7 Continuous-Time Fourier Series

Solutions to Recommended Problems

S7.1

(a) For the LTI system indicated in Figure S7.1, the output $y(t)$ is expressed as

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)\, d\tau,$$

where $h(t)$ is the impulse response and $x(t)$ is the input.

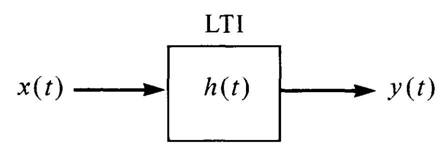

Figure S7.1

For $x(t) = e^{j\omega t}$,

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)e^{j\omega(t-\tau)}\, d\tau \\ &= e^{j\omega t}\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\, d\tau \\ &= e^{+j\omega t}H(\omega) \end{aligned}$$

(b) We are given that the first-order differential equation is of the form

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

From part (a), when $x(t) = e^{j\omega t}$, then $y(t) = e^{j\omega t}H(\omega)$. Also, by differentiating $y(t)$, we have

$$\frac{dy(t)}{dt} = j\omega e^{j\omega t}H(\omega)$$

Substituting, we get

$$j\omega e^{j\omega t}H(\omega) + ae^{j\omega t}H(\omega) = e^{j\omega t}$$

Hence,

$$j\omega H(\omega) + aH(\omega) = 1, \quad \text{or}$$

$$H(\omega) = \frac{1}{a + j\omega}$$

S7.2

(a) The output of a discrete-time LTI system is given by the discrete-time convolution sum

$$y[n] = \sum_k h[k]x[n - k]$$

If $x[n] = z^n$, then

$$\begin{aligned} y[n] &= \sum_k h[k] z^{n-k} \\ &= z^n \sum_k h[k] z^{-k} \\ &= z^n H(z) \end{aligned}$$

(b) We are given that the first-order difference equation is of the form

$$y[n] + ay[n-1] = x[n]$$

From part (a), if $x[n] = z^n$, then $y[n] = z^n H(z)$. Hence,

$$y[n-1] = z^{n-1} H(z).$$

By substitution,

$$z^n H(z) + az^{n-1} H(z) = z^n,$$

which implies

$$\begin{aligned} (1 + az^{-1}) H(z) &= 1, \\ H(z) &= \frac{1}{1 + az^{-1}} \end{aligned}$$

S7.3

(a) $x(t) = \sin\left(10\pi t + \frac{\pi}{6}\right)$

$$= \frac{e^{j\pi/6}}{2j} e^{j2\pi t5} - \frac{e^{-j\pi/6}}{2j} e^{-j2\pi t5}$$

We choose ω_0, the fundamental frequency, to be 2π.

$$x(t) = \sum_k a_k e^{jk\omega_0 t},$$

where

$$a_5 = \frac{e^{j\pi/6}}{2j}, \qquad a_{-5} = \frac{-e^{-j\pi/6}}{2j}$$

Otherwise $a_k = 0$.

(b) $x(t) = 1 + \cos(2\pi t)$

$$= 1 + \frac{e^{j2\pi t}}{2} + \frac{e^{-j2\pi t}}{2}$$

For $\omega_0 = 2\pi$, $a_{-1} = a_1 = \frac{1}{2}$, and $a_0 = 1$. All other a_k's $= 0$.

(c) $x(t) = [1 + \cos(2\pi t)]\left[\sin\left(10\pi t + \frac{\pi}{6}\right)\right]$

$$\begin{aligned} &= \sin\left(10\pi t + \frac{\pi}{6}\right) + \cos(2\pi t)\sin\left(10\pi t + \frac{\pi}{6}\right) \\ &= \left(\frac{e^{j\pi/6}}{2j} e^{j2\pi t5} - \frac{e^{-j\pi/6}}{2j} e^{-j2\pi t5}\right) + \left(\tfrac{1}{2}e^{j2\pi t} + \tfrac{1}{2}e^{-j2\pi t}\right)\left(\frac{e^{j\pi/6}}{2j} e^{j2\pi t5} - \frac{e^{-j\pi/6}}{2j} e^{-j2\pi t5}\right) \\ &= \frac{e^{j\pi/6}}{2j} e^{j2\pi t5} - \frac{e^{-j\pi/6}}{2j} e^{-j2\pi t5} + \frac{e^{j\pi/6}}{4j} e^{j2\pi t6} - \frac{e^{-j\pi/6}}{4j} e^{-j2\pi t4} \\ &\quad + \frac{e^{j\pi/6}}{4j} e^{j2\pi t4} - \frac{e^{-j\pi/6}}{4j} e^{-j2\pi t6} \end{aligned}$$

