chapter 8

guided electromagnetic waves

The uniform plane wave solutions developed in Chapter 7 cannot in actuality exist throughout all space, as an infinite amount of energy would be required from the sources. However, TEM waves can also propagate in the region of finite volume between electrodes. Such electrode structures, known as transmission lines, are used for electromagnetic energy flow from power (60 Hz) to microwave frequencies, as delay lines due to the finite speed c of electromagnetic waves, and in pulse forming networks due to reflections at the end of the line. Because of the electrode boundaries, more general wave solutions are also permitted where the electric and magnetic fields are no longer perpendicular. These new solutions also allow electromagnetic power flow in closed single conductor structures known as waveguides.

8-1 THE TRANSMISSION LINE EQUATIONS

8-1-1 The Parallel Plate Transmission Line

The general properties of transmission lines are illustrated in Figure 8-1 by the parallel plate electrodes a small distance d apart enclosing linear media with permittivity ε and permeability μ. Because this spacing d is much less than the width w or length l, we neglect fringing field effects and assume that the fields only depend on the z coordinate.

The perfectly conducting electrodes impose the boundary conditions:

(i) The tangential component of **E** is zero.
(ii) The normal component of **B** (and thus **H** in the linear media) is zero.

With these constraints and the neglect of fringing near the electrode edges, the fields cannot depend on x or y and thus are of the following form:

$$\begin{aligned}\mathbf{E} &= E_x(z, t)\mathbf{i}_x \\ \mathbf{H} &= H_y(z, t)\mathbf{i}_y\end{aligned} \tag{1}$$

which when substituted into Maxwell's equations yield

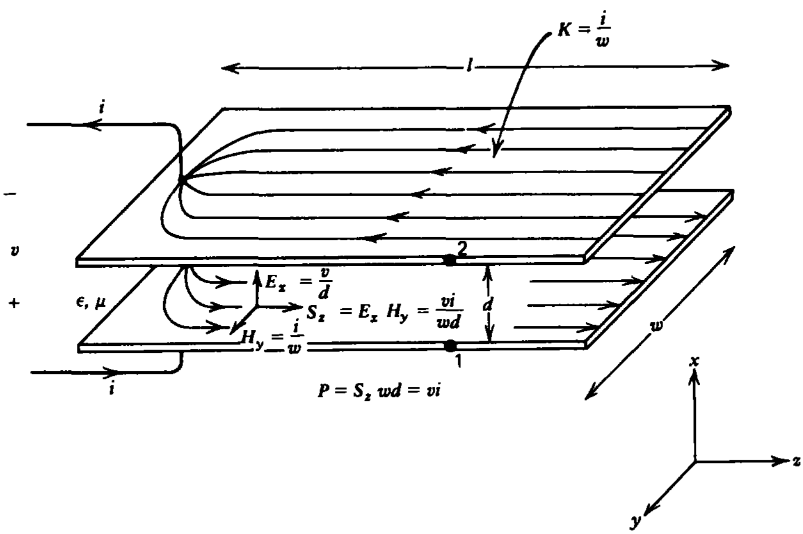


Figure 8-1 The simplest transmission line consists of two parallel perfectly conducting plates a small distance d apart.

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \Rightarrow \frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t}$$
$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} \Rightarrow \frac{\partial H_y}{\partial z} = -\varepsilon \frac{\partial E_x}{\partial t} \tag{2}$$

We recognize these equations as the same ones developed for plane waves in Section 7-3-1. The wave solutions found there are also valid here. However, now it is more convenient to introduce the circuit variables of voltage and current along the transmission line, which will depend on z and t.

Kirchoff's voltage and current laws will not hold along the transmission line as the electric field in (2) has nonzero curl and the current along the electrodes will have a divergence due to the time varying surface charge distribution, $\sigma_f = \pm \varepsilon E_x(z, t)$. Because $\mathbf{E}$ has a curl, the voltage difference measured between any two points is not unique, as illustrated in Figure 8-2, where we see time varying magnetic flux passing through the contour L_1. However, no magnetic flux passes through the path L_2, where the potential difference is measured between the two electrodes at the same value of z, as the magnetic flux is parallel to the surface. Thus, the voltage can be uniquely defined between the two electrodes at the same value of z:

$$v(z, t) = \int_{\substack{1 \\ z = \text{const}}}^{2} \mathbf{E} \cdot d\mathbf{l} = E_x(z, t) d \tag{3}$$

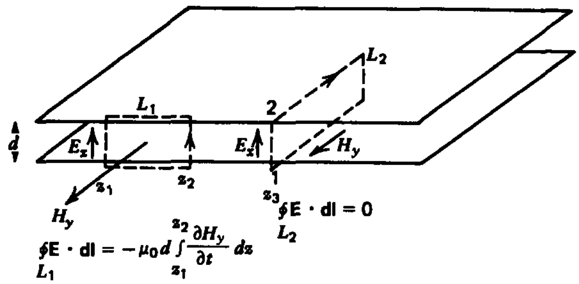


Figure 8-2 The potential difference measured between any two arbitrary points at different positions z_1 and z_2 on the transmission line is not unique—the line integral L_1 of the electric field is nonzero since the contour has magnetic flux passing through it. If the contour L_2 lies within a plane of constant z such as at z_3, no magnetic flux passes through it so that the voltage difference between the two electrodes at the same value of z is unique.

Similarly, the tangential component of **H** is discontinuous at each plate by a surface current $\pm\mathbf{K}$. Thus, the total current $i(z, t)$ flowing in the z direction on the lower plate is

$$i(z, t) = K_z w = H_y w \tag{4}$$

Substituting (3) and (4) back into (2) results in the transmission line equations:

$$\begin{aligned} \frac{\partial v}{\partial z} &= -L\frac{\partial i}{\partial t} \\ \frac{\partial i}{\partial z} &= -C\frac{\partial v}{\partial t} \end{aligned} \tag{5}$$

where L and C are the inductance and capacitance per unit length of the parallel plate structure:

$$L = \frac{\mu d}{w}\ \text{henry/m}, \qquad C = \frac{\varepsilon w}{d}\ \text{farad/m} \tag{6}$$

If both quantities are multiplied by the length of the line l, we obtain the inductance of a single turn plane loop if the line were short circuited, and the capacitance of a parallel plate capacitor if the line were open circuited.

It is no accident that the LC product

$$LC = \varepsilon\mu = 1/c^2 \tag{7}$$

is related to the speed of light in the medium.

8-1-2 General Transmission Line Structures

The transmission line equations of (5) are valid for any two-conductor structure of arbitrary shape in the transverse

xy plane but whose cross-sectional area does not change along its axis in the z direction. L and C are the inductance and capacitance per unit length as would be calculated in the quasi-static limits. Various simple types of transmission lines are shown in Figure 8-3. Note that, in general, the field equations of (2) must be extended to allow for x and y components but still no z components:

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_T(x, y, z, t) = E_x\mathbf{i}_x + E_y\mathbf{i}_y, \qquad E_z = 0 \\ \mathbf{H} &= \mathbf{H}_T(x, y, z, t) = H_x\mathbf{i}_x + H_y\mathbf{i}_y, \qquad H_z = 0 \end{aligned} \tag{8}$$

We use the subscript T in (8) to remind ourselves that the fields lie purely in the transverse xy plane. We can then also distinguish between spatial derivatives along the z axis ($\partial/\partial z$) from those in the transverse plane ($\partial/\partial x$, $\partial/\partial y$):

$$\nabla = \underbrace{\nabla_T}_{\mathbf{i}_x\frac{\partial}{\partial x}+\mathbf{i}_y\frac{\partial}{\partial y}} + \mathbf{i}_z\frac{\partial}{\partial z} \tag{9}$$

We may then write Maxwell's equations as

$$\begin{aligned} \nabla_T\times\mathbf{E}_T + \frac{\partial}{\partial z}(\mathbf{i}_z\times\mathbf{E}_T) &= -\mu\frac{\partial \mathbf{H}_T}{\partial t} \\ \nabla_T\times\mathbf{H}_T + \frac{\partial}{\partial z}(\mathbf{i}_z\times\mathbf{H}_T) &= \varepsilon\frac{\partial \mathbf{E}_T}{\partial t} \\ \nabla_T\cdot\mathbf{E}_T &= 0 \\ \nabla_T\cdot\mathbf{H}_T &= 0 \end{aligned} \tag{10}$$

The following vector properties for the terms in (10) apply:

(i) $\nabla_T\times\mathbf{H}_T$ and $\nabla_T\times\mathbf{E}_T$ lie purely in the z direction.
(ii) $\mathbf{i}_z\times\mathbf{E}_T$ and $\mathbf{i}_z\times\mathbf{H}_T$ lie purely in the xy plane.

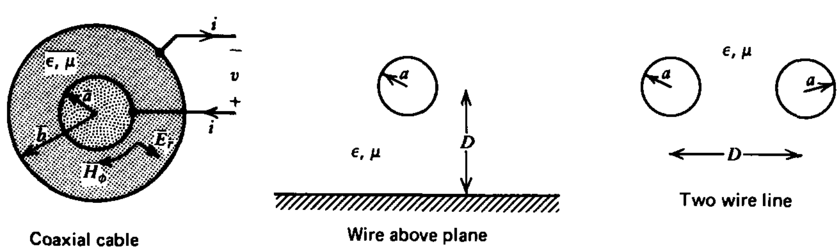


Figure 8-3 Various types of simple transmission lines.

Thus, the equations in (10) may be separated by equating vector components:

$$\begin{aligned} \nabla_T \times \mathbf{E}_T = 0, \qquad \nabla_T \times \mathbf{H}_T = 0 \\ \nabla_T \cdot \mathbf{E}_T = 0, \qquad \nabla_T \cdot \mathbf{H}_T = 0 \end{aligned} \tag{11}$$

$$\begin{aligned} &\frac{\partial}{\partial z}(\mathbf{i}_z \times \mathbf{E}_T) = -\mu \frac{\partial \mathbf{H}_T}{\partial t} \Rightarrow \frac{\partial \mathbf{E}_T}{\partial z} = \mu \frac{\partial}{\partial t}(\mathbf{i}_z \times \mathbf{H}_T) \\ &\frac{\partial}{\partial z}(\mathbf{i}_z \times \mathbf{H}_T) = \varepsilon \frac{\partial \mathbf{E}_T}{\partial t} \end{aligned} \tag{12}$$

where the Faraday's law equalities are obtained by crossing with $\mathbf{i}_z$ and expanding the double cross product

$$\mathbf{i}_z \times (\mathbf{i}_z \times \mathbf{E}_T) = \mathbf{i}_z \overbrace{(\mathbf{i}_z \cdot \mathbf{E}_T)}^{0} - \mathbf{E}_T(\mathbf{i}_z \cdot \mathbf{i}_z) = -\mathbf{E}_T \tag{13}$$

and remembering that $\mathbf{i}_z \cdot \mathbf{E}_T = 0$.

The set of equations in (11) tell us that the field dependences on the transverse coordinates are the same as if the system were static and source free. Thus, all the tools developed for solving static field solutions, including the two-dimensional Laplace's equations and the method of images, can be used to solve for $\mathbf{E}_T$ and $\mathbf{H}_T$ in the transverse xy plane.

We need to relate the fields to the voltage and current defined as a function of z and t for the transmission line of arbitrary shape shown in Figure 8-4 as

$$\begin{aligned} v(z, t) &= \int_{1\ (z=\text{const})}^{2} \mathbf{E}_T \cdot d\mathbf{l} \\ i(z, t) &= \oint_{\substack{\text{contour } L \\ \text{at constant } z \\ \text{enclosing the} \\ \text{inner conductor}}} \mathbf{H}_T \cdot d\mathbf{s} \end{aligned} \tag{14}$$

The related quantities of charge per unit length q and flux per unit length λ along the transmission line are

$$\begin{aligned} q(z, t) &= \varepsilon \oint_{L\ (z=\text{const})} \mathbf{E}_T \cdot \mathbf{n}\, ds \\ \lambda(z, t) &= \mu \int_{1\ (z=\text{const})}^{2} \mathbf{H}_T \cdot (\mathbf{i}_z \times d\mathbf{l}) \end{aligned} \tag{15}$$

The capacitance and inductance per unit length are then defined as the ratios:

$$\begin{aligned} C &= \frac{q(z, t)}{v(z, t)} = \left.\frac{\varepsilon \oint_L \mathbf{E}_T \cdot d\mathbf{s}}{\int_1^2 \mathbf{E}_T \cdot d\mathbf{l}}\right|_{z=\text{const}} \\ L &= \frac{\lambda(z, t)}{i(z, t)} = \left.\frac{\mu \int_1^2 \mathbf{H}_T \cdot (\mathbf{i}_z \times d\mathbf{l})}{\oint_L \mathbf{H}_T \cdot d\mathbf{s}}\right|_{z=\text{const}} \end{aligned} \tag{16}$$

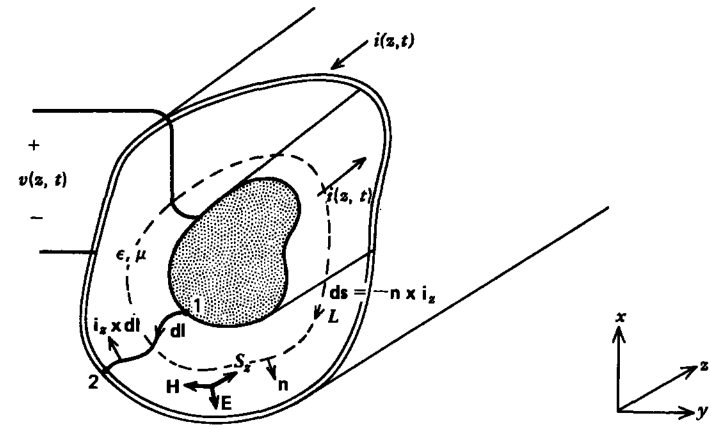


Figure 8-4 A general transmission line has two perfect conductors whose cross-sectional area does not change in the direction along its z axis, but whose shape in the transverse xy plane is arbitrary. The electric and magnetic fields are perpendicular, lie in the transverse xy plane, and have the same dependence on x and y as if the fields were static.

which are constants as the geometry of the transmission line does not vary with z. Even though the fields change with z, the ratios in (16) do not depend on the field amplitudes.

To obtain the general transmission line equations, we dot the upper equation in (12) with **dl**, which can be brought inside the derivatives since **dl** only varies with x and y and not z or t. We then integrate the resulting equation over a line at constant z joining the two electrodes:

$$\frac{\partial}{\partial z}\left(\int_1^2 \mathbf{E}_T\cdot d\mathbf{l}\right)=\frac{\partial}{\partial t}\left(\mu\int_1^2 (\mathbf{i}_z\times\mathbf{H}_T)\cdot d\mathbf{l}\right)$$
$$=-\frac{\partial}{\partial t}\left(\mu\int_1^2 \mathbf{H}_T\cdot(\mathbf{i}_z\times d\mathbf{l})\right) \qquad (17)$$

where the last equality is obtained using the scalar triple product allowing the interchange of the dot and the cross:

$$(\mathbf{i}_z\times\mathbf{H}_T)\cdot d\mathbf{l}=-(\mathbf{H}_T\times\mathbf{i}_z)\cdot d\mathbf{l}=-\mathbf{H}_T\cdot(\mathbf{i}_z\times d\mathbf{l}) \qquad (18)$$

We recognize the left-hand side of (17) as the z derivative of the voltage defined in (14), while the right-hand side is the negative time derivative of the flux per unit length defined in (15):

$$\frac{\partial v}{\partial z}=-\frac{\partial\lambda}{\partial t}=-L\frac{\partial i}{\partial t} \qquad (19)$$

We could also have derived this last relation by dotting the upper equation in (12) with the normal **n** to the inner conductor and then integrating over the contour L surrounding the inner conductor:

$$\frac{\partial}{\partial z}\left(\oint_L \mathbf{n}\cdot\mathbf{E}_T\,ds\right)=\frac{\partial}{\partial t}\left(\mu\oint_L \mathbf{n}\cdot(\mathbf{i}_z\times\mathbf{H}_T)\,ds\right)=-\frac{\partial}{\partial t}\left(\mu\oint_L \mathbf{H}_T\cdot d\mathbf{s}\right) \tag{20}$$

where the last equality was again obtained by interchanging the dot and the cross in the scalar triple product identity:

$$\mathbf{n}\cdot(\mathbf{i}_z\times\mathbf{H}_T)=(\mathbf{n}\times\mathbf{i}_z)\cdot\mathbf{H}_T=-\mathbf{H}_T\cdot d\mathbf{s} \tag{21}$$

The left-hand side of (20) is proportional to the charge per unit length defined in (15), while the right-hand side is proportional to the current defined in (14):

$$\frac{1}{\varepsilon}\frac{\partial q}{\partial z}=-\mu\frac{\partial i}{\partial t}\Rightarrow C\frac{\partial v}{\partial z}=-\varepsilon\mu\frac{\partial i}{\partial t} \tag{22}$$

Since (19) and (22) must be identical, we obtain the general result previously obtained in Section 6-5-6 that the inductance and capacitance per unit length of any arbitrarily shaped transmission line are related as

$$LC=\varepsilon\mu \tag{23}$$

We obtain the second transmission line equation by dotting the lower equation in (12) with $d\mathbf{l}$ and integrating between electrodes:

$$\frac{\partial}{\partial t}\left(\varepsilon\int_1^2 \mathbf{E}_T\cdot d\mathbf{l}\right)=\frac{\partial}{\partial z}\left(\int_1^2(\mathbf{i}_z\times\mathbf{H}_T)\cdot d\mathbf{l}\right)=-\frac{\partial}{\partial z}\left(\int_1^2 \mathbf{H}_T\cdot(\mathbf{i}_z\times d\mathbf{l})\right) \tag{24}$$

to yield from (14)–(16) and (23)

$$\varepsilon\frac{\partial v}{\partial t}=-\frac{1}{\mu}\frac{\partial \lambda}{\partial z}=-\frac{L}{\mu}\frac{\partial i}{\partial z}\Rightarrow\frac{\partial i}{\partial z}=-C\frac{\partial v}{\partial t} \tag{25}$$

EXAMPLE 8-1 THE COAXIAL TRANSMISSION LINE

Consider the coaxial transmission line shown in Figure 8-3 composed of two perfectly conducting concentric cylinders of radii a and b enclosing a linear medium with permittivity ε and permeability μ. We solve for the transverse dependence of the fields as if the problem were static, independent of time. If the voltage difference between cylinders is v with the inner cylinder carrying a total current i the static fields are

$$E_r=\frac{v}{r\ln(b/a)},\qquad H_\phi=\frac{i}{2\pi r}$$

The surface charge per unit length q and magnetic flux per unit length λ are

$$q = \varepsilon E_r(r = a)2\pi a = \frac{2\pi\varepsilon v}{\ln (b/a)}$$

$$\lambda = \int_a^b \mu H_\phi \, dr = \frac{\mu i}{2\pi} \ln \frac{b}{a}$$

so that the capacitance and inductance per unit length of this structure are

$$C = \frac{q}{v} = \frac{2\pi\varepsilon}{\ln (b/a)}, \qquad L = \frac{\lambda}{i} = \frac{\mu}{2\pi} \ln \frac{b}{a}$$

where we note that as required

$$LC = \varepsilon\mu$$

Substituting E_r and H_ϕ into (12) yields the following transmission line equations:

$$\frac{\partial E_r}{\partial z} = -\mu \frac{\partial H_\phi}{\partial t} \Rightarrow \frac{\partial v}{\partial z} = -L \frac{\partial i}{\partial t}$$

$$\frac{\partial H_\phi}{\partial z} = -\varepsilon \frac{\partial E_r}{\partial t} \Rightarrow \frac{\partial i}{\partial z} = -C \frac{\partial v}{\partial t}$$

8-1-3 Distributed Circuit Representation

Thus far we have emphasized the field theory point of view from which we have derived relations for the voltage and current. However, we can also easily derive the transmission line equations using a distributed equivalent circuit derived from the following criteria:

(i) The flow of current through a lossless medium between two conductors is entirely by displacement current, in exactly the same way as a capacitor.
(ii) The flow of current along lossless electrodes generates a magnetic field as in an inductor.

