8 Continuous-Time Fourier Transform

Solutions to Recommended Problems

S8.1

(a)

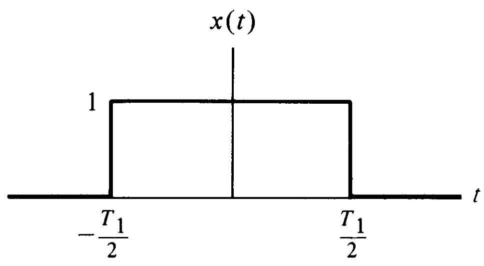

Figure S8.1-1

Note that the *total* width is T_1.

(b)

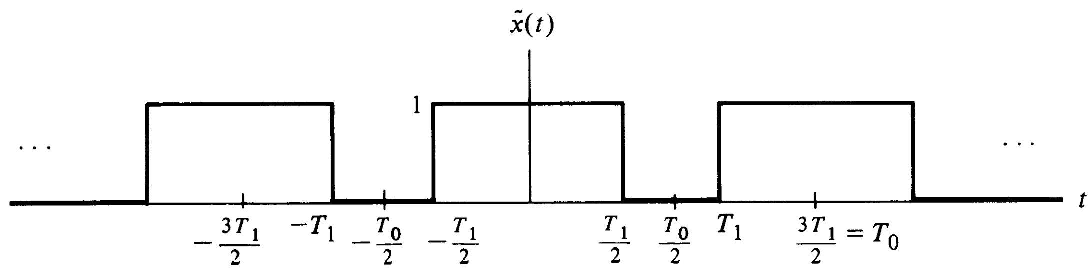

Figure S8.1-2

(c) Using the definition of the Fourier transform, we have

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-T_1/2}^{T_1/2} 1e^{-j\omega t}\,dt \quad \text{since } x(t) = 0 \quad \text{for} \quad |t| > \frac{T_1}{2}$$

$$= \frac{-1}{j\omega} e^{-j\omega t}\Big|_{-T_1/2}^{T_1/2} = \frac{-1}{j\omega}(e^{-j\omega T_1/2} - e^{j\omega T_1/2}) = \frac{2\sin\frac{\omega T_1}{2}}{\omega}$$

See Figure S8.1-3.

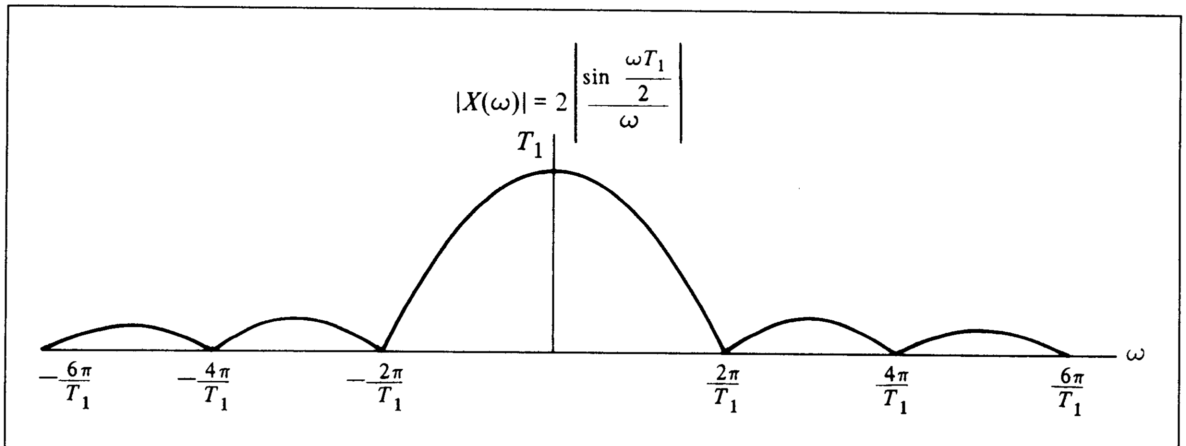

Figure S8.1-3

(d) Using the analysis formula, we have

$$a_k = \frac{1}{T_0}\int_{T_0} \tilde{x}(t)e^{-jk\omega_0 t}\,dt,$$

where we integrate over *any* period.