Therefore,

$$x(t) = \sum_k a_k e^{jk\omega_0 t},$$

where $\omega_0 = 2\pi$.

$$a_4 = \frac{e^{j\pi/6}}{4j}, \qquad a_{-4} = \frac{-e^{-j\pi/6}}{4j},$$
$$a_5 = \frac{e^{j\pi/6}}{2j}, \qquad a_{-5} = \frac{-e^{-j\pi/6}}{2j},$$
$$a_6 = \frac{e^{j\pi/6}}{4j}, \qquad a_{-6} = \frac{-e^{-j\pi/6}}{4j}$$

All other a_k's = 0.

S7.4

(a)

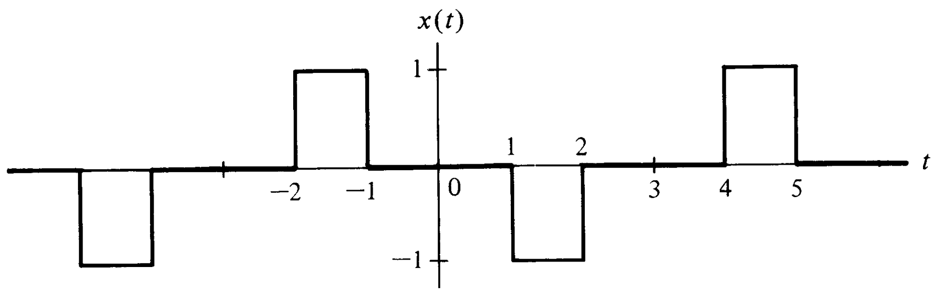

Figure S7.4-1

Note that the period is $T_0 = 6$. Fourier coefficients are given by

$$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,dt$$

We take $\omega_0 = 2\pi/T_0 = \pi/3$. Choosing the period of integration as -3 to 3, we have

$$\begin{aligned}
a_k &= \frac{1}{6}\int_{-2}^{-1} e^{-jk(\pi/3)t}\,dt - \frac{1}{6}\int_{1}^{2} e^{-jk(\pi/3)t}\,dt \\
&= \frac{1}{6}\,\frac{1}{-jk(\pi/3)}\,e^{-jk(\pi/3)t}\Big|_{-2}^{-1} - \frac{1}{6}\,\frac{1}{-jk(\pi/3)}\,e^{-jk(\pi/3)t}\Big|_{1}^{2} \\
&= \frac{1}{-j2\pi k}\left[e^{+j(\pi/3)k} - e^{+j(2\pi/3)k} - e^{-j(2\pi/3)k} + e^{-j(\pi/3)k}\right] \\
&= \frac{\cos(2\pi/3)k}{j\pi k} - \frac{\cos(\pi/3)k}{j\pi k}
\end{aligned}$$

Therefore,

$$x(t) = \sum_k a_k e^{jk\omega_0 t}, \qquad \omega_0 = \frac{\pi}{3}$$

and

$$a_k = \frac{\cos(2\pi/3)k - \cos(\pi/3)k}{j\pi k}$$

Note that $a_0 = 0$, as can be determined either by applying L'Hôpital's rule or by noting that

$$a_0 = (1/T_0) \int_{T_0} x(t)dt.$$

(b)

