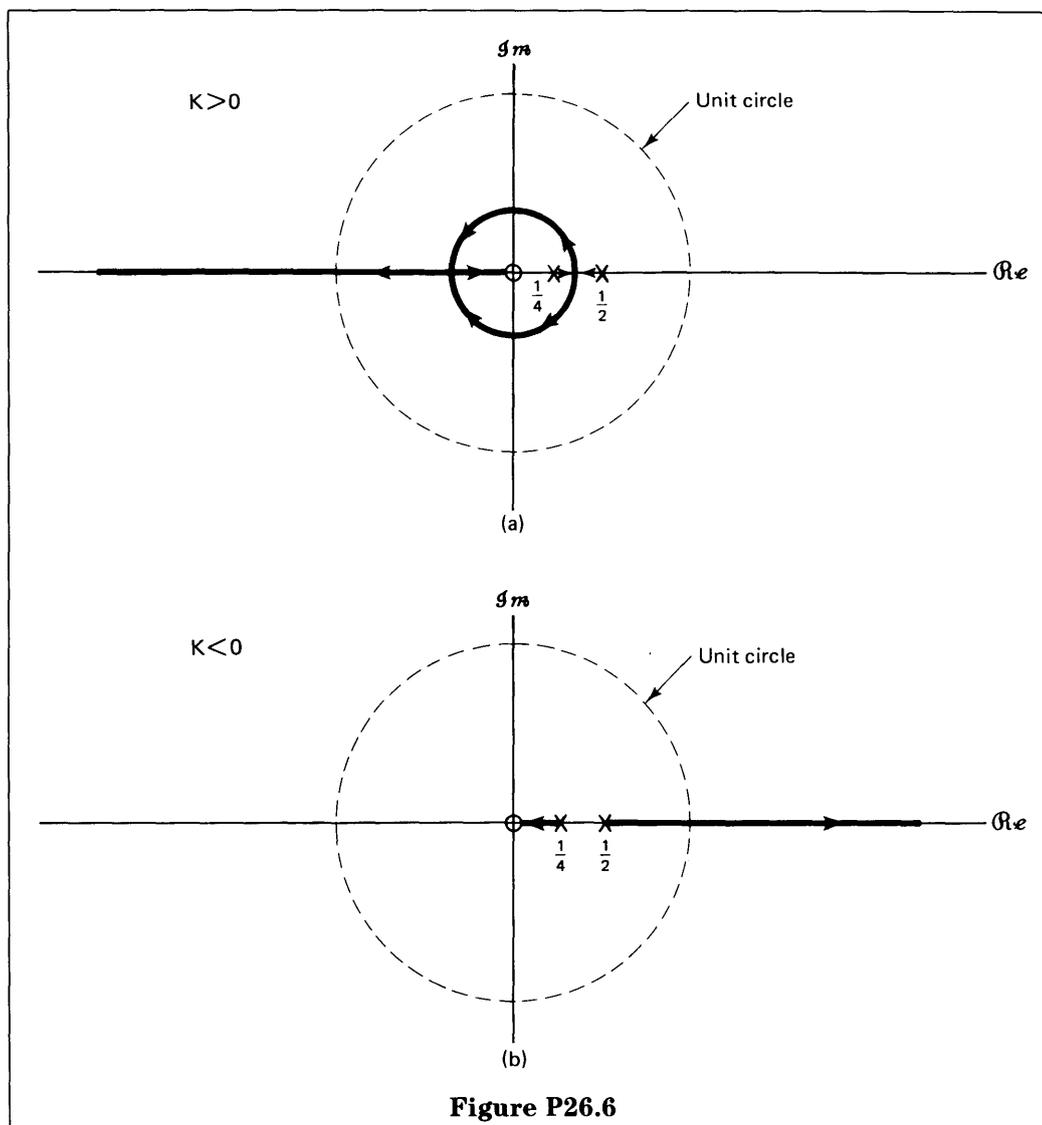
P26.6

Consider again the discrete-time feedback system of Example 11.3 of the text (page 712):

$$G(z)H(z) = \frac{z}{(z - \frac{1}{2})(z - \frac{1}{4})}$$

The root loci for $K > 0$ and $K < 0$ are depicted in Figure P26.6.

