3 Signals and Systems: Part II

Solutions to Recommended Problems

S3.1

(a)

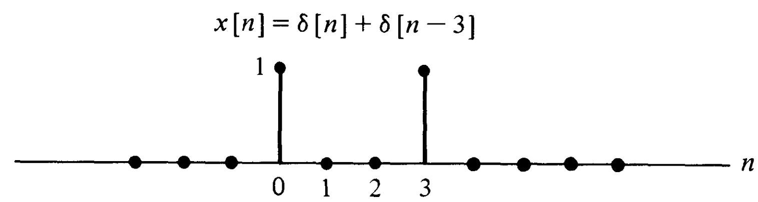

Figure S3.1-1

(b)

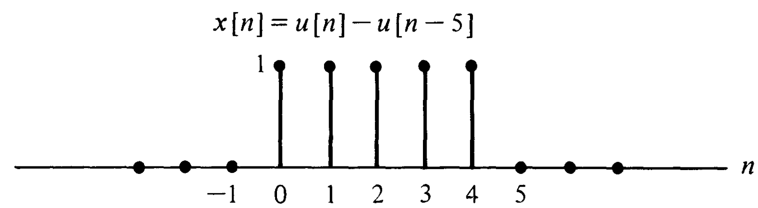

Figure S3.1-2

(c)

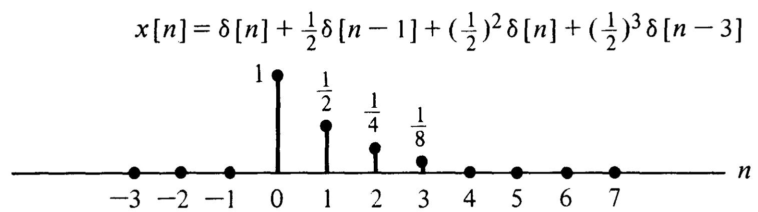

Figure S3.1-3

(d)

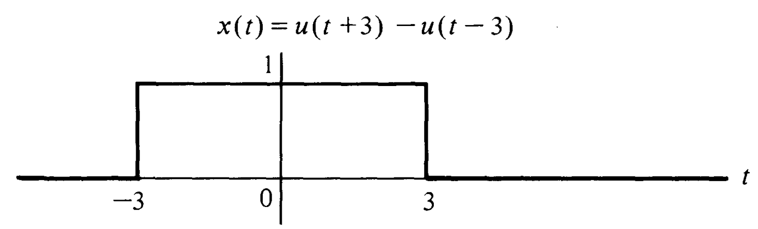

Figure S3.1-4

(e)

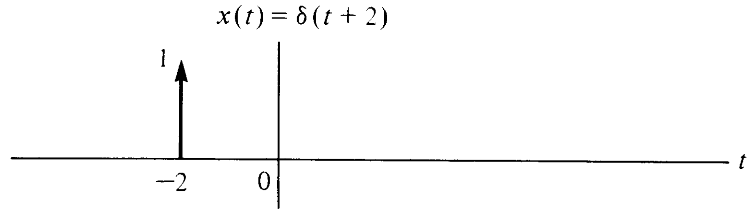

Figure S3.1-5

(f)

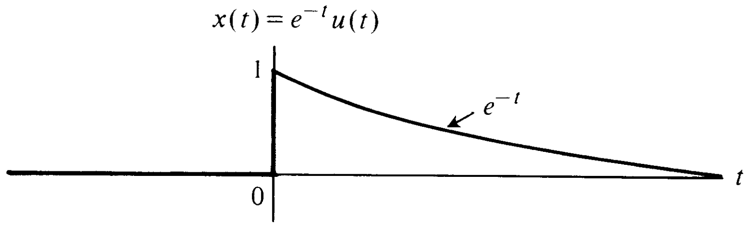

Figure S3.1-6

S3.2

(1) h
(2) d
(3) b
(4) e
(5) a, f
(6) None

S3.3

(a) $x[n] = \delta[n-1] - 2\delta[n-2] + 3\delta[n-3] - 2\delta[n-4] + \delta[n-5]$

(b) $s[n] = -u[n+3] + 4u[n+1] - 4u[n-2] + u[n-4]$

S3.4

We are given Figure S3.4-1.

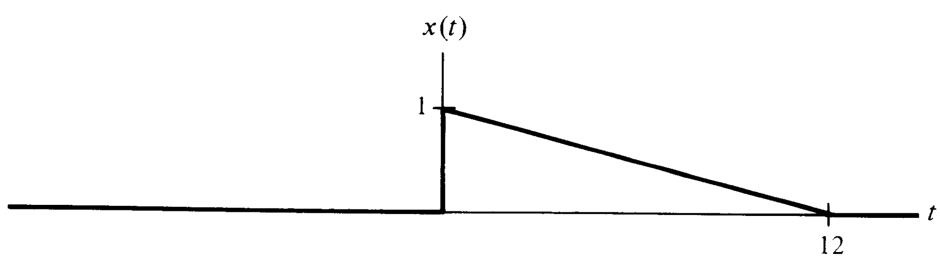

Figure S3.4-1

$x(-t)$ and $x(1-t)$ are as shown in Figures S3.4-2 and S3.4-3.