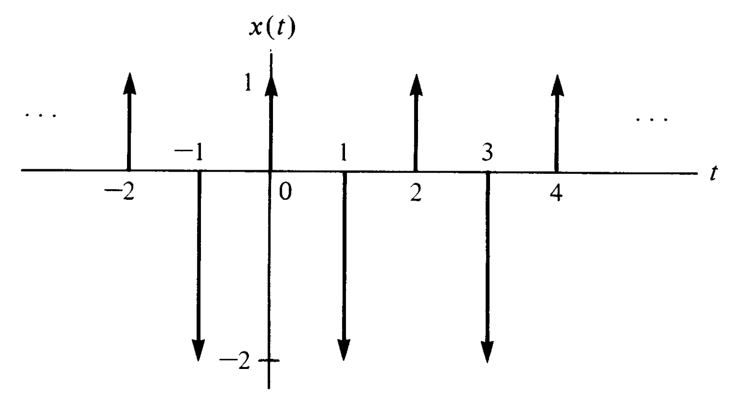

Figure S7.4-2

The period is $T_0 = 2$, with $\omega_0 = 2\pi/2 = \pi$. The Fourier coefficients are

$$a_k = \frac{1}{T_0} \int_{T_0} x(t)e^{-jk\omega_0 t}\, dt$$

Choosing the period of integration as $-\frac{1}{2}$ to $\frac{3}{2}$, we have

$$\begin{aligned} a_k &= \frac{1}{2} \int_{-1/2}^{3/2} x(t)e^{-jk\omega_0 t}\, dt \\ &= \frac{1}{2} \int_{-1/2}^{3/2} [\delta(t) - 2\delta(t-1)]e^{-jk\omega_0 t}\, dt \\ &= \frac{1}{2} - e^{-jk\omega_0} = \frac{1}{2} - (e^{-j\pi})^k \end{aligned}$$

Therefore,

$$a_0 = -\tfrac{1}{2}, \qquad a_k = \tfrac{1}{2} - (-1)^k$$

It is instructive to plot a_k, which we have done in Figure S7.4-3.

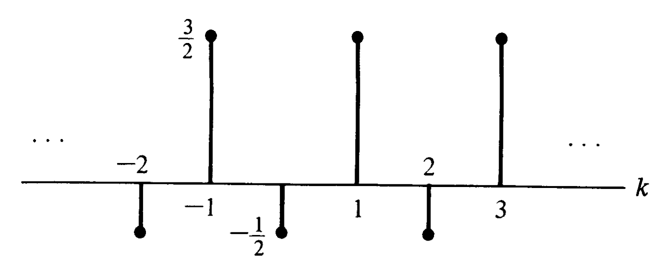

Figure S7.4-3

S7.5

(a) (i) and (ii)
From Problem 4.12 of the text (page 260), we have

$$x\left(t - \frac{T}{2}\right) = -x(t),$$

which means odd harmonics. Since $x(t)$ is real and even, the waveform has real coefficients.

(b) (i) and (iii)

$$-x(t) = x\left(t - \frac{T}{2}\right),$$

which means odd harmonics. Since $x(t)$ is real and odd, the waveform has imaginary coefficients.

(c) (i)

$$-x(t) = x\left(t - \frac{T}{2}\right),$$

which means odd harmonics. Also, $x(t)$ is neither even nor odd.

Solutions to Optional Problems

S7.6

$x(t)$ is specified in the interval $0 < t < T/4$, as shown in Figure S7.6-1.

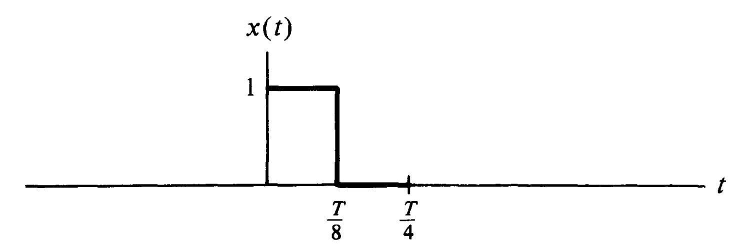

Figure S7.6-1

(a) Since $x(t)$ is even, we can extend Figure S7.6-1 as indicated in Figure S7.6-2.

