13 Continuous-Time Modulation

Solutions to Recommended Problems

S13.1

(a) By the shifting property,

$$x(t)e^{j3\omega_c t} \xrightarrow{\mathcal{F}} X(\omega - 3\omega_c) = Y(\omega)$$

The magnitude and phase of $Y(\omega)$ are given in Figure S13.1-1.

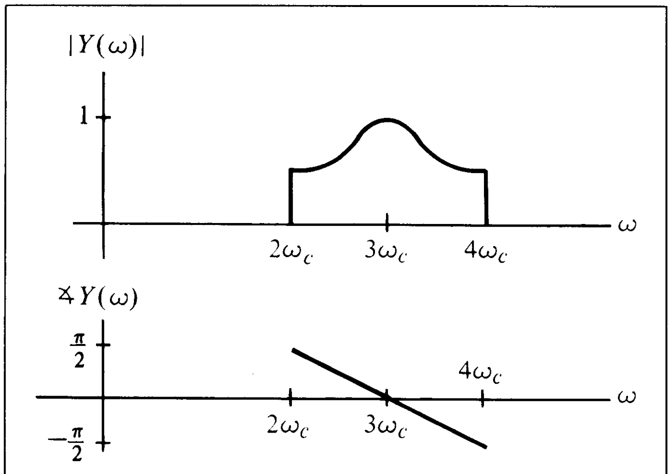

Figure S13.1-1

(b) Since $e^{j3\omega_c + j\pi/2} = e^{j\pi/2}e^{j3\omega_c t}$, we are modulating the same carrier as in part (a) except that we multiply the result by $e^{j\pi/2}$. Thus

$$Y(\omega) = e^{j\pi/2}X(\omega - 3\omega_c)$$

Note in Figure S13.1-2 that the magnitude of $Y(\omega)$ is unaffected and that the phase is shifted by $\pi/2$.

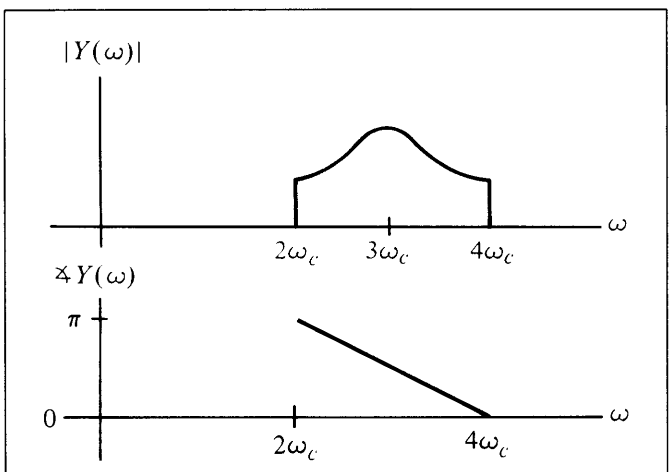

Figure S13.1-2

(c) Since

$$\cos 3\omega_c t = \frac{e^{j3\omega_c t}}{2} + \frac{e^{-j3\omega_c t}}{2},$$

we can think of modulation by cos $3\omega_c t$ as the sum of modulation by

$$\frac{e^{j3\omega_c t}}{2} \quad \text{and} \quad \frac{e^{-j3\omega_c t}}{2}$$

Thus, the magnitude and phase of $Y(\omega)$ are as shown in Figure S13.1-3. Note the scaling in the magnitude.

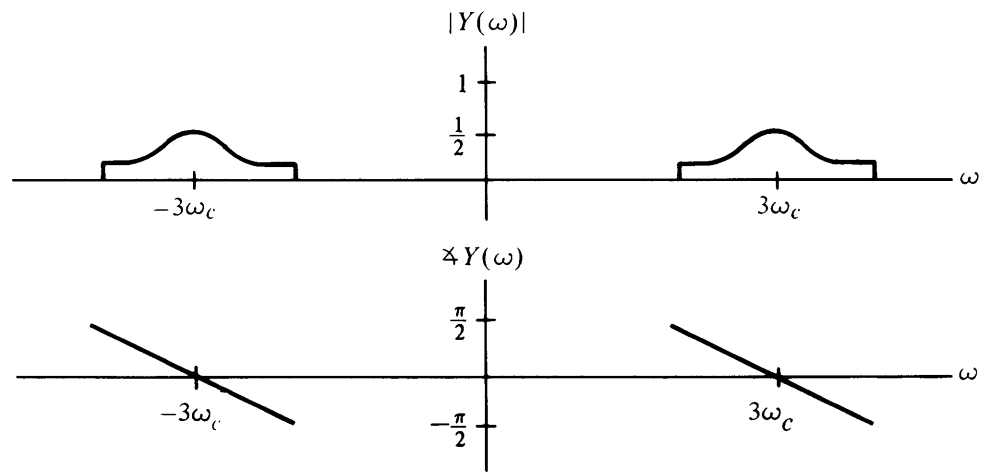

Figure S13.1-3

(d) We can think of modulation by sin $3\omega_c t$ as the sum of modulation by

$$\frac{e^{j3\omega_c t - j\pi/2}}{2} \quad \text{and} \quad \frac{e^{-j3\omega_c t - j\pi/2}}{2}$$

Thus, the magnitude and phase of $Y(\omega)$ are as given in Figure S13.1-4. Note the scaling by $\frac{1}{2}$ in the magnitude.