Thus, we may discretize the transmission line into many small incremental sections of length Δz with series inductance $L\,\Delta z$ and shunt capacitance $C\,\Delta z$, where L and C are the inductance and capacitance per unit lengths. We can also take into account the small series resistance of the electrodes $R\,\Delta z$, where R is the resistance per unit length (ohms per meter) and the shunt conductance loss in the dielectric $G\,\Delta z$, where G is the conductance per unit length (siemens per meter). If the transmission line and dielectric are lossless, $R = 0$, $G = 0$.

The resulting equivalent circuit for a lossy transmission line shown in Figure 8-5 shows that the current at $z+\Delta z$ and z differ by the amount flowing through the shunt capacitance and conductance:

$$i(z,t)-i(z+\Delta z,t)=C\,\Delta z\frac{\partial v(z,t)}{\partial t}+G\,\Delta z\,v(z,t) \qquad (26)$$

Similarly, the voltage difference at $z+\Delta z$ from z is due to the drop across the series inductor and resistor:

$$v(z,t)-v(z+\Delta z,t)=L\Delta z\frac{\partial i(z+\Delta z,t)}{\partial t}+i(z+\Delta z,t)R\,\Delta z \qquad (27)$$

By dividing (26) and (27) through by Δz and taking the limit as $\Delta z\to 0$, we obtain the lossy transmission line equations:

$$\begin{aligned}\lim_{\Delta z\to 0}\frac{i(z+\Delta z,t)-i(z,t)}{\Delta z}&=\frac{\partial i}{\partial z}=-C\frac{\partial v}{\partial t}-Gv\\ \lim_{\Delta z\to 0}\frac{v(z+\Delta z,t)-v(z,t)}{\Delta z}&=\frac{\partial v}{\partial z}=-L\frac{\partial i}{\partial t}-iR\end{aligned} \qquad (28)$$

which reduce to (19) and (25) when R and G are zero.

8-1-4 Power Flow

Multiplying the upper equation in (28) by v and the lower by i and then adding yields the circuit equivalent form of Poynting's theorem:

$$\frac{\partial}{\partial z}(vi)=-\frac{\partial}{\partial t}\left(\tfrac{1}{2}Cv^2+\tfrac{1}{2}Li^2\right)-Gv^2-i^2R \qquad (29)$$

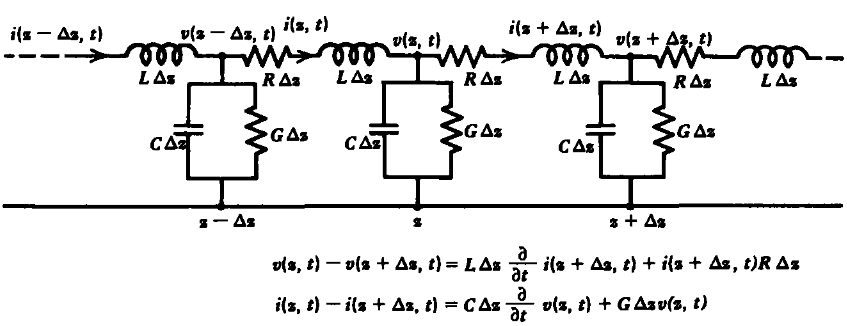


Figure 8-5 Distributed circuit model of a transmission line including small series and shunt resistive losses.

The power flow vi is converted into energy storage $(\frac{1}{2}Cv^2+\frac{1}{2}Li^2)$ or is dissipated in the resistance and conductance per unit lengths.

From the fields point of view the total electromagnetic power flowing down the transmission line at any position z is

$$P(z,t)=\int_S (\mathbf{E}_T\times\mathbf{H}_T)\cdot\mathbf{i}_z\,dS=\int_S \mathbf{E}_T\cdot(\mathbf{H}_T\times\mathbf{i}_z)\,dS \tag{30}$$

where S is the region between electrodes in Figure 8-4. Because the transverse electric field is curl free, we can define the scalar potential

$$\nabla\times\mathbf{E}_T=0\Rightarrow\mathbf{E}_T=-\nabla_T V \tag{31}$$

so that (30) can be rewritten as

$$P(z,t)=\int_S (\mathbf{i}_z\times\mathbf{H}_T)\cdot\nabla_T V\,dS \tag{32}$$

It is useful to examine the vector expansion

$$\nabla_T\cdot[V(\mathbf{i}_z\times\mathbf{H}_T)]=(\mathbf{i}_z\times\mathbf{H}_T)\cdot\nabla_T V+V\overbrace{\nabla_T\cdot(\mathbf{i}_z\times\mathbf{H}_T)}^{0} \tag{33}$$

where the last term is zero because $\mathbf{i}_z$ is a constant vector and $\mathbf{H}_T$ is also curl free:

$$\nabla_T\cdot(\mathbf{i}_z\times\mathbf{H}_T)=\mathbf{H}_T\cdot(\nabla_T\times\mathbf{i}_z)-\mathbf{i}_z\cdot(\nabla_T\times\mathbf{H}_T)=0 \tag{34}$$

Then (32) can be converted to a line integral using the two-dimensional form of the divergence theorem:

$$\begin{aligned}P(z,t)&=\int_S \nabla_T\cdot[V(\mathbf{i}_z\times\mathbf{H}_T)]\,dS\\&=-\int_{\substack{\text{contours on}\\\text{the surfaces of}\\\text{both electrodes}}} V(\mathbf{i}_z\times\mathbf{H}_T)\cdot\mathbf{n}\,ds\end{aligned} \tag{35}$$

where the line integral is evaluated at cónstant z along the surface of both electrodes. The minus sign arises in (35) because $\mathbf{n}$ is defined inwards in Figure 8-4 rather than outwards as is usual in the divergence theorem. Since we are free to pick our zero potential reference anywhere, we take the outer conductor to be at zero voltage. Then the line integral in (35) is only nonzero over the inner conductor, where $V=v$:

$$\begin{aligned} P(z,t) &= -v \oint_{\substack{\text{inner}\\ \text{conductor}}} (\mathbf{i}_z \times \mathbf{H}_T)\cdot \mathbf{n}\, ds \\ &= v \oint_{\substack{\text{inner}\\ \text{conductor}}} (\mathbf{H}_T \times \mathbf{i}_z)\cdot \mathbf{n}\, ds \\ &= v \oint_{\substack{\text{inner}\\ \text{conductor}}} \mathbf{H}_T \cdot (\mathbf{i}_z \times \mathbf{n})\, ds \\ &= v \oint_{\substack{\text{inner}\\ \text{conductor}}} \mathbf{H}_T \cdot \mathbf{ds} \\ &= vi \end{aligned} \tag{36}$$

where we realized that $(\mathbf{i}_z \times \mathbf{n})\, ds = \mathbf{ds}$, defined in Figure 8-4 if L lies along the surface of the inner conductor. The electromagnetic power flowing down a transmission line just equals the circuit power.

8-1-5 The Wave Equation

Restricting ourselves now to lossless transmission lines so that $R=G=0$ in (28), the two coupled equations in voltage and current can be reduced to two single wave equations in v and i:

$$\begin{aligned} \frac{\partial^2 v}{\partial t^2} &= c^2 \frac{\partial^2 v}{\partial z^2} \\ \frac{\partial^2 i}{\partial t^2} &= c^2 \frac{\partial^2 i}{\partial z^2} \end{aligned} \tag{37}$$

where the speed of the waves is

$$c = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\varepsilon\mu}} \text{ m/sec} \tag{38}$$

As we found in Section 7-3-2 the solutions to (37) are propagating waves in the $\pm z$ directions at the speed c:

$$\begin{aligned} v(z,t) &= V_+(t - z/c) + V_-(t + z/c) \\ i(z,t) &= I_+(t - z/c) + I_-(t + z/c) \end{aligned} \tag{39}$$

where the functions V_+, V_-, I_+, and I_- are determined by boundary conditions imposed by sources and transmission line terminations. By substituting these solutions back into (28) with $R = G = 0$, we find the voltage and current functions related as

$$\begin{aligned} V_+ &= I_+ Z_0 \\ V_- &= -I_- Z_0 \end{aligned} \tag{40}$$

where

$$Z_0 = \sqrt{L/C} \text{ ohm} \tag{41}$$

is known as the characteristic impedance of the transmission line, analogous to the wave impedance η in Chapter 7. Its inverse $Y_0 = 1/Z_0$ is also used and is termed the characteristic admittance. In practice, it is difficult to measure L and C of a transmission line directly. It is easier to measure the wave speed c and characteristic impedance Z_0 and then calculate L and C from (38) and (41).

The most useful form of the transmission line solutions of (39) that we will use is

$$\begin{aligned} v(z,t) &= V_+(t - z/c) + V_-(t + z/c) \\ i(z,t) &= Y_0[V_+(t - z/c) - V_-(t + z/c)] \end{aligned} \tag{42}$$

Note the complete duality between these voltage–current solutions and the plane wave solutions in Section 7-3-2 for the electric and magnetic fields.

8-2 TRANSMISSION LINE TRANSIENT WAVES

The easiest way to solve for transient waves on transmission lines is through use of physical reasoning as opposed to mathematical rigor. Since the waves travel at a speed c, once generated they cannot reach any position z until a time z/c later. Waves traveling in the positive z direction are described by the function $V_+(t - z/c)$ and waves traveling in the $-z$ direction by $V_-(t + z/c)$. However, at any time t and position z, the voltage is equal to the sum of both solutions while the current is proportional to their difference.

8-2-1 Transients on Infinitely Long Transmission Lines

The transmission line shown in Figure 8-6a extends to infinity in the positive z direction. A time varying voltage source $V(t)$ that is turned on at $t = 0$ is applied at $z = 0$ to the line which is initially unexcited. A positively traveling wave $V_+(t - z/c)$ propagates away from the source. There is no negatively traveling wave, $V_-(t + z/c) = 0$. These physical

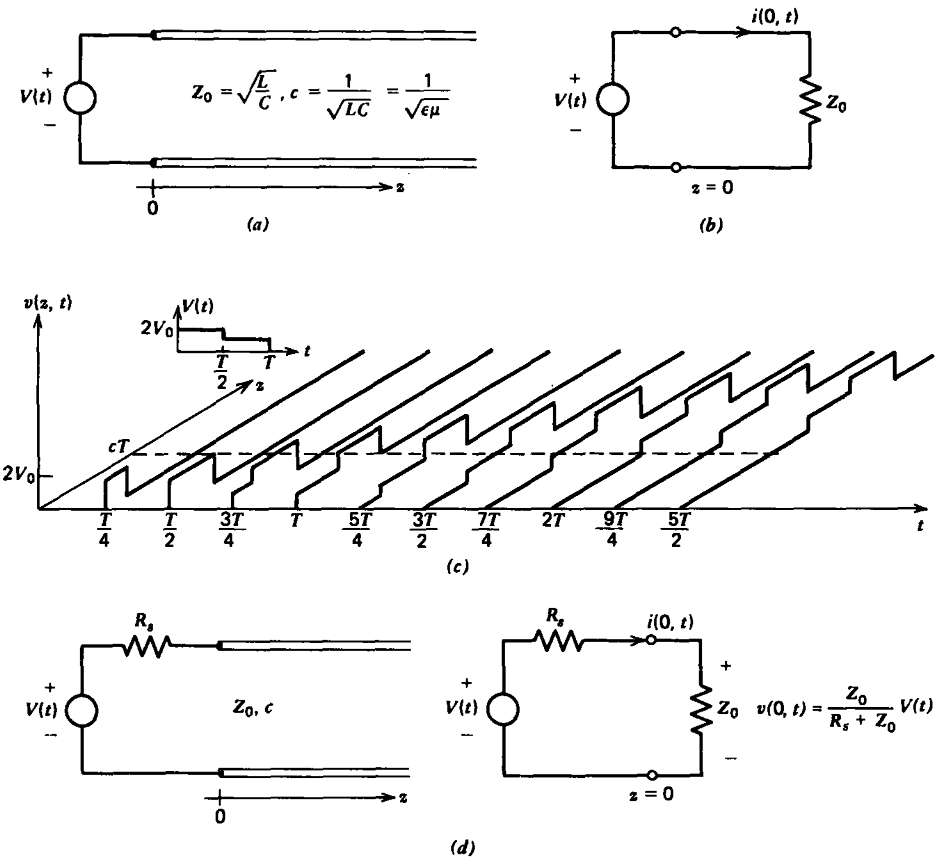


Figure 8-6 (*a*) A semi-infinite transmission line excited by a voltage source at $z = 0$. (*b*) To the source, the transmission line looks like a resistor Z_0 equal to the characteristic impedance. (*c*) The spatial distribution of the voltage $v(z, t)$ at various times for a staircase pulse of $V(t)$. (*d*) If the voltage source is applied to the transmission line through a series resistance R_s, the voltage across the line at $z = 0$ is given by the voltage divider relation.

arguments are verified mathematically by realizing that at $t = 0$ the voltage and current are zero for $z > 0$,

$$\begin{aligned} v(z, t=0) &= V_+(-z/c) + V_-(z/c) = 0 \\ i(z, t=0) &= Y_0[V_+(-z/c) - V_-(z/c)] = 0 \end{aligned} \tag{1}$$

which only allows the trivial solutions

$$V_+(-z/c) = 0, \qquad V_-(z/c) = 0 \tag{2}$$

Since z can only be positive, whenever the argument of V_+ is negative and of V_- positive, the functions are zero. Since t can only be positive, the argument of $V_-(t + z/c)$ is always positive so that the function is always zero. The argument of $V_+(t-z/c)$ can be positive, allowing a nonzero solution if $t>z/c$ agreeing with our conclusions reached by physical arguments.

With $V_-(t+z/c)=0$, the voltage and current are related as

$$\begin{aligned} v(z,t) &= V_+(t-z/c) \\ i(z,t) &= Y_0 V_+(t-z/c) \end{aligned} \tag{3}$$

The line voltage and current have the same shape as the source, delayed in time for any z by z/c with the current scaled in amplitude by Y_0. Thus as far as the source is concerned, the transmission line looks like a resistor of value Z_0 yielding the equivalent circuit at $z=0$ shown in Figure 8-6*b*. At $z=0$, the voltage equals that of the source

$$v(0,t)=V(t)=V_+(t) \tag{4}$$

If $V(t)$ is the staircase pulse of total duration T shown in Figure 8-6*c*, the pulse extends in space over the spatial interval:

$$\begin{aligned} 0\leq z\leq ct, &\qquad 0\leq t\leq T \\ c(t-T)\leq z\leq ct, &\qquad t>T \end{aligned} \tag{5}$$

The analysis is the same even if the voltage source is in series with a source resistance R_s, as in Figure 8-6*d*. At $z=0$ the transmission line still looks like a resistor of value Z_0 so that the transmission line voltage divides in the ratio given by the equivalent circuit shown:

$$\begin{aligned} v(z=0,t) &= \frac{Z_0}{R_s+Z_0}V(t)=V_+(t) \\ i(z=0,t) &= Y_0V_+(t)=\frac{V(t)}{R_s+Z_0} \end{aligned} \tag{6}$$

The total solution is then identical to that of (3) and (4) with the voltage and current amplitudes reduced by the voltage divider ratio $Z_0/(R_s+Z_0)$.

8-2-2 Reflections from Resistive Terminations

(a) Reflection Coefficient

All transmission lines must have an end. In Figure 8-7 we see a positively traveling wave incident upon a load resistor R_L at $z=l$. The reflected wave will travel back towards the source at $z=0$ as a V_- wave. At the $z=l$ end the following circuit

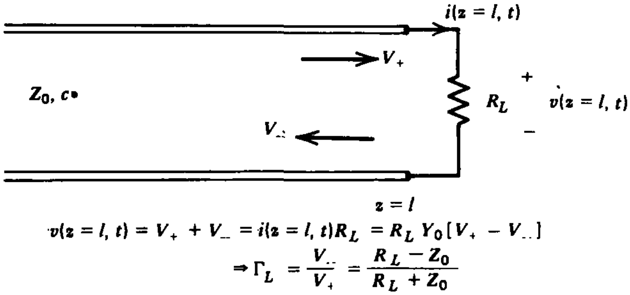


Figure 8-7 A V_+ wave incident upon the end of a transmission line with a load resistor R_L is reflected as a V_- wave.

relations hold:

$$\begin{aligned} v(l, t) &= V_+(t - l/c) + V_-(t + l/c) \\ &= i(l, t)R_L \\ &= Y_0 R_L[V_+(t - l/c) - V_-(t + l/c)] \end{aligned} \tag{7}$$

We then find the amplitude of the negatively traveling wave in terms of the incident positively traveling wave as

$$\Gamma_L = \frac{V_-(t + l/c)}{V_+(t - l/c)} = \frac{R_L - Z_0}{R_L + Z_0} \tag{8}$$

where Γ_L is known as the reflection coefficient that is of the same form as the reflection coefficient R in Section 7-6-1 for normally incident uniform plane waves on a dielectric.

The reflection coefficient gives us the relative amplitude of the returning V_- wave compared to the incident V_+ wave. There are several important limits of (8):

(i) If $R_L = Z_0$, the reflection coefficient is zero ($\Gamma_L = 0$) so that there is no reflected wave and the line is said to be matched.

(ii) If the line is short circuited ($R_L = 0$), then $\Gamma_L = -1$. The reflected wave is equal in amplitude but opposite in sign to the incident wave. In general, if $R_L < Z_0$, the reflected voltage wave has its polarity reversed.

(iii) If the line is open circuited ($R_L = \infty$), then $\Gamma_L = +1$. The reflected wave is identical to the incident wave. In general, if $R_L > Z_0$, the reflected voltage wave is of the same polarity as the incident wave.