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \tilde{x}(t)e^{-jk(2\pi/T_0)t}\,dt = \frac{1}{T_0}\int_{-T_1/2}^{T_1/2} e^{-jk(2\pi/T_0)t}\,dt,$$

$$a_k = \frac{1}{T_0}\left(\frac{1}{-jk\frac{2\pi}{T_0}}\right)(e^{-jk\pi T_1/T_0} - e^{jk\pi T_1/T_0}) = \frac{\sin k\pi(T_1/T_0)}{\pi k} = \frac{\sin \pi(2k/3)}{\pi k}$$

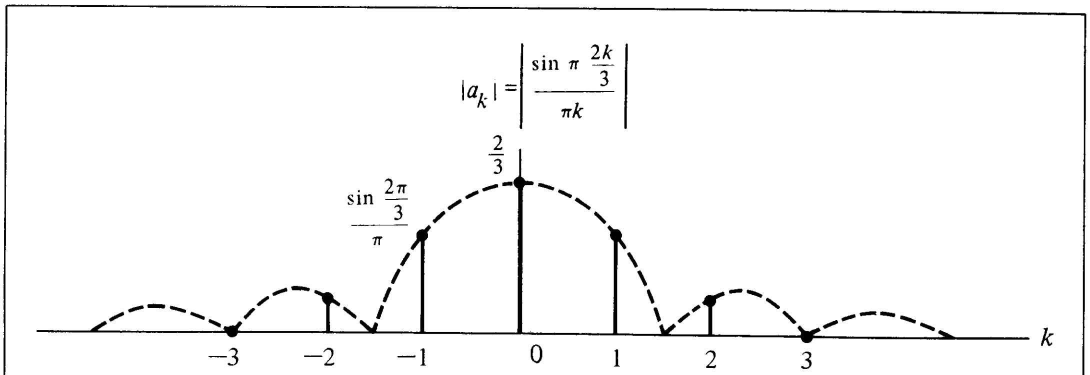

Figure S8.1-4

Note that $a_k = 0$ whenever $(2\pi k)/3 = \pi m$ for m a nonzero integer.

(e) Substituting $(2\pi k)/T_0$ for ω, we obtain

$$\frac{1}{T_0}X(\omega)\Big|_{\omega=(2\pi k)/T_0} = \frac{1}{T_0}\frac{2\sin(\pi k T_1/T_0)}{2\pi k/T_0} = \frac{\sin \pi k(T_1/T_0)}{\pi k} = a_k$$

(f) From the result of part (e), we sample the Fourier transform of $x(t)$, $X(\omega)$, at $\omega = 2\pi k/T_0$ and then scale by $1/T_0$ to get a_k.

S8.2

(a) $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} \delta(t-5)e^{-j\omega t}\,dt = e^{-j5\omega} = \cos 5\omega - j\sin 5\omega,$

by the sifting property of the unit impulse.

$$|X(\omega)| = |e^{j5\omega}| = 1 \qquad \text{for all } \omega,$$

$$\sphericalangle X(\omega) = \tan^{-1}\left[\frac{Im\{X(\omega)\}}{Re\{X(\omega)\}}\right] = \tan^{-1}\left(\frac{-\sin 5\omega}{\cos 5\omega}\right) = -5\omega$$

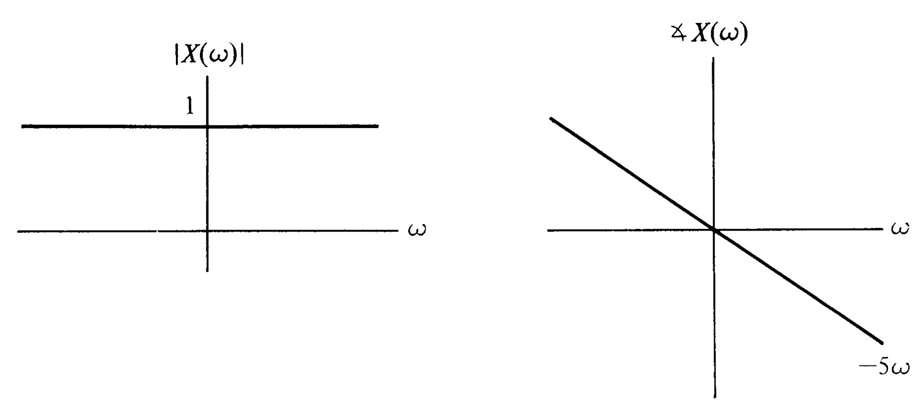

Figure S8.2-1

(b) $X(\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,dt$

$$= \int_0^{\infty} e^{-(a+j\omega)t}\,dt = \frac{-1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^{\infty}$$

Since $Re\{a\} > 0$, e^{-at} goes to zero as t goes to infinity. Therefore,

$$X(\omega) = \frac{-1}{a+j\omega}(0-1) = \frac{1}{a+j\omega},$$

$$|X(\omega)| = [X(\omega)X^*(\omega)]^{1/2} = \left[\frac{1}{a+j\omega}\left(\frac{1}{a-j\omega}\right)\right]^{1/2} \frac{1}{\sqrt{a^2+\omega^2}},$$

$$Re\{X(\omega)\} = \frac{X(\omega)+X^*(\omega)}{2} = \frac{a}{a^2+\omega^2},$$

$$Im\{X(\omega)\} = \frac{X(\omega)-X^*(\omega)}{2} = \frac{-\omega}{a^2+\omega^2},$$

$$\sphericalangle X(\omega) = \tan^{-1}\left[\frac{Im\{X(\omega)\}}{Re\{X(\omega)\}}\right] = -\tan^{-1}\frac{\omega}{a}$$

The magnitude and angle of $X(\omega)$ are shown in Figure S8.2-2.