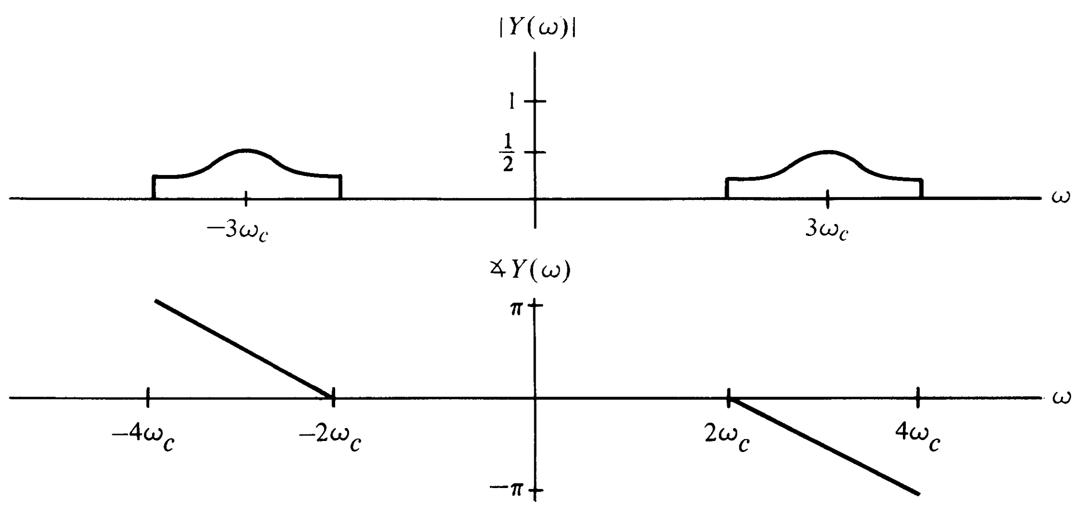

Figure S13.1-4

(e) Since the phase terms are different in parts (c) and (d), we cannot just add spectra. We need to convert cos $3\omega_c t$ + sin $3\omega_c t$ into the form $A \cos(3\omega_c t + \theta)$. Note that

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Let $\alpha = 3\omega_c t$ and $\beta = \pi/4$. Then

$$\cos\left(3\omega_c t - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}(\cos 3\omega_c t + \sin 3\omega_c t)$$

Thus

$$\cos 3\omega_c t + \sin 3\omega_c t = \sqrt{2}\cos\left(3\omega_c t - \frac{\pi}{4}\right)$$

Now we write $c(t)$ as

$$\frac{\sqrt{2}}{2} e^{j[3\omega_c t - (\pi/4)]} + \frac{\sqrt{2}}{2} e^{-j[3\omega_c t - (\pi/4)]}$$

Modulating by each exponential separately and then adding yields the magnitude and phase given in Figure S13.1-5. (Note the scaling in the magnitude.)

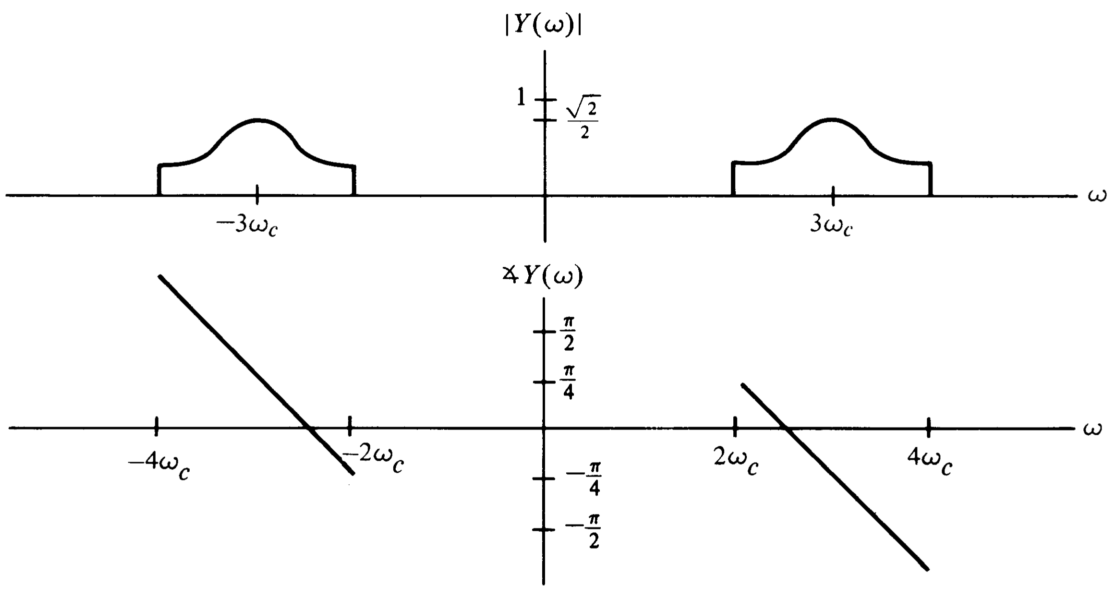

Figure S13.1-5

S13.2

In Figure S13.2-1 we redraw the system with some auxiliary signals labeled.

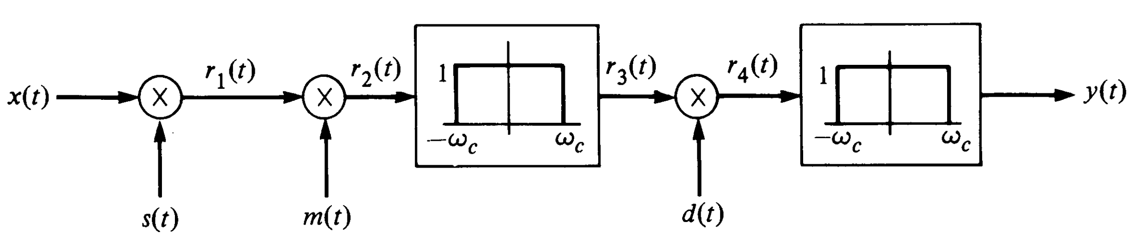

Figure S13.2-1

By the modulation property, $R_1(\omega)$, the Fourier transform of $r_1(t)$, is

$$R_1(\omega) = \frac{1}{2\pi}[X(\omega) * S(\omega)]$$

Since $S(\omega)$ is composed of impulses, $R_1(\omega)$ is a repetition of $X(\omega)$ centered at $-2\omega_c$, 0, and $2\omega_c$, and scaled by $1/(2\pi)$. See Figure S13.2-2.

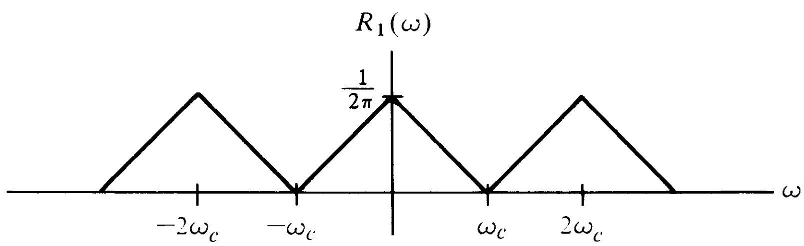

Figure S13.2-2

(a) Since $m(t) = d(t) = 1$, $y(t)$ is $r_1(t)$ filtered twice by the same ideal lowpass filter with cutoff at ω_c. Thus, comparing the resulting Fourier transform of $y(t)$, shown in Figure S13.2-3, we see that $y(t) = 1/(2\pi)x(t)$, which is nonzero.