- (a) Consider the root locus for $K > 0$. In this case the system becomes unstable when one of the closed-loop poles is less than or equal to -1 . Find the value of K for which $z = -1$ is a closed-loop pole.
- (b) Consider the root locus for $K < 0$. In this case the system becomes unstable when one of the closed-loop poles is greater than or equal to 1 . Find the value of K for which $z = 1$ is a closed-loop pole.
- (c) What is the full range of values of K for which the closed-loop system is stable?

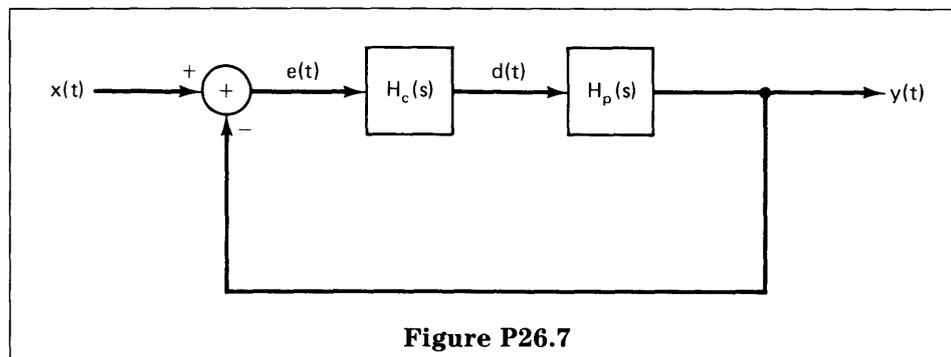

P26.7

In this problem we consider several examples of the design of tracking systems. Consider the system depicted in Figure P26.7. Here $H_p(s)$ is the system whose output is to be controlled, and $H_c(s)$ is the compensator to be designed. Our objective in choosing $H_c(s)$ is that we would like the output $y(t)$ to follow the input $x(t)$. Specifically, in addition to stabilizing the system, we would also like to design the system so that the error $e(t)$ decays to zero for certain specified inputs.

(a) Suppose that

$$H_p(s) = \frac{\alpha}{s + \alpha}, \quad \alpha \neq 0 \quad (\text{P26.7-1})$$

Show that if $H_c(s) = K$ (which is known as *proportional* or P control), we can choose K to stabilize the system so that $e(t)$ goes to 0 if $x(t) = \delta(t)$. Show that we *cannot* get $e(t) \rightarrow 0$ if $x(t) = u(t)$.



- (b) Again, let $H_p(s)$ be as in eq. (P26.7-1) and suppose that we use *proportional plus integral* (PI) control:

$$H_c(s) = K_1 + \frac{K_2}{s}$$

Show that we can choose K_1 and K_2 to stabilize the system, and we can also get $e(t) \rightarrow 0$ if $x(t) = u(t)$. Thus, this system can track a step. In fact, this illustrates a basic and important fact in feedback system design: To track a step ($X(s) = 1/s$), we need an integrator ($1/s$) in the feedback system.

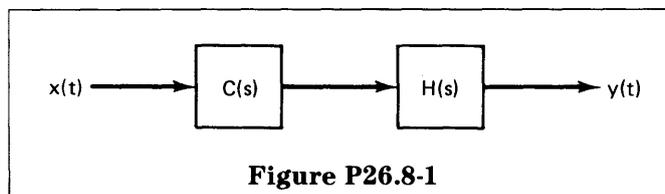
P26.8

Consider a system with system function

$$H(s) = \frac{1}{(s + 1)(s - 2)} \tag{P26.8-1}$$

This system is unstable, and we would like to devise some method for its stabilization.