(b) Step Voltage

A dc battery of voltage V_0 with series resistance R_s is switched onto the transmission line at $t = 0$, as shown in Figure 8-8*a*. At $z = 0$, the source has no knowledge of the

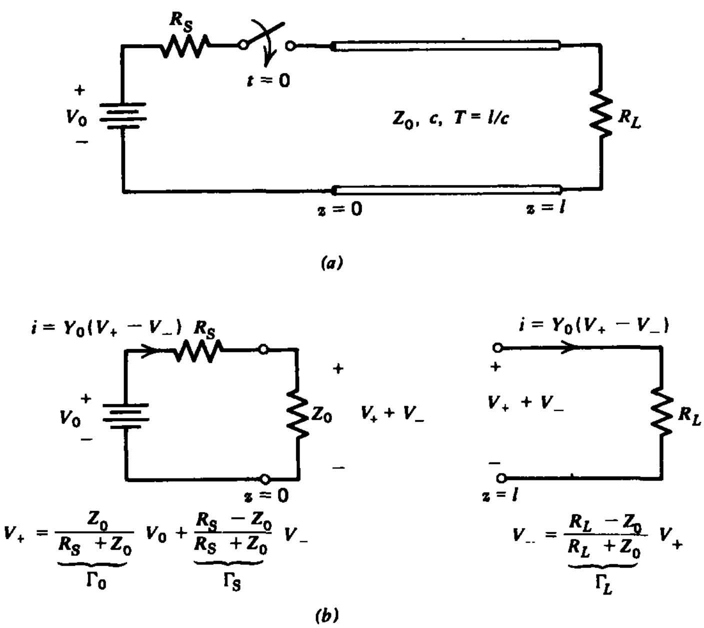


Figure 8-8 (*a*) A dc voltage V_0 is switched onto a resistively loaded transmission line through a source resistance R_s. (*b*) The equivalent circuits at $z = 0$ and $z = l$ allow us to calculate the reflected voltage wave amplitudes in terms of the incident waves.

line's length or load termination, so as for an infinitely long line the transmission line looks like a resistor of value Z_0 to the source. There is no V_- wave initially. The V_+ wave is determined by the voltage divider ratio of the series source resistance and transmission line characteristic impedance as given by (6).

This V_+ wave travels down the line at speed c where it is reflected at $z = l$ for $t > T$, where $T = l/c$ is the transit time for a wave propagating between the two ends. The new V_- wave generated is related to the incident V_+ wave by the reflection coefficient Γ_L. As the V_+ wave continues to propagate in the positive z direction, the V_- wave propagates back towards the source. The total voltage at any point on the line is equal to the sum of V_+ and V_- while the current is proportional to their difference.

When the V_- wave reaches the end of the transmission line at $z = 0$ at time $2T$, in general a new V_+ wave is generated, which can be found by solving the equivalent circuit shown in Figure 8-8*b*:

$$v(0, t) + i(0, t)R_s = V_0 \Rightarrow V_+(0, t) + V_-(0, t) + Y_0 R_s[V_+(0, t) - V_-(0, t)] = V_0 \quad (9)$$

to yield

$$V_+(0,t)=\Gamma_s V_-(0,t)+\frac{Z_0 V_0}{Z_0+R_s},\qquad \Gamma_s=\frac{R_s-Z_0}{R_s+Z_0} \tag{10}$$

where Γ_s is just the reflection coefficient at the source end. This new V_+ wave propagates towards the load again generating a new V_- wave as the reflections continue.

If the source resistance is matched to the line, $R_s=Z_0$ so that $\Gamma_s=0$, then V_+ is constant for all time and the steady state is reached for $t>2T$. If the load was matched, the steady state is reached for $t>T$ no matter the value of R_s. There are no further reflections from the end of a matched line. In Figure 8-9 we plot representative voltage and current spatial distributions for various times assuming the source is matched to the line for the load being matched, open, or short circuited.

(i) Matched Line

When $R_L=Z_0$ the load reflection coefficient is zero so that $V_+=V_0/2$ for all time. The wavefront propagates down the line with the voltage and current being identical in shape. The system is in the dc steady state for $t\geq T$.

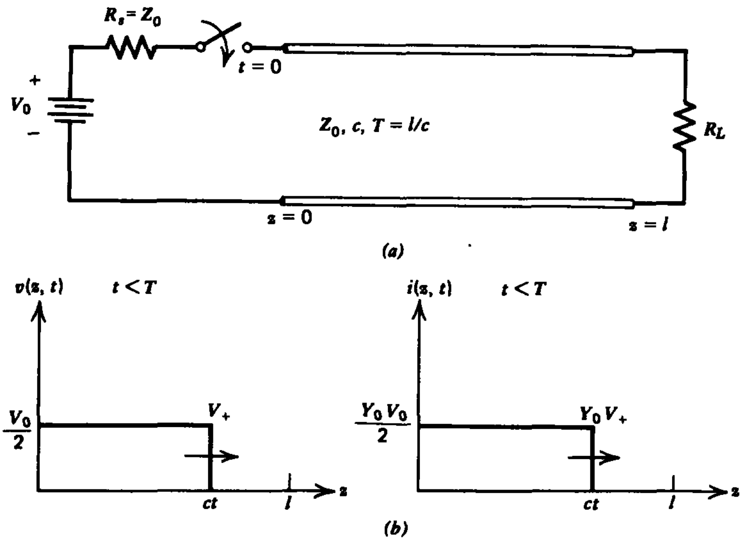
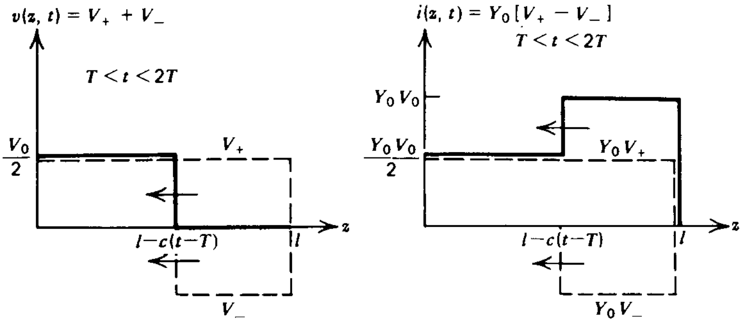
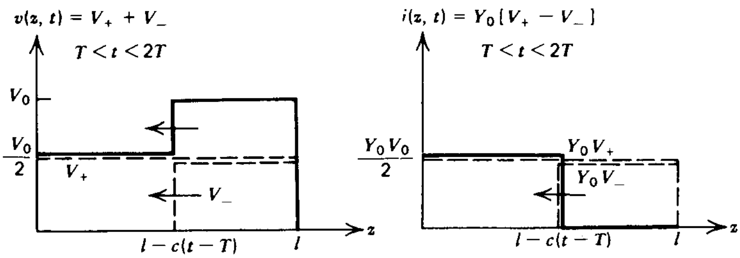


Figure 8-9 (*a*) A dc voltage is switched onto a transmission line with load resistance R_L through a source resistance R_s matched to the line. (*b*) Regardless of the load resistance, half the source voltage propagates down the line towards the load. If the load is also matched to the line ($R_L=Z_0$), there are no reflections and the steady state of $v(z,t\geq T)=V_0/2$, $i(z,t\geq T)=Y_0V_0/2$ is reached for $t\geq T$. (*c*) If the line is short circuited ($R_L=0$), then $\Gamma_L=-1$ so that the V_+ and V_- waves cancel for the voltage but add for the current wherever they overlap in space. Since the source end is matched, no further reflections arise at $z=0$ so that the steady state is reached for $t\geq 2T$. (*d*) If the line is open circuited ($R_L=\infty$) so that $\Gamma_L=+1$, the V_+ and V_- waves add for the voltage but cancel for the current.



Short circuited line, $R_L = 0$, $(v(z, t \geq 2T) = 0, i(z, t \geq 2T) = Y_0 V_0)$

(c)



Open circuited line, $R_L = \infty$, $(v(z, t \geq 2T) = V_0, i(z, t \geq 2T) = 0)$

(d)

Figure 8-9

(ii) Open Circuited Line

When $R_L = \infty$ the reflection coefficient is unity so that $V_+ = V_-$. When the incident and reflected waves overlap in space the voltages add to a stairstep pulse shape while the current is zero. For $t \geq 2T$, the voltage is V_0 everywhere on the line while the current is zero.

(iii) Short Circuited Line

When $R_L = 0$ the load reflection coefficient is -1 so that $V_+ = -V_-$. When the incident and reflected waves overlap in space, the total voltage is zero while the current is now a stairstep pulse shape. For $t \geq 2T$ the voltage is zero everywhere on the line while the current is V_0/Z_0.

8-2-3 Approach to the dc Steady State

If the load end is matched, the steady state is reached after one transit time $T = l/c$ for the wave to propagate from the source to the load. If the source end is matched, after one round trip $2T=2l/c$ no further reflections occur. If neither end is matched, reflections continue on forever. However, for nonzero and noninfinite source and load resistances, the reflection coefficient is always less than unity in magnitude so that each successive reflection is reduced in amplitude. After a few round-trips, the changes in V_+ and V_- become smaller and eventually negligible. If the source resistance is zero and the load resistance is either zero or infinite, the transient pulses continue to propagate back and forth forever in the lossless line, as the magnitude of the reflection coefficients are unity.

Consider again the dc voltage source in Figure 8-8*a* switched through a source resistance R_s at $t=0$ onto a transmission line loaded at its $z=l$ end with a load resistor R_L. We showed in (10) that the V_+ wave generated at the $z=0$ end is related to the source and an incoming V_- wave as

$$V_+=\Gamma_0 V_0+\Gamma_s V_-, \qquad \Gamma_0=\frac{Z_0}{R_s+Z_0}, \qquad \Gamma_s=\frac{R_s-Z_0}{R_s+Z_0} \tag{11}$$

Similarly, at $z=l$, an incident V_+ wave is converted into a V_- wave through the load reflection coefficient:

$$V_-=\Gamma_L V_+, \qquad \Gamma_L=\frac{R_L-Z_0}{R_L+Z_0} \tag{12}$$

We can now tabulate the voltage at $z=l$ using the following reasoning:

(i) For the time interval $t<T$ the voltage at $z=l$ is zero as no wave has yet reached the end.

(ii) At $z=0$ for $0\leq t\leq 2T$, $V_-=0$ resulting in a V_+ wave emanating from $z=0$ with amplitude $V_+=\Gamma_0 V_0$.

(iii) When this V_+ wave reaches $z=l$, a V_- wave is generated with amplitude $V_-=\Gamma_L V_+$. The incident V_+ wave at $z=l$ remains unchanged until another interval of $2T$, whereupon the just generated V_- wave after being reflected from $z=0$ as a new V_+ wave given by (11) again returns to $z=l$.

(iv) Thus, the voltage at $z=l$ only changes at times $(2n-1)T$, $n=1, 2, \ldots$, while the voltage at $z=0$ changes at times $2(n-1)T$. The resulting voltage waveforms at the ends are stairstep patterns with steps at these times.

The nth traveling V_+ wave is then related to the source and the $(n-1)$th V_- wave at $z=0$ as

$$V_{+n}=\Gamma_0 V_0+\Gamma_s V_{-(n-1)} \tag{13}$$

while the $(n-1)$th V_- wave is related to the incident $(n-1)$th V_+ wave at $z=l$ as

$$V_{-(n-1)}=\Gamma_L V_{+(n-1)} \tag{14}$$

Using (14) in (13) yields a single linear constant coefficient difference equation in V_{+n}:

$$V_{+n}-\Gamma_s\Gamma_L V_{+(n-1)}=\Gamma_0 V_0 \tag{15}$$

For a particular solution we see that V_{+n} being a constant satisfies (15):

$$V_{+n}=C\Rightarrow C(1-\Gamma_s\Gamma_L)=\Gamma_0 V_0\Rightarrow C=\frac{\Gamma_0}{1-\Gamma_s\Gamma_L}V_0 \tag{16}$$

To this solution we can add any homogeneous solution assuming the right-hand side of (15) is zero:

$$V_{+n}-\Gamma_s\Gamma_L V_{+(n-1)}=0 \tag{17}$$

We try a solution of the form

$$V_{+n}=A\lambda^n \tag{18}$$

which when substituted into (17) requires

$$A\lambda^{n-1}(\lambda-\Gamma_s\Gamma_L)=0\Rightarrow\lambda=\Gamma_s\Gamma_L \tag{19}$$

The total solution is then a sum of the particular and homogeneous solutions:

$$V_{+n}=\frac{\Gamma_0}{1-\Gamma_s\Gamma_L}V_0+A(\Gamma_s\Gamma_L)^n \tag{20}$$

The constant A is found by realizing that the first transient wave is

$$V_{+1}=\Gamma_0 V_0=\frac{\Gamma_0}{1-\Gamma_s\Gamma_L}V_0+A(\Gamma_s\Gamma_L) \tag{21}$$

which requires A to be

$$A=-\frac{\Gamma_0 V_0}{1-\Gamma_s\Gamma_L} \tag{22}$$

so that (20) becomes

$$V_{+n}=\frac{\Gamma_0 V_0}{1-\Gamma_s\Gamma_L}[1-(\Gamma_s\Gamma_L)^n] \tag{23}$$

Raising the index of (14) by one then gives the nth V_- wave as

$$V_{-n}=\Gamma_L V_{+n} \tag{24}$$

so that the total voltage at $z=l$ after n reflections at times $(2n-1)T$, $n=1,2,\ldots$, is

$$\mathrm{V}_n=\mathrm{V}_{+n}+\mathrm{V}_{-n}=\frac{\mathrm{V}_0\Gamma_0(1+\Gamma_L)}{1-\Gamma_s\Gamma_L}[1-(\Gamma_s\Gamma_L)^n] \tag{25}$$

or in terms of the source and load resistances

$$\mathrm{V}_n=\frac{R_L}{R_L+R_s}\,V_0[1-(\Gamma_s\Gamma_L)^n] \tag{26}$$

The steady-state results as $n\to\infty$. If either R_s or R_L are nonzero or noninfinite, the product of $\Gamma_s\Gamma_L$ must be less than unity. Under these conditions

$$\lim_{\substack{n\to\infty\\(|\Gamma_s\Gamma_L|<1)}}(\Gamma_s\Gamma_L)^n=0 \tag{27}$$

so that in the steady state

$$\lim_{n\to\infty}\mathrm{V}_n=\frac{R_L}{R_s+R_L}\mathrm{V}_0 \tag{28}$$

which is just the voltage divider ratio as if the transmission line was just a pair of zero-resistance connecting wires. Note also that if either end is matched so that either Γ_s or Γ_L is zero, the voltage at the load end is immediately in the steady state after the time T.

In Figure 8-10 the load is plotted versus time with $R_s=0$ and $R_L=3Z_0$ so that $\Gamma_s\Gamma_L=-\frac{1}{2}$ and with $R_L=\frac{1}{3}Z_0$ so that

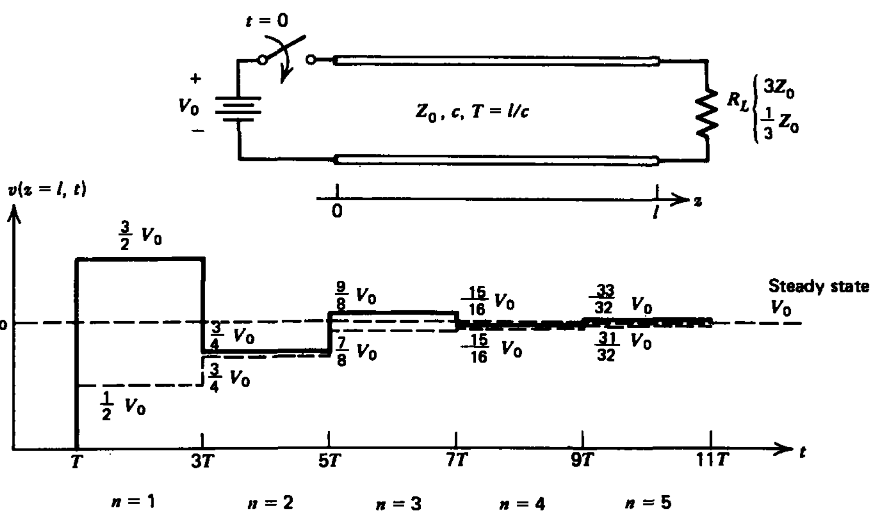


Figure 8-10 The load voltage as a function of time when $R_s=0$ and $R_L=3Z_0$ so that $\Gamma_s\Gamma_L=-\frac{1}{2}$ (solid) and with $R_L=\frac{1}{3}Z_0$ so that $\Gamma_s\Gamma_L=\frac{1}{2}$ (dashed). The dc steady state is the same as if the transmission line were considered a pair of perfectly conducting wires in a circuit.

$\Gamma_s\Gamma_L = +\frac{1}{2}$. Then (26) becomes

$$V_n = \begin{cases} V_0[1-(-\frac{1}{2})^n], & R_L = 3Z_0 \\ V_0[1-(\frac{1}{2})^n], & R_L = \frac{1}{3}Z_0 \end{cases} \tag{29}$$

The step changes in load voltage oscillate about the steady-state value $V_\infty = V_0$. The steps rapidly become smaller having less than one-percent variation for $n > 7$.

If the source resistance is zero and the load resistance is either zero or infinite (short or open circuits), a lossless transmission line never reaches a dc steady state as the limit of (27) does not hold with $\Gamma_s\Gamma_L = \pm 1$. Continuous reflections with no decrease in amplitude results in pulse waveforms for all time. However, in a real transmission line, small losses in the conductors and dielectric allow a steady state to be eventually reached.

Consider the case when $R_s = 0$ and $R_L = \infty$ so that $\Gamma_s\Gamma_L = -1$. Then from (26) we have

$$V_n = \begin{cases} 0, & n \text{ even} \\ 2V_0, & n \text{ odd} \end{cases} \tag{30}$$

which is sketched in Figure 8-11*a*.

For any source and load resistances the current through the load resistor at $z = l$ is

$$\begin{aligned} I_n &= \frac{V_n}{R_L} = \frac{V_0\Gamma_0(1+\Gamma_L)}{R_L(1-\Gamma_s\Gamma_L)}[1-(\Gamma_s\Gamma_L)^n] \\ &= \frac{2V_0\Gamma_0}{R_L+Z_0}\frac{[1-(\Gamma_s\Gamma_L)^n]}{(1-\Gamma_s\Gamma_L)} \end{aligned} \tag{31}$$

If both R_s and R_L are zero so that $\Gamma_s\Gamma_L = 1$, the short circuit current in (31) is in the indeterminate form 0/0, which can be evaluated using l'Hôpital's rule:

$$\begin{aligned} \lim_{\Gamma_s\Gamma_L \to 1} I_n &= \frac{2V_0\Gamma_0}{R_L+Z_0}\frac{[-n(\Gamma_s\Gamma_L)^{n-1}]}{(-1)} \\ &= \frac{2V_0 n}{Z_0} \end{aligned} \tag{32}$$

As shown by the solid line in Figure 8-11*b*, the current continually increases in a stepwise fashion. As n increases to infinity, the current also becomes infinite, which is expected for a battery connected across a short circuit.

