11 Discrete-Time Fourier Transform

Solutions to Recommended Problems

S11.1

(a)
$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$
$$= \sum_{n=-\infty}^{\infty} (\tfrac{1}{4})^n u[n]e^{-j\Omega n}$$
$$= \sum_{n=0}^{\infty} (\tfrac{1}{4}e^{-j\Omega})^n$$
$$= \frac{1}{1 - \frac{1}{4}e^{-j\Omega}}$$

Here we have used the fact that

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \qquad \text{for } |a| < 1$$

(b) $x[n] = (a^n \sin \Omega_0 n)u[n]$

We can use the modulation property to evaluate this signal. Since

$$\sin \Omega_0 n \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{2\pi}{2j}[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)],$$

periodically repeated, then

$$X(\Omega) = \frac{1}{2j}\left[\frac{1}{1 - ae^{-j(\Omega-\Omega_0)}} - \frac{1}{1 - ae^{-j(\Omega+\Omega_0)}}\right]$$

periodically repeated.

(c)
$$X(\Omega) = \sum_{n=0}^{3} e^{-j\Omega n}$$
$$= \frac{1 - e^{-j4\Omega}}{1 - e^{-j\Omega}},$$

using the identity

$$\sum_{n=0}^{N-1} a^n = \frac{1 - a^N}{1 - a}$$

Alternatively, we can use the fact that $x[n] = u[n] - u[n-4]$, so

$$X(\Omega) = \frac{1}{1 - e^{-j\Omega}} - \frac{e^{-j4\Omega}}{1 - e^{-j\Omega}} = \frac{1 - e^{-j4\Omega}}{1 - e^{-j\Omega}}$$

(d)
$$x[n] = (\tfrac{1}{4})^n u[n+2]$$
$$= (\tfrac{1}{4})^{n+2}(\tfrac{1}{4})^{-2}u[n+2]$$
$$= 16(\tfrac{1}{4})^{n+2}u[n+2]$$

We know that

$$16\left(\frac{1}{4}\right)^n u[n] \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{16}{1 - \frac{1}{4}e^{-j\Omega}},$$

so

$$16\left(\frac{1}{4}\right)^{n+2} u[n+2] \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{16e^{j2\Omega}}{1 - \frac{1}{4}e^{-j\Omega}}$$

S11.2

(a) The difference equation $y[n] - \frac{1}{2}y[n-1] = x[n]$, which is initially at rest, has a system transfer function that can be obtained by taking the Fourier transform of both sides of the equation. This yields

$$Y(\Omega)(1 - \tfrac{1}{2}e^{-j\Omega}) = X(\Omega),$$

so

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1}{1 - (\frac{1}{2})^{-j\Omega}}$$

(b) (i) If $x[n] = \delta[n]$, then $X(\Omega) = 1$ and

$$Y(\Omega) = H(\Omega)X(\Omega) = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}},$$

so

$$y[n] = (\tfrac{1}{2})^n u[n]$$

(ii) $X(\Omega) = e^{-j\Omega n_0}$, so

$$Y(\Omega) = \frac{e^{-j\Omega n_0}}{1 - \frac{1}{2}e^{-j\Omega}}$$

and, using the delay property of the Fourier transform,

$$y[n] = (\tfrac{1}{2})^{n-n_0}u[n - n_0]$$

(iii) If $x[n] = (\frac{3}{4})^n u[n]$, then

$$X(\Omega) = \frac{1}{1 - \frac{3}{4}e^{-j\Omega}},$$

$$Y(\Omega) = \left(\frac{1}{1 - \frac{1}{2}e^{-j\Omega}}\right)\left(\frac{1}{1 - \frac{3}{4}e^{-j\Omega}}\right) = \frac{-2}{1 - \frac{1}{2}e^{-j\Omega}} + \frac{3}{1 - \frac{3}{4}e^{-j\Omega}},$$

so

$$y[n] = -2(\tfrac{1}{2})^n u[n] + 3(\tfrac{3}{4})^n u[n]$$

S11.3

(a) We are given a system with impulse response

$$h[n] = \left[\left(\frac{1}{2}\right)^n \cos\frac{\pi n}{2}\right] u[n]$$

The signal $h_1[n] = (\frac{1}{2})^n u[n]$ has the Fourier transform

$$H_1(\Omega) = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}}$$

Using the modulation theorem, we have

$$H(\Omega) = \frac{1}{2}\left[\frac{1}{1 - \frac{1}{2}e^{-j(\Omega-\pi/2)}} + \frac{1}{1 - \frac{1}{2}e^{-j(\Omega+\pi/2)}}\right]$$