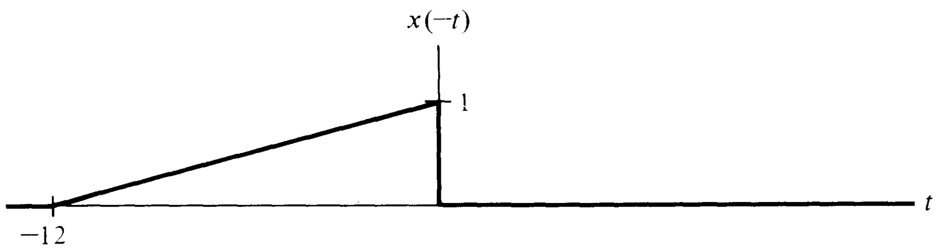

Figure S3.4-2

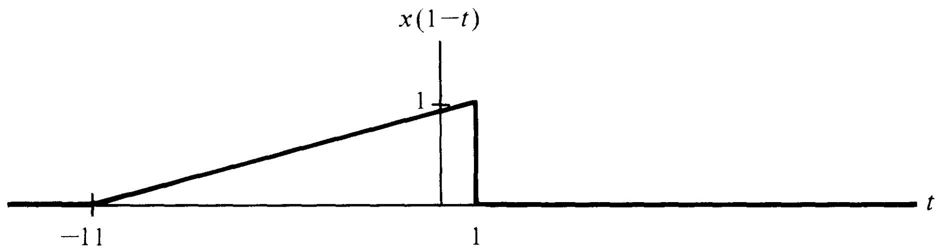

Figure S3.4-3

(a) $u(t+1) - u(t-2)$ is as shown in Figure S3.4-4.

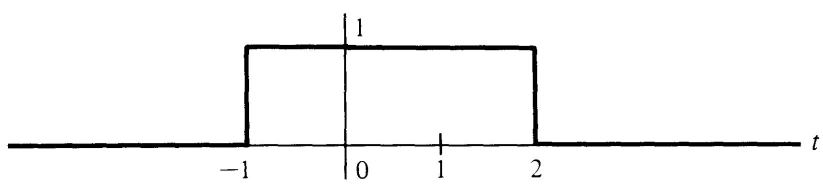

Figure S3.4-4

Hence, $x(1-t)[u(t+1) - u(t-2)]$ looks as in Figure S3.4-5.

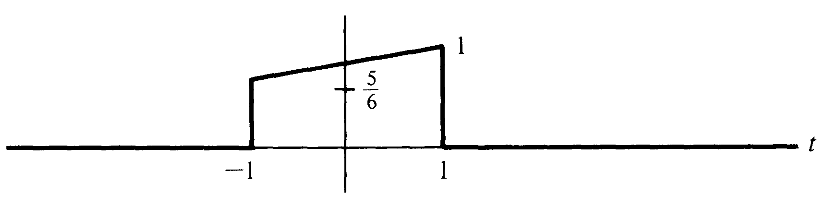

Figure S3.4-5

(b) $-u(2 - 3t)$ looks as in Figure S3.4-6.

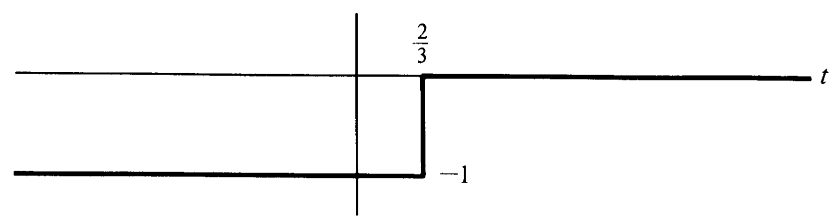

Figure S3.4-6

Hence, $u(t + 1) - u(2 - 3t)$ is given as in Figure S3.4-7.

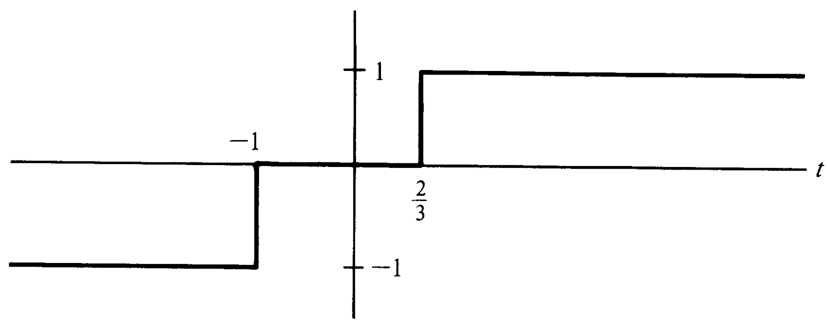

Figure S3.4-7

So $x(1 - t)[u(t + 1) - u(2 - 3t)]$ is given as in Figure S3.4-8.