8-2-4 Inductors and Capacitors as Quasi-static Approximations to Transmission Lines

If the transmission line was one meter long with a free space dielectric medium, the round trip transit time $2T = 2l/c$

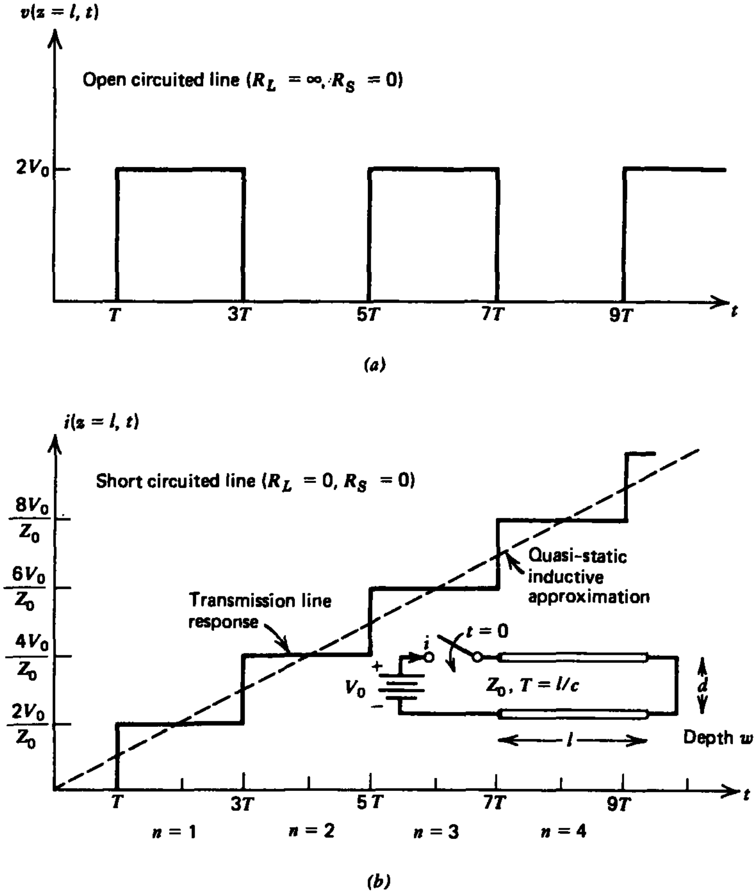


Figure 8-11 The (*a*) open circuit voltage and (*b*) short circuit current at the $z = l$ end of the transmission line for $R_s = 0$. No dc steady state is reached because the system is lossless. If the short circuited transmission line is modeled as an inductor in the quasi-static limit, a step voltage input results in a linearly increasing current (shown dashed). The exact transmission line response is the solid staircase waveform.

is approximately 6 nsec. For many circuit applications this time is so fast that it may be considered instantaneous. In this limit the quasi-static circuit element approximation is valid.

For example, consider again the short circuited transmission line ($R_L = 0$) of length l with zero source resistance. In the magnetic quasi-static limit we would call the structure an inductor with inductance Ll (remember, L is the inductance per unit length) so that the terminal voltage and current are related as

$$v = (Ll)\frac{di}{dt} \tag{33}$$

If a constant voltage V_0 is applied at $t=0$, the current is obtained by integration of (33) as

$$i=\frac{V_0}{Ll}t \tag{34}$$

where we use the initial condition of zero current at $t=0$. The linear time dependence of the current, plotted as the dashed line in Figure 8-11*b*, approximates the rising staircase waveform obtained from the exact transmission line analysis of (32).

Similarly, if the transmission line were open circuited with $R_L=\infty$, it would be a capacitor of value Cl in the electric quasi-static limit so that the voltage on the line charges up through the source resistance R_s with time constant $\tau=R_sCl$ as

$$v(t)=V_0(1-e^{-t/\tau}) \tag{35}$$

The exact transmission line voltage at the $z=l$ end is given by (26) with $R_L=\infty$ so that $\Gamma_L=1$:

$$V_n=V_0(1-\Gamma_s^n) \tag{36}$$

where the source reflection coefficient can be written as

$$\begin{aligned}\Gamma_s&=\frac{R_s-Z_0}{R_s+Z_0}\\&=\frac{R_s-\sqrt{L/C}}{R_s+\sqrt{L/C}}\end{aligned} \tag{37}$$

If we multiply the numerator and denominator of (37) through by Cl, we have

$$\begin{aligned}\Gamma_s&=\frac{R_sCl-l\sqrt{LC}}{R_sCl+l\sqrt{LC}}\\&=\frac{\tau-T}{\tau+T}=\frac{1-T/\tau}{1+T/\tau}\end{aligned} \tag{38}$$

where

$$T=l\sqrt{LC}=l/c \tag{39}$$

For the quasi-static limit to be valid, the wave transit time T must be much faster than any other time scale of interest so that $T/\tau\ll 1$. In Figure 8-12 we plot (35) and (36) for two values of T/τ and see that the quasi-static and transmission line results approach each other as T/τ becomes small.

When the roundtrip wave transit time is so small compared to the time scale of interest so as to appear to be instantaneous, the circuit treatment is an excellent approximation.

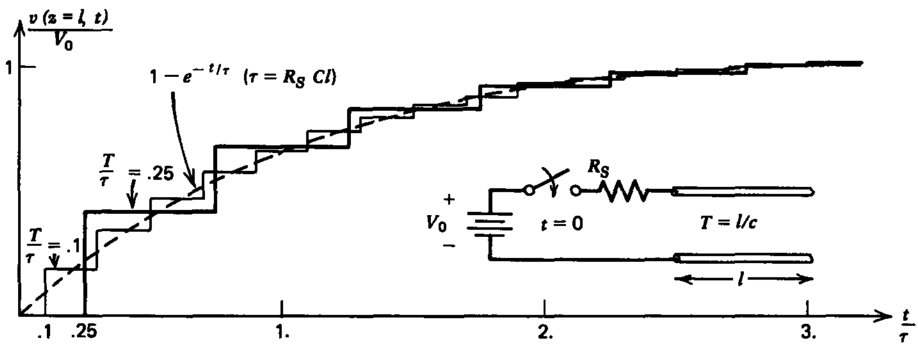


Figure 8-12 The open circuit voltage at $z=l$ for a step voltage applied at $t=0$ through a source resistance R_s for various values of T/τ, which is the ratio of propagation time $T=l/c$ to quasi-static charging time $\tau = R_sCl$. The dashed curve shows the exponential rise obtained by a circuit analysis assuming the open circuited transmission line is a capacitor.

If this propagation time is significant, then the transmission line equations must be used.

8-2-5 Reflections from Arbitrary Terminations

For resistive terminations we have been able to relate reflected wave amplitudes in terms of an incident wave amplitude through the use of a reflection coefficient because the voltage and current in the resistor are algebraically related. For an arbitrary termination, which may include any component such as capacitors, inductors, diodes, transistors, or even another transmission line with perhaps a different characteristic impedance, it is necessary to solve a circuit problem at the end of the line. For the arbitrary element with voltage V_L and current I_L at $z=l$, shown in Figure 8-13a, the voltage and current at the end of line are related as

$$v(z=l, t) = V_L(t) = V_+(t-l/c) + V_-(t+l/c) \tag{40}$$

$$i(z=l, t) = I_L(t) = Y_0[V_+(t-l/c) - V_-(t+l/c)] \tag{41}$$

We assume that we know the incident V_+ wave and wish to find the reflected V_- wave. We then eliminate the unknown V_- in (40) and (41) to obtain

$$2V_+(t-l/c) = V_L(t) + I_L(t)Z_0 \tag{42}$$

which suggests the equivalent circuit in Figure 8-13b.

For a particular lumped termination we solve the equivalent circuit for $V_L(t)$ or $I_L(t)$. Since $V_+(t-l/c)$ is already known as it is incident upon the termination, once $V_L(t)$ or

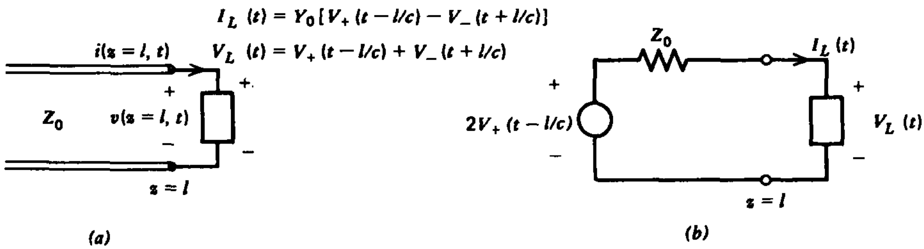


Figure 8-13 A transmission line with an (*a*) arbitrary load at the $z=l$ end can be analyzed from the equivalent circuit in (*b*). Since V_+ is known, calculation of the load current or voltage yields the reflected wave V_-.

$I_L(t)$ is calculated from the equivalent circuit, $V_-(t+l/c)$ can be calculated as $V_- = V_L - V_+$.

For instance, consider the lossless transmission lines of length l shown in Figure 8-14*a* terminated at the end with either a lumped capacitor C_L or an inductor L_L. A step voltage at $t=0$ is applied at $z=0$ through a source resistor matched to the line.

The source at $z=0$ is unaware of the termination at $z=l$ until a time $2T$. Until this time it launches a V_+ wave of amplitude $V_0/2$. At $z=l$, the equivalent circuit for the capacitive termination is shown in Figure 8-14*b*. Whereas resistive terminations just altered wave amplitudes upon reflection, inductive and capacitive terminations introduce differential equations.

From (42), the voltage across the capacitor v_c obeys the differential equation

$$Z_0C_L\frac{dv_c}{dt} + v_c = 2V_+ = V_0, \qquad t>T \tag{43}$$

with solution

$$v_c(t) = V_0[1 - e^{-(t-T)/Z_0C_L}], \qquad t>T \tag{44}$$

Note that the voltage waveform plotted in Figure 8-14*b* begins at time $T = l/c$.

Thus, the returning V_- wave is given as

$$V_- = v_c - V_+ = V_0/2 + V_0\, e^{-(t-T)/Z_0C_L} \tag{45}$$

This reflected wave travels back to $z=0$, where no further reflections occur since the source end is matched. The current at $z=l$ is then

$$i_c = C_L\frac{dv_c}{dt} = \frac{V_0}{Z_0}e^{-(t-T)/Z_0C_L}, \qquad t>T \tag{46}$$

and is also plotted in Figure 8-14*b*.

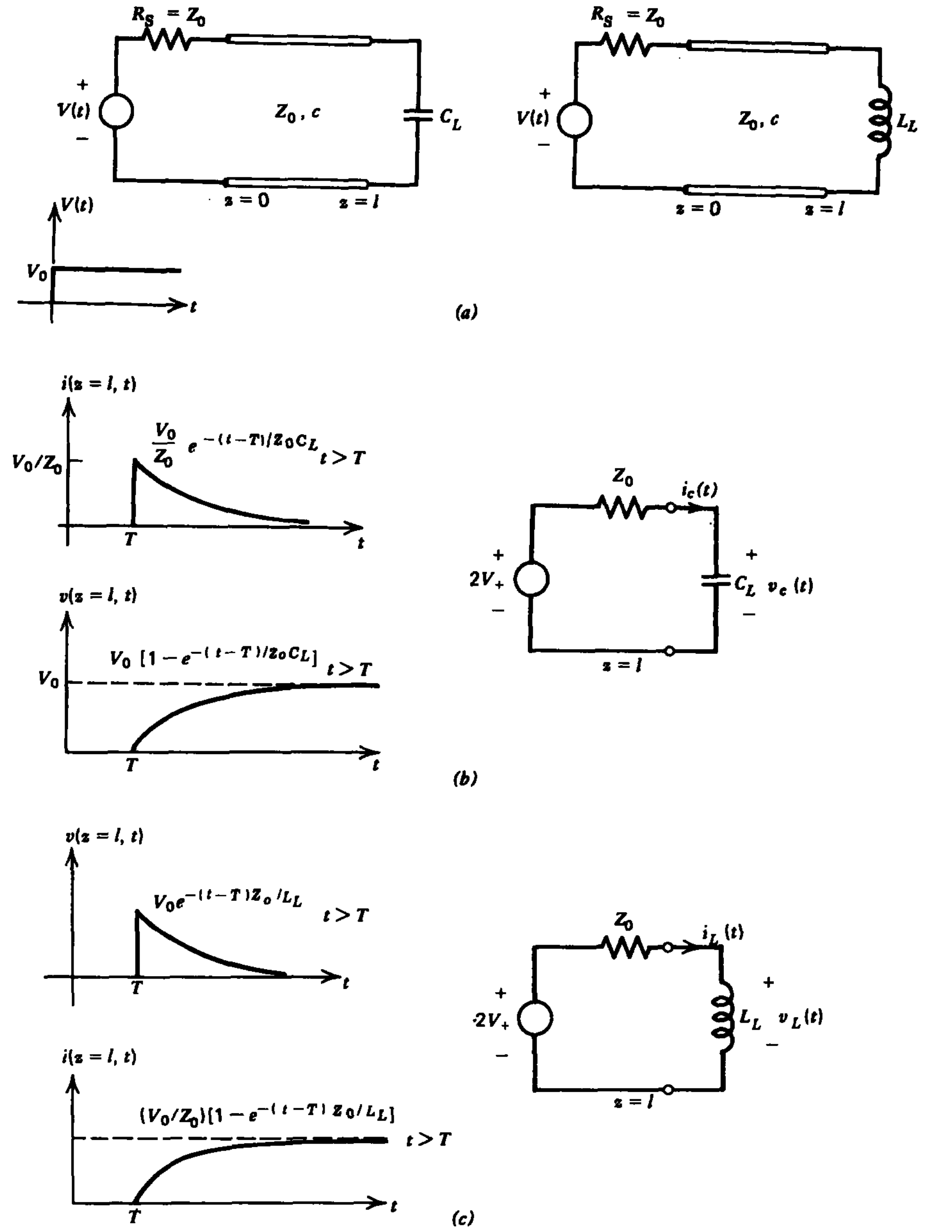


Figure 8-14 (a) A step voltage is applied to transmission lines loaded at $z = l$ with a capacitor C_L or inductor L_L. The load voltage and current are calculated from the (b) resistive-capacitive or (c) resistive-inductive equivalent circuits at $z = l$ to yield exponential waveforms with respective time constants $\tau = Z_0 C_L$ and $\tau = L_L/Z_0$ as the solutions approach the dc steady state. The waveforms begin after the initial V_+ wave arrives at $z = l$ after a time $T = l/c$. There are no further reflections as the source end is matched.

If the end at $z=0$ were not matched, a new V_+ would be generated. When it reached $z=l$, we would again solve the RC circuit with the capacitor now initially charged. The reflections would continue, eventually becoming negligible if R_s is nonzero.

Similarly, the governing differential equation for the inductive load obtained from the equivalent circuit in Figure 8-14c is

$$L_L\frac{di_L}{dt}+i_LZ_0=2V_+=V_0, \qquad t>T \tag{47}$$

with solution

$$i_L=\frac{V_0}{Z_0}(1-e^{-(t-T)Z_0/L_L}), \qquad t>T \tag{48}$$

The voltage across the inductor is

$$v_L=L_L\frac{di_L}{dt}=V_0\,e^{-(t-T)Z_0/L_L}, \qquad t>T \tag{49}$$

Again since the end at $z=0$ is matched, the returning V_- wave from $z=l$ is not reflected at $z=0$. Thus the total voltage and current for all time at $z=l$ is given by (48) and (49) and is sketched in Figure 8-14c.

8-3 SINUSOIDAL TIME VARIATIONS

8-3-1 Solutions to the Transmission Line Equations

Often transmission lines are excited by sinusoidally varying sources so that the line voltage and current also vary sinusoidally with time:

$$\begin{aligned} v(z,t)&=\operatorname{Re}[\hat{v}(z)\,e^{j\omega t}]\\ i(z,t)&=\operatorname{Re}[\hat{\imath}(z)\,e^{j\omega t}]\end{aligned} \tag{1}$$

Then as we found for TEM waves in Section 7-4, the voltage and current are found from the wave equation solutions of Section 8-1-5 as linear combinations of exponential functions with arguments $t-z/c$ and $t+z/c$:

$$\begin{aligned} v(z,t)&=\operatorname{Re}[\hat{V}_+\,e^{j\omega(t-z/c)}+\hat{V}_-\,e^{j\omega(t+z/c)}]\\ i(z,t)&=Y_0\operatorname{Re}[\hat{V}_+\,e^{j\omega(t-z/c)}-\hat{V}_-\,e^{j\omega(t+z/c)}]\end{aligned} \tag{2}$$

Now the phasor amplitudes $\hat{V}_+$ and $\hat{V}_-$ are complex numbers and do not depend on z or t.

By factoring out the sinusoidal time dependence in (2), the spatial dependences of the voltage and current are

$$\hat{v}(z) = \hat{V}_+ e^{-jkz} + \hat{V}_- e^{+jkz}$$
$$\hat{\imath}(z) = Y_0(\hat{V}_+ e^{-jkz} - \hat{V}_- e^{+jkz}) \tag{3}$$

where the wavenumber is again defined as

$$k = \omega/c \tag{4}$$

8-3-2 Lossless Terminations

(a) Short Circuited Line

The transmission line shown in Figure 8-15*a* is excited by a sinusoidal voltage source at $z=-l$ imposing the boundary condition

$$v(z=-l, t) = V_0 \cos \omega t$$
$$= \text{Re}\,(V_0\, e^{j\omega t}) \Rightarrow \hat{v}(z=-l) = V_0 = \hat{V}_+ e^{jkl} + \hat{V}_- e^{-jkl} \tag{5}$$

Note that to use (3) we must write all sinusoids in complex notation. Then since all time variations are of the form $e^{j\omega t}$, we may suppress writing it each time and work only with the spatial variations of (3).