(b) We expect the system output to be a sinusoid modified in amplitude and phase. Using the results in part (a) and the fact that

$$x[n] = \tfrac{1}{2}e^{j(\pi n/2)} + \tfrac{1}{2}e^{-j(\pi n/2)},$$

we have

$$H(\Omega)\Big|_{\Omega=\pi/2} = \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}} + \frac{1}{1+\frac{1}{2}}\right)$$
$$= \frac{1}{2}\left(2 + \frac{2}{3}\right) = \frac{4}{3},$$
$$H(\Omega)\Big|_{\Omega=-\pi/2} = H^*(\Omega)\Big|_{\Omega=\pi/2} = \frac{4}{3}$$

so

$$y[n] = \frac{2}{3}e^{j(\pi n/2)} + \frac{2}{3}e^{-j(\pi n/2)}$$
$$= \frac{4}{3}\cos\frac{\pi}{2}n$$

S11.4

(a) The use of the Fourier transform simplifies the analysis of the difference equation.

$$y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] - x[n-1],$$
$$Y(\Omega)(1 + \frac{1}{4}e^{-j\Omega} - \frac{1}{8}e^{-j2\Omega}) = X(\Omega)(1 - e^{-j\Omega}),$$
$$\frac{Y(\Omega)}{X(\Omega)} = H(\Omega) = \frac{1 - e^{-j\Omega}}{(1 + \frac{1}{2}e^{-j\Omega})(1 - \frac{1}{4}e^{-j\Omega})}$$

We want to put this in a form that is easily invertible to get the impulse response $h[n]$. Using a partial fraction expansion, we see that

$$H(\Omega) = \frac{2}{1 + \frac{1}{2}e^{-j\Omega}} + \frac{-1}{1 - \frac{1}{4}e^{-j\Omega}},$$

so

$$h[n] = 2(-\tfrac{1}{2})^n u[n] - (\tfrac{1}{4})^n u[n]$$

(b) At $\Omega = 0$, $H(\Omega) = 0$. At $\Omega = \pi/4$, $H(\Omega) = 0.65e^{j(1.22)}$. Since $h[n]$ is real, $H(\Omega) = H^*(-\Omega)$, so $H(-\Omega) = H^*(\Omega)$ and $H(-\pi/4) = 0.65e^{-j(1.22)}$. Since $H(\Omega)$ is periodic in 2π,

$$H\left(\frac{9\pi}{4}\right) = H\left(\frac{\pi}{4}\right) = 0.65e^{j(1.22)}$$

S11.5

(a) $x[n]$ is an aperiodic signal with extent $[0, N-1]$. The periodic signal

$$\tilde{y}[n] = \sum_{r=-\infty}^{\infty} x[n + rN]$$

is periodic with period N. To get the Fourier series coefficients for $\tilde{y}[n]$, we sum over one period of $\tilde{y}[n]$ to get

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk(2\pi/N)n}$$

(b) The Fourier transform of $x[n]$ is

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=0}^{N-1} x[n]e^{-j\Omega n}$$

since $x[n] = 0$ for $n < 0$, $n > N - 1$.

We can now easily see the relation between a_k and $X(\Omega)$ since

$$\frac{1}{N}X(\Omega)\bigg|_{\Omega=(2\pi k)/N} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk(2\pi/N)n}$$

Therefore,

$$\frac{1}{N}X\left(\frac{2\pi k}{N}\right) = a_k$$

S11.6

(a)

Signal Description			Transform
Continuous time	Infinite duration	Periodic	I, III
Continuous time	Infinite duration	Aperiodic	III
Continuous time	Finite duration	Aperiodic	III, I*
Discrete time	Infinite duration	Periodic	II, IV
Discrete time	Infinite duration	Aperiodic	IV
Discrete time	Finite duration	Aperiodic	IV, II*

*Because these two signals are aperiodic, we know that they do not possess a Fourier series. However, since they are both finite duration, the Fourier series can be used to express a periodic signal that is formed by periodically replicating the finite-duration signal.