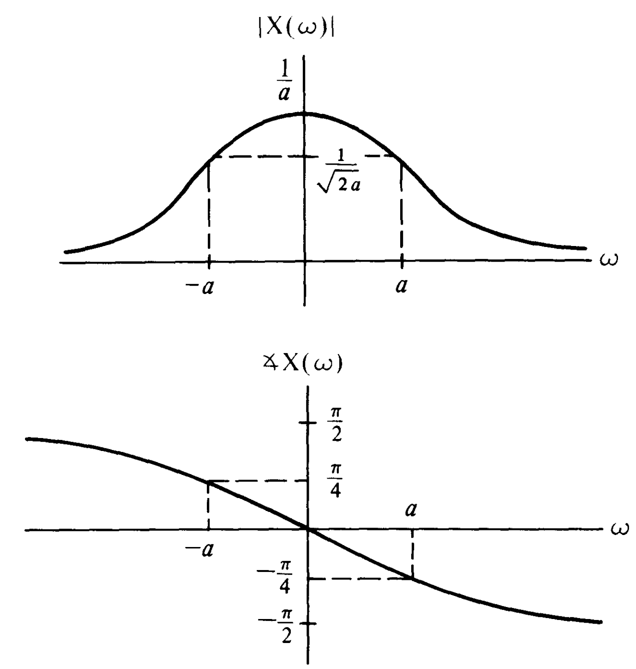

Figure S8.2-2

(c) $X(\omega) = \int_{-\infty}^{\infty} e^{(-1+j2)t}u(t)e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{(-1+j2)t}e^{-j\omega t}\,dt$

$$= \frac{1}{-1+j(2-\omega)} e^{[-1+j(2-\omega)]t}\Big|_0^{\infty}$$

Since $Re\{-1+j(2-\omega)\} < 0$, $\lim_{t\to\infty} e^{[-1+j(2-\omega)]t} = 0$. Therefore,

$$X(\omega) = \frac{1}{1+j(\omega-2)}$$

$$|X(\omega)| = [X(\omega)X^*(\omega)]^{1/2} = \frac{1}{\sqrt{1+(\omega-2)^2}}$$

$$Re\{X(\omega)\} = \frac{X(\omega)+X^*(\omega)}{2} = \frac{1}{1+(\omega-2)^2}$$

$$Im\{X(\omega)\} = \frac{X(\omega)-X^*(\omega)}{2}\,\frac{-(\omega-2)}{1+(\omega-2)^2}$$

$$\measuredangle X(\omega) = \tan^{-1}\left[\frac{Im\{X(\omega)\}}{Re\{X(\omega)\}}\right] = -\tan^{-1}(\omega-2)$$

The magnitude and angle of $X(\omega)$ are shown in Figure S8.2-3.

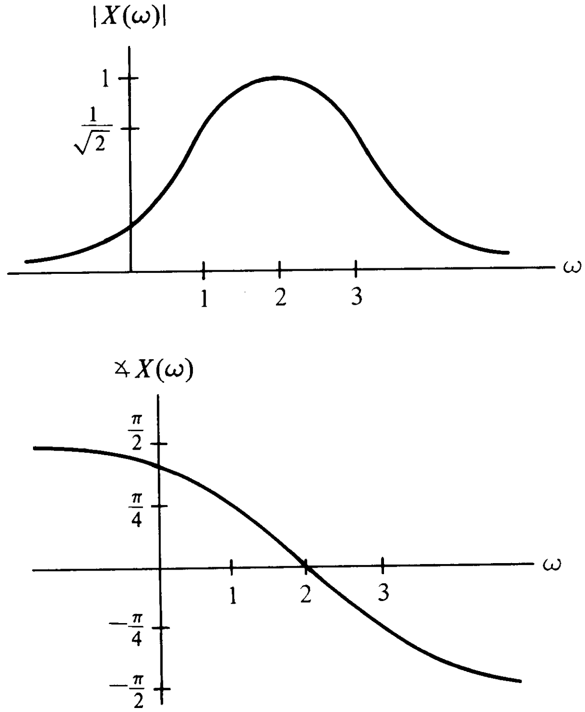

Figure S8.2-3

Note that there is no symmetry about $\omega = 0$ since $x(t)$ is not real.