Because the transmission line is short circuited, we have the additional boundary condition

$$v(z=0, t) = 0 \Rightarrow \hat{v}(z=0) = 0 = \hat{V}_+ + \hat{V}_- \tag{6}$$

which when simultaneously solved with (5) yields

$$\hat{V}_+ = -\hat{V}_- = \frac{V_0}{2j \sin kl} \tag{7}$$

The spatial dependences of the voltage and current are then

$$\hat{v}(z) = \frac{V_0(e^{-jkz} - e^{jkz})}{2j \sin kl} = -\frac{V_0 \sin kz}{\sin kl}$$
$$\hat{\imath}(z) = \frac{V_0 Y_0(e^{-jkz} + e^{+jkz})}{2j \sin kl} = -j\frac{V_0 Y_0 \cos kz}{\sin kl} \tag{8}$$

The instantaneous voltage and current as functions of space and time are then

$$v(z, t) = \text{Re}\,[\hat{v}(z)\, e^{j\omega t}] = -V_0 \frac{\sin kz}{\sin kl} \cos \omega t$$
$$i(z, t) = \text{Re}\,[\hat{\imath}(z)\, e^{j\omega t}] = \frac{V_0 Y_0 \cos kz \sin \omega t}{\sin kl} \tag{9}$$

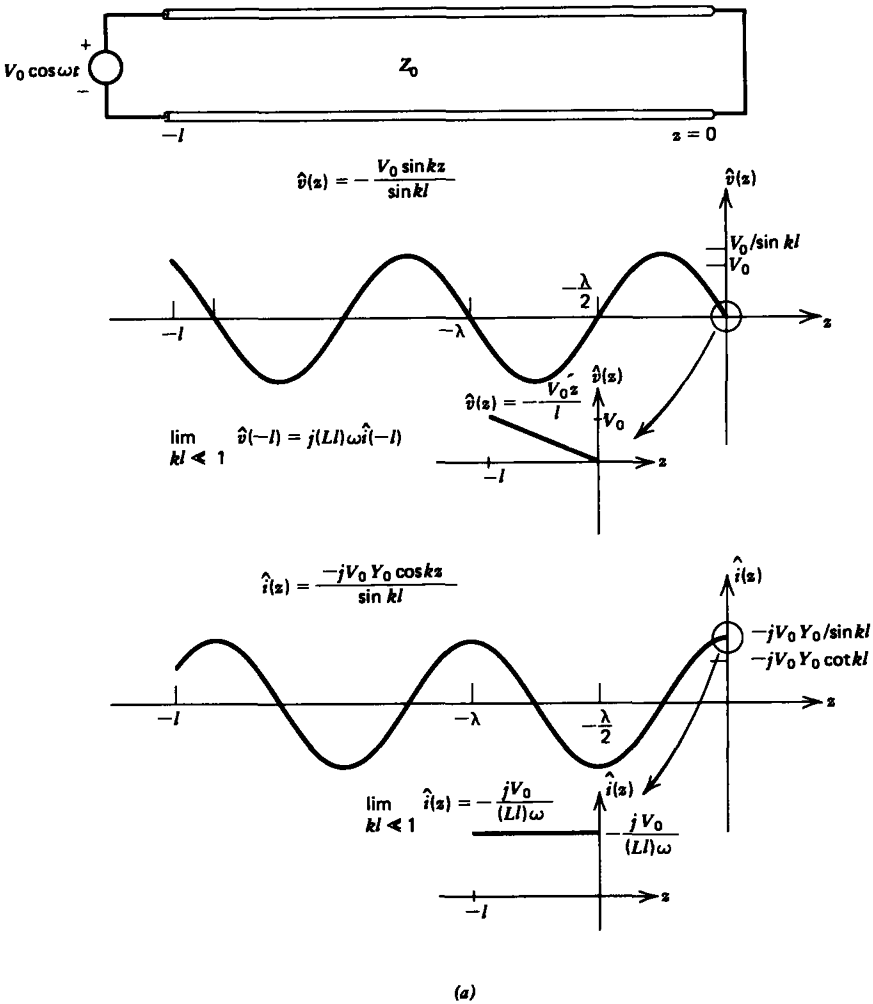
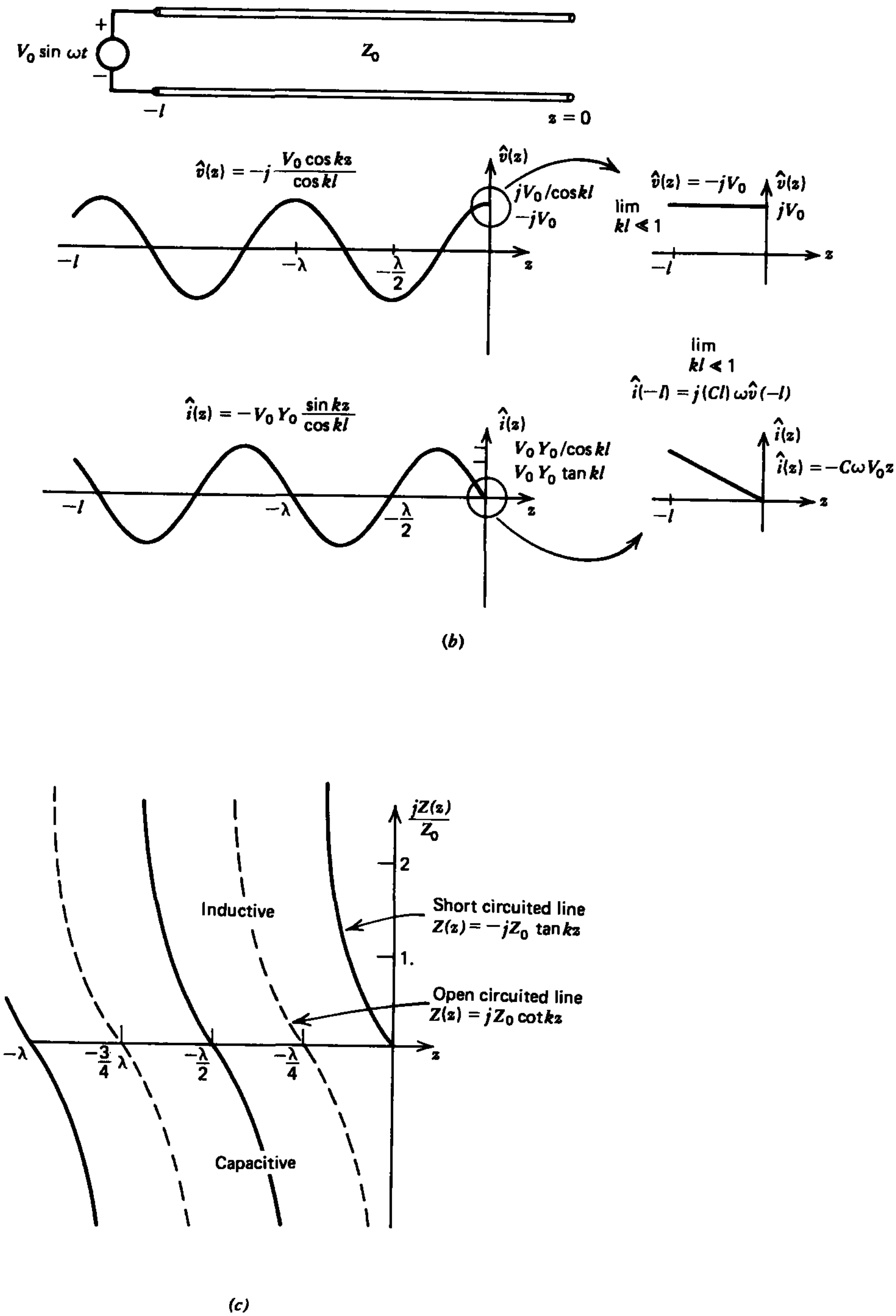


Figure 8-15 The voltage and current distributions on a (*a*) short circuited and (*b*) open circuited transmission line excited by sinusoidal voltage sources at $z=-l$. If the lines are much shorter than a wavelength, they act like reactive circuit elements. (*c*) As the frequency is raised, the impedance reflected back as a function of z can look capacitive or inductive making the transition through open or short circuits.

The spatial distributions of voltage and current as a function of z at a specific instant of time are plotted in Figure 8-15*a* and are seen to be 90° out of phase with one another in space with their distributions periodic with wavelength λ given by

$$\lambda = \frac{2\pi}{k} = \frac{2\pi c}{\omega} \tag{10}$$



$V_0 \sin \omega t$
+
−
Z_0
$-l$
$z = 0$
$\hat{v}(z) = -j\frac{V_0 \cos kz}{\cos kl}$
$\hat{v}(z)$
$jV_0/\cos kl$
$-jV_0$
$-l$
$-\lambda$
$-\frac{\lambda}{2}$
z
lim
$kl \ll 1$
$\hat{v}(z) = -jV_0$
$\hat{v}(z)$
jV_0
$\hat{i}(-l) = j(Cl)\omega\hat{v}(-l)$
$\hat{i}(z) = -V_0 Y_0 \frac{\sin kz}{\cos kl}$
$\hat{i}(z)$
$V_0 Y_0/\cos kl$
$V_0 Y_0 \tan kl$
$\hat{i}(z) = -C\omega V_0 z$
(b)
$\frac{jZ(z)}{Z_0}$
2
1.
Inductive
Capacitive
Short circuited line
$Z(z) = -jZ_0 \tan kz$
Open circuited line
$Z(z) = jZ_0 \cot kz$
$-\lambda$
$-\frac{3}{4}\lambda$
$-\frac{\lambda}{2}$
$-\frac{\lambda}{4}$
(c)

Figure 8-15

The complex impedance at any position z is defined as

$$Z(z)=\frac{\hat{v}(z)}{\hat{i}(z)} \tag{11}$$

which for this special case of a short circuited line is found from (8) as

$$Z(z)=-jZ_0 \tan kz \tag{12}$$

In particular, at $z=-l$, the transmission line appears to the generator as an impedance of value

$$Z(z=-l)=jZ_0 \tan kl \tag{13}$$

From the solid lines in Figure 8-15c we see that there are various regimes of interest:

(i) When the line is an integer multiple of a half wavelength long so that $kl=n\pi$, $n=1,2,3,\ldots$, the impedance at $z=-l$ is zero and the transmission line looks like a short circuit.
(ii) When the line is an odd integer multiple of a quarter wavelength long so that $kl=(2n-1)\pi/2$, $n=1,2,\ldots$, the impedance at $z=-l$ is infinite and the transmission line looks like an open circuit.
(iii) Between the short and open circuit limits $(n-1)\pi<kl<(2n-1)\pi/2$, $n=1,2,3,\ldots$, $Z(z=-l)$ has a positive reactance and hence looks like an inductor.
(iv) Between the open and short circuit limits $(n-\frac{1}{2})\pi<kl<n\pi$, $n=1,2,\ldots$, $Z(z=-l)$ has a negative reactance and so looks like a capacitor.

Thus, the short circuited transmission line takes on all reactive values, both positive (inductive) and negative (capacitive), including open and short circuits as a function of kl. Thus, if either the length of the line l or the frequency is changed, the impedance of the transmission line is changed.

Examining (8) we also notice that if $\sin kl=0$, $(kl=n\pi, n=1,2,\ldots)$, the voltage and current become infinite (in practice the voltage and current become large limited only by losses). Under these conditions, the system is said to be resonant with the resonant frequencies given by

$$\omega_n=n\pi c/l, \qquad n=1,2,3,\ldots \tag{14}$$

Any voltage source applied at these frequencies will result in very large voltages and currents on the line.

(b) Open Circuited Line

If the short circuit is replaced by an open circuit, as in Figure 8-15b, and for variety we change the source at $z=-l$ to

$V_0 \sin \omega t$ the boundary conditions are

$$\begin{aligned} i(z=0,t)&=0 \\ v(z=-l,t)&=V_0 \sin \omega t = \mathrm{Re}\,(-jV_0\, e^{j\omega t}) \end{aligned} \tag{15}$$

Using (3) the complex amplitudes obey the relations

$$\begin{aligned} \hat{i}(z=0)&=0=Y_0(\hat{V}_+-\hat{V}_-) \\ \hat{v}(z=-l)&=-jV_0=\hat{V}_+\, e^{jkl}+\hat{V}_-\, e^{-jkl} \end{aligned} \tag{16}$$

which has solutions

$$\hat{V}_+=\hat{V}_-=\frac{-jV_0}{2\cos kl} \tag{17}$$

The spatial dependences of the voltage and current are then

$$\begin{aligned} \hat{v}(z)&=\frac{-jV_0}{2\cos kl}(e^{-jkz}+e^{jkz})=\frac{-jV_0}{\cos kl}\cos kz \\ \hat{i}(z)&=\frac{-jV_0Y_0}{2\cos kl}(e^{-jkz}-e^{+jkz})=-\frac{V_0Y_0}{\cos kl}\sin kz \end{aligned} \tag{18}$$

with instantaneous solutions as a function of space and time:

$$\begin{aligned} v(z,t)&=\mathrm{Re}\,[\hat{v}(z)\, e^{j\omega t}]=\frac{V_0\cos kz}{\cos kl}\sin \omega t \\ i(z,t)&=\mathrm{Re}\,[\hat{i}(z)\, e^{j\omega t}]=-\frac{V_0Y_0}{\cos kl}\sin kz\cos \omega t \end{aligned} \tag{19}$$

The impedance at $z=-l$ is

$$Z(z=-l)=\frac{\hat{v}(-l)}{\hat{i}(-l)}=-jZ_0\cot kl \tag{20}$$

Again the impedance is purely reactive, as shown by the dashed lines in Figure 8-15c, alternating signs every quarter wavelength so that the open circuit load looks to the voltage source as an inductor, capacitor, short or open circuit depending on the frequency and length of the line.

Resonance will occur if

$$\cos kl=0 \tag{21}$$

or

$$kl=(2n-1)\pi/2, \qquad n=1,2,3,\ldots \tag{22}$$

so that the resonant frequencies are

$$\omega_n=\frac{(2n-1)\pi c}{2l} \tag{23}$$

8-3-3 Reactive Circuit Elements as Approximations to Short Transmission Lines

Let us re-examine the results obtained for short and open circuited lines in the limit when l is much shorter than the wavelength λ so that in this long wavelength limit the spatial trigonometric functions can be approximated as

$$\lim_{kl\ll 1}\begin{cases}\sin kz \approx kz\\ \cos kz \approx 1\end{cases} \tag{24}$$

Using these approximations, the voltage, current, and impedance for the short circuited line excited by a voltage source $V_0 \cos \omega t$ can be obtained from (9) and (13) as

$$\lim_{kl\ll 1}\begin{cases} v(z,t) = -\dfrac{V_0 z}{l}\cos \omega t, & v(-l,t) = V_0 \cos \omega t\\ i(z,t) = \dfrac{V_0 Y_0}{kl}\sin \omega t, & i(-l,t) = \dfrac{V_0 \sin \omega t}{(Ll)\omega}\\ Z(-l) = jZ_0 kl = j\dfrac{\omega Z_0 l}{c} = j\omega(Ll) \end{cases} \tag{25}$$

We see that the short circuited transmission line acts as an inductor of value (Ll) (remember that L is the inductance per unit length), where we used the relations

$$Z_0 = \frac{1}{Y_0} = \sqrt{\frac{L}{C}}, \qquad c = \frac{1}{\sqrt{LC}} \tag{26}$$

Note that at $z=-l$,

$$v(-l,t) = (Ll)\frac{di(-l,t)}{dt} \tag{27}$$

Similarly for the open circuited line we obtain:

$$\lim_{kl\ll 1}\begin{cases} v(z,t) = V_0 \sin \omega t\\ i(z,t) = -V_0 Y_0 kz \cos \omega t, & i(-l,t) = (Cl)\omega V_0 \cos \omega t\\ Z(-l) = \dfrac{-jZ_0}{kl} = \dfrac{-j}{(Cl)\omega} \end{cases} \tag{28}$$

For the open circuited transmission line, the terminal voltage and current are simply related as for a capacitor,

$$i(-l,t) = (Cl)\frac{dv(-l,t)}{dt} \tag{29}$$

with capacitance given by (Cl).

In general, if the frequency of excitation is low enough so that the length of a transmission line is much shorter than the wavelength, the circuit approximations of inductance and capacitance are appropriate. However, it must be remembered that if the frequencies of interest are so high that the length of a circuit element is comparable to the wavelength, it no longer acts like that element. In fact, as found in Section 8-3-2, a capacitor can even look like an inductor, a short circuit, or an open circuit at high enough frequency while vice versa an inductor can also look capacitive, a short or an open circuit.

In general, if the termination is neither a short nor an open circuit, the voltage and current distribution becomes more involved to calculate and is the subject of Section 8-4.

8-3-4 Effects of Line Losses

(a) Distributed Circuit Approach

If the dielectric and transmission line walls have Ohmic losses, the voltage and current waves decay as they propagate. Because the governing equations of Section 8-1-3 are linear with constant coefficients, in the sinusoidal steady state we assume solutions of the form

$$\begin{aligned} v(z,t) &= \mathrm{Re}\,(\hat{V}\,e^{j(\omega t-kz)}) \\ i(z,t) &= \mathrm{Re}\,(\hat{I}\,e^{j(\omega t-kz)}) \end{aligned} \tag{30}$$

where now ω and k are not simply related as the nondispersive relation in (4). Rather we substitute (30) into Eq. (28) in Section 8-1-3:

$$\begin{aligned} \frac{\partial i}{\partial z} &= -C\frac{\partial v}{\partial t} - Gv \Rightarrow -jk\hat{I} = -(Cj\omega + G)\hat{V} \\ \frac{\partial v}{\partial z} &= -L\frac{\partial i}{\partial t} - iR \Rightarrow -jk\hat{V} = -(Lj\omega + R)\hat{I} \end{aligned} \tag{31}$$

which requires that

$$\frac{\hat{V}}{\hat{I}} = \frac{jk}{(Cj\omega + G)} = \frac{Lj\omega + R}{jk} \tag{32}$$

We solve (32) self-consistently for k as

$$k^2 = -(Lj\omega + R)(Cj\omega + G) = LC\omega^2 - j\omega(RC + LG) - RG \tag{33}$$

The wavenumber is thus complex so that we find the real and imaginary parts from (33) as

$$\begin{aligned} k = k_r + jk_i \Rightarrow k_r^2 - k_i^2 &= LC\omega^2 - RG \\ 2k_rk_i &= -\omega(RC + LG) \end{aligned} \tag{34}$$

In the low loss limit where $\omega RC \ll 1$ and $\omega LG \ll 1$, the spatial decay of k_i is small compared to the propagation wavenumber k_r. In this limit we have the following approximate solution:

$$\lim_{\substack{\omega RC \ll 1 \\ \omega LG \ll 1}} \begin{cases} k_r \approx \pm\omega\sqrt{LC} = \pm\omega/c \\ k_i = -\dfrac{\omega(RC+LG)}{2k_r} \approx \mp\dfrac{1}{2}\left[R\sqrt{\dfrac{C}{L}} + G\sqrt{\dfrac{L}{C}}\right] \\ \qquad \approx \mp\frac{1}{2}(RY_0 + GZ_0) \end{cases} \tag{35}$$

We use the upper sign for waves propagating in the $+z$ direction and the lower sign for waves traveling in the $-z$ direction.

(b) Distortionless lines

Using the value of k of (33),

$$k = \pm[-(Lj\omega + R)(Cj\omega + G)]^{1/2} \tag{36}$$

in (32) gives us the frequency dependent wave impedance for waves traveling in the $\pm z$ direction as

$$\frac{\hat{V}}{\hat{I}} = \pm\left(\frac{Lj\omega + R}{Cj\omega + G}\right)^{1/2} = \pm\sqrt{\frac{L}{C}}\left(\frac{j\omega + R/L}{j\omega + G/C}\right)^{1/2} \tag{37}$$

If the line parameters are adjusted so that

$$\frac{R}{L} = \frac{G}{C} \tag{38}$$

the impedance in (37) becomes frequency independent and equal to the lossless line impedance. Under the conditions of (38) the complex wavenumber reduces to

$$k_r = \pm\omega\sqrt{LC}, \qquad k_i = \mp\sqrt{RG} \tag{39}$$

Although the waves are attenuated, all frequencies propagate at the same phase and group velocities as for a lossless line

$$\begin{aligned} v_p &= \frac{\omega}{k_r} = \pm\frac{1}{\sqrt{LC}} \\ v_g &= \frac{d\omega}{dk_r} = \pm\frac{1}{\sqrt{LC}} \end{aligned} \tag{40}$$

Since all the Fourier components of a pulse excitation will travel at the same speed, the shape of the pulse remains unchanged as it propagates down the line. Such lines are called distortionless.

(c) Fields Approach

If $R=0$, we can directly find the TEM wave solutions using the same solutions found for plane waves in Section 7-4-3. There we found that a dielectric with permittivity ε and small Ohmic conductivity σ has a complex wavenumber:

$$\lim_{\sigma/\omega\varepsilon \ll 1} k \approx \pm\left(\frac{\omega}{c} - \frac{j\sigma\eta}{2}\right) \tag{41}$$

Equating (41) to (35) with $R=0$ requires that $GZ_0 = \sigma\eta$.