(b) The discrete-time Fourier series has time- and frequency-domain duality. Both the analysis and synthesis equations are summations. The continuous-time Fourier transform has time- and frequency-domain duality. Both the analysis and synthesis equations are integrals.

(c) The discrete-time Fourier series and Fourier transform are periodic with periods N and 2π respectively.

Solutions to Optional Problems

S11.7

Because of the discrete nature of a discrete-time signal, the time/frequency scaling property does not hold. A result that closely parallels this property but does hold for discrete-time signals can be developed. Define

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k, \\ 0, & \text{otherwise} \end{cases}$$

$x_{(k)}[n]$ is a "slowed-down" version of $x[n]$ with zeros interspersed. By analysis in the frequency domain,

$$X_{(k)}(\Omega) = X(k\Omega),$$

which indicates that $X_{(k)}(\Omega)$ is compressed in the frequency domain.

S11.8

(a) $X(\Omega - \Omega_0)$ is a shift in frequency of the spectrum $X(\Omega)$. We will see later that this is the result of modulating $x[n]$ with an exponential carrier. To derive the modification $x_m[n]$, we use the synthesis equation:

$$x_m[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega - \Omega_0) e^{j\Omega n} \, d\Omega$$

Changing variables so that $\Omega - \Omega_0 = \Omega'$, we have

$$x_m[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega') e^{j(\Omega' + \Omega_0)n} \, d\Omega = x[n] e^{j\Omega_0 n}$$

(b) Using the synthesis equation, we have

$$\begin{aligned} \frac{1}{2\pi} \int_{2\pi} Re\{X(\Omega)\} e^{j\Omega n} \, d\Omega &= \frac{1}{2\pi} \int_{2\pi} \frac{1}{2} [X(\Omega) + X^*(\Omega)] e^{j\Omega n} \, d\Omega \\ &= \frac{1}{2} x[n] + \frac{1}{2\pi} \left(\int_{2\pi} \frac{1}{2} X(\Omega) e^{-j\Omega n} \, d\Omega \right)^* \\ &= \frac{1}{2} \{x[n] + x^*[-n]\} \end{aligned}$$

(c)

$$\begin{aligned} \frac{1}{2\pi} \int_{2\pi} Im\{X(\Omega)\} e^{j\Omega n} \, d\Omega &= \frac{1}{2\pi} \int_{2\pi} \left[\frac{X(\Omega) - X^*(\Omega)}{2j} \right] e^{j\Omega n} \, d\Omega \\ &= \frac{1}{2j} x[n] - \frac{1}{2j} \left(\frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{-j\Omega n} \, d\Omega \right)^* \\ &= \frac{1}{2j} \{x[n] - x^*[-n]\} \end{aligned}$$

(d) Since $|X(\Omega)|^2 = X(\Omega) X^*(\Omega)$, we see that the inverse transform will be in the form of a convolution. Since

$$\begin{aligned} \frac{1}{2\pi} \int_{2\pi} X^*(\Omega) e^{j\Omega n} \, d\Omega &= \left(\frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{-j\Omega n} \, d\Omega \right)^* \\ &= x^*[-n], \end{aligned}$$

then

$$\frac{1}{2\pi} \int_{2\pi} |X(\Omega)|^2 e^{j\Omega n} \, d\Omega = x[n] * x^*[-n]$$

S11.9

We are given an LTI system with impulse response

$$h[n] = \frac{\sin(\pi n/3)}{\pi n}$$

(a) We know from duality that $H(\Omega)$ is a pulse sequence that is periodic with period 2π. Suppose we assume this and adjust the parameters of the pulse so that

$$\frac{1}{2\pi}\int H(\Omega)e^{j\Omega n}\,d\Omega = h[n]$$

Let a be the pulse amplitude and let $2W$ be the pulse width. Then

$$\frac{a}{2\pi}\int_{-W}^{W} e^{j\Omega n}\,d\Omega = \frac{a}{2\pi}\left(\frac{e^{j\Omega W} - e^{-j\Omega W}}{jn}\right)$$
$$= \frac{a}{2\pi}\frac{2\sin Wn}{n},$$

so $a = 1$ and $W = \pi/3$, as indicated in Figure S11.9-1.