S8.3

(a) $X_3(\omega) = \int_{-\infty}^{\infty} x_3(t)e^{-j\omega t}\,dt$

Substituting for $x_3(t)$, we obtain

$$\begin{aligned} X_3(\omega) &= \int_{-\infty}^{\infty} [ax_1(t) + bx_2(t)]e^{-j\omega t}\,dt \\ &= \int_{-\infty}^{\infty} ax_1(t)e^{-j\omega t}\,dt + \int_{-\infty}^{\infty} bx_2(t)e^{-j\omega t}\,dt \\ &= a\int_{-\infty}^{\infty} x_1(t)e^{-j\omega t}\,dt + b\int_{-\infty}^{\infty} x_2(t)e^{-j\omega t}\,dt = aX_1(\omega) + bX_2(\omega) \end{aligned}$$

(b) Recall the sifting property of the unit impulse function:

$$\int_{-\infty}^{\infty} h(t)\delta(t - t_0)\,dt = h(t_0)$$

Therefore,

$$\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}\,d\omega = 2\pi e^{j\omega_0 t}$$

Thus,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}\,d\omega = e^{j\omega_0 t}$$

Note that the integral relating $2\pi\delta(\omega - \omega_0)$ and $e^{j\omega_0 t}$ is exactly of the form

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega,$$

where $x(t) = e^{j\omega_0 t}$ and $X(\omega) = 2\pi\delta(\omega - \omega_0)$. Thus, we can think of $e^{j\omega_0 t}$ as the inverse Fourier transform of $2\pi\delta(\omega - \omega_0)$. Therefore, $2\pi\delta(\omega - \omega_0)$ is the Fourier transform of $e^{j\omega_0 t}$.

(c) Using the result of part (a), we have

$$X(\omega) = \mathcal{F}\{\tilde{x}(t)\} = \mathcal{F}\left\{\sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}\right\} = \sum_{k=-\infty}^{\infty} a_k\, \mathcal{F}\left\{e^{jk(2\pi/T)t}\right\}$$

From part (b),

$$\mathcal{F}\{e^{jk(2\pi/T)t}\} = 2\pi\delta\left(\omega - \frac{2\pi k}{T}\right)$$

Therefore,

$$\tilde{X}(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{T}\right)$$

(d)

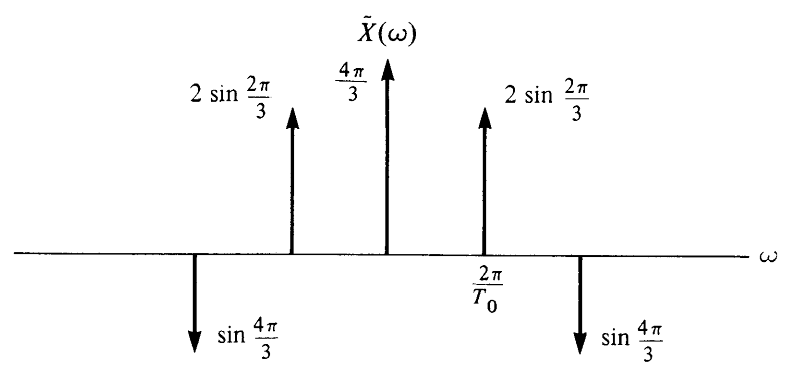

Figure S8.3

S8.4

(a) We see that the new transform is

$$X_a(f) = X(\omega)\Big|_{\omega=2\pi f}$$

We know that

$$x(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

Let $\omega = 2\pi f$. Then $d\omega = 2\pi df$, and

$$x(t) = \frac{1}{2\pi}\int_{f=-\infty}^{\infty} X(2\pi f)e^{j2\pi ft}2\pi\, df = \int_{f=-\infty}^{\infty} X_a(f)e^{j2\pi ft}\, df$$

Thus, there is no factor of 2π in the inverse relation.

