5 Properties of Linear, Time-Invariant Systems

Solutions to Recommended Problems

S5.1

The inverse system for a continuous-time accumulation (or integration) is a differentiator. This can be verified because

$$\frac{d}{dt}\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = x(t)$$

Therefore, the input-output relation for the inverse system in Figure S5.1 is

$$x(t) = \frac{dy(t)}{dt}$$

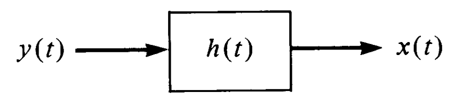

Figure S5.1

S5.2

(a) We want to show that

$$h[n] - ah[n-1] = \delta[n]$$

Substituting $h[n] = a^n u[n]$, we have

$$a^n u[n] - aa^{n-1}u[n-1] = a^n(u[n] - u[n-1])$$

But

$$u[n] - u[n-1] = \delta[n] \quad \text{and} \quad a^n\delta[n] = a^0\delta[n] = \delta[n]$$

(b) (i) The system is not memoryless since $h[n] \neq k\delta[n]$.

(ii) The system is causal since $h[n] = 0$ for $n < 0$.

(iii) The system is stable for $|a| < 1$ since

$$\sum_{n=0}^{\infty} |a|^n = \frac{1}{1 - |a|}$$

is bounded.

(c) The system is not stable for $|a| > 1$ since $\Sigma_{n=0}^{\infty} |a|^n$ is not finite.

S5.3

(a) Consider $x(t) = \delta(t) \rightarrow y(t) = h(t)$. We want to verify that $h(t) = e^{-2t}u(t)$, so

$$\frac{dy(t)}{dt} = -2e^{-2t}u(t) + e^{-2t}\,\delta(t), \quad \text{or}$$

$$\frac{dy(t)}{dt} + 2y(t) = e^{-2t}\,\delta(t),$$

but $e^{-2t}\delta(t) = \delta(t)$ because both functions have the same effect on a test function within an integral. Therefore, the impulse response is verified to be correct.

(b) (i) The system is not memoryless since $h(t) \neq k\delta(t)$.

(ii) The system is causal since $h(t) = 0$ for $t < 0$.

(iii) The system is stable since $h(t)$ is absolutely integrable.

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_0^{\infty} e^{-2t}\, dt = -\tfrac{1}{2}e^{-2t}\Big|_0^{\infty}$$
$$= \frac{1}{2}$$

S5.4

By using the commutative property of convolution we can exchange the two systems to yield the system in Figure S5.4.

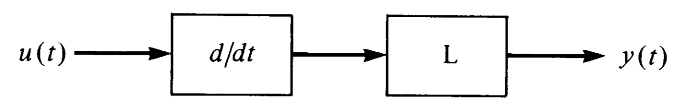

Figure S5.4

Now we note that the input to system L is

$$\frac{du(t)}{dt} = \delta(t),$$

so $y(t)$ is the impulse response of system L. From the original diagram,

$$\frac{ds(t)}{dt} = y(t)$$

Therefore,

$$h(t) = \frac{ds(t)}{dt}$$

S5.5

(a) By definition, an inverse system cascaded with the original system is the identity system, which has an impulse response $h(t) = \delta(t)$. Therefore, if the cascaded system has an input of $\delta(t)$, the output $w(t) = h(t) = \delta(t)$.

(b) Because the system is an identity system, an input of $x(t)$ produces an output $w(t) = x(t)$.

Solutions to Optional Problems

S5.6

(a) If $y(t) = ay_1(t) + by_2(t)$, we know that since system A is linear, $x(t) = ax_1(t) + bx_2(t)$. Since the cascaded system is an identity system, the output $w(t) = ax_1(t) + bx_2(t)$.