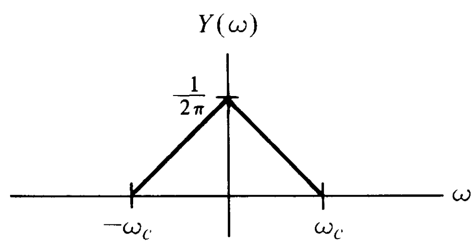

Figure S13.2-3

(b) Modulating $r_1(t)$ by $e^{j\omega_c t}$ yields $R_1(\omega - \omega_c)$ as shown in Figure S13.2-4.

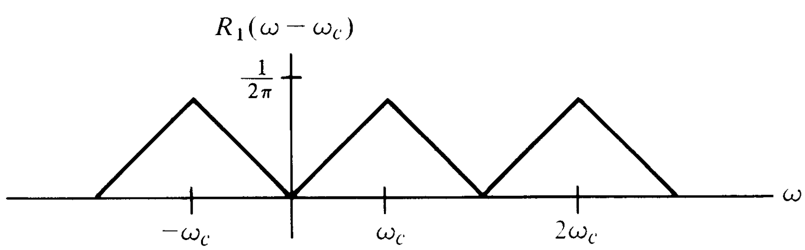

Figure S13.2-4

Similarly, modulating by $e^{-j\omega_c t}$ yields $R_1(\omega + \omega_c)$ as shown in Figure S13.2-5.

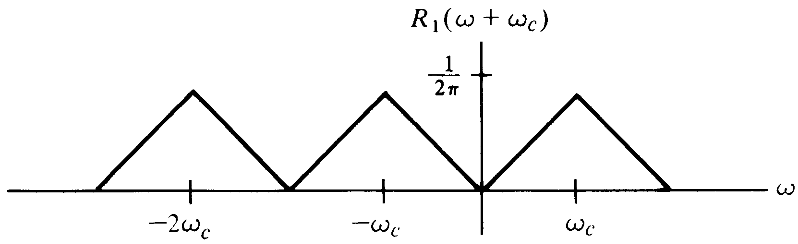

Figure S13.2-5

Since $\cos \omega_c t = (e^{j\omega_c t} + e^{-j\omega_c t})/2$, modulating $r_1(t)$ by $\cos \omega_c t$ yields a Fourier transform of $r_2(t)$ given by

$$\frac{R_1(\omega - \omega_c) + R_1(\omega + \omega_c)}{2}$$

Thus, $R_2(\omega)$ is as given in Figure S13.2-6.

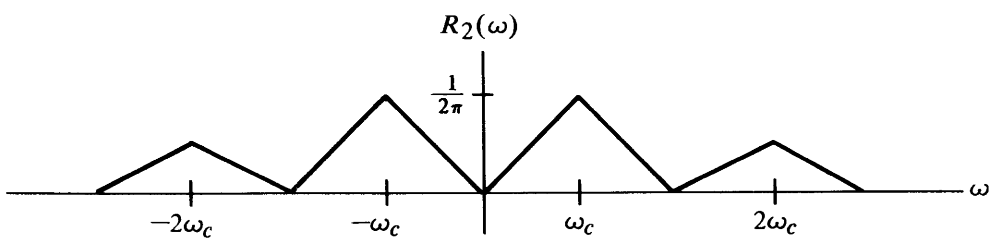

Figure S13.2-6

After filtering, $R_3(\omega)$ is given as in Figure S13.2-7.

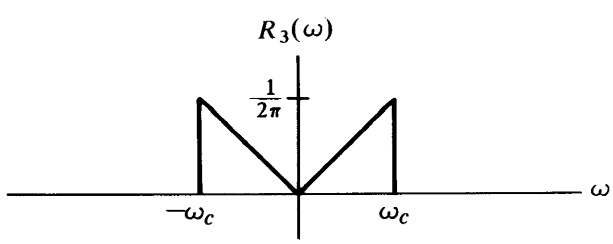

Figure S13.2-7

$R_4(\omega)$ is given by shifting $R_3(\omega)$ up and down by ω_c and dividing by 2. See Figure S13.2-8.

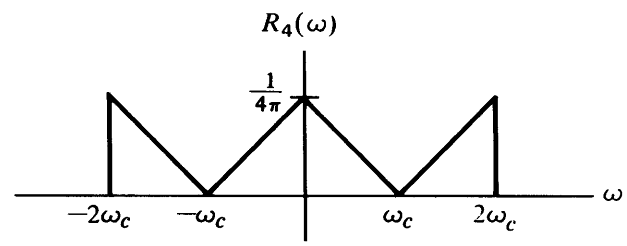

Figure S13.2-8

After filtering, $Y(\omega)$ is as shown in Figure S13.2-9.

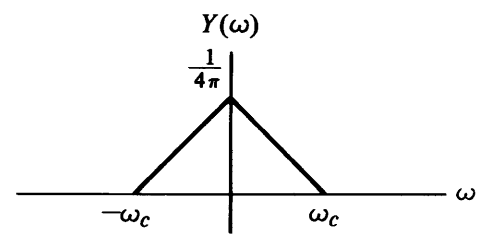

Figure S13.2-9

Comparing $Y(\omega)$ and $X(\omega)$ yields

$$y(t) = \frac{1}{4\pi} x(t)$$

(c) Since

$$\sin \omega_c t = \frac{e^{j\omega_c t} - e^{-j\omega_c t}}{2j},$$

then

$$R_2(\omega) = \frac{R_1(\omega - \omega_c) - R_1(\omega + \omega_c)}{2j},$$

which is drawn in Figure S13.2-10.