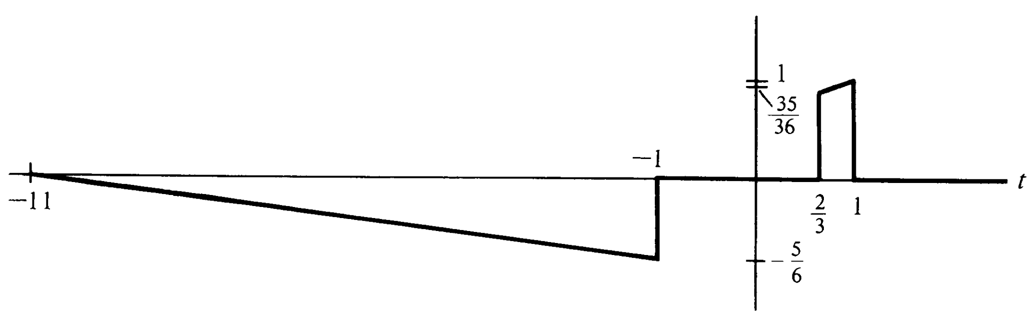

Figure S3.4-8

S3.5

(a) $y[n] = x^2[n] + x[n] - x[n - 1]$

(b) $y[n] = x^2[n] + x[n] - x[n - 1]$

(c) $y[n] = \mathrm{H}[x[n] - x[n - 1]]$
$= x^2[n] + x^2[n - 1] - 2x[n]x[n - 1]$

(d) $y[n] = \mathrm{G}[x^2[n]]$
$= x^2[n] - x^2[n - 1]$

(e) $y[n] = F[x[n] - x[n-1]]$
$= 2(x[n] - x[n-1]) + (x[n-1] - x[n-2])$
$y[n] = 2x[n] - x[n-1] - x[n-2]$

(f) $y[n] = G[2x[n] + x[n-1]]$
$= 2x[n] + x[n-1] - 2x[n-1] - x[n-2]$
$= 2x[n] - x[n-1] - x[n-2]$

(a) and (b) are equivalent. (e) and (f) are equivalent.

S3.6

Memoryless:

(a) $y(t) = (2 + \sin t)x(t)$ is memoryless because $y(t)$ depends only on $x(t)$ and not on prior values of $x(t)$.

(d) $y[n] = \Sigma_{k=-\infty}^{n} x[n]$ is not memoryless because $y[n]$ does depend on values of $x[\cdot]$ before the time instant n.

(f) $y[n] = \max\{x[n], x[n-1], \dots, x[-\infty]\}$ is clearly not memoryless.

Linear:

(a)
$$\begin{aligned} y(t) &= (2 + \sin t)x(t) = T[x(t)], \\ T[ax_1(t) + bx_2(t)] &= (2 + \sin t)[ax_1(t) + bx_2(t)] \\ &= a(2 + \sin t)x_1(t) + b(2 + \sin t)x_2(t) \\ &= aT[x_1(t)] + bT[x_2(t)] \end{aligned}$$

Therefore, $y(t) = (2 + \sin t)x(t)$ is linear.

(b)
$$\begin{aligned} y(t) &= x(2t) = T[x(t)], \\ T[ax_1(t) + bx_2(t)] &= ax_1(2t) + bx_2(t) \\ &= aT[x_1(t)] + bT[x_2(t)] \end{aligned}$$

Therefore, $y(t) = x(2t)$ is linear.

(c)
$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k] = T[x[n]], \\ T[ax_1[n] + bx_2[n]] &= a\sum_{k=-\infty}^{\infty} x_1[k] + b\sum_{k=-\infty}^{\infty} x_2[k] \\ &= aT[x_1[n]] + bT[x_2[n]] \end{aligned}$$

Therefore, $y[n] = \Sigma_{k=-\infty}^{\infty} x[k]$ is linear.

(d) $y[n] = \displaystyle\sum_{k=-\infty}^{n} x[k]$ is linear (see part c).

(e)
$$\begin{aligned} y(t) &= \frac{dx(t)}{dt} = T[x(t)], \\ T[ax_1(t) + bx_2(t)] &= \frac{d}{dt}[ax_1(t) + bx_2(t)] \\ &= a\frac{dx_1(t)}{dt} + b\frac{dx_2(t)}{dt} = aT[x_1(t)] + bT[x_2(t)] \end{aligned}$$

Therefore, $y(t) = dx(t)/dt$ is linear.

(f)
$$\begin{aligned} y[n] &= \max\{x[n], \dots, x[-\infty]\} = T[x[n]], \\ T[ax_1[n] + bx_2[n]] &= \max\{ax_1[n] + bx_2[n], \dots, ax_1[-\infty] + bx_2[-\infty]\} \\ &\neq a\max\{x_1[n], \dots, x_1[-\infty]\} + b\max\{x_2[n], \dots, x_2[-\infty]\} \end{aligned}$$

Therefore, $y[n] = \max\{x[n], \dots, x[-\infty]\}$ is not linear.

Time-invariant:

(a)
$$y(t) = (2 + \sin t)x(t) = T[x(t)],$$
$$T[x(t - T_0)] = (2 + \sin t)x(t - T_0)$$
$$\neq y(t - T_0) = (2 + \sin (t - T_0))x(t - T_0)$$

Therefore, $y(t) = (2 + \sin t)x(t)$ is not time-invariant.

(b)
$$y(t) = x(2t) = T[x(t)],$$
$$T[x(t - T_0)] = x(2t - 2T_0) \neq x(2t - T_0) = y(t - T_0)$$

Therefore, $y(t) = x(2t)$ is not time-invariant.

(c)
$$y[n] = \sum_{k=-\infty}^{\infty} x[k] = T[x[n]],$$
$$T[x[n - N_0]] = \sum_{k=-\infty}^{\infty} x[k - N_0] = y[n - N_0]$$

Therefore, $y[n] = \Sigma_{k=-\infty}^{\infty} x[k]$ is time-invariant.

(d)
$$y[n] = \sum_{k=-\infty}^{n} x[k] = T[x[n]],$$
$$T[x[n - N_0]] = \sum_{k=-\infty}^{n} x[k - N_0] = \sum_{l=-\infty}^{n-N_0} x[l] = y[n - N_0]$$

Therefore, $y[n] = \Sigma_{k=-\infty}^{n} x[k]$ is time-invariant.