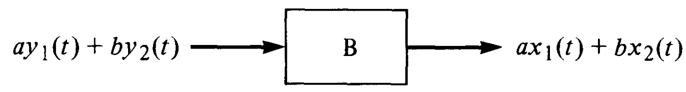

Figure S5.6-1

(b) If $y(t) = y_1(t - \tau)$, then since system A is time-invariant, $x(t) = x_1(t - \tau)$ and also $w(t) = x_1(t - \tau)$.

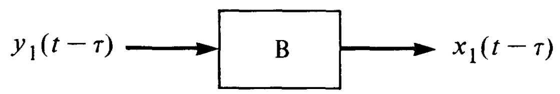

Figure S5.6-2

(c) From the solutions to parts (a) and (b), we see that system B is linear and time-invariant.

S5.7

(a) The following signals are obtained by addition and graphical convolution:

$(x[n] + w[n]) * y[n]$ (see Figure S5.7-1)
$x[n] * y[n] + w[n] * y[n]$ (see Figure S5.7-2)

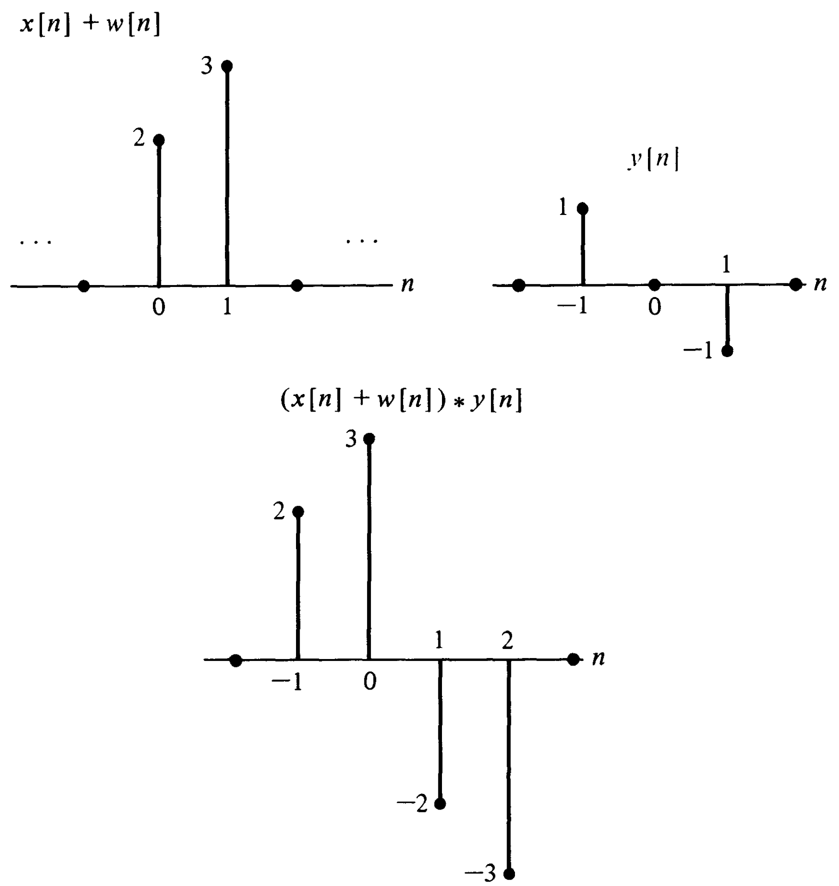

Figure S5.7-1

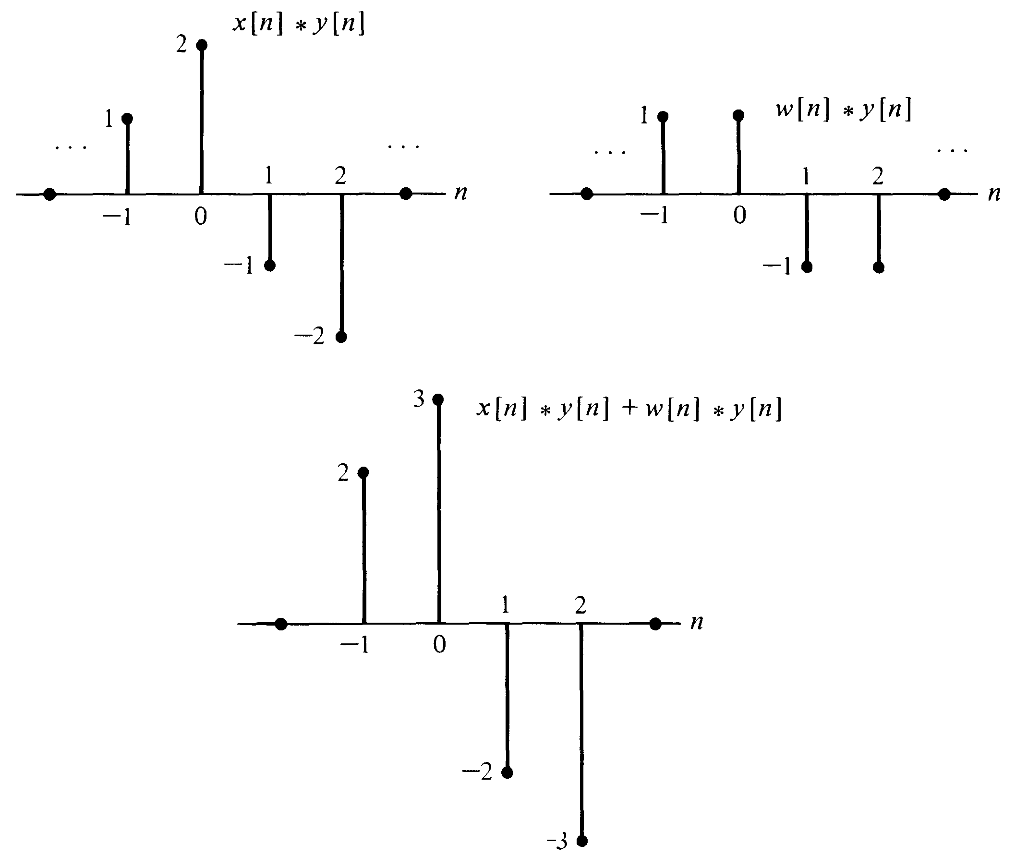