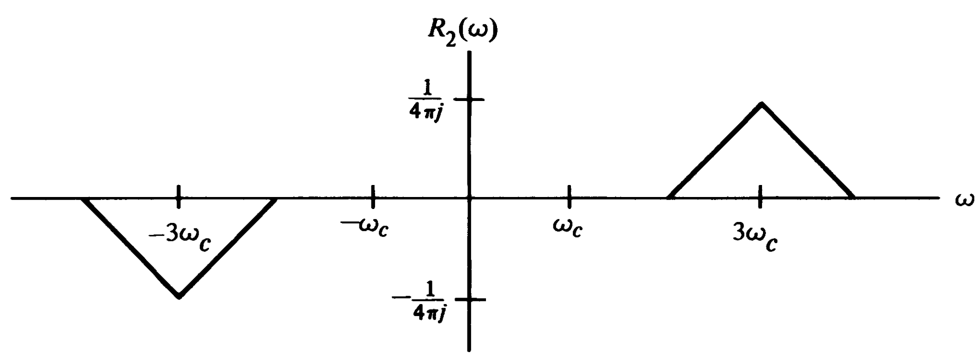

Figure S13.2-10

After filtering, $R_3(\omega) = 0$. Therefore, $y(t) = 0$.

(d) In this case, it is not necessary to know $r_3(t)$ exactly. Suppose $r_3(t)$ is nonzero, with $R_3(\omega)$ given as in Figure S13.2-11.

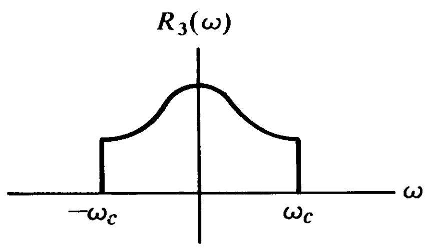

Figure S13.2-11

After modulating by $d(t) = \cos 2\omega_c t$, $R_4(\omega)$ is given as in Figure S13.2-12.

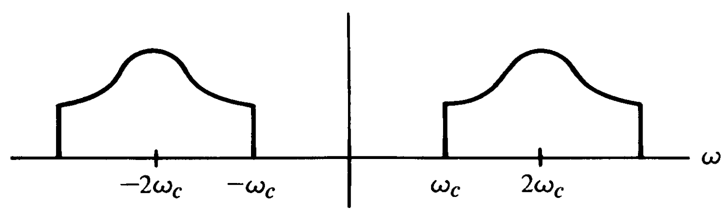

Figure S13.2-12

After filtering, $y(t) = 0$ since $R_4(\omega)$ has no energy from $-\omega_c$ to ω_c.

(e) For this part, let us calculate $R_2(\omega)$ explicitly.

$$R_2(\omega) = \frac{R_1(\omega - 2\omega_c) + R_1(\omega + 2\omega_c)}{2},$$

which is drawn in Figure S13.2-13.

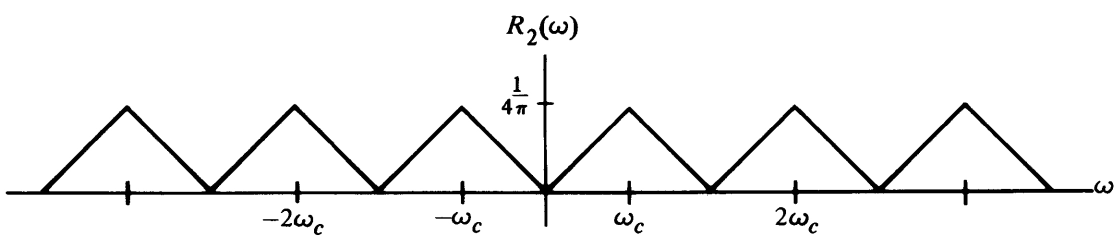

Figure S13.2-13

After filtering, $R_3(\omega)$ is as shown in Figure S13.2-14.

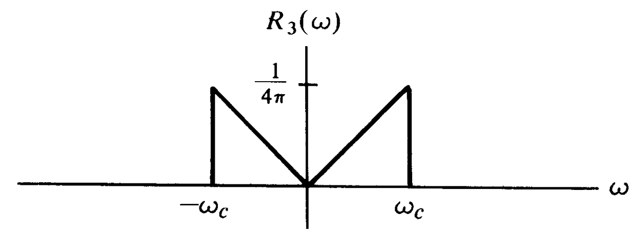

Figure S13.2-14

Modulating again yields $R_4(\omega)$ as shown in Figure S13.2-15.

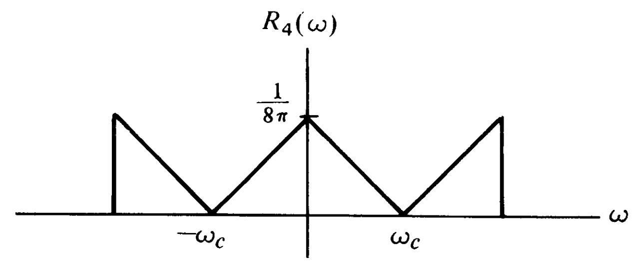

Figure S13.2-15

Finally, filtering $R_4(\omega)$ gives the Fourier transform of $y(t)$, shown in Figure S13.2-16.

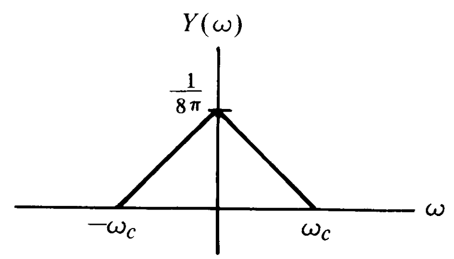

Figure S13.2-16

Thus,

$$y(t) = \frac{1}{8\pi} x(t)$$

S13.3

(a) The demodulator signal $w(t)$ is related to $x(t)$ via

$$w(t) = (\cos \omega_d t)(\cos \omega_c t)x(t)$$

Since $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$,

$$w(t) = \tfrac{1}{2}[\cos(\Delta\omega)t + \cos(\Delta\omega + 2\omega_c)t]x(t)$$
$$= \tfrac{1}{2}[\cos(\Delta\omega)t]x(t) + \tfrac{1}{2}[\cos(\Delta\omega + 2\omega_c)t]x(t)$$

The first term is bandlimited to $\pm(\omega_M + |\Delta\omega|)$, while the second term is bandlimited from $\Delta\omega + 2\omega_c - \omega_M$ to $\Delta\omega + 2\omega_c + \omega_M$. Thus after filtering, only the first term remains. Therefore, the output of the demodulator lowpass filter is given by $\frac{1}{2}x(t)\cos \Delta\omega t$.