The tangential component of **H** at the perfectly conducting transmission line walls is discontinuous by a surface current. However, if the wall has a large but noninfinite Ohmic conductivity σ_w, the fields penetrate in with a characteristic distance equal to the skin depth $\delta = \sqrt{2/\omega\mu\sigma_w}$. The resulting z-directed current gives rise to a z-directed electric field so that the waves are no longer purely TEM.

Because we assume this loss to be small, we can use an approximate perturbation method to find the spatial decay rate of the fields. We assume that the fields between parallel plane electrodes are essentially the same as when the system is lossless except now being exponentially attenuated as $e^{-\alpha z}$, where $\alpha = -k_i$:

$$\begin{aligned} E_x(z,t) &= \text{Re}\,[\hat{E}\,e^{j(\omega t - k_r z)}\,e^{-\alpha z}] \\ H_y(z,t) &= \text{Re}\left[\frac{\hat{E}}{\eta}\,e^{j(\omega t - k_r z)}\,e^{-\alpha z}\right], \qquad k_r = \frac{\omega}{c} \end{aligned} \tag{42}$$

From the real part of the complex Poynting's theorem derived in Section 7-2-4, we relate the divergence of the time-average electromagnetic power density to the time-average dissipated power:

$$\nabla\cdot \langle \mathbf{S}\rangle = -\langle P_d\rangle \tag{43}$$

Using the divergence theorem we integrate (43) over a volume of thickness Δz that encompasses the entire width and thickness of the line, as shown in Figure 8-16:

$$\begin{aligned} \int_V \nabla\cdot\langle \mathbf{S}\rangle\, dV &= \oint_S \langle \mathbf{S}\rangle\cdot d\mathbf{S} \\ &= \int_{z+\Delta z} \langle S_z(z+\Delta z)\rangle\, dS \\ &\quad - \int_z \langle S_z(z)\rangle\, dS = -\int_V \langle P_d\rangle\, dV \end{aligned} \tag{44}$$

The power $\langle P_d\rangle$ is dissipated in the dielectric and in the walls. Defining the total electromagnetic power as

$$\langle P(z)\rangle = \int_z \langle S_z(z)\rangle\, dS \tag{45}$$

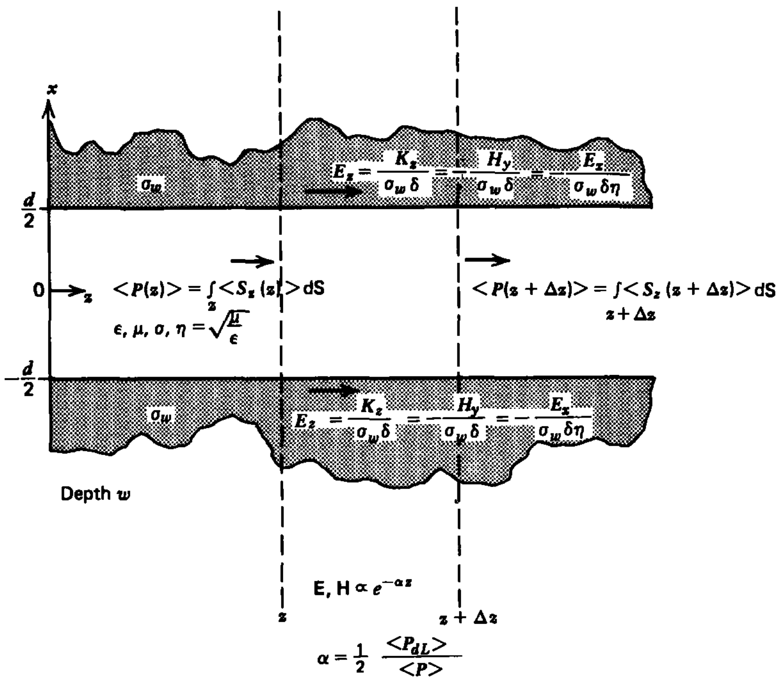


Figure 8-16 A transmission line with lossy walls and dielectric results in waves that decay as they propagate. The spatial decay rate α of the fields is approximately proportional to the ratio of time average dissipated power per unit length $<P_{dL}>$ to the total time average electromagnetic power flow $<P>$ down the line.

(44) can be rewritten as

$$<P(z+\Delta z)>-<P(z)>=-\int <P_d>\, dx\, dy\, dz \tag{46}$$

Dividing through by $dz = \Delta z$, we have in the infinitesimal limit

$$\lim_{\Delta z\to 0}\frac{<P(z+\Delta z)>-<P(z)>}{\Delta z}=\frac{d<P>}{dz}=-\int_S <P_d>\, dx\, dy$$
$$=-<P_{dL}> \tag{47}$$

where $<P_{dL}>$ is the power dissipated per unit length. Since the fields vary as $e^{-\alpha z}$, the power flow that is proportional to the square of the fields must vary as $e^{-2\alpha z}$ so that

$$\frac{d<P>}{dz}=-2\alpha<P>=-<P_{dL}> \tag{48}$$

which when solved for the spatial decay rate is proportional to the ratio of dissipated power per unit length to the total electromagnetic power flowing down the transmission line:

$$\alpha = \frac{1}{2}\frac{<P_{dL}>}{<P>} \tag{49}$$

For our lossy transmission line, the power is dissipated both in the walls and in the dielectric. Fortunately, it is not necessary to solve the complicated field problem within the walls because we already approximately know the magnetic field at the walls from (42). Since the wall current is effectively confined to the skin depth δ, the cross-sectional area through which the current flows is essentially $w\delta$ so that we can define the surface conductivity as $\sigma_w\delta$, where the electric field at the wall is related to the lossless surface current as

$$\mathbf{K}_w = \sigma_w \delta \mathbf{E}_w \tag{50}$$

The surface current in the wall is approximately found from the magnetic field in (42) as

$$K_z = -H_y = -E_x/\eta \tag{51}$$

The time-average power dissipated in the wall is then

$$<P_{dL}>_{\text{wall}} = \frac{w}{2}\text{Re}\,(\mathbf{E}_w \cdot \mathbf{K}_w^*) = \frac{1}{2}\frac{|\mathbf{K}_w|^2 w}{\sigma_w \delta} = \frac{1}{2}\frac{|\hat{E}|^2 w}{\sigma_w \delta \eta^2} \tag{52}$$

The total time-average dissipated power in the walls and dielectric per unit length for a transmission line system of depth w and plate spacing d is then

$$\begin{aligned} <P_{dL}> &= 2<P_{dL}>_{\text{wall}} + \tfrac{1}{2}\sigma|\hat{E}|^2 wd \\ &= \tfrac{1}{2}|\hat{E}|^2 w\left(\frac{2}{\eta^2\sigma_w\delta} + \sigma d\right) \end{aligned} \tag{53}$$

where we multiply (52) by two because of the losses in both electrodes. The time-average electromagnetic power is

$$<P> = \frac{1}{2}\frac{|\hat{E}|^2}{\eta} wd \tag{54}$$

so that the spatial decay rate is found from (49) as

$$\alpha = -k_i = \frac{1}{2}\left(\frac{2}{\eta^2\sigma_w\delta} + \sigma d\right)\frac{\eta}{d} = \frac{1}{2}\left(\sigma\eta + \frac{2}{\eta\sigma_w\delta d}\right) \tag{55}$$

Comparing (55) to (35) we see that

$$GZ_0 = \sigma\eta, \qquad RY_0 = \frac{2}{\eta\sigma_w\delta d}$$

$$\Rightarrow Z_0 = \frac{1}{Y_0} = \frac{d}{w}\eta, \qquad G = \frac{\sigma w}{d}, \qquad R = \frac{2}{\sigma_w w\delta} \tag{56}$$

8-4 ARBITRARY IMPEDANCE TERMINATIONS

8-4-1 The Generalized Reflection Coefficient

A lossless transmission line excited at $z=-l$ with a sinusoidal voltage source is now terminated at its other end at $z=0$ with an arbitrary impedance Z_L, which in general can be a complex number. Defining the load voltage and current at $z=0$ as

$$\begin{aligned} v(z=0,t) &= v_L(t) = \text{Re}\,(\text{V}_L\, e^{j\omega t}) \\ i(z=0,t) &= i_L(t) = \text{Re}\,(I_L\, e^{j\omega t}), \qquad I_L = \text{V}_L/Z_L \end{aligned} \tag{1}$$

where V_L and I_L are complex amplitudes, the boundary conditions at $z=0$ are

$$\begin{aligned} \text{V}_+ + \text{V}_- &= \text{V}_L \\ Y_0(\text{V}_+ - \text{V}_-) &= I_L = \text{V}_L/Z_L \end{aligned} \tag{2}$$

We define the reflection coefficient as the ratio

$$\Gamma_L = \text{V}_-/\text{V}_+ \tag{3}$$

and solve as

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{4}$$

Here in the sinusoidal steady state with reactive loads, Γ_L can be a complex number as Z_L may be complex. For transient pulse waveforms, Γ_L was only defined for resistive loads. For capacitative and inductive terminations, the reflections were given by solutions to differential equations in time. Now that we are only considering sinusoidal time variations so that time derivatives are replaced by $j\omega$, we can generalize Γ_L for the sinusoidal steady state.

It is convenient to further define the generalized reflection coefficient as

$$\Gamma(z) = \frac{\text{V}_-\, e^{jkz}}{\text{V}_+\, e^{-jkz}} = \frac{\text{V}_-}{\text{V}_+}\, e^{2jkz} = \Gamma_L\, e^{2jkz} \tag{5}$$

where Γ_L is just $\Gamma(z=0)$. Then the voltage and current on the line can be expressed as

$$\begin{aligned} \hat{v}(z) &= \text{V}_+\, e^{-jkz}[1+\Gamma(z)] \\ \hat{\imath}(z) &= Y_0 \text{V}_+\, e^{-jkz}[1-\Gamma(z)] \end{aligned} \tag{6}$$

The advantage to this notation is that now the impedance along the line can be expressed as

$$Z_n(z) = \frac{Z(z)}{Z_0} = \frac{\hat{v}(z)}{\hat{\imath}(z) Z_0} = \frac{1+\Gamma(z)}{1-\Gamma(z)} \tag{7}$$

where Z_n is defined as the normalized impedance. We can now solve (7) for $\Gamma(z)$ as

$$\Gamma(z)=\frac{Z_n(z)-1}{Z_n(z)+1} \tag{8}$$

Note the following properties of $Z_n(z)$ and $\Gamma(z)$ for passive loads:

(i) $Z_n(z)$ is generally complex. For passive loads its real part is allowed over the range from zero to infinity while its imaginary part can extend from negative to positive infinity.
(ii) The magnitude of $\Gamma(z)$, $|\Gamma_L|$ must be less than or equal to 1 for passive loads.
(iii) From (5), if z is increased or decreased by a half wavelength, $\Gamma(z)$ and hence $Z_n(z)$ remain unchanged. Thus, if the impedance is known at any position, the impedance of all-points integer multiples of a half wavelength away have the same impedance.
(iv) From (5), if z is increased or decreased by a quarter wavelength, $\Gamma(z)$ changes sign, while from (7) $Z_n(z)$ goes to its reciprocal $\Rightarrow 1/Z_n(z)=Y_n(z)$.
(v) If the line is matched, $Z_L=Z_0$, then $\Gamma_L=0$ and $Z_n(z)=1$. The impedance is the same everywhere along the line.

8-4-2 Simple Examples

(a) Load Impedance Reflected Back to the Source

Properties (iii)–(v) allow simple computations for transmission line systems that have lengths which are integer multiples of quarter or half wavelengths. Often it is desired to maximize the power delivered to a load at the end of a transmission line by adding a lumped admittance Y across the line. For the system shown in Figure 8-17a, the impedance of the load is reflected back to the generator and then added in parallel to the lumped reactive admittance Y. The normalized load impedance of $(R_L+jX_L)/Z_0$ inverts when reflected back to the source by a quarter wavelength to $Z_0/(R_L+jX_L)$. Since this is the normalized impedance the actual impedance is found by multiplying by Z_0 to yield $Z(z=-\lambda/4)=Z_0^2/(R_L+jX_L)$. The admittance of this reflected load then adds in parallel to Y to yield a total admittance of $Y+(R_L+jX_L)/Z_0^2$. If Y is pure imaginary and of opposite sign to the reflected load susceptance with value $-jX_L/Z_0^2$, maximum power is delivered to the line if the source resistance R_S also equals the resulting line input impedance, $R_S=Z_0^2/R_L$. Since Y is purely

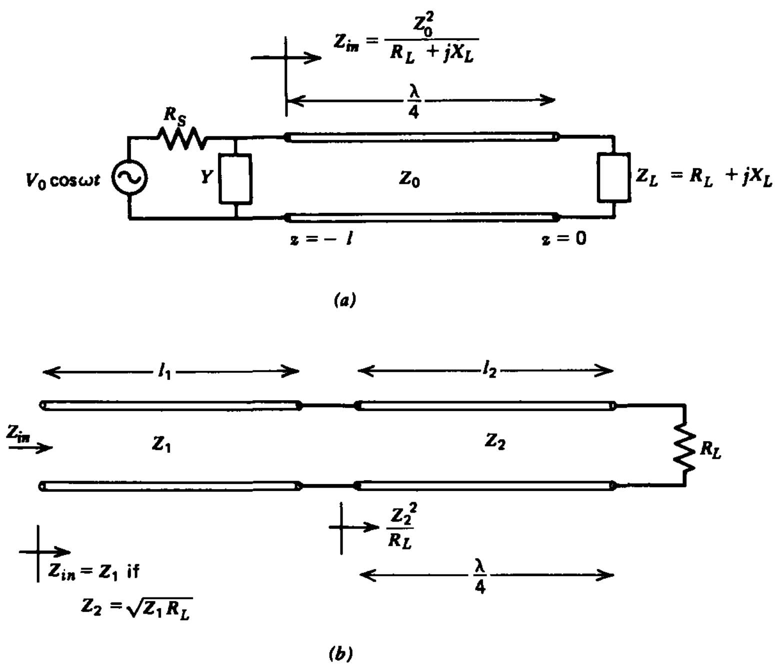


Figure 8-17 The normalized impedance reflected back through a quarter-wave-long line inverts. (*a*) The time-average power delivered to a complex load can be maximized if Y is adjusted to just cancel the reactive admittance of the load reflected back to the source with R_s equaling the resulting input resistance. (*b*) If the length l_2 of the second transmission line shown is a quarter wave long or an odd integer multiple of $\lambda/4$ and its characteristic impedance is equal to the geometric average of Z_1 and R_L, the input impedance Z_{in} is matched to Z_1.

reactive and the transmission line is lossless, half the time-average power delivered by the source is dissipated in the load:

$$<P> = \frac{1}{8}\frac{V_0^2}{R_S} = \frac{1}{8}\frac{R_L V_0^2}{Z_0^2} \tag{9}$$

Such a reactive element Y is usually made from a variable length short circuited transmission line called a stub. As shown in Section 8-3-2*a*, a short circuited lossless line always has a pure reactive impedance.

To verify that the power in (9) is actually dissipated in the load, we write the spatial distribution of voltage and current along the line as

$$\begin{aligned} \hat{v}(z) &= V_+ e^{-jkz}(1 + \Gamma_L e^{2jkz}) \\ \hat{i}(z) &= Y_0 V_+ e^{-jkz}(1 - \Gamma_L e^{2jkz}) \end{aligned} \tag{10}$$

where the reflection coefficient for this load is given by (4) as

$$\Gamma_L = \frac{R_L + jX_L - Z_0}{R_L + jX_L + Z_0} \tag{11}$$

At $z = -l = -\lambda/4$ we have the boundary condition

$$\begin{aligned} \hat{v}(z=-l) = V_0/2 &= \mathrm{V}_+ e^{jkl}(1 + \Gamma_L e^{-2jkl}) \\ &= j\mathrm{V}_+(1-\Gamma_L) \end{aligned} \tag{12}$$

which allows us to solve for V_+ as

$$\mathrm{V}_+ = \frac{-jV_0}{2(1-\Gamma_L)} = \frac{-jV_0}{4Z_0}(R_L + jX_L + Z_0) \tag{13}$$

The time-average power dissipated in the load is then

$$\begin{aligned} \langle P_L \rangle &= \tfrac{1}{2}\,\mathrm{Re}\,[\hat{v}(z=0)\hat{\imath}^*(z=0)] \\ &= \tfrac{1}{2}|\hat{\imath}(z=0)|^2 R_L \\ &= \tfrac{1}{2}|\mathrm{V}_+|^2|1-\Gamma_L|^2 Y_0^2 R_L \\ &= \tfrac{1}{8} V_0^2 Y_0^2 R_L \end{aligned} \tag{14}$$

which agrees with (9).

(b) Quarter Wavelength Matching

It is desired to match the load resistor R_L to the transmission line with characteristic impedance Z_1 for any value of its length l_1. As shown in Figure 8-17*b*, we connect the load to Z_1 via another transmission line with characteristic impedance Z_2. We wish to find the values of Z_2 and l_2 necessary to match R_L to Z_1.

This problem is analogous to the dielectric coating problem of Example 7-1, where it was found that reflections could be eliminated if the coating thickness between two different dielectric media was an odd integer multiple of a quarter wavelength and whose wave impedance was equal to the geometric average of the impedance in each adjacent region. The normalized load on Z_2 is then $Z_{n2} = R_L/Z_2$. If l_2 is an odd integer multiple of a quarter wavelength long, the normalized impedance Z_{n2} reflected back to the first line inverts to Z_2/R_L. The actual impedance is obtained by multiplying this normalized impedance by Z_2 to give Z_2^2/R_L. For Z_{in} to be matched to Z_1 for any value of l_1, this impedance must be matched to Z_1:

$$Z_1 = Z_2^2/R_L \Rightarrow Z_2 = \sqrt{Z_1 R_L} \tag{15}$$

8-4-3 The Smith Chart

Because the range of allowed values of Γ_L must be contained within a unit circle in the complex plane, all values of $Z_n(z)$ can be mapped by a transformation within this unit circle using (8). This transformation is what makes the substitutions of (3)–(8) so valuable. A graphical aid of this mathematical transformation was developed by P. H. Smith in 1939 and is known as the Smith chart. Using the Smith chart avoids the tedium in problem solving with complex numbers.