Figure S5.7-2

Therefore, the distributive property $(x + w) * y = x * y + w * y$ is verified.

(b) Figure S5.7-3 shows the required convolutions and multiplications.

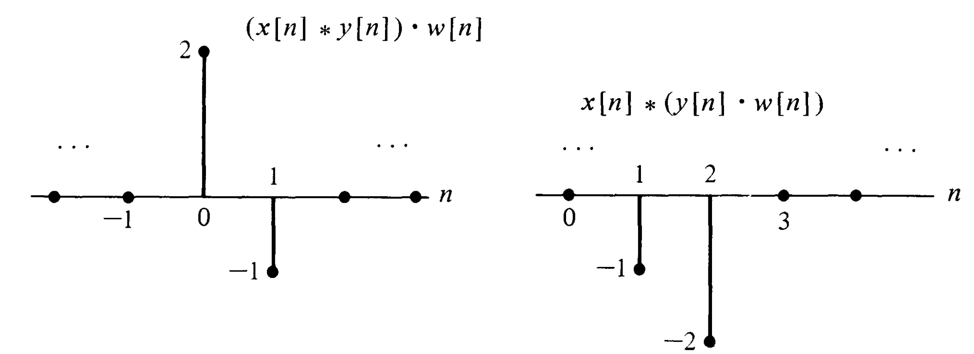

Figure S5.7-3

Note, therefore, that $(x[n] * y[n]) \cdot w[n] \neq x[n] * (y[n] \cdot w[n])$.

S5.8

Consider

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau$$
$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

(a) $$y'(t) = \int_{-\infty}^{\infty} x'(t-\tau)h(\tau)\,d\tau = x'(t) * h(t)$$
$$= \int_{-\infty}^{\infty} x(\tau)h'(t-\tau)\,d\tau = x(t) * h'(t),$$

where the primes denote d/dt.