(b) Consider first $|\Delta\omega| > \omega_M$. Then for $X(\omega)$ as given, $\frac{1}{2}x(t)\cos \Delta\omega t$ has a Fourier transform as shown in Figure S13.3-1.

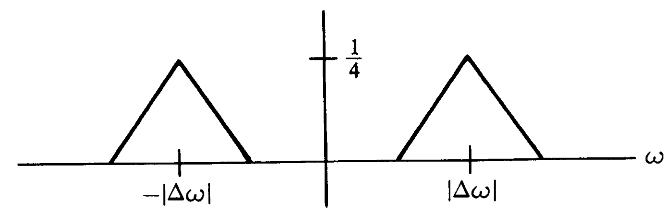

Figure S13.3-1

For $|\Delta\omega| < \omega_M$, there is some overlap. See Figure S13.3-2.

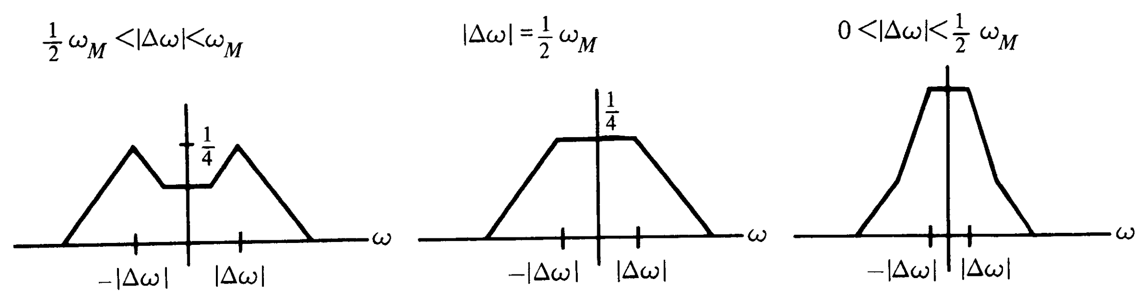

Figure S13.3-2

S13.4

(a) In this case,

$$y(t) = [A + \cos \omega_M t]\cos(\omega_c t + \theta_c)$$

But

$$\cos \omega_M t \cos(\omega_c t + \theta_c) = \tfrac{1}{2}[\cos((\omega_M - \omega_c)t - \theta_c) + \cos((\omega_M + \omega_c)t + \theta_c)]$$

Thus,

$$y(t) = A\cos(\omega_c t + \theta_c) + \frac{1}{2}\cos((\omega_M - \omega_c)t - \theta_c) + \frac{1}{2}\cos((\omega_M + \omega_c)t + \theta_c)$$
$$= \frac{Ae^{j\theta_c}}{2}e^{j\omega_c t} + \frac{Ae^{-j\theta_c}}{2}e^{-j\omega_c t} + \frac{1}{4}e^{-j\theta_c}e^{j(\omega_M - \omega_c)t}$$
$$+ \frac{1}{4}e^{j\theta_c}e^{-j(\omega_M - \omega_c)t} + \frac{1}{4}e^{j\theta_c}e^{j(\omega_M + \omega_c)t} + \frac{1}{4}e^{-j\theta_c}e^{-j(\omega_M + \omega_c)t}$$

We recognize that the preceding expression is a Fourier series expansion. Using Parseval's theorem for the Fourier series, we have

$$\frac{1}{T_0}\int_{T_0} |y(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |a_k|^2 = P_y$$

Thus,

$$P_y = 2\left(\frac{A}{2}\right)^2 + 4\left(\frac{1}{4}\right)^2 = \frac{A^2}{2} + \frac{1}{4}$$

Since

$$m = \frac{\max |x(t)|}{A} = \frac{1}{A},$$

then

$$P_y = \frac{1}{2m^2} + \frac{1}{4},$$

as shown in Figure S13.4-1.

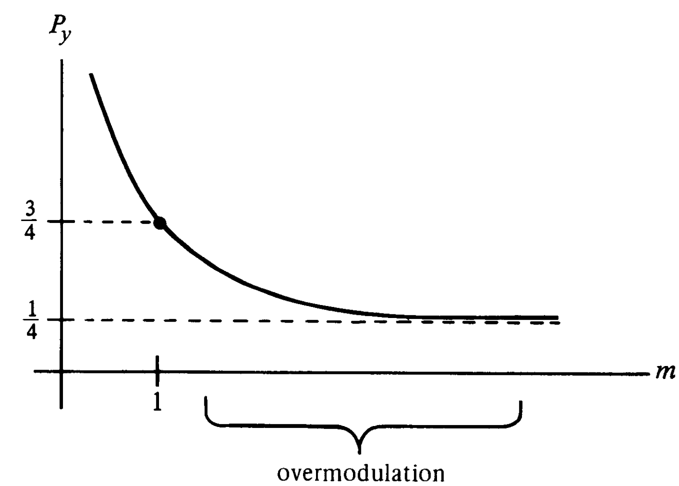

Figure S13.4-1

(b) The power in the sidebands is found from P_y when $A = 0$. Thus, $P_y = \frac{1}{4}$ and the efficiency is

$$E = \frac{\frac{1}{4}}{1/(2m^2) + \frac{1}{4}} = \frac{m^2}{2 + m^2},$$

which is sketched in Figure S13.4-2.