- (a) Consider first a series compensation scheme as illustrated in Figure P26.8-1.

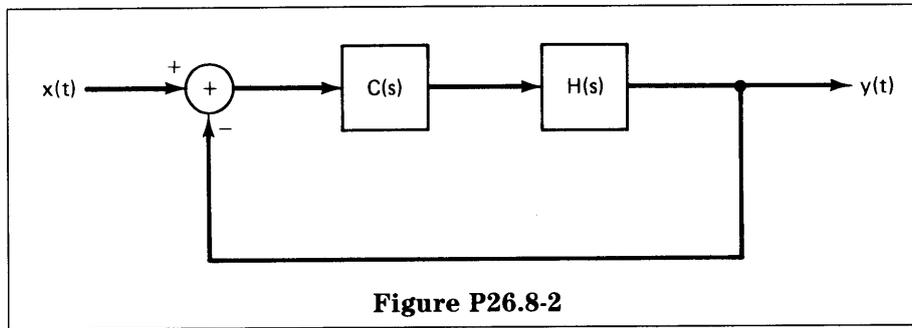


Show that the overall system of this figure is stable if the system function $C(s)$ is taken as

$$C(s) = \frac{s - 2}{s + 3}$$

In practice, this is *not* considered a particularly useful way to attempt to stabilize a system. Explain why.

- (b) Suppose that instead we use a feedback system as depicted in Figure P26.8-2. Is it possible to stabilize this system using a constant gain for the compensating element, i.e., $C(s) = K$? Justify your answer.



- (c) Show that the system of Figure P26.8-2 can be stabilized if $C(s)$ is a proportional plus derivative system:

$$C(s) = K(s + a)$$

Consider both $0 < a < 1$ and $a > 1$.

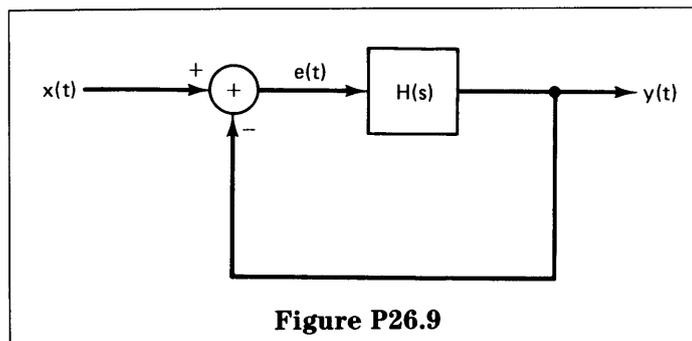
- (d) Suppose that $C(s) = K(s + 2)$. Choose the value of K so that the closed-loop system has a pair of complex poles with a damping ratio of $\zeta = \frac{1}{2}$. [Hint: In this case the denominator of the closed-loop system function must have the form $s^2 + \omega_n s + \omega_n^2$.]

P26.9

In Problem P26.7 we discussed how the presence of an integrator in a feedback system can make it possible for the system to track a step input with zero error in steady state. This problem extends this idea. Specifically, consider the feedback system depicted in Figure P26.9, and assume that the overall closed-loop system is stable. Suppose also that $H(s)$ has the form

$$H(s) = \frac{K \prod_{k=1}^m (s - \beta_k)}{s^l \prod_{k=1}^{n-l} (s - \alpha_k)},$$

where α_k and β_k are given, nonzero numbers and l is a positive integer. The feedback system of Figure P26.9 is often referred to as a Type l feedback system.



- (a) Use the final value theorem (Section 9.5.9 of the text) to show that a Type 1 feedback system can track a step, i.e., that

$$e(t) \rightarrow 0 \quad \text{if } x(t) = u(t)$$

- (b) Similarly, show that a Type 1 system cannot track a ramp but rather that

$$e(t) \rightarrow \text{a finite constant} \quad \text{if } x(t) = u_{-2}(t)$$

- (c) Show that for a Type 1 system, unbounded errors result if $x(t) = u_{-k}(t)$, with $k > 2$.