Let us define the real and imaginary parts of the normalized impedance at some value of z as

$$Z_n(z) = r + jx \tag{16}$$

The reflection coefficient similarly has real and imaginary parts given as

$$\Gamma(z) = \Gamma_r + j\Gamma_i \tag{17}$$

Using (7) we have

$$r + jx = \frac{1 + \Gamma_r + j\Gamma_i}{1 - \Gamma_r - j\Gamma_i} \tag{18}$$

Multiplying numerator and denominator by the complex conjugate of the denominator $(1 - \Gamma_r + j\Gamma_i)$ and separating real and imaginary parts yields

$$\begin{aligned} r &= \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1 - \Gamma_r)^2 + \Gamma_i^2} \\ x &= \frac{2\Gamma_i}{(1 - \Gamma_r)^2 + \Gamma_i^2} \end{aligned} \tag{19}$$

Since we wish to plot (19) in the $\Gamma_r - \Gamma_i$ plane we rewrite these equations as

$$\begin{aligned} \left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 &= \frac{1}{(1+r)^2} \\ (\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 &= \frac{1}{x^2} \end{aligned} \tag{20}$$

Both equations in (20) describe a family of orthogonal circles. The upper equation is that of a circle of radius $1/(1+r)$ whose center is at the position $\Gamma_i = 0$, $\Gamma_r = r/(1+r)$. The lower equation is a circle of radius $|1/x|$ centered at the position $\Gamma_r = 1$, $\Gamma_i = 1/x$. Figure 8-18*a* illustrates these circles for a particular value of r and x, while Figure 8-18*b* shows a few representative values of r and x. In Figure 8-19, we have a complete Smith chart. Only those parts of the circles that lie within the unit circle in the Γ plane are considered for passive

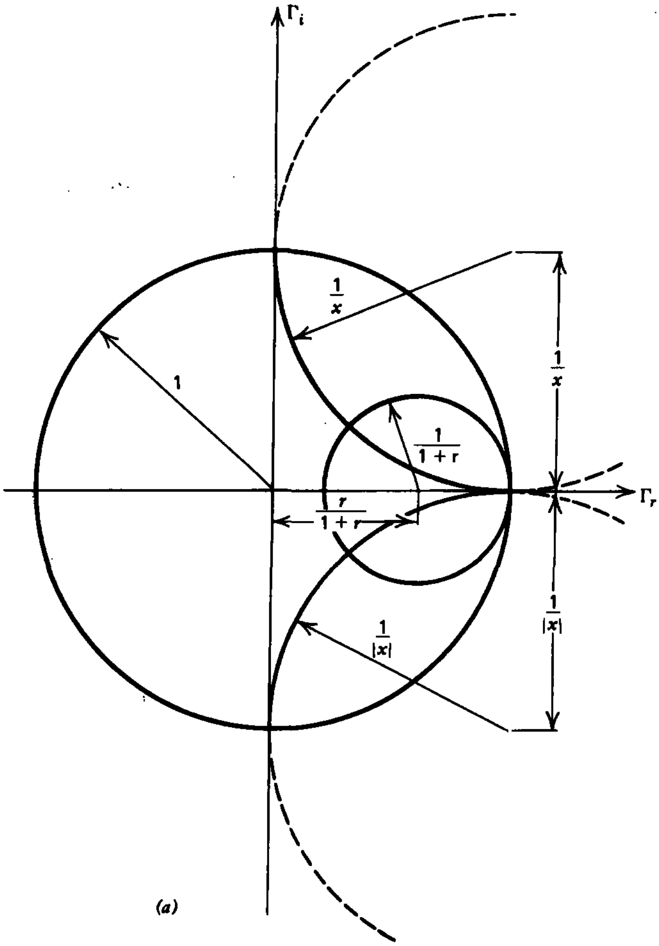
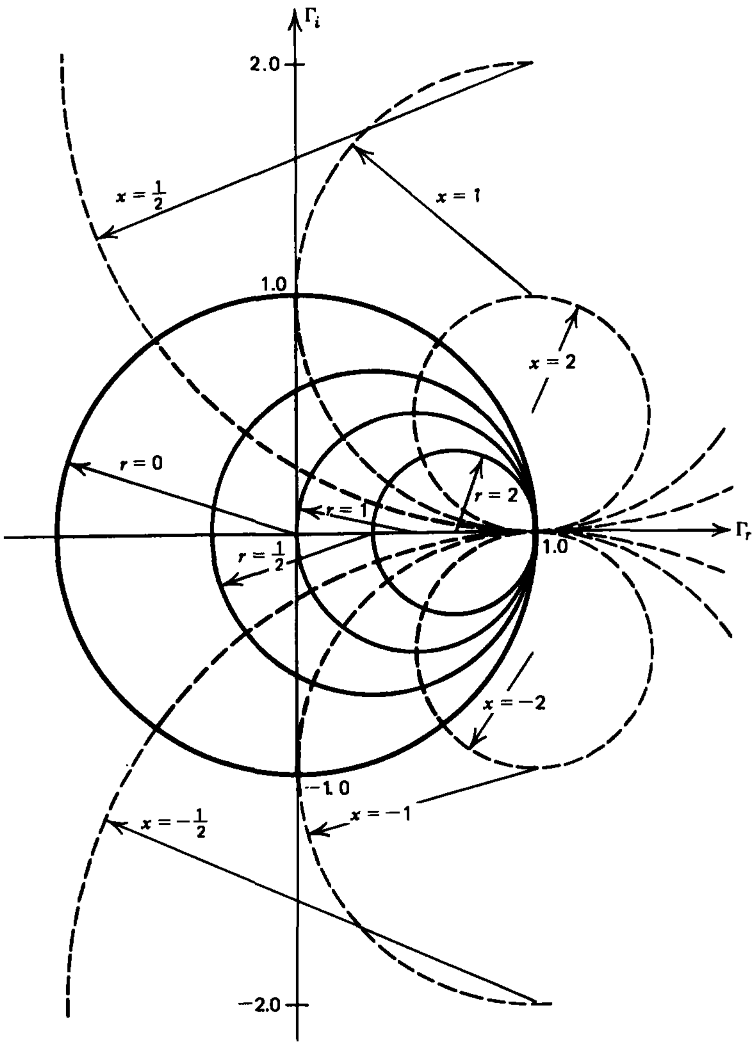


Figure 8-18 For passive loads the Smith chart is constructed within the unit circle in the complex Γ plane. (*a*) Circles of constant normalized resistance r and reactance x are constructed with the centers and radii shown. (*b*) Smith chart construction for various values of r and x.

resistive-reactive loads. The values of $\Gamma(z)$ themselves are usually not important and so are not listed, though they can be easily found from (8). Note that all circles pass through the point $\Gamma_r = 1$, $\Gamma_i = 0$.

The outside of the circle is calibrated in wavelengths toward the generator, so if the impedance is known at any point on the transmission line (usually at the load end), the impedance at any other point on the line can be found using just a compass and a ruler. From the definition of $\Gamma(z)$ in (5) with z negative, we move clockwise around the Smith chart when heading towards the source and counterclockwise when moving towards the load.



(b)

Figure 8-18

In particular, consider the transmission line system in Figure 8-20*a*. The normalized load impedance is $Z_n = 1 + j$. Using the Smith chart in Figure 8-20*b*, we find the load impedance at position *A*. The effective impedance reflected back to $z = -l$ must lie on the circle of constant radius returning to *A* whenever *l* is an integer multiple of a half wavelength. The table in Figure 8-20 lists the impedance at $z = -l$ for various line lengths. Note that at point *C*, where $l = \lambda/4$, that the normalized impedance is the reciprocal of

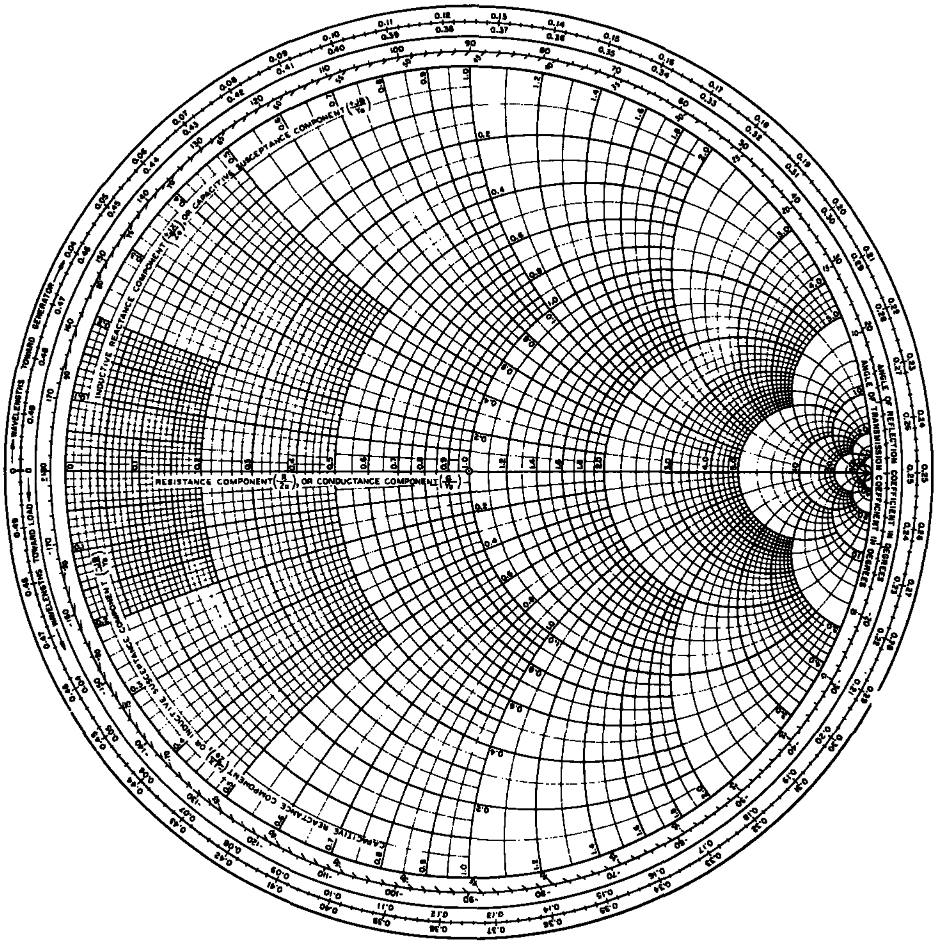


Figure 8-19 A complete Smith chart.

that at A. Similarly the normalized impedance at B is the reciprocal of that at D.

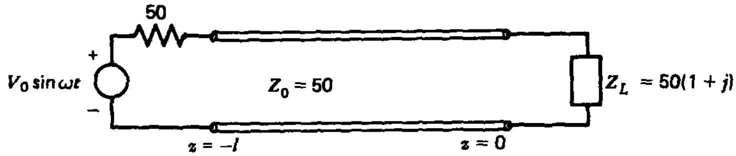
The current from the voltage source is found using the equivalent circuit shown in Figure 8-20c as

$$i = |\hat{I}| \sin(\omega t - \phi) \tag{21}$$

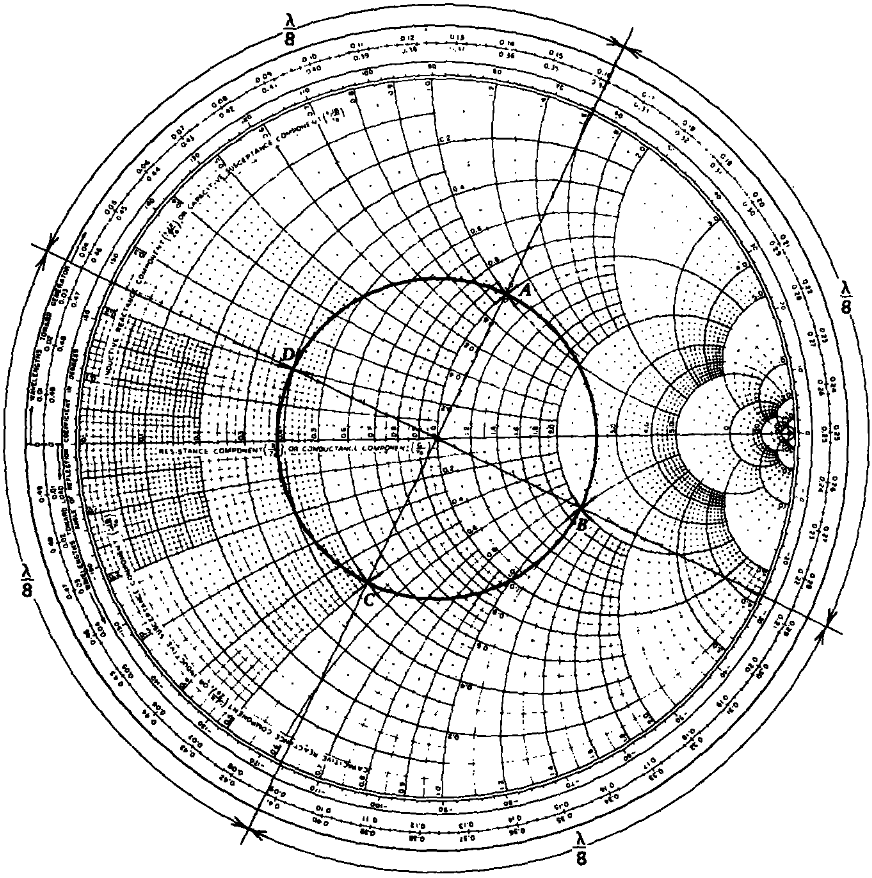
where the current magnitude and phase angle are

$$|\hat{I}| = \frac{V_0}{|50 + Z(z=-l)|}, \qquad \phi = \tan^{-1} \frac{\text{Im}\,[Z(z=-l)]}{50 + \text{Re}\,[Z(z=-l)]} \tag{22}$$

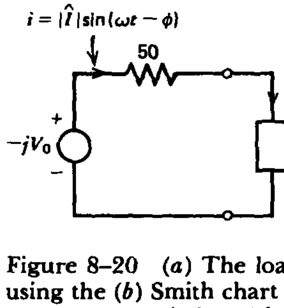
Representative numerical values are listed in Figure 8-20.



(*a*)



(*b*)



$$\hat{I} = \frac{-jV_0}{50 + Z(z = -l)}$$

Point	l	$Z_n(z = -l)$	$\lvert\hat{I}\rvert Z_0/V_0$	ϕ
A	$0, \lambda/2$	$1 + j$	0.447	26.6°
B	$\lambda/8$	$2 - j$	0.316	−18.4°
C	$\lambda/4$	$\frac{1}{2}(1 - j)$	0.632	−18.4°
D	$\frac{3}{8}\lambda$	$.4 + .2j$	0.707	8.1°

$$\phi = \tan^{-1}\left[\frac{\text{Im}(Z(z = -l))}{50 + \text{Re}(Z(z = -l))}\right]$$

(*c*)

Figure 8–20 (*a*) The load impedance at $z = 0$ reflected back to the source is found using the (*b*) Smith chart for various line lengths. Once this impedance is known the source current is found by solving the simple series circuit in (*c*).

8-4-4 Standing Wave Parameters

The impedance and reflection coefficient are not easily directly measured at microwave frequencies. In practice, one slides an ac voltmeter across a slotted transmission line and measures the magnitude of the peak or rms voltage and not its phase angle.

From (6) the magnitude of the voltage and current at any position z is

$$\begin{aligned} |\hat{v}(z)| &= |V_+|\,|1+\Gamma(z)| \\ |\hat{\imath}(z)| &= Y_0|V_+|\,|1-\Gamma(z)| \end{aligned} \tag{23}$$

From (23), the variations of the voltage and current magnitudes can be drawn by a simple construction in the Γ plane, as shown in Figure 8-21. Note again that $|V_+|$ is just a real number independent of z and that $|\Gamma(z)| \le 1$ for a passive termination. We plot $|1+\Gamma(z)|$ and $|1-\Gamma(z)|$ since these terms are proportional to the voltage and current magnitudes, respectively. The following properties from this con-

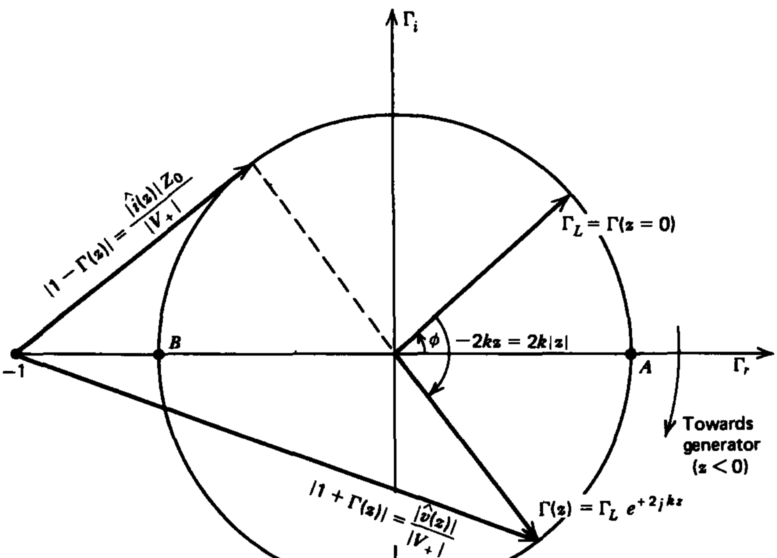


Figure 8-21 The voltage and current magnitudes along a transmission line are respectively proportional to the lengths of the vectors $|1+\Gamma(z)|$ and $|1-\Gamma(z)|$ in the complex Γ plane.

struction are apparent:

(i) The magnitude of the current is smallest and the voltage magnitude largest when $\Gamma(z) = 1$ at point A and vice versa when $\Gamma(z) = -1$ at point B.
(ii) The voltage and current are in phase at the points of maximum or minimum magnitude of either at points A or B.
(iii) A rotation of $\Gamma(z)$ by an angle π corresponds to a change of $\lambda/4$ in z, thus any voltage (or current) maximum is separated by $\lambda/4$ from its nearest minima on either side.

By plotting the lengths of the phasors $|1 \pm \Gamma(z)|$, as in Figure 8-22, we obtain a plot of what is called the standing wave pattern on the line. Observe that the curves are not sinusoidal. The minima are sharper than the maxima so the minima are usually located in position more precisely by measurement than the maxima.