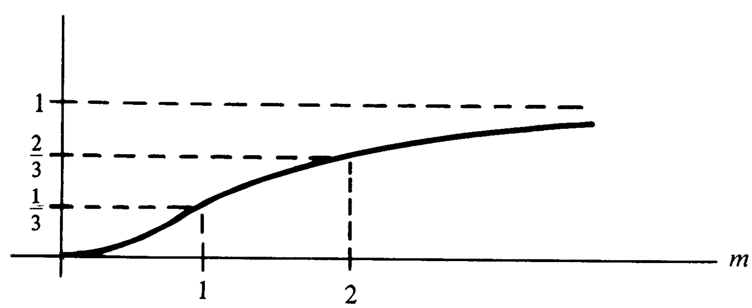

Figure S13.4-2

Solutions to Optional Problems

S13.5

(a) Using the identity for $\cos(A + B)$, we have

$$A(t)\cos(\omega_c t + \theta_c) = A(t)\,(\cos\theta_c \cos\omega_c t - \sin\theta_c \sin\omega_c t)$$

Thus, we see that

$$x(t) = A(t)\cos\theta_c,$$
$$y(t) = -A(t)\sin\theta_c$$

Therefore,

$$\begin{aligned} z(t) &= A(t)\cos(\omega_c t + \theta_c) \\ &= x(t)\cos\omega_c t + y(t)\sin\omega_c t \end{aligned}$$

(b) Consider modulating $z(t)$ by $\cos\omega_c t$. Then

$$z(t)\cos\omega_c t = x(t)\cos^2\omega_c t + y(t)\sin\omega_c t\cos\omega_c t$$

Using trigonometric identities, we have

$$z(t)\cos\omega_c t = \frac{x(t)}{2} + \frac{x(t)}{2}\cos 2\omega_c t + \frac{y(t)}{2}\sin 2\omega_c t$$

If we use an ideal lowpass filter with cutoff ω_c and if $A(t)$, and thus $x(t)$, is bandlimited to $\pm\omega_c$, then we recover the term $x(t)/2$. Thus the processing is as shown in Figure S13.5-1.

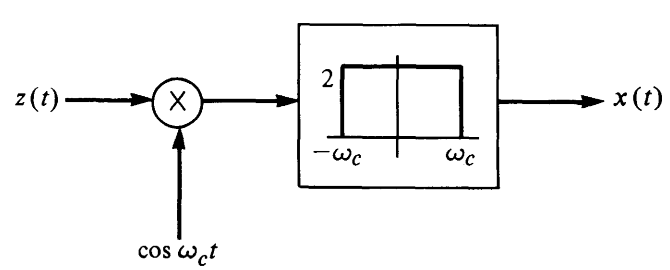

Figure S13.5-1

(c) Similarly, consider

$$\begin{aligned} z(t)\sin\omega_c t &= x(t)\cos\omega_c t\sin\omega_c t + y(t)\sin^2\omega_c t \\ &= \frac{x(t)}{2}\sin 2\omega_c t + \frac{y(t)}{2} - \frac{y(t)}{2}\cos 2\omega_c t \end{aligned}$$

Filtering $z(t)\sin\omega_c t$ with the same filter as in part (b) yields $y(t)$, as shown in Figure S13.5-2.

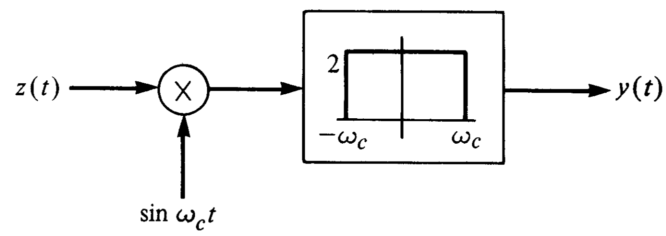

Figure S13.5-2

(d) We can readily see that

$$x^2(t) + y^2(t) = A^2(t)(\cos^2\theta_c + \sin^2\theta_c) = A^2(t)$$

Therefore, $A(t) = \sqrt{x^2(t) + y^2(t)}$. The block diagram in Figure S13.5-3 summarizes how to recover $A(t)$ from $z(t)$.

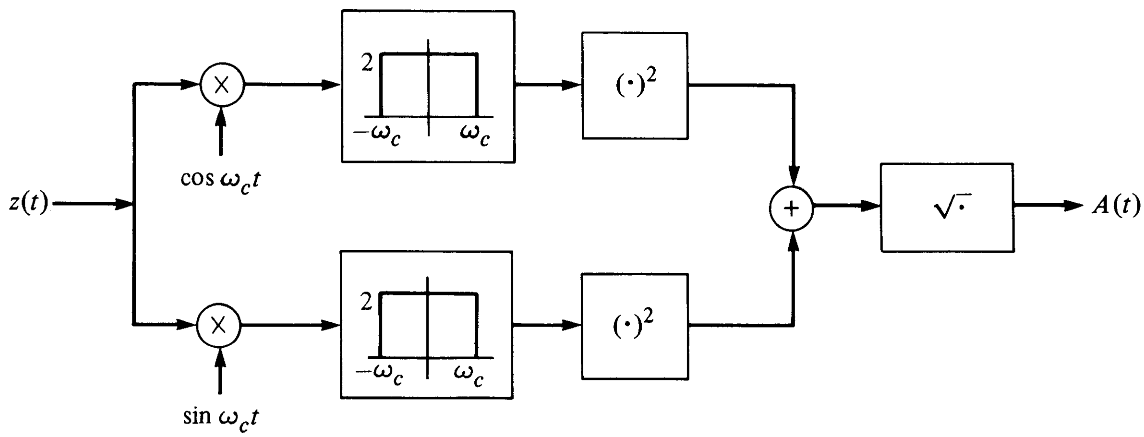

Figure S13.5-3

Note that to be able to recover $A(t)$ in this way, the Fourier transform of $A(t)$ must be zero for $\omega > |\omega_c|$ and $A(t) > 0$. Also note that we are implicitly assuming that $A(t)$ is a real signal.

S13.6

From Figures P13.6-1 to P13.6-3, we can relate the Fourier transforms of all the signals concerned.

$$S_1(\omega) = \frac{1}{2j}\left[X\left(\omega - \frac{\omega_0}{2}\right) - X\left(\omega + \frac{\omega_0}{2}\right)\right]$$

$$S_2(\omega) = \frac{1}{2}\left[X\left(\omega - \frac{\omega_0}{2}\right) + X\left(\omega + \frac{\omega_0}{2}\right)\right]$$

Thus, $S_1(\omega)$ and $S_2(\omega)$ appear as in Figure S13.6-1.