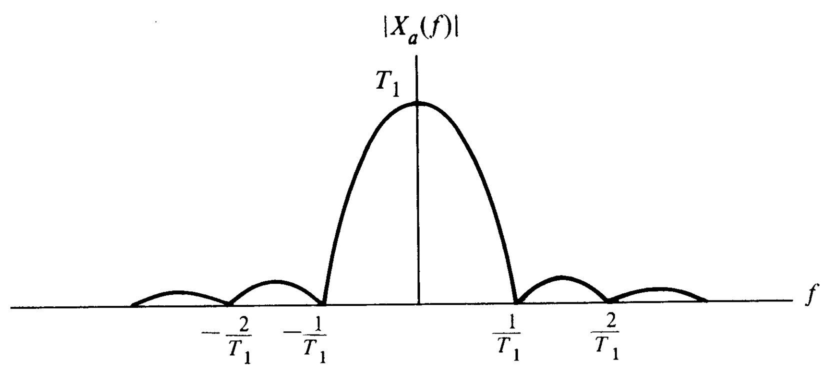

Figure S8.4

(b) Comparing

$$X_b(v) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x(t)e^{-jvt}\, dt \qquad \text{and} \qquad X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt,$$

we see that

$$X_b(v) = \frac{1}{\sqrt{2\pi}} X(\omega)\bigg|_{\omega=v} \qquad \text{or} \qquad X(\omega) = \sqrt{2\pi}\, X_b(\omega)$$

The inverse transform relation for $X(\omega)$ is

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \sqrt{2\pi}\, X_b(\omega)e^{j\omega t}\, d\omega$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} X_b(v)e^{jvt}\, dv,$$

where we have substituted v for ω. Thus, the factor of $1/2\pi$ has been distributed among the forward and inverse transforms.

S8.5

(a) By inspection, $T_0 = 6$.

(b) $a_k = \frac{1}{T_0}\int_{T_0} \tilde{x}(t)\, e^{-jk(2\pi/T_0)t}\, dt$

We integrate from -3 to 3:

$$a_k = \frac{1}{6}\int_{-3}^{3}\left[\frac{1}{2}\delta(t+1) + \delta(t) + \frac{1}{2}\delta(t-1)\right]e^{-jk(2\pi/6)t}dt$$
$$= \frac{1}{6}\left(\frac{1}{2}e^{j2\pi k/6} + 1 + \frac{1}{2}e^{-j2\pi k/6}\right) = \frac{1}{6}\left(1 + \cos\frac{2\pi k}{6}\right)$$

so

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{6}\left(1 + \cos\frac{2\pi k}{6}\right) e^{jk(2\pi/6)t}$$

(c) (i) $X_1(\omega) = \int_{-\infty}^{\infty} x_1(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} [\tfrac{1}{2}\delta(t+1) + \delta(t) + \tfrac{1}{2}\delta(t-1)]e^{-j\omega t}\,dt$

$= \tfrac{1}{2}e^{j\omega} + 1 + \tfrac{1}{2}e^{-j\omega} = 1 + \cos\omega$

(ii) $X_2(\omega) = \int_{-\infty}^{\infty} x_2(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} [\delta(t) + \tfrac{1}{2}\delta(t-1) + \tfrac{1}{2}\delta(t-5)]e^{-j\omega t}\,dt$

$= 1 + \tfrac{1}{2}e^{-j\omega} + \tfrac{1}{2}e^{-j5\omega}$

(d) We see that by periodically repeating $x_1(t)$ with period $T_1 = 6$, we get $\tilde{x}(t)$, as shown in Figure S8.5-1.