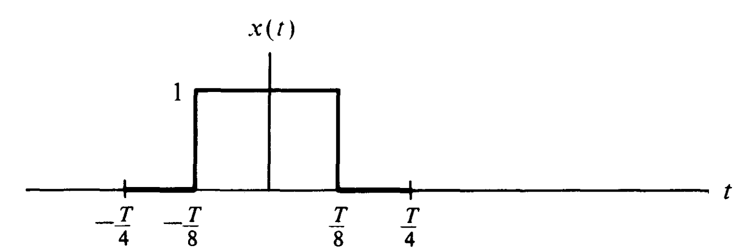

Figure S7.6-2

Since $x(t)$ has only odd harmonics, it must have the property that $x(t - T/2) = -x(t)$, as shown in Figure S7.6-3.

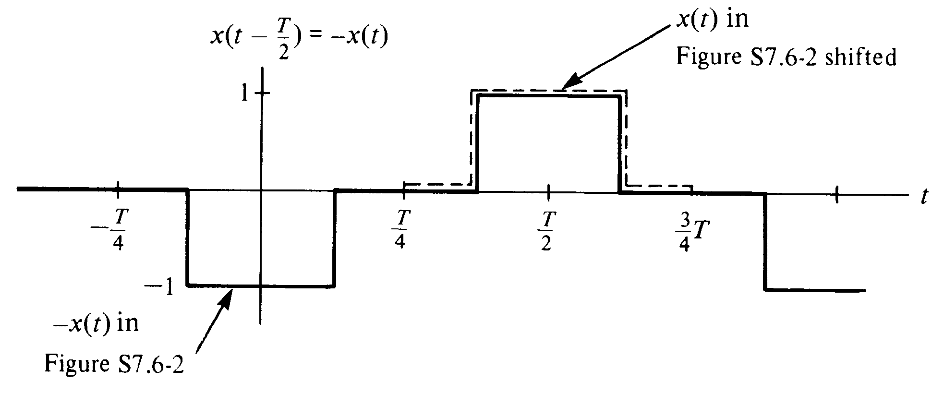

Figure S7.6-3

So we have $x(t)$ as in Figure S7.6-4.

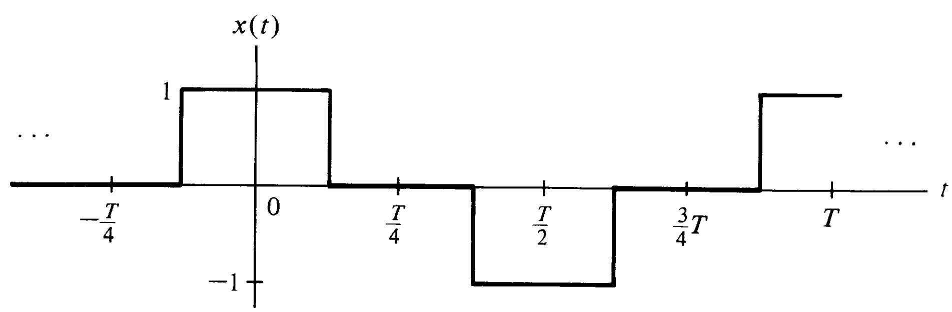

Figure S7.6-4

(b) In the interval from $t = 0$ to $t = T/4$, $x(t)$ is given as in Figure S7.6-5.

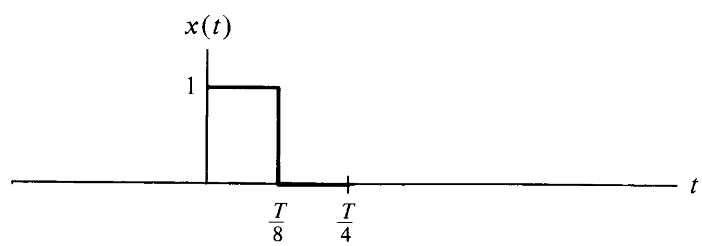

Figure S7.6-5

Since $x(t)$ is odd, for $-T/4 < t < T/4$ it must be as indicated in Figure S7.6-6.