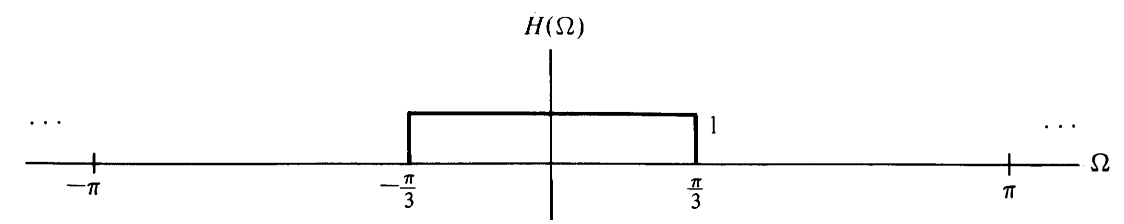

Figure S11.9-1

(b) We know that

$$\cos\frac{3\pi}{4}n \overset{\mathcal{F}}{\longleftrightarrow} \pi\left[\delta\left(\Omega - \frac{3\pi}{4}\right) + \delta\left(\Omega + \frac{3\pi}{4}\right)\right],$$

periodically repeated, and that multiplication by $(-1)^n$ shifts the periodic spectrum by π, so the spectrum $Y(\Omega)$ is as shown in Figure S11.9-2.

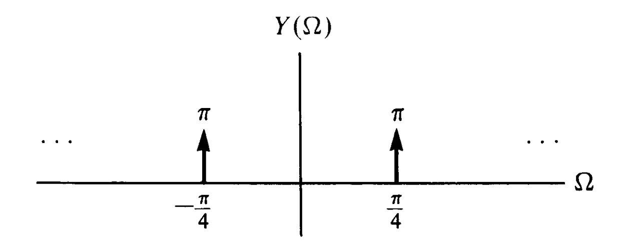

Figure S11.9-2

From Figures S11.9-1 and S11.9-2, we can see that

$$Y(\Omega) = H(\Omega)X(\Omega) = X(\Omega)$$

Therefore,

$$y[n] = x[n] = (-1)^n \cos \frac{3\pi}{4} n = \cos \frac{\pi n}{4}$$

S11.10

Here

$$Y(\Omega) = 2X(\Omega) + e^{-j\Omega}X(\Omega) - \frac{dX(\Omega)}{d\Omega}$$

(a) (i) The system is linear because if

$$x[n] = ax_1[n] + bx_2[n],$$

then

$$y[n] = ay_1[n] + by_2[n],$$

where $y_1[n]$ is obtained from $x_1[n]$ via the given transfer function. The similar result applies for $y_2[n]$.

(ii) The system is time-varying by the following argument.
If $x[n] \to y[n]$, does $x[n-1] \to y[n-1]$?

$$x[n-1] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\Omega}X(\Omega)$$

The corresponding $Y(\Omega)$ is

$$2e^{j\Omega}X(\Omega) + e^{-j\Omega}X(\Omega)e^{-j\Omega} + je^{-j\Omega}X(\Omega) - e^{-j\Omega}\frac{dX(\Omega)}{d\Omega}$$
$$\neq e^{-j\Omega}\left[2X(\Omega) + e^{-j\Omega}X(\Omega) - \frac{dX(\Omega)}{d\Omega}\right]$$

(iii) If $x[n] = \delta[n]$, $X(\Omega) = 1$. Then

$$Y(\Omega) = 2 + e^{-j\Omega},$$
$$y[n] = 2\delta[n] + \delta[n-1]$$

S11.11

$$\tilde{x}[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$

(a) If we multiply both sides of this equation by $e^{-jl(2\pi/N)n}$ and sum over $\langle N\rangle$, we obtain

$$\sum_{n=\langle N\rangle} \tilde{x}[n]e^{-jl(2\pi/N)n} = \sum_{k=\langle N\rangle}\sum_{n=\langle N\rangle} a_k e^{j(k-l)(2\pi/N)n}$$