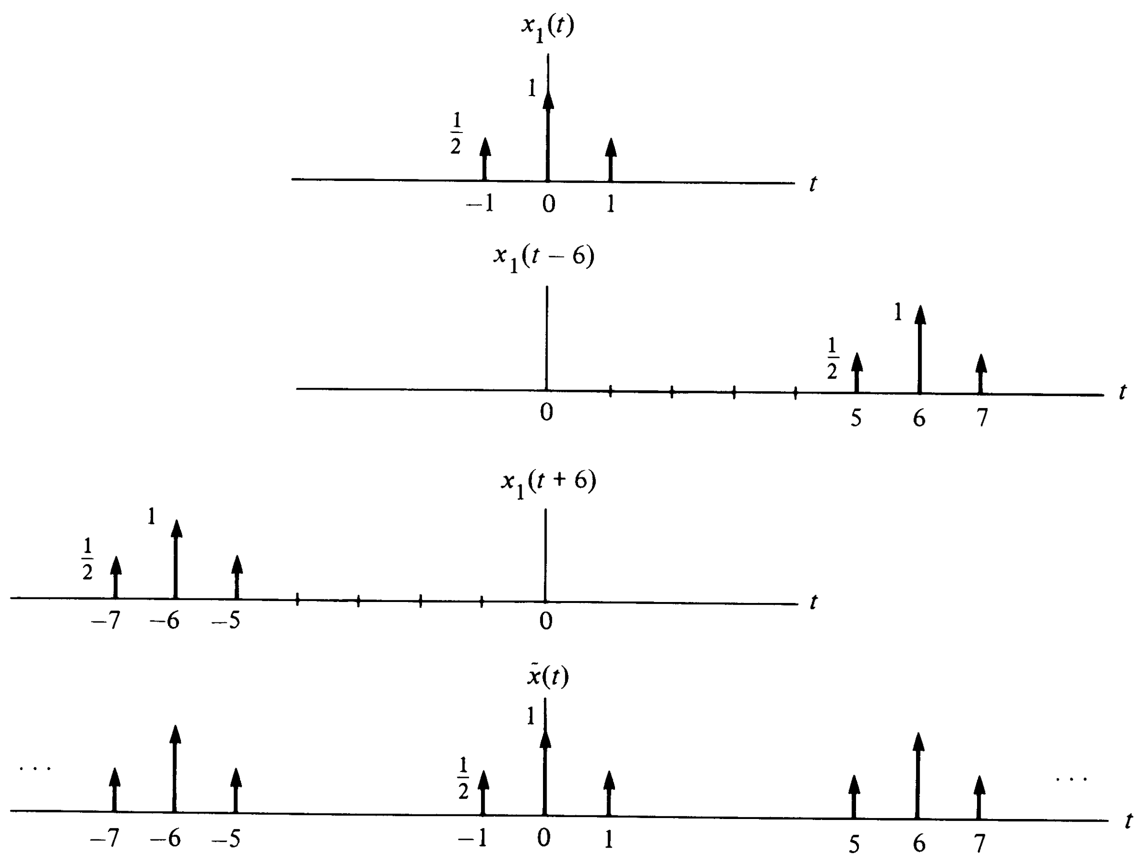

Figure S8.5-1

Similarly, we can periodically repeat $x_2(t)$ to get $\tilde{x}(t)$. Thus $T_2 = 6$. See Figure S8.5-2.