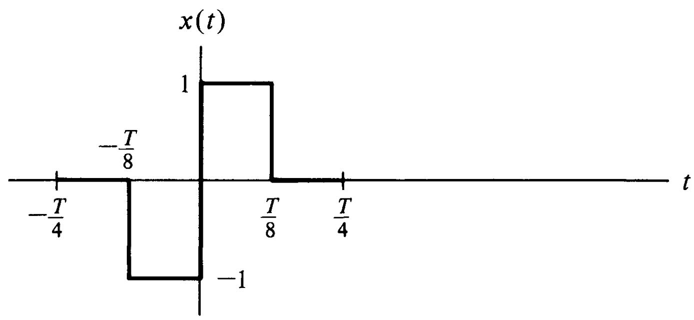

Figure S7.6-6

Since $x(t)$ has odd harmonics, $x[t - (T/2)] = -x(t)$. Consequently $x(t)$ is as shown in Figure S7.6-7.

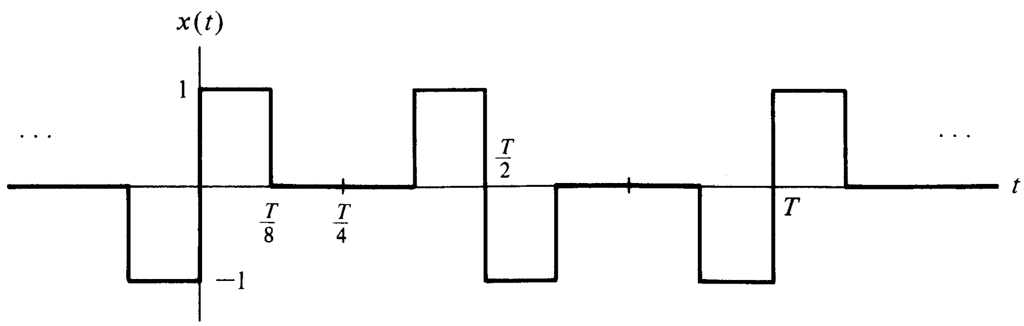

Figure S7.6-7

S7.7

$$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,dt$$

(a) $$\hat{a}_k = \frac{1}{T_0}\int_{T_0} x(t - t_0)e^{-jk\omega_0 t}\,dt$$

Substituting $\tau = t - t_0$, we obtain

$$\begin{aligned}\hat{a}_k &= \frac{1}{T_0}\int_{T_0} x(\tau)e^{jk\omega_0\tau}\,d\tau \cdot e^{-jk\omega_0 t_0}\\ &= a_k e^{-jk\omega_0 t_0}\end{aligned}$$

(b) $$\hat{a}_k = \frac{1}{T_0}\int_{T_0} x(-t)e^{-jk\omega_0 t}\,dt$$

Substituting $\tau = -t$, we have

$$\hat{a}_k = \frac{1}{T_0}\int_{T_0} x(\tau)e^{jk\omega_0\tau}\,d\tau = a_{-k}$$

(c) $$\hat{a}_k = \frac{1}{T_0}\int_{T_0} x^*(t)e^{-jk\omega_0 t}\,dt$$

$$\hat{a}_k^* = \frac{1}{T_0}\int_{T_0} x(t)e^{jk\omega_0 t}\,dt = a_{-k},$$

$$\hat{a}_k = a_{-k}^*$$

(d) $$\hat{a}_k = \frac{\alpha}{T_0}\int_{T_0/\alpha} x(\alpha t)e^{-jk(2\pi\alpha/T_0)t}\,dt$$

Let $\tau = \alpha t$. Then

$$\hat{a}_k = \frac{1}{T_0}\int_{T_0} x(\tau)e^{-jk(2\pi/T_0)\tau}\,d\tau = a_k$$

Therefore,

$$\hat{T}_0 = \frac{T_0}{\alpha}$$

S7.8

(a) Since $\phi_k(t)$ are eigenfunctions and the system is linear, the output is

$$y(t) = \sum_{k=-\infty}^{\infty} \lambda_k c_k \phi_k(t).$$

(b) $$y(t) = t^2\frac{d^2x(t)}{dt^2} + t\frac{dx(t)}{dt},$$

$$\phi_k(t) = t^k,$$

$$\frac{d\phi_k(t)}{dt} = kt^{k-1},$$

$$\frac{d^2\phi_k(t)}{dt^2} = k(k-1)t^{k-2}$$

So if $\phi_k(t) = x(t)$, then

$$\begin{aligned} y(t) &= t^2k(k-1)t^{k-2} + tkt^{k-1} \\ &= k(k-1)t^k + kt^k \\ &= k^2t^k = k^2\phi_k(t) \end{aligned}$$

The eigenfunction $\phi_k(t)$ has eigenvalue $\lambda_k = k^2$.