(b) $y(t) = x(t) * h(t),$

$y(t) = x(t) * u_{-1}(t) * u_1(t) * h(t),$

$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau * h'(t)$$

(c) $y(t) = x(t) * h(t),$

$y(t) = x(t) * u_1(t) * h(t) * u_{-1}(t),$

$$y(t) = \int_{-\infty}^{t} x'(\tau) * h(\tau)\,d\tau$$

(d) $y(t) = x(t) * h(t)$

$= x(t) * u_1(t) * h(t) * u_{-1}(t),$

$$y(t) = x'(t) * \int_{-\infty}^{t} h(\tau)\,d\tau$$

S5.9

(a) True.

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \sum_{k=-\infty}^{\infty} \int_{0}^{T} |h(t)|\,dt = \infty$$

(b) False. If $h(t) = \delta(t - t_0)$ for $t_0 > 0$, then the inverse system impulse response is $\delta(t + t_0)$, which is noncausal.

(c) False. Suppose $h[n] = u[n]$. Then

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} u[n] = \infty$$

(d) True, assuming $h[n]$ is finite-amplitude.

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-K}^{L} |h[n]| = M \quad \text{(a number)}$$

(e) False. $h(t) = u(t)$ implies causality, but $\int_{-\infty}^{\infty} u(t)\,dt = \infty$ implies that the system is not stable.

(f) False.

$$h_1(t) = \delta(t - t_1), \qquad t_1 > 0 \qquad \text{Causal}$$
$$h_2(t) = \delta(t + t_2), \qquad t_2 > 0 \qquad \text{Noncausal}$$
$$h(t) = h_1(t) * h_2(t) = \delta(t + t_2 - t_1), \qquad t_2 \le t_1 \qquad \text{Causal}$$

(g) False. Suppose $h(t) = e^{-t}u(t)$. Then

$$\int_{-\infty}^{\infty} e^{-t}u(t)\,dt = -e^{-t}\Big|_0^{\infty} = 1 \qquad \text{Stable}$$

The step response is

$$\int_{-\infty}^{\infty} u(t-\tau)e^{-\tau}u(\tau)\,d\tau = \int_0^t e^{-\tau}\,d\tau$$
$$= (1 - e^{-t})u(t),$$
$$\int_0^{\infty} (1 - e^{-t})\,dt = t + e^{-t}\Big|_0^{\infty} = \infty$$

(h) True. We know that $u[n] = \Sigma_{k=0}^{\infty} \delta[n-k]$ and, from superposition, $s[n] = \Sigma_{k=0}^{\infty} h[n-k]$. If $s[n] \neq 0$ for some $n < 0$, there exists some value of $h[k] \neq 0$ for some $k < 0$. If $s[n] = 0$ for all $n < 0$, $h[k] = 0$ for all $k < 0$.

S5.10

(a)

$$\int_{-\infty}^{\infty} g(\tau)u_1(\tau)\,d\tau = -g'(0),$$
$$g(\tau) = x(t-\tau), \qquad t \text{ fixed},$$
$$\int_{-\infty}^{\infty} x(t-\tau)u_1(\tau)\,d\tau = -\frac{dg(\tau)}{d\tau}\Big|_{\tau=0} = -\frac{dx(t-\tau)}{d\tau}\Big|_{\tau=0}$$
$$= \frac{dx(t-\tau)}{dt}\Big|_{\tau=0} = \frac{dx(t)}{dt}$$

(b)

$$\int_{-\infty}^{\infty} g(t)f(t)u_1(t)\,dt = -\frac{d}{dt}[g(t)f(t)]\Big|_{t=0}$$
$$= -[g'(t)f(t) + g(t)f'(t)]\Big|_{t=0}$$
$$= -[g'(0)f(0) + g(0)f'(0)],$$
$$\int g(t)[f(0)u_1(t) - f'(0)\delta(t)]\,dt = -f(0)g'(0) - f'(0)g(0)$$

So when we use a test function $g(t)$, $f(t)u_1(t)$ and $f(0)u_1(t) - f'(0)\delta(t)$ both produce the same operational effect.