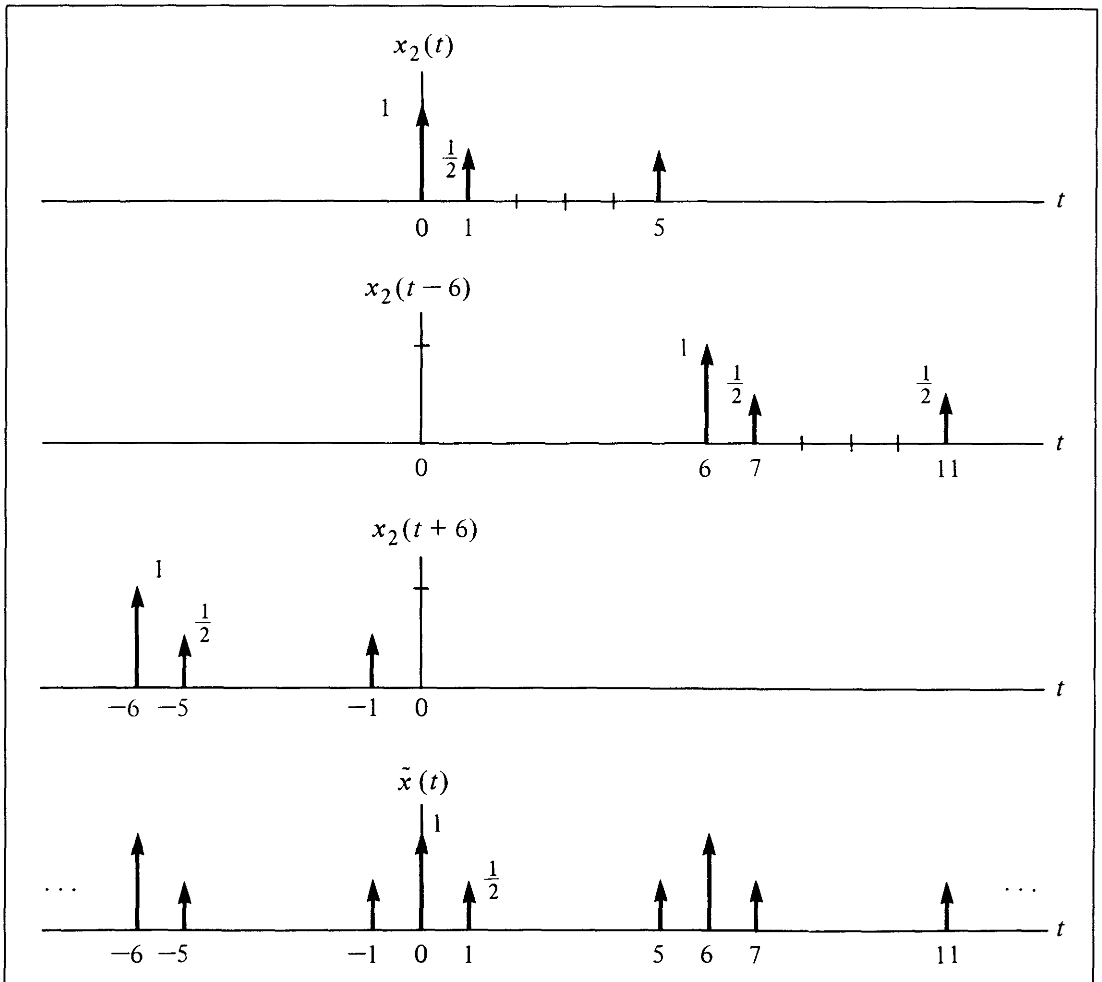

Figure S8.5-2

(e) Since $\tilde{x}(t)$ is a periodic repetition of $x_1(t)$ or $x_2(t)$, the Fourier series coefficients of $\tilde{x}(t)$ should be expressible as scaled samples of $X_1(\omega)$. Evaluate $X_1(\omega)$ at $\omega = 2\pi k/6$. Then

$$X_1(\omega)\Big|_{\omega=2\pi k/6} = 1 + \cos\frac{2\pi k}{6} = 6a_k \Rightarrow a_k = \frac{1}{6}X_1\left(\frac{2\pi k}{6}\right)$$

Similarly, we can get a_k as a scaled sample of $X_2(\omega)$. Consider $X_2(2\pi k/6)$:

$$X_2\left(\frac{2\pi k}{6}\right) = 1 + \frac{1}{2}e^{-j2\pi k/6} + \frac{1}{2}e^{-j10\pi k/6}$$

But $e^{-j10\pi k/6} = e^{-j(10\pi k/6 - 2\pi k)} = e^{j2\pi k/6}$. Thus,

$$X_2\left(\frac{2\pi k}{6}\right) = 1 + \cos\frac{2\pi k}{6} = 6a_k.$$

Although $X_1(\omega) \neq X_2(\omega)$, they are equal for $\omega = 2\pi k/6$.

S8.6

(a) By inspection,

$$e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a + j\omega}$$

Thus,

$$e^{-7t}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{7 + j\omega}$$

Direct inversion using the inverse Fourier transform formula is very difficult.

(b) $X_b(\omega) = 2\delta(\omega + 7) + 2\delta(\omega - 7)$,

$$x_b(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_b(\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2[\delta(\omega + 7) + \delta(\omega - 7)]e^{j\omega t}\,d\omega$$

$$= \frac{1}{\pi}e^{-j7t} + \frac{1}{\pi}e^{j7t} = \frac{2}{\pi}\cos 7t$$

(c) From Example 4.8 of the text (page 191), we see that

$$e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2a}{a^2 + \omega^2}$$

However, note that

$$\alpha x(t) \overset{\mathcal{F}}{\longleftrightarrow} \alpha X(\omega)$$

since

$$\int_{-\infty}^{\infty} \alpha x(t)e^{-j\omega t}\,dt = \alpha\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \alpha X(\omega)$$

Thus,

$$\frac{1}{2a}e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a^2 + \omega^2} \quad \text{or} \quad \frac{1}{9 + \omega^2} \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{6}e^{-3|t|}$$

(d) $X_a(\omega)X_b(\omega) = X_a(\omega)[2\delta(\omega + 7) + 2\delta(\omega - 7)]$

$$= 2X_a(-7)\delta(\omega + 7) + 2X_a(7)\delta(\omega - 7)$$

$$X_d(\omega) = \frac{2}{7 - j7}\delta(\omega + 7) + \frac{2}{7 + j7}\delta(\omega - 7)$$

$$x_d(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{2}{7 - j7}\delta(\omega + 7) + \frac{2}{7 + j7}\delta(\omega - 7)\right]e^{j\omega t}\,d\omega$$

$$x_d(t) = \frac{1}{\pi}\frac{1}{7 - j7}e^{-j7t} + \frac{1}{\pi}\frac{1}{7 + j7}e^{j7t}$$

Note that

$$\frac{1}{7 + j7} = \frac{1}{7}\left(\frac{\sqrt{2}}{2}\right)e^{-j\pi/4}, \qquad \frac{1}{7 - j7} = \frac{1}{7}\left(\frac{\sqrt{2}}{2}\right)e^{+j\pi/4}$$

Thus

$$x_d(t) = \frac{1}{\pi}\left(\frac{1}{7}\right)\frac{\sqrt{2}}{2}[e^{-j(7t - \pi/4)} + e^{j(7t - \pi/4)}] = \frac{\sqrt{2}}{7\pi}\cos\left(7t - \frac{\pi}{4}\right)$$