- (d) More generally, show that for a Type l system,

(i) $e(t) \rightarrow 0$ if $x(t) = u_{-k}(t)$, with $k \leq l$

(ii) $e(t) \rightarrow \text{a finite constant}$ if $x(t) = u_{-(l+1)}(t)$

(iii) $e(t) \rightarrow \infty$ if $x(t) = u_{-k}(t)$, with $k > l + 1$

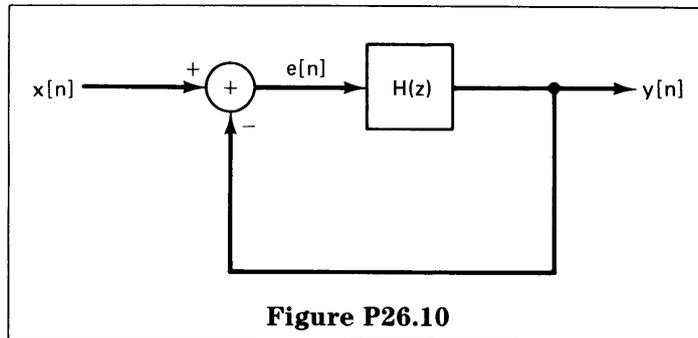
P26.10

- (a) Consider the discrete-time feedback system of Figure P26.10. Suppose that

$$H(z) = \frac{1}{(z - 1)(z + \frac{1}{2})}$$

Show that this system can track a unit step in the sense that if $x[n] = u[n]$, then

$$\lim_{n \rightarrow \infty} e[n] = 0 \tag{P26.10-1}$$



- (b) More generally, consider the feedback system of Figure P26.10 and assume that the closed-loop system is stable. Suppose that $H(z)$ has a pole at $z = 1$. Show that the system can track a unit step. [Hint: Express the transform $E(z)$ of $e[n]$ in terms of $H(z)$ and the transform of $u[n]$; explain why all the poles of $E(z)$ are inside the unit circle.]

The results of parts (a) and (b) are discrete-time counterparts of the results for continuous-time systems discussed in Problems P26.7 and P26.9. In discrete time, we can also consider the design of systems that track specified inputs *perfectly* after a finite number of steps. Such systems are known as *deadbeat feedback systems*, several examples of which are illustrated in the remainder of this problem.

- (c) Consider the discrete-time system of Figure P26.10, with

$$H(z) = \frac{z^{-1}}{1 - z^{-1}}$$

Show that the overall closed-loop system has the property that it tracks a step input exactly after one step. That is, if $x[n] = u[n]$, then $e[n] = 0, n \geq 1$.

- (d) Show that the feedback system of Figure P26.10, with

$$H(z) = \frac{\frac{3}{4}z^{-1} + \frac{1}{4}z^{-2}}{(1 + \frac{1}{4}z^{-1})(1 - z^{-1})},$$

has the property that the output tracks a unit step perfectly after a finite number of steps. At what time step does the error $e[n]$ first settle to zero?

- (e) More generally, consider the feedback system of Figure P26.10. Find $H(z)$ such that $y[n]$ perfectly tracks a unit step for $n \geq N$ and in fact such that

$$e[n] = \sum_{k=0}^{N-1} a_k \delta[n - k], \quad (\text{P26.10-2})$$

where the a_k are specified constants. [*Hint:* Use the relation between $H(z)$ and $E(z)$ when the input is a unit step and $e[n]$ is given by eq. (P26.10-2).]

- (f) Consider the system of Figure P26.10, with

$$H(z) = \frac{z^{-1} + z^{-2} - z^{-3}}{(1 + z^{-1})(1 - z^{-1})^2}$$

Show that this system tracks a ramp $x[n] = (n + 1)u[n]$ exactly after two time steps.

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