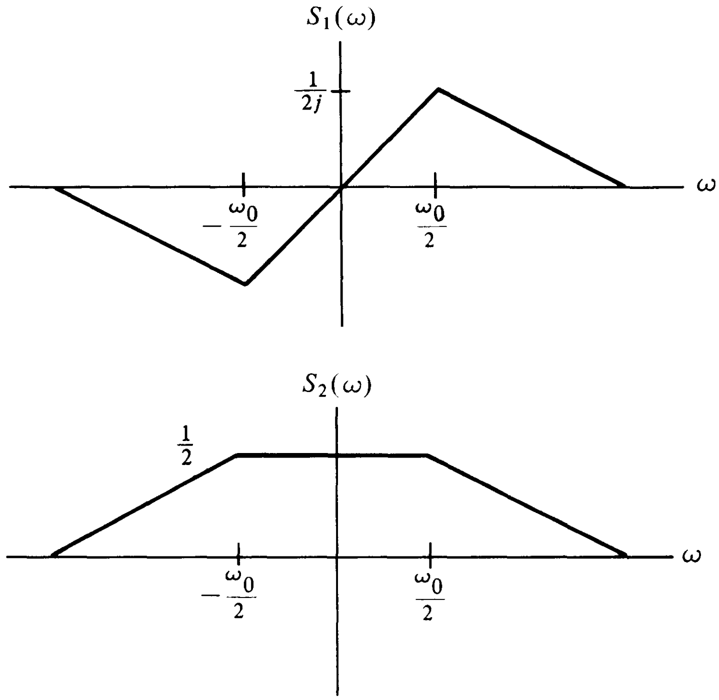

Figure S13.6-1

After filtering, $S_3(\omega)$ and $S_4(\omega)$ are given as in Figure S13.6-2.

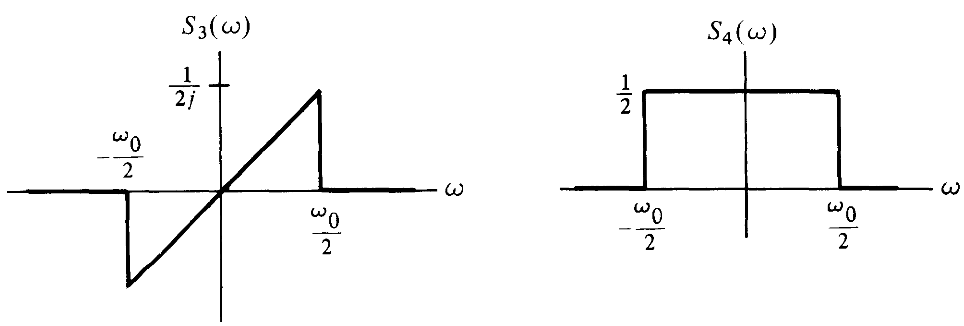

Figure S13.6-2

$S_5(\omega)$ is as follows (see Figure S13.6-3):

$$S_5(\omega) = \frac{1}{2j}\left[S_3\left(\omega - \omega_c - \frac{\omega_0}{2}\right) - S_3\left(\omega + \omega_c + \frac{\omega_0}{2}\right)\right]$$

Note that the amplitude is reversed since $(1/2j)(1/2j) = -\frac{1}{4}$.

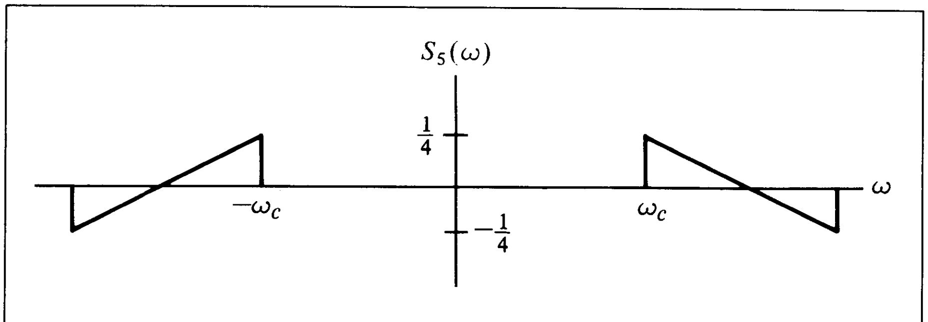

Figure S13.6-3

$S_6(\omega)$ is as follows and as shown in Figure S13.6-4.

$$S_6(\omega) = \frac{1}{2}\left[S_4\left(\omega - \omega_c - \frac{\omega_0}{2}\right) + S_4\left(\omega + \omega_c + \frac{\omega_0}{2}\right)\right]$$

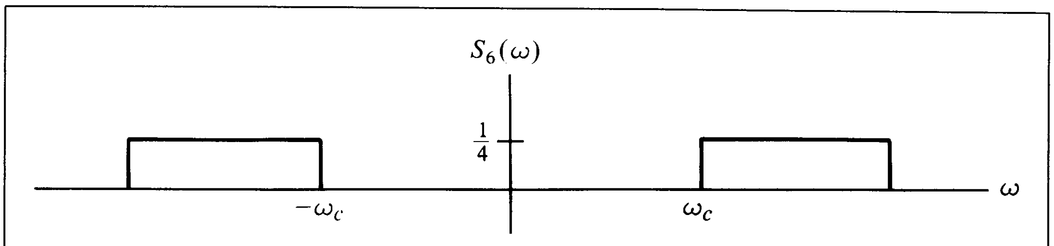

Figure S13.6-4

Finally, $Y(\omega) = S_5(\omega) + S_6(\omega)$, as shown in Figure S13.6-5.

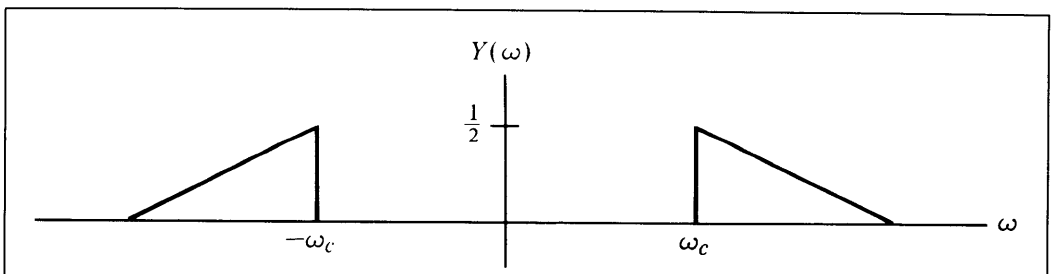

Figure S13.6-5

Thus, $y(t)$ is a single-sideband modulation of $x(t)$.