(e) $$X_e(\omega) = \begin{cases} \omega e^{-j3\omega}, & 0 \le \omega \le 1, \\ -\omega e^{-j3\omega}, & -1 \le \omega \le 0, \\ 0, & \text{elsewhere,} \end{cases}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\left[\int_0^1 \omega e^{-j3\omega}e^{j\omega t}\,d\omega - \int_{-1}^0 \omega e^{-j3\omega}e^{j\omega t}\,d\omega\right]$$

Note that

$$\int xe^{\alpha x}\,dx = \frac{e^{\alpha x}}{\alpha^2}(\alpha x - 1)$$

Substituting $\alpha = j(t-3)$ into the integrals, we obtain

$$x(t) = \frac{1}{2\pi}\left[\frac{e^{j(t-3)\omega}}{(j(t-3))^2}(j(t-3)\omega - 1)\Big|_0^1 - \frac{e^{j(t-3)\omega}}{(j(t-3))^2}(j(t-3)\omega - 1)\Big|_{-1}^0\right],$$

which can be simplified to yield

$$x(t) = \frac{1}{\pi}\left[\frac{\cos(t-3) - 1}{(t-3)^2} + \frac{\sin(t-3)}{(t-3)}\right]$$

Solutions to Optional Problems

S8.7

(a) $$Y(\omega) = \int_{t=-\infty}^{\infty} y(t)e^{-j\omega t}\,dt = \int_{t=-\infty}^{\infty}\int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau\, e^{-j\omega t}\,dt$$
$$= \int_{t=-\infty}^{\infty}\int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)e^{-j\omega t}\,d\tau\,dt$$

(b) Let $r = t - \tau$ and integrate for all τ and r. Then

$$Y(\omega) = \int_{\tau=-\infty}^{\infty}\int_{r=-\infty}^{\infty} x(\tau)h(r)e^{-j\omega(r+\tau)}\,dr\,d\tau$$
$$= \int_{\tau=-\infty}^{\infty} x(\tau)e^{-j\omega\tau}\,d\tau \int_{r=-\infty}^{\infty} h(r)e^{-j\omega r}\,dr$$
$$= X(\omega)H(\omega)$$

S8.8

(a) Using the analysis equation, we obtain

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jk(2\pi/T)t}\,dt = \frac{1}{T}$$

Thus all the Fourier series coefficients are equal to $1/T$.

(b) For periodic signals, the Fourier transform can be calculated from a_k as

$$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k\delta\left(\omega - \frac{2\pi k}{T}\right)$$

In this case,

$$P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

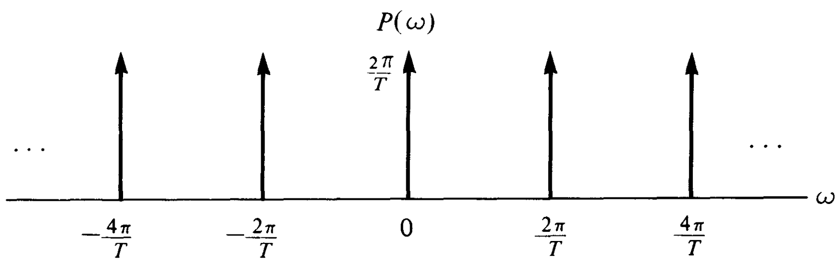

Figure S8.8

(c) We are required to show that

$$\tilde{x}(t) = x(t) * p(t)$$

Substituting for $p(t)$, we have

$$x(t) * p(t) = x(t) * \left[\sum_{k=-\infty}^{\infty} \delta(t - kT) \right]$$

Using the associative property of convolution, we obtain

$$x(t) * p(t) = \sum_{k=-\infty}^{\infty} [x(t) * \delta(t - kT)]$$

From the sifting property of $\delta(t)$, it follows that

$$x(t) * p(t) = \sum_{k=-\infty}^{\infty} x(t - kT) = \tilde{x}(t)$$

Thus, $x(t) * p(t)$ is a periodic repetition of $x(t)$ with period T.

(d) From Problem P8.7, we have

$$\begin{aligned} \tilde{X}(\omega) &= X(\omega)P(\omega) \\ &= X(\omega) \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta\left(\omega - \frac{2\pi k}{T}\right) \\ &= \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} X(\omega) \delta\left(\omega - \frac{2\pi k}{T}\right) \end{aligned}$$

Since each summation term is nonzero *only* at $\omega = 2\pi k/T$,

$$\tilde{X}(\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} X\left(\frac{2\pi k}{T}\right) \delta\left(\omega - \frac{2\pi k}{T}\right)$$

From this expression we see that the Fourier series coefficients of $\tilde{x}(t)$ are

$$a_k = \frac{1}{T} X\left(\frac{2\pi k}{T}\right),$$

which is consistent with our previous discussions.