S7.9

(a) $$\begin{aligned} \tilde{y}(t) &= \tilde{x}_1(t) \circledast \tilde{x}_2(t) \\ &= \int_{T_0} \tilde{x}_1(\tau)\tilde{x}_2(t-\tau)\,d\tau \end{aligned}$$

The Fourier coefficients for $\tilde{y}(t)$ are given by

$$\begin{aligned} c_k &= \frac{1}{T_0}\int_{T_0}\int_{T_0} \tilde{x}_1(\tau)\tilde{x}_2(t-\tau)\,d\tau e^{-jk(2\pi/T_0)t}\,dt \\ &= \frac{1}{T_0}\int_{T_0} \tilde{x}_1(\tau)e^{-jk(2\pi/T_0)\tau}\,d\tau \int_{T_0} \tilde{x}_2(t-\tau)e^{-jk(2\pi/T_0)(t-\tau)}\,dt \\ &= T_0 a_k b_k \end{aligned}$$

(b) Since $z(t) * z(t) = x(t)$, as shown in Figure S7.9-1, then $\tilde{z}(t)$ is shown in Figure S7.9-2.

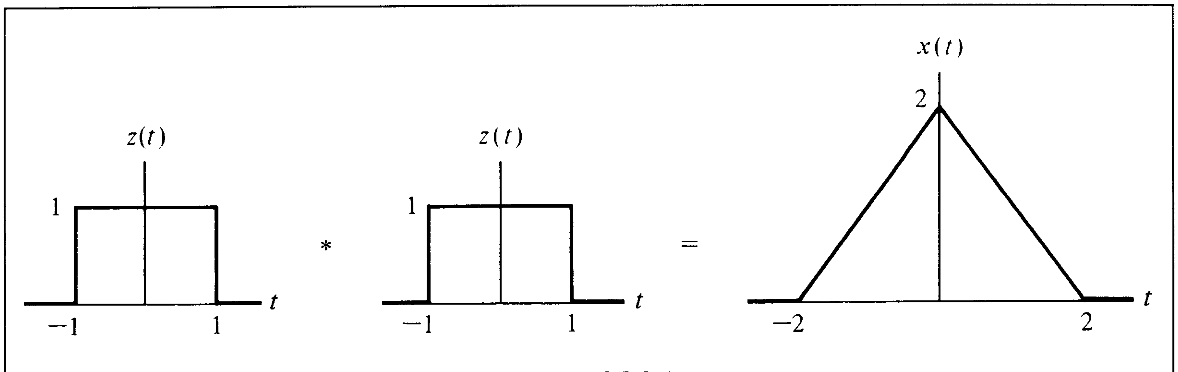

Figure S7.9-1

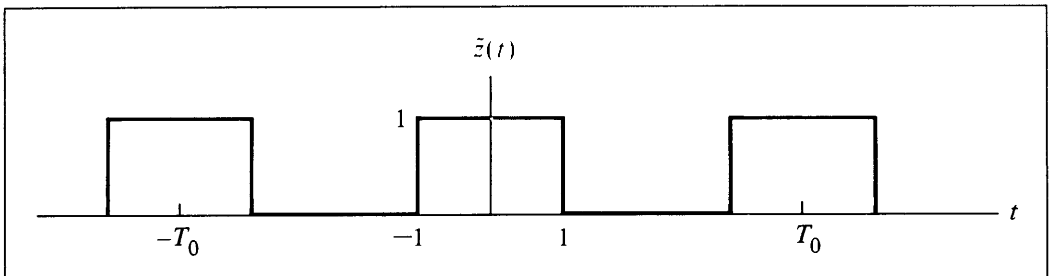

Figure S7.9-2

In Figure S7.9-2, $T_0 = 5$. Hence,

$$\tilde{x}(t) \longleftrightarrow T_0 z_k^2 = \frac{4}{5}\left[\text{sinc}\left(\frac{2\pi k}{5}\right)\right]^2$$