(e)
$$y(t) = \frac{dx(t)}{dt} = T[x(t)],$$
$$T[x(t - T_0)] = \frac{d}{dt}x(t - T_0) = y(t - T_0)$$

Therefore, $y(t) = dx(t)/dt$ is time-invariant.

Causal:

(b) $y(t) = x(2t)$,
$y(1) = x(2)$

The value of $y(\cdot)$ at time = 1 depends on $x(\cdot)$ at a future time = 2. Therefore, $y(t) = x(2t)$ is not causal.

(d) $y[n] = \sum_{k=-\infty}^{n} x[k]$

Yes, $y[n] = \Sigma_{k=-\infty}^{n} x[k]$ is causal because the value of $y[\cdot]$ at any instant n depends only on the previous (past) values of $x[\cdot]$.

Invertible:

(b) $y(t) = x(2t)$ is invertible; $x(t) = y(t/2)$.

(c) $y[n] = \Sigma_{k=-\infty}^{\infty} x[k]$ is not invertible. Summation is not generally an invertible operation.

(e) $y(t) = dx(t)/dt$ is invertible to within a constant.

Stable:

(a) If $|x(t)| < M$, $|y(t)| < (2 + \sin t)M$. Therefore, $y(t) = (2 + \sin t)x(t)$ is stable.

(b) If $|x(t)| < M$, $|x(2t)| < M$ and $|y(t)| < M$. Therefore, $y(t) = x(2t)$ is stable.

(d) If $|x[k]| \leq M$, $|y[n]| \leq M \cdot \Sigma_{-\infty}^{n} 1$, which is unbounded. Therefore, $y[n] = \Sigma_{-\infty}^{n} x[k]$ is not stable.

S3.7

(a) Since H is an integrator, H^{-1} must be a differentiator.

$$H^{-1}: \quad y(t) = \frac{dx(t)}{dt}$$
$$G: \quad y(t) = x(2t)$$
$$G^{-1}: \quad y(t) = x(t/2)$$

(b)