If k is held fixed, the summation over $\langle N\rangle$ is zero unless $k = l$, which yields Na_l. Thus

$$a_l = \frac{1}{N}\sum_{n=\langle N\rangle} \tilde{x}[n]e^{-jl(2\pi/N)n}$$

and therefore

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}$$

(b) We are given that $x[n]$ is an aperiodic signal

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} \, d\Omega$$

(i) By multiplying both sides by $e^{-j\Omega_1 n}$ and summing over all n, we have

$$\sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega_1 n} = \frac{1}{2\pi} \int_{2\pi} X(\Omega) \sum_{n=-\infty}^{\infty} e^{j(\Omega - \Omega_1)n} \, d\Omega$$

(ii) $\sum_{n=-\infty}^{\infty} e^{j(\Omega-\Omega_1)n}$ needs to be evaluated. We can recognize that this summation is a Fourier series representation

$$\sum_{n=-\infty}^{\infty} e^{j(\Omega-\Omega_1)n} = \sum_{n=-\infty}^{\infty} a_n e^{j[(2\pi(\Omega-\Omega_1))/T]n},$$

where $T = 2\pi$ and $a_n = 1$. The periodic function represented by this series is a periodic impulse train with period $T = 2\pi$, so

$$\sum_{n=-\infty}^{\infty} e^{j(\Omega-\Omega_1)n} = 2\pi \sum_{n=-\infty}^{\infty} \delta(\Omega - \Omega_1 + 2\pi n)$$

(iii) Only a single impulse in the train appears in the integration interval of one period. So

$$\frac{1}{2\pi} \int_{2\pi} X(\Omega) \sum_{n=-\infty}^{\infty} e^{j(\Omega-\Omega_1)n} = X(\Omega_1 + 2\pi n)$$
$$= X(\Omega_1)$$

Therefore, the analysis formula for aperiodic discrete signals has been verified to be analogous to the analysis formula in part (a).

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

S11.12

(a) The Fourier transform of $e^{jk(2\pi/N)n}$ can be performed by inspection using the synthesis formula

$$e^{jk(2\pi/N)n} = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} \, d\Omega,$$
$$X(\Omega) = 2\pi\delta\left(\Omega - \frac{2\pi k}{N}\right), \qquad |\Omega| < \pi$$

and since we know that $X(\Omega)$ is periodic in $\Omega = 2\pi$, we have

$$e^{jk(2\pi/N)n} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi \sum_{m=-\infty}^{\infty} \delta\left(\Omega - \frac{2\pi k}{N} + 2\pi m\right)$$

(b) By using superposition and the result in part (a), we have

$$\sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n} \overset{\mathcal{F}}{\longleftrightarrow} \sum_{m=-\infty}^{\infty} 2\pi \sum_{k=\langle N \rangle} a_k \delta\left(\Omega - \frac{2\pi k}{N} + 2\pi m\right)$$

(c) We can change the double summation to a single summation since a_k is periodic:

$$\sum_{n=-\infty}^{\infty} 2\pi \sum_{k=\langle N\rangle} a_k \delta\left(\Omega - \frac{2\pi k}{N} + 2\pi n\right) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\Omega - \frac{2\pi k}{N}\right)$$

So we have established the Fourier transform of a periodic signal via the use of a Fourier series:

$$\tilde{x}[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\Omega - \frac{2\pi k}{N}\right)$$

(d) We have

$$\tilde{x}[n] = \sum_{k=-\infty}^{\infty} x[n - kN] \longleftrightarrow \sum_{k=-\infty}^{\infty} X(\Omega) e^{-j\Omega kN}$$

As in S11.11(b)(ii), we can show that

$$\sum_{k=-\infty}^{\infty} e^{-j\Omega kN} = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - \frac{2\pi k}{N}\right)$$

Therefore,

$$\begin{aligned}\tilde{x}[n] \longleftrightarrow 2\pi \sum_{k=-\infty}^{\infty} \frac{1}{N} X(\Omega) \delta\left(\Omega - \frac{2\pi k}{N}\right) \\ = 2\pi \sum_{k=-\infty}^{\infty} \frac{1}{N} X\left(\frac{2\pi k}{N}\right) \delta\left(\Omega - \frac{2\pi k}{N}\right)\end{aligned}$$

Comparing with the result of part (c), we see that

$$a_k = \frac{1}{N} X(\Omega) \bigg|_{\Omega = (2\pi k)/N}$$