(c) Without explicitly carrying out the convolutions, we can argue that the aperiodic convolution of $x_1(t)$ and $x_2(t)$ will be symmetric about the origin and is nonzero from $t = -2T$ to $t = 2T$. Now, if $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ are periodic with period T_0, then the periodic convolution, $\tilde{y}(t)$, will be periodic with period T_0. If T_0 is large enough, then $\tilde{y}(t)$ is the periodic version of $y(t)$ with period T_0. Hence, to recover $y(t)$ from $\tilde{y}(t)$ we should extract only one period of $\tilde{y}(t)$ from $t = -T_0/2$ to $t = T_0/2$ and set $y(t) = 0$ for $|t| > T_0/2$, where $T_0/2 \geq 2T$, or $T_0 \geq 4T$.

S7.10

(a) The approximation is

$$\hat{x}_N(t) = \sum_{k=-N}^{N} a_k \phi_k(t)$$

with the corresponding error signal

$$\begin{aligned} e_N(t) &= x(t) - \hat{x}_N(t) \\ &= x(t) - \sum_{k=-N}^{N} a_k \phi_k(t) \end{aligned}$$

Hence,

$$|e_N(t)|^2 = \left[x(t) - \sum_k a_k\phi_k(t)\right]\left[x^*(t) - \sum_k a_k^*\phi_k^*(t)\right]$$
$$= |x(t)|^2 - \sum_k a_k^* x(t)\phi_k^*(t) - \sum_k a_k x^*(t)\phi_k(t) + \sum_k \sum_l a_k a_l^* \phi_k(t)\phi_l^*(t)$$

If we integrate, $\int_a^b |e_N(t)|^2\,dt$, and use the property that

$$\int_a^b \phi_k(t)\phi_l^*(t)\,dt = \begin{cases} 1, & k = l, \\ 0, & \text{otherwise,} \end{cases}$$

we get

$$E = \int_a^b |x(t)|^2\,dt - \sum_k a_k^* \int_a^b x(t)\phi_k^*(t)\,dt$$
$$- \sum_k a_k \int_a^b x^*(t)\phi_k(t)\,dt + \sum_k |a_k|^2$$

Since $a_i = b_i + jc_i$,

$$\frac{\partial E}{\partial b_i} = -\int_a^b x(t)\phi_i^*(t)\,dt - \int_a^b x^*(t)\phi_i(t)\,dt + 2b_i$$

and

$$\frac{\partial E}{\partial c_i} = j\int_a^b x(t)\phi_i^*(t)\,dt - j\int_a^b x^*(t)\phi_i(t)\,dt + 2c_i$$

Setting

$$\frac{\partial E}{\partial b_i} = 0 \qquad \text{and} \qquad \frac{\partial E}{\partial c_i} = 0,$$

we can multiply the second equation by j and add the two equations to get

$$\frac{\partial E}{\partial b_i} + j\frac{\partial E}{\partial c_i} = 0$$

By substitution, we get

$$b_i + jc_i = \int_a^b x(t)\phi_i^*(t)\,dt$$
$$= a_i$$

(b) If $\{\phi_i(t)\}$ are orthogonal but not orthonormal, then the only thing that changes from the result of part (a) is

$$\int_a^b \sum_k \sum_l a_k a_l^* \phi_k(t)\phi_l^*(t)\,dt = \sum_k |a_k|^2 A_k$$

It is easy to see that we will now get

$$a_i = \frac{1}{A_i}\int_a^b x(t)\phi_i^*(t)\,dt$$

(c) Since

$$\int_{a}^{T_0+a} e^{jn\omega_0 t}e^{-jn\omega_0 t}\,dt = T_0$$

for all values of a, using parts (a) and (b) we can write

$$\begin{aligned} a_i &= \frac{1}{T_0}\int_{a}^{T_0+a} x(t)e^{-jn\omega_0 t}\,dt \\ &= \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}\,dt \end{aligned}$$