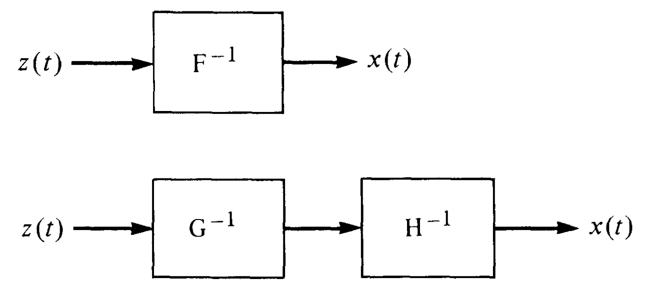

Figure S3.7

Solutions to Optional Problems

S3.8

(a) $x_2(t) = x_1(t) - x_1(t - 2)$

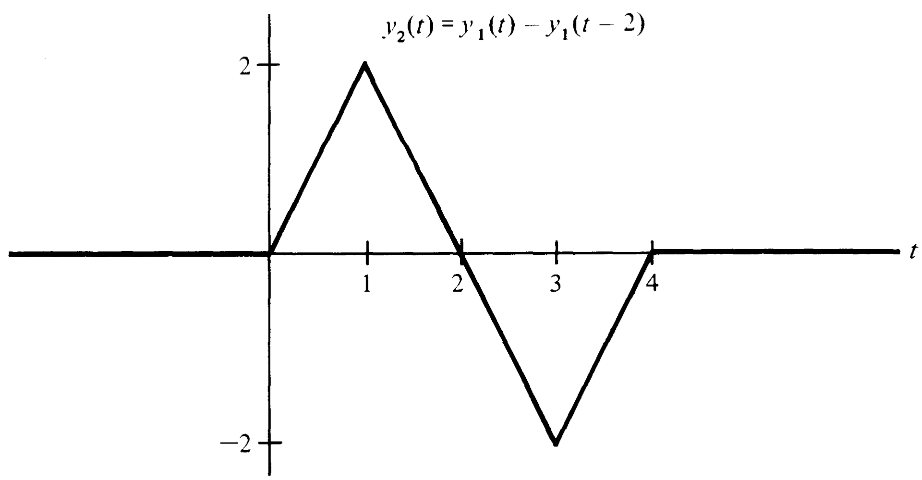

Figure S3.8-1

(b) $x_3(t) = x_1(t) + x_1(t+1)$

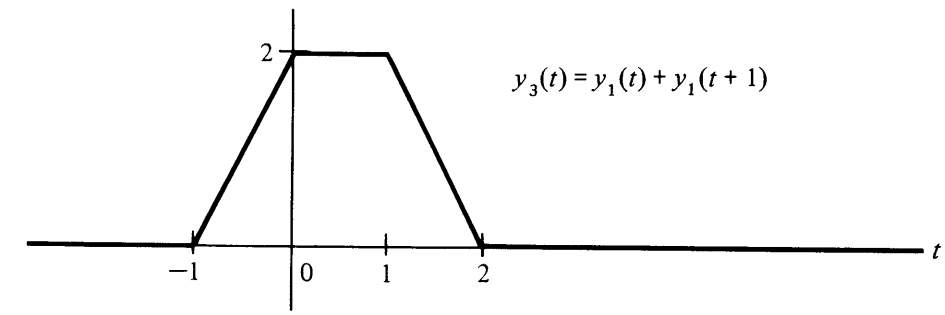

Figure S3.8-2

(c) $x(t) = u(t-1) - u(t-2)$

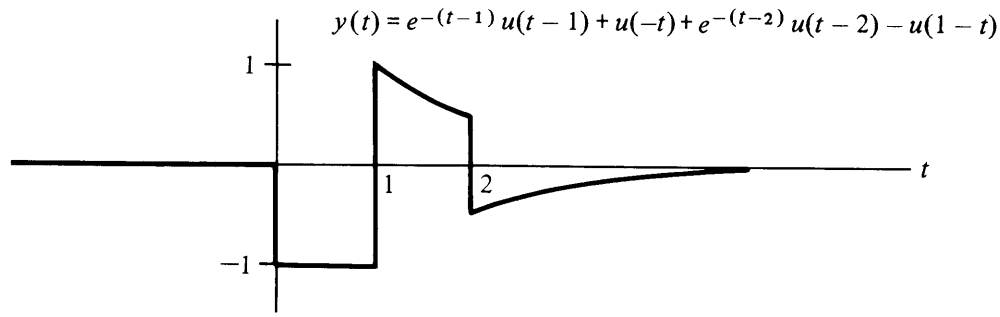

Figure S3.8-3

(d) $y[n] = 3y_1[n] - 2y_2[n] + 2y_3[n]$

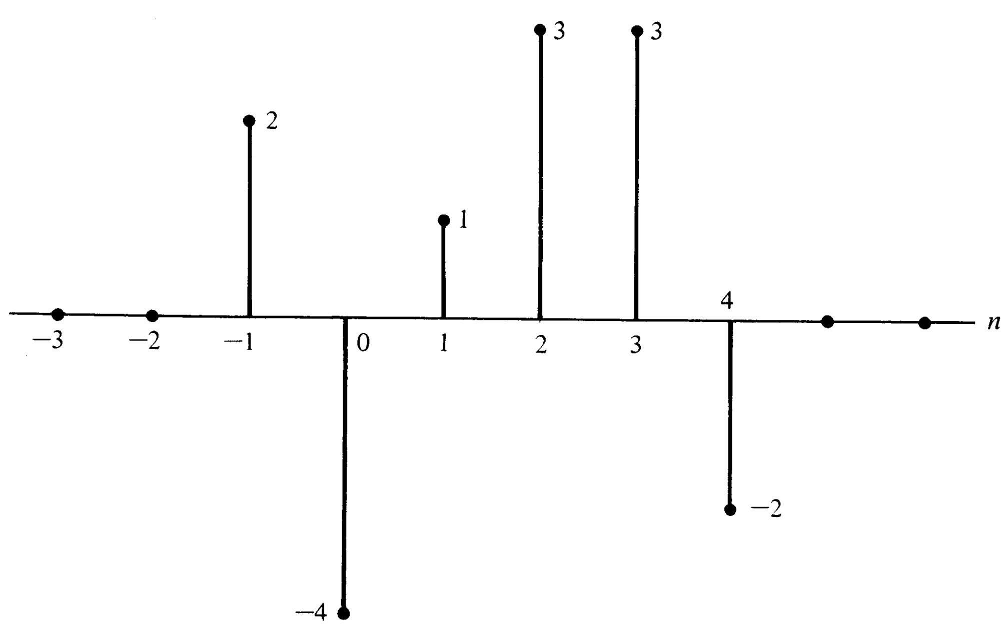

Figure S3.8-4

(e) $y_2[n] = y_1[n] + y_1[n-1]$

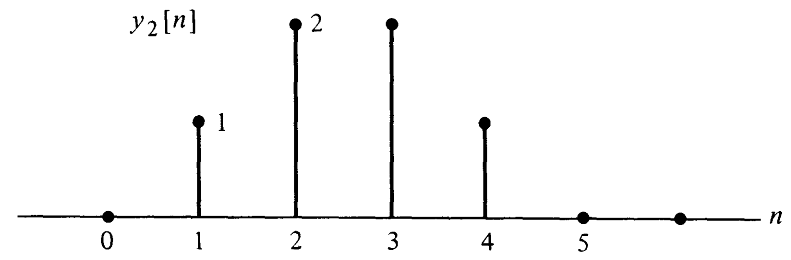

Figure S3.8-5

$y_3[n] = y_1[n+1]$

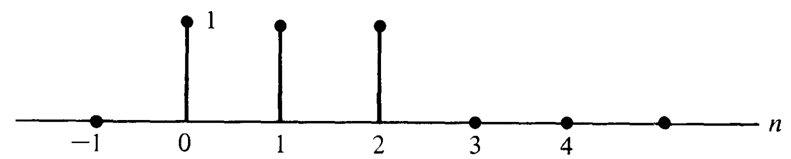

Figure S3.8-6

(f) From linearity,

$$y_1(t) = \pi + 6\cos(2t) - 47\cos(5t) + \sqrt{e}\cos(6t),$$
$$x_2(t) = \frac{1+t^{10}}{1+t^2} = \sum_{n=0}^{4}(-t^2)^n.$$

So $y_2(t) = 1 - \cos(2t) + \cos(4t) - \cos(6t) + \cos(8t)$.