S13.7

Note that

$$\begin{aligned} q_1(t) &= [s_1(t)\cos \omega_0 t + s_2(t)\sin \omega_0 t]\cos \omega_0 t \\ &= s_1(t)\cos^2\omega_0 t + s_2(t)\sin \omega_0 t \cos \omega_0 t \end{aligned}$$

Using trigonometric identities, we have

$$q_1(t) = \tfrac{1}{2}s_1(t) + \tfrac{1}{2}s_1(t)\cos 2\omega_0 t + \tfrac{1}{2}s_2(t)\sin 2\omega_0 t$$

Thus, if $s_1(t)$ is bandlimited to $\pm\omega_0$ and we use the filter $H(\omega)$ as given in Figure S13.7, $y_1(t)$ will then equal $s_1(t)$.

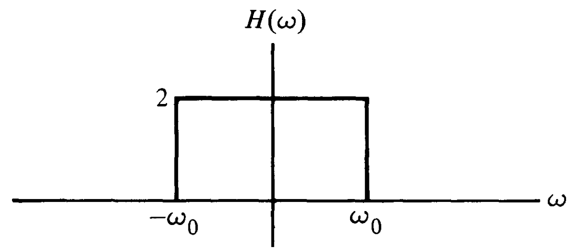

Figure S13.7

Similarly,

$$q_2(t) = s_1(t)\cos \omega_0 t \sin \omega_0 t + s_2(t)\sin^2 \omega_0 t$$
$$= \frac{s_1(t)}{2}\sin 2\omega_0 t + \frac{s_2(t)}{2} - \frac{s_2(t)}{2}\cos 2\omega_0 t$$

Using the same filter and imposing the same restrictions on $s_2(t)$, we obtain $y_2(t) = s_2(t)$.

S13.8

(a) $X(\omega)$ is given as in Figure S13.8-1.

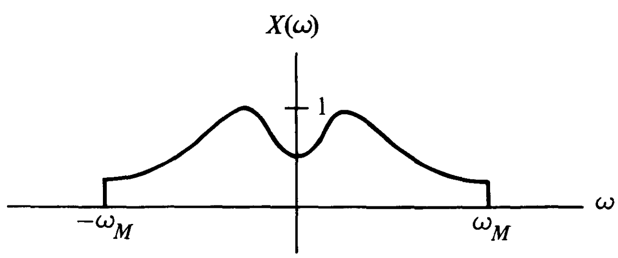

Figure S13.8-1

For $Y(\omega)$, the spectrum of the scrambled signal is as shown in Figure S13.8-2.

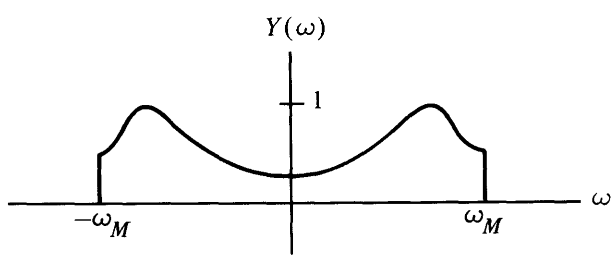

Figure S13.8-2

Thus, $X(\omega)$ is reversed for $\omega > 0$ and $\omega < 0$.

(b) Suppose we multiply $x(t)$ by $\cos \omega_M t$. Denoting $z(t) = x(t)\cos \omega_M t$, we find that $Z(\omega)$ is composed of scaled versions of $X(\omega)$ centered at $\pm\omega_M$. See Figure S13.8-3.

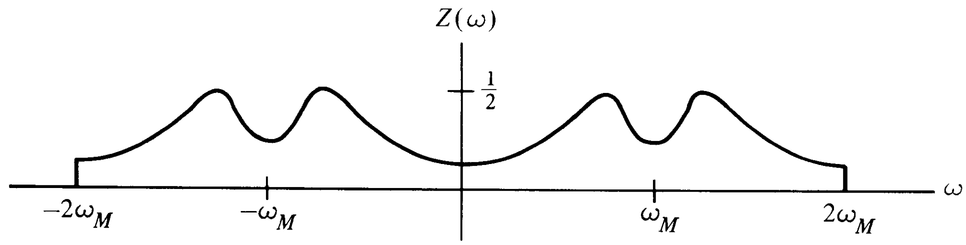

Figure S13.8-3

Filtering $z(t)$ with an ideal lowpass filter with a gain of 2 yields $y(t)$, as shown in Figure S13.8-4.

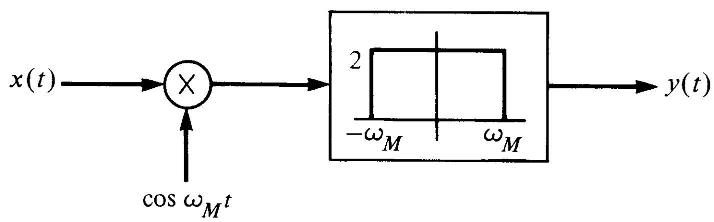

Figure S13.8-4

(c) Suppose we use the same system to recover $x(t)$. Let $y(t)\cos \omega_M t = r(t)$. Then $R(\omega)$ is as given in Figure S13.8-5.

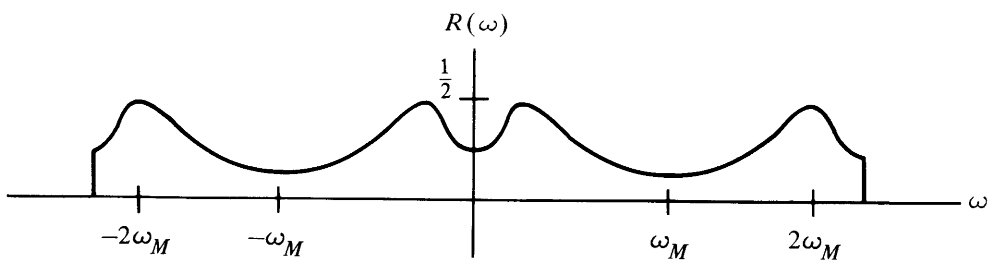

Figure S13.8-5

Filtering with the same lowpass filter yields $x(t)$.