(c)

$$\int_{-\infty}^{\infty} x(\tau)u_2(\tau)\,d\tau = x(\tau)\cancelto{0}{u_1(\tau)}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{dx}{d\tau}u_1(\tau)\,d\tau$$
$$= -\int_{-\infty}^{\infty} \frac{dx}{d\tau}u_1(\tau)\,d\tau = -\frac{dx}{d\tau}\cancelto{0}{u_0(\tau)}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{d^2x}{d\tau^2}u_0(\tau)\,d\tau$$
$$= \frac{d^2x}{d\tau^2}\Big|_{\tau=0}$$

(d)

$$\int g(\tau)f(\tau)u_2(\tau)\,d\tau = g''(\tau)f(\tau) + 2g'(\tau)f'(\tau) + g(\tau)f''(\tau)\Big|_{\tau=0}$$

Noting that $2g'(\tau)f'(\tau)|_{\tau=0} = -2f'(0)\int g(\tau)u_1(\tau)\,d\tau$, we have an equivalent operational definition:

$$f(\tau)u_2(\tau) = f(0)u_2(\tau) - 2f'(0)u_1(\tau) + f''(0)\delta(\tau)$$

S5.11

(a) $h(t) * g(t) = \int_{-\infty}^{\infty} h(t - \tau)g(\tau)\,d\tau = \int_0^t h(t - \tau)g(\tau)\,d\tau$ since $h(t) = 0$ for $t < 0$ and $g(t) = 0$ for $t < 0$. But if $t < 0$, this integral is obviously zero. Therefore, the cascaded system is causal.

(b) By the definition of stability we know that for any bounded input to H, the output of H is also bounded. This output is also the input to system G. Since the input to G is bounded and G is stable, the output of G is bounded. Therefore, a bounded input to the cascaded system produces a bounded output. Hence, this system is stable.

S5.12

We have a total system response of

$$h = \{[(h_1 * h_2) + (h_2 * h_2) - (h_2 * h_1)] * h_1 + h_1^{-1}\} * h_2^{-1}$$
$$h = (h_2 * h_1) + (h_1^{-1} * h_2^{-1})$$

S5.13

We are given that $y[n] = x[n] * h[n]$.

$$y[n] = \sum_{k=-\infty}^{\infty} x[n - k]h[k]$$
$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} x[n - k]h[k]\right|$$
$$\max\{|y[n]|\} = \max\left\{\left|\sum_{k=-\infty}^{\infty} x[n - k]h[k]\right|\right\}$$
$$\leq \max \sum_{k=-\infty}^{\infty} |x[n - k]|\,|h[k]|$$
$$\leq \sum_{k=-\infty}^{\infty} \max\{|x[n - k]|\}|h[k]|$$
$$= \max\{|x[n]|\} \sum_{k=-\infty}^{\infty} |h[k]|$$

We can see from the inequality

$$\max\{|y[n]|\} \leq \max\{|x[n]|\} \sum_{k=-\infty}^{\infty} |h[k]|$$

that $\Sigma_{k=-\infty}^{\infty} |h[k]| \leq 1 \Rightarrow \max\{|y[n]|\} \leq \max\{|x[n]|\}$. This means that $\Sigma_{k=-\infty}^{\infty} |h[k]| \leq 1$ is a sufficient condition. It is necessary because some $x[n]$ always exists that yields $y[n] = \Sigma_{k=-\infty}^{\infty} |h[k]|$. ($x[n]$ consists of a sequence of $+1$'s and -1's.) Therefore, since $\max\{x[n]\} = 1$, it is necessary that $\Sigma_{k=-\infty}^{\infty} |h[k]| \leq 1$ to ensure that $y[n] \leq \max\{|x[n]|\} = 1$.