S3.9

(a) (i) The system is linear because

$$\begin{aligned} T[ax_1(t) + bx_2(t)] &= \sum_{n=-\infty}^{\infty}[ax_1(t) + bx_2(t)]\delta(t-nT) \\ &= a\sum_{n=-\infty}^{\infty}x_1(t)\delta(t-nT) + b\sum_{n=-\infty}x_2(t)\delta(t-nT) \\ &= aT[x_1(t)] + bT[x_2(t)] \end{aligned}$$

(ii) The system is not time-invariant. For example, let $x_1(t) = \sin(2\pi t/T)$. The corresponding output $y_1(t) = 0$. Now let us shift the input $x_1(t)$ by $\pi/2$ to get

$$x_2(t) = \sin\left(\frac{2\pi t}{T} + \frac{\pi}{2}\right) = \cos\left(\frac{2\pi t}{T}\right)$$

Now the output

$$y_2(t) = \sum_{n=-\infty}^{+\infty}\delta(t-nT) \neq y_1\left(t + \frac{\pi}{2}\right) = 0$$

$$\text{(b)}\quad y(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} \cos(2\pi t)\delta(t - nT)$$

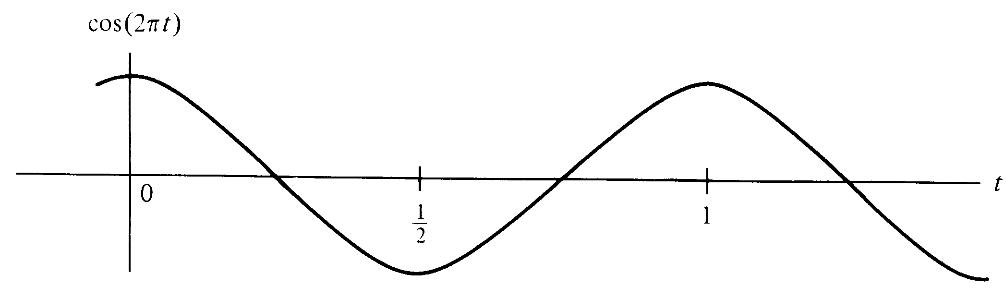

Figure S3.9-1

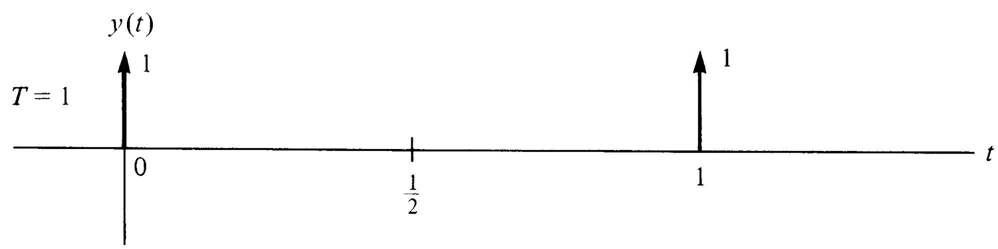

Figure S3.9-2

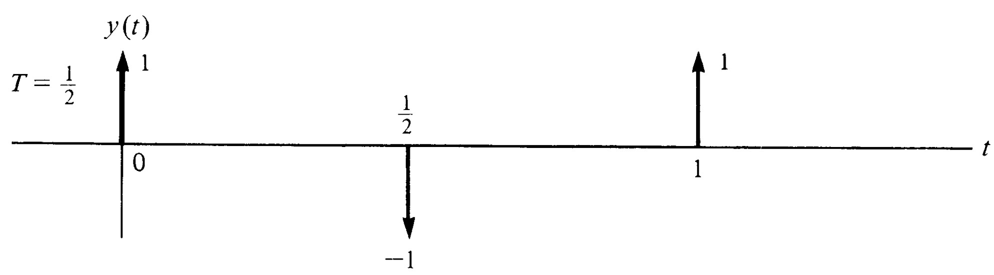

Figure S3.9-3

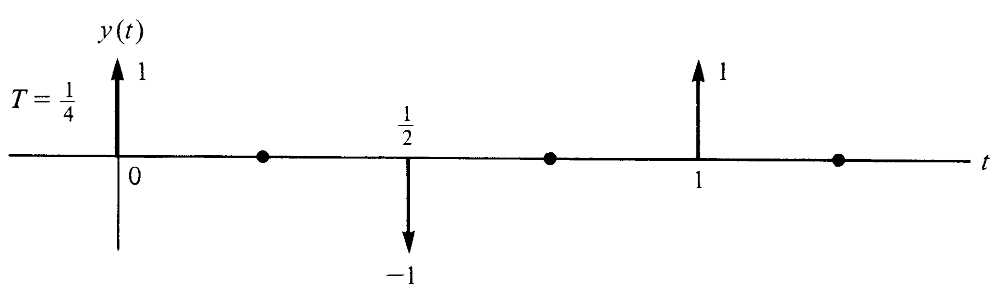

Figure S3.9-4

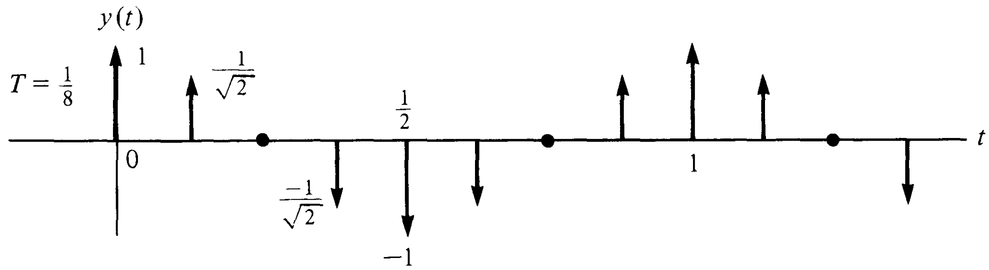

Figure S3.9-5

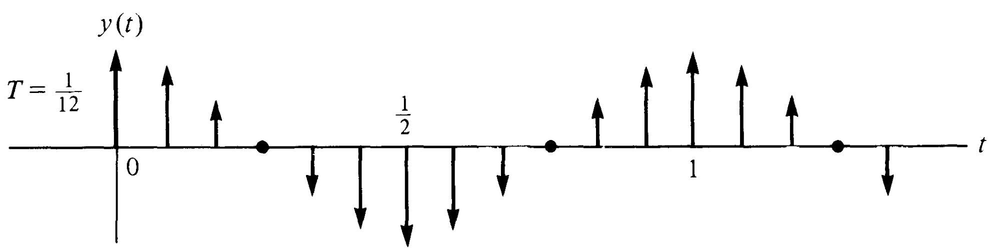

Figure S3.9-6

(c) $y(t) = \sum_{n=-\infty}^{\infty} e^t \cos(2\pi t)\delta(t - nT)$

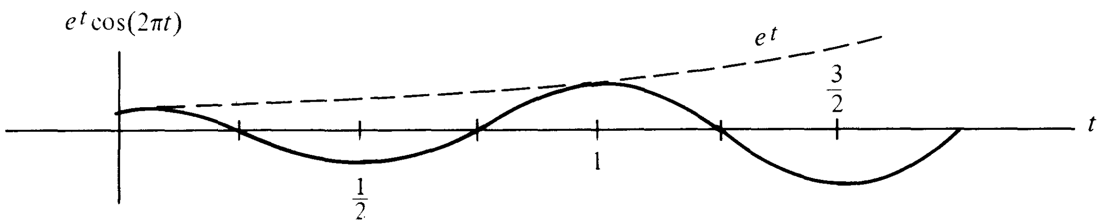

Figure S3.9-7

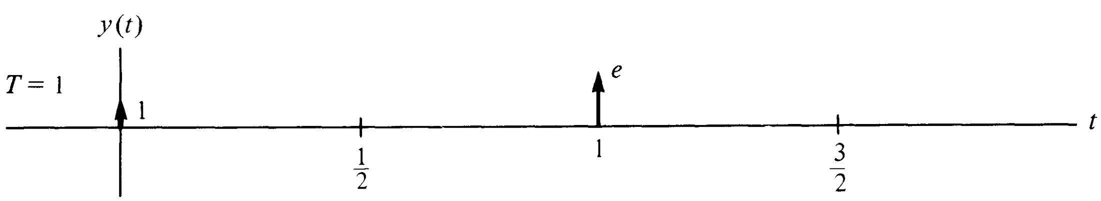

Figure S3.9-8

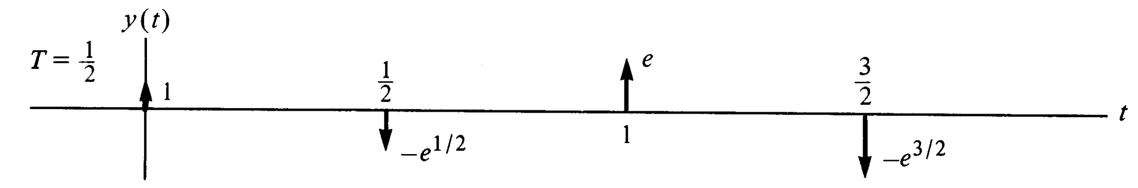

Figure S3.9-9

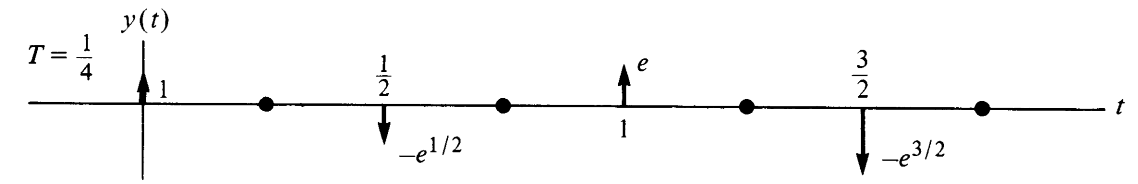

Figure S3.9-10

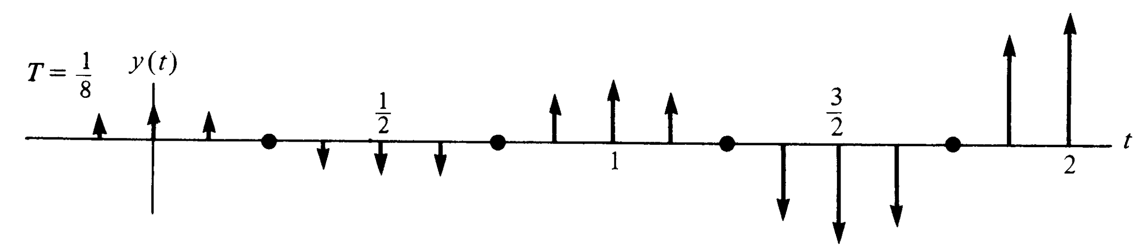

Figure S3.9-11

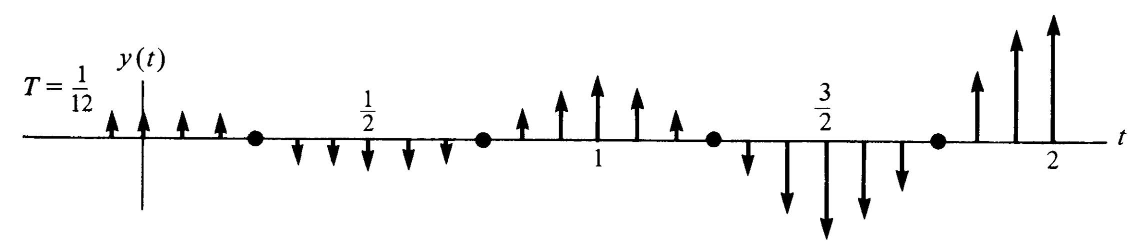

Figure S3.9-12

S3.10

(a) True. To see that the system is linear, write

$$y_2(t) = T_2[T_1[x(t)]] \triangleq T[x(t)],$$

$$T_1[ax_1(t) + bx_2(t)] = aT_1[x_1(t)] + bT_1[x_2(t)]$$

$$\Rightarrow T_2[T_1[ax_1(t) + bx_2(t)]] = T_2[aT_1[x_1(t)] + bT_1[x_2(t)]]$$

$$= aT_2[T_1[x_1(t)]] + bT_2[T_1[x_2(t)]]$$

$$= aT[x_1(t)] + bT[x_2(t)]$$

We see that the system is time-invariant from

$$\begin{aligned} T_2[T_1[x(t-T)]] &= T_2[y_1(t-T)] \\ &= y_2(t-T), \\ T[x(t-T)] &= y_2(t-T) \end{aligned}$$

(b) False. Two nonlinear systems in cascade can be linear, as shown in Figure S3.10. The overall system is identity, which is a linear system.

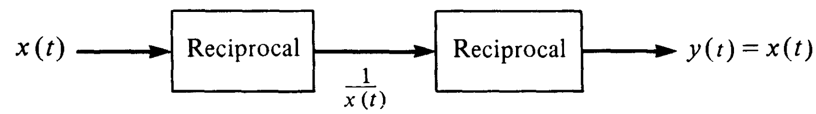

Figure S3.10

(c) $y[n] = z[2n] = w[2n] + \frac{1}{2}w[2n-1] + \frac{1}{4}w[2n-2]$
$\quad = x[n] + \frac{1}{4}x[n-1]$

The system is linear and time-invariant.

(d) $y[n] = z[-n] = aw[-n-1] + bw[-n] + cw[-n+1]$
$\quad = ax[n+1] + bx[n] + cx[n-1]$

(i) The overall system is linear and time-invariant for any choice of a, b, and c.

(ii) $a = c$

(iii) $a = 0$

S3.11

(a) $y[n] = x[n] + x[n-1] = T[x[n]]$. The system is linear because

$$\begin{aligned} T[ax_1[n] + bx_2[n]] &= ax_1[n] + ax_1[n-1] + bx_2[n] + bx_2[n-1] \\ &= aT[x_1[n]] + bT[x_2[n-1]] \end{aligned}$$

The system is time-invariant because

$$\begin{aligned} y[n] &= x[n] + x[n-1] = T[x[n]], \\ T[x[n-N]] &= x[n-N] + x[n-1-N] \\ &= y[n-N] \end{aligned}$$

(b) The system is linear, shown by similar steps to those in part (a). It is not time-invariant because

$$\begin{aligned} T[x[n-N]] &= x[n-N] + x[n-N-1] + x[0] \\ &\neq y[n-N] = x[n-N] + x[n-N-1] + x[-N] \end{aligned}$$

S3.12

(a) To show that causality implies the statement, suppose

$x_1(t) \rightarrow y_1(t)$ (input $x_1(t)$ results in output $y_1(t)$),
$x_2(t) \rightarrow y_2(t)$,

where $y_1(t)$ and $y_2(t)$ depend on $x_1(t)$ and $x_2(t)$ for $t < t_0$. By linearity,

$$x_1(t) - x_2(t) \rightarrow y_1(t) - y_2(t)$$

If $x_1(t) = x_2(t)$ for $t < t_0$, then $y_1(t) = y_2(t)$ for $t < t_0$. Hence, if $x(t) = 0$ for $t < t_0$, $y(t) = 0$ for $t < t_0$.

(b) $y(t) = x(t)x(t + 1)$,
$x(t) = 0 \quad \text{for } t < t_0 \quad \Rightarrow \quad y(t) = 0, \quad \text{for } t < t_0$

This is a nonlinear, noncausal system.

(c) $y(t) = x(t) + 1$ is a nonlinear, causal system.

(d) We want to show the equivalence of the following two statements:

Statement 1 (S1): The system is invertible.

Statement 2 (S2): The only input that produces the output $y[n] = 0$ for all n is $x[n] = 0$ for all n.

To show the equivalence, we will show that

$$\text{S2 false} \Rightarrow \text{S1 false} \quad \text{and}$$
$$\text{S1 false} \Rightarrow \text{S2 false}$$

S2 false $\Rightarrow$ S1 false: Let $x[n] \neq 0$ produce $y[n] = 0$. Then $cx[n] \Rightarrow y[n] = 0$.

S1 false $\Rightarrow$ S2 false: Let $x_1 \Rightarrow y_1$ and $x_2 \Rightarrow y_2$. If $x_1 \neq x_2$ but $y_1 = y_2$, then $x_1 - x_2 \neq 0$ but $y_1 - y_1 = 0$.

(e) $y[n] = x^2[n]$ is nonlinear and not invertible.