From Figures 8-21 and 8-22, the ratio of the maximum voltage magnitude to the minimum voltage magnitude is defined as the voltage standing wave ratio, or VSWR for short:

$$\frac{|\hat{v}(z)|_{\max}}{|\hat{v}(z)|_{\min}} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} = \text{VSWR} \tag{24}$$

The VSWR is measured by simply recording the largest and smallest readings of a sliding voltmeter. Once the VSWR is measured, the reflection coefficient magnitude can be calculated from (24) as

$$|\Gamma_L| = \frac{\text{VSWR}-1}{\text{VSWR}+1} \tag{25}$$

The angle ϕ of the reflection coefficient

$$\Gamma_L = |\Gamma_L|\, e^{j\phi} \tag{26}$$

can also be determined from these standing wave measurements. According to Figure 8-21, $\Gamma(z)$ must swing clockwise through an angle $\phi + \pi$ as we move from the load at $z = 0$ toward the generator to the first voltage minimum at B. The shortest distance $d_{\min}$ that we must move to reach the first voltage minimum is given by

$$2kd_{\min} = \phi + \pi \tag{27}$$

or

$$\frac{\phi}{\pi} = 4\frac{d_{\min}}{\lambda} - 1 \tag{28}$$

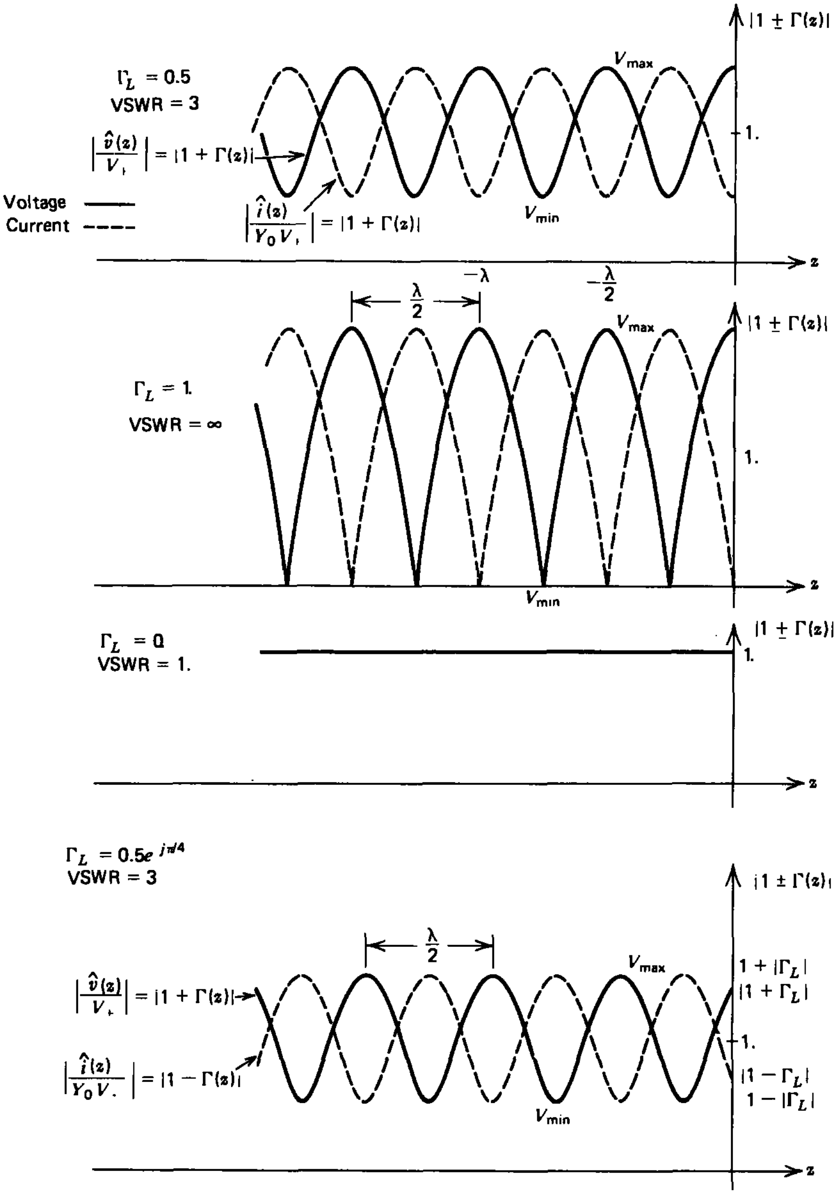


Figure 8-22 Voltage and current standing wave patterns plotted for various values of the VSWR.

A measurement of d_{min}, as well as a determination of the wavelength (the distance between successive minima or maxima is $\lambda/2$) yields the complex reflection coefficient of the load using (25) and (28). Once we know the complex reflection coefficient we can calculate the load impedance from (7). These standing wave measurements are sufficient to determine the terminating load impedance Z_L. These measurement properties of the load reflection coefficient and its relation to the load impedance are of great importance at high frequencies where the absolute measurement of voltage or current may be difficult. Some special cases of interest are:

(i) Matched line—If $\Gamma_L = 0$, then VSWR = 1. The voltage magnitude is constant everywhere on the line.
(ii) Short or open circuited line—If $|\Gamma_L| = 1$, then VSWR = ∞. The minimum voltage on the line is zero.
(iii) The peak normalized voltage $|\hat{v}(z)/V_+|$ is $1+|\Gamma_L|$ while the minimum normalized voltage is $1-|\Gamma_L|$.
(iv) The normalized voltage at $z=0$ is $|1+\Gamma_L|$ while the normalized current $|\hat{\imath}(z)/Y_0 V_+|$ at $z=0$ is $|1-\Gamma_L|$.
(v) If the load impedance is real ($Z_L = R_L$), then (4) shows us that Γ_L is real. Then evaluating (7) at $z=0$, where $\Gamma(z=0)=\Gamma_L$, we see that when $Z_L > Z_0$ that VSWR = Z_L/Z_0 while if $Z_L < Z_0$, VSWR = Z_0/Z_L.

For a general termination, if we know the VSWR and d_{min}, we can calculate the load impedance from (7) as

$$Z_L = Z_0 \frac{1+|\Gamma_L|\, e^{j\phi}}{1-|\Gamma_L|\, e^{j\phi}}$$

$$= Z_0 \frac{[\text{VSWR}+1+(\text{VSWR}-1)\, e^{j\phi}]}{[\text{VSWR}+1-(\text{VSWR}-1)\, e^{j\phi}]} \tag{29}$$

Multiplying through by $e^{-j\phi/2}$ and then simplifying yields

$$Z_L = \frac{Z_0[\text{VSWR}-j\tan(\phi/2)]}{[1-j\,\text{VSWR}\tan(\phi/2)]}$$

$$= \frac{Z_0[1-j\,\text{VSWR}\tan kd_{min}]}{[\text{VSWR}-j\tan kd_{min}]} \tag{30}$$

EXAMPLE 8-2 VOLTAGE STANDING WAVE RATIO

The VSWR on a 50-Ohm (characteristic impedance) transmission line is 2. The distance between successive voltage minima is 40 cm while the distance from the load to the first minima is 10 cm. What is the reflection coefficient and load impedance?

SOLUTION

We are given

$$\text{VSWR}=2$$

$$kd_{\min}=\frac{2\pi(10)}{2(40)}=\frac{\pi}{4}$$

The reflection coefficient is given from (25)–(28) as

$$\Gamma_L=\tfrac{1}{3}e^{-j\pi/2}=\frac{-j}{3}$$

while the load impedance is found from (30) as

$$\begin{aligned}Z_L&=\frac{50(1-2j)}{2-j}\\&=40-30j\text{ ohm}\end{aligned}$$

8-5 STUB TUNING

In practice, most sources are connected to a transmission line through a series resistance matched to the line. This eliminates transient reflections when the excitation is turned on or off. To maximize the power flow to a load, it is also necessary for the load impedance reflected back to the source to be equal to the source impedance and thus equal to the characteristic impedance of the line, Z_0. This matching of the load to the line for an arbitrary termination can only be performed by adding additional elements along the line.

Usually these elements are short circuited transmission lines, called stubs, whose lengths can be varied. The reactance of the stub can be changed over the range from $-j\infty$ to $j\infty$ simply by varying its length, as found in Section 8-3-2, for the short circuited line. Because stubs are usually connected in parallel to a transmission line, it is more convenient to work with admittances rather than impedances as admittances in parallel simply add.

8-5-1 Use of the Smith Chart for Admittance Calculations

Fortunately the Smith chart can also be directly used for admittance calculations where the normalized admittance is defined as

$$Y_n(z)=\frac{Y(z)}{Y_0}=\frac{1}{Z_n(z)} \tag{1}$$

If the normalized load admittance Y_{nL} is known, straightforward impedance calculations first require the computation

$$Z_{nL} = 1/Y_{nL} \tag{2}$$

so that we could enter the Smith chart at Z_{nL}. Then we rotate by the required angle corresponding to $2kz$ and read the new $Z_n(z)$. Then we again compute its reciprocal to find

$$Y_n(z) = 1/Z_n(z) \tag{3}$$

The two operations of taking the reciprocal are tedious. We can use the Smith chart itself to invert the impedance by using the fact that the normalized impedance is inverted by a $\lambda/4$ section of line, so that a rotation of $\Gamma(z)$ by 180° changes a normalized impedance into its reciprocal. Hence, if the admittance is given, we enter the Smith chart with a given value of normalized admittance Y_n and rotate by 180° to find Z_n. We then rotate by the appropriate number of wavelengths to find $Z_n(z)$. Finally, we again rotate by 180° to find $Y_n(z) = 1/Z_n(z)$. We have actually rotated the reflection coefficient by an angle of $2\pi + 2kz$. Rotation by 2π on the Smith chart, however, brings us back to wherever we started, so that only the $2kz$ rotation is significant. As long as we do an even number of π rotations by entering the Smith chart with an admittance and leaving again with an admittance, we can use the Smith chart with normalized admittances exactly as if they were normalized impedances.

EXAMPLE 8-3 USE OF THE SMITH CHART FOR ADMITTANCE CALCULATIONS

The load impedance on a 50-Ohm line is

$$Z_L = 50(1+j)$$

What is the admittance of the load?

SOLUTION

By direct computation we have

$$Y_L = \frac{1}{Z_L} = \frac{1}{50(1+j)} = \frac{(1-j)}{100}$$

To use the Smith chart we find the normalized impedance at A in Figure 8-23:

$$Z_{nL} = 1 + j$$

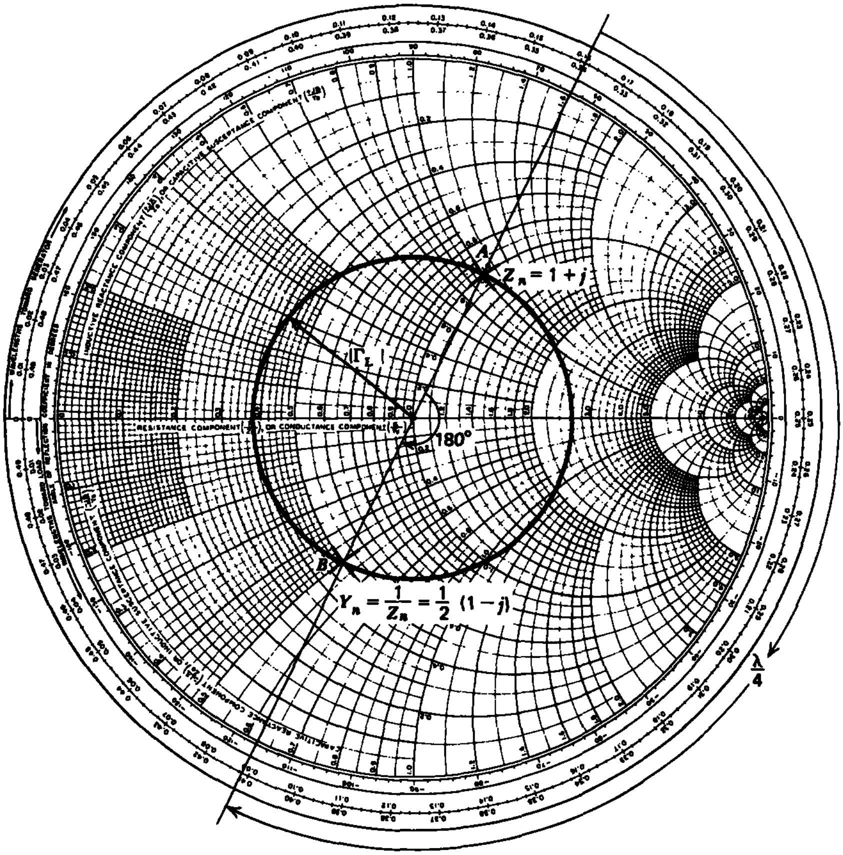


Figure 8-23 The Smith chart offers a convenient way to find the reciprocal of a complex number using the property that the normalized impedance reflected back by a quarter wavelength inverts. Thus, the normalized admittance is found by locating the normalized impedance and rotating this point by 180° about the constant $|\Gamma_L|$ circle.

The normalized admittance that is the reciprocal of the normalized impedance is found by locating the impedance a distance $\lambda/4$ away from the load end at B:

$$Y_{nL} = 0.5(1-j) \Rightarrow Y_L = Y_n Y_0 = (1-j)/100$$

Note that the point B is just 180° away from A on the constant $|\Gamma_L|$ circle. For more complicated loads the Smith chart is a convenient way to find the reciprocal of a complex number.

8-5-2 Single-Stub Matching

A termination of value $Z_L = 50(1+j)$ on a 50-Ohm transmission line is to be matched by means of a short circuited stub at a distance l_1 from the load, as shown in Figure 8-24*a*. We need to find the line length l_1 and the length of the stub l_2 such that the impedance at the junction is matched to the line ($Z_{in} = 50$ Ohm). Then we know that all further points to the left of the junction have the same impedance of 50 Ohms.

Because of the parallel connection, it is simpler to use the Smith chart as an admittance transformation. The normalized load admittance can be computed using the Smith chart by rotating by 180° from the normalized load impedance at A, as was shown in Figure 8-23 and Example 8-3,

$$Z_{nL} = 1+j \tag{4}$$

to yield

$$Y_{nL} = 0.5(1-j) \tag{5}$$

at the point B.

Now we know from Section 8-3-2 that the short circuited stub can only add an imaginary component to the admittance. Since we want the total normalized admittance to be unity to the left of the stub in Figure 8-24

$$Y_{in} = Y_1 + Y_2 = 1 \tag{6}$$

when Y_{nL} is reflected back to be Y_1 it must wind up on the circle whose real part is 1 (as Y_2 can only be imaginary), which occurs either at C or back at A allowing l_1 to be either 0.25λ at A or $(0.25+0.177)\lambda = 0.427\lambda$ at C (or these values plus any integer multiple of $\lambda/2$). Then Y_1 is either of the following two conjugate values:

$$Y_1 = \begin{cases} 1+j, & l_1 = 0.25\lambda \ (A) \\ 1-j, & l_1 = 0.427\lambda \ (C) \end{cases} \tag{7}$$

For Y_{in} to be unity we must pick Y_2 to have an imaginary part to just cancel the imaginary part of Y_1:

$$Y_2 = \begin{cases} -j, & l_1 = 0.25\lambda \\ +j, & l_1 = 0.427\lambda \end{cases} \tag{8}$$

which means, since the shorted end has an infinite admittance at D that the stub must be of length such as to rotate the admittance to the points E or F requiring a stub length l_2 of $(\lambda/8)(E)$ or $(3\lambda/8)(F)$ (or these values plus any integer multiple

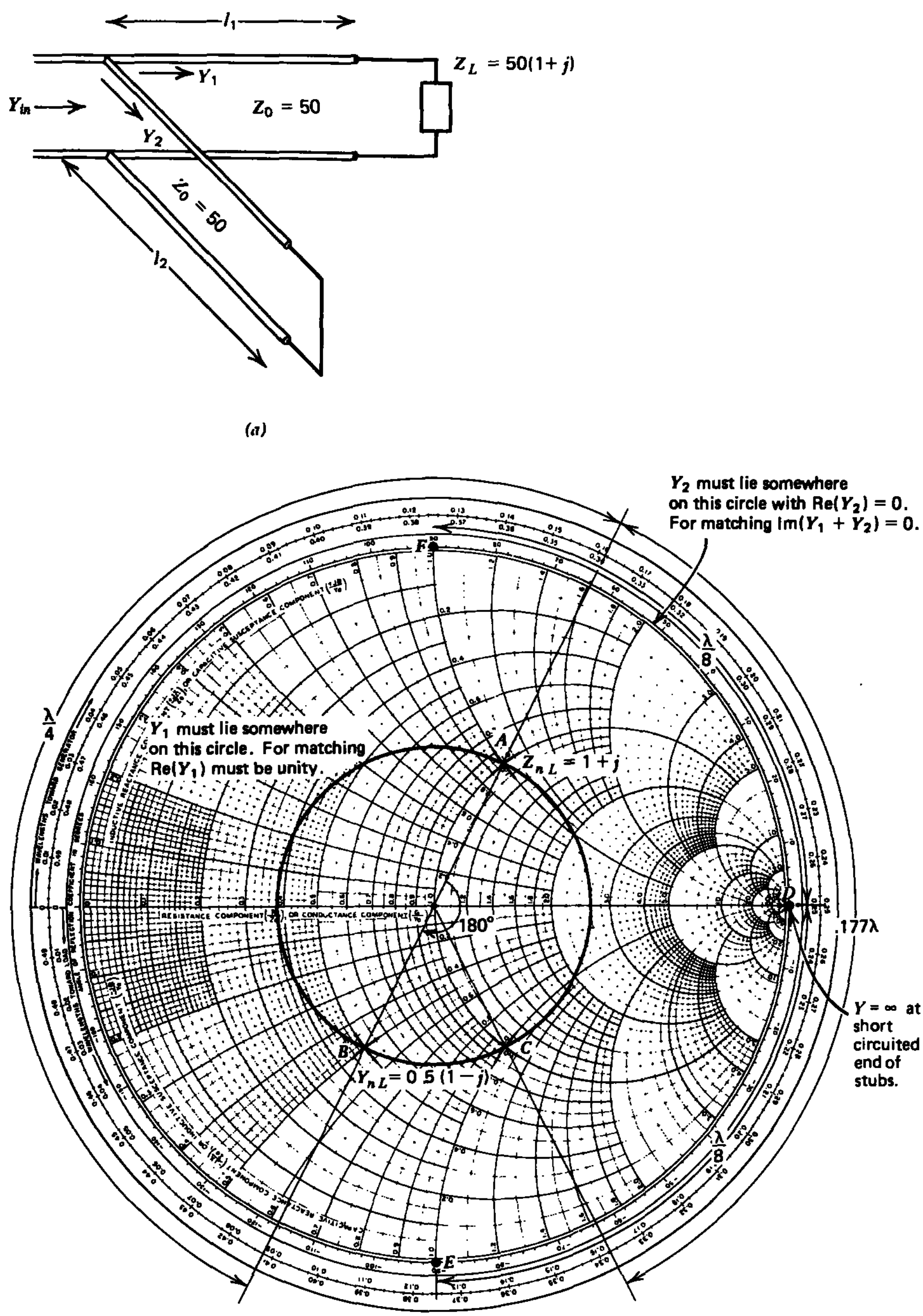


Figure 8-24 (*a*) A single stub tuner consisting of a variable length short circuited line l_2 can match any load to the line by putting the stub at the appropriate distance l_1 from the load. (*b*) Smith chart construction. (*c*) Voltage standing wave pattern.

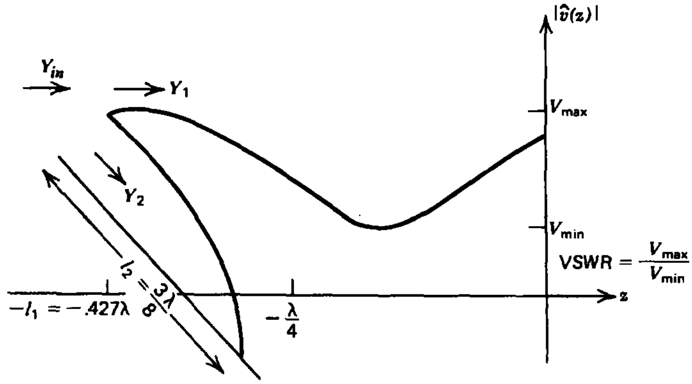


Figure 8-24

of $\lambda/2$). Thus, the solutions can be summarized as

$$l_1 = 0.25\lambda + n\lambda/2, \qquad l_2 = \lambda/8 + m\lambda/2$$

or

$$l_1 = 0.427\lambda + n\lambda/2, \qquad l_2 = 3\lambda/8 + m\lambda/2 \tag{9}$$

where n and m are any nonnegative integers (including zero).

When the load is matched by the stub to the line, the VSWR to the left of the stub is unity, while to the right of the stub over the length l_1 the reflection coefficient is

$$\Gamma_L = \frac{Z_{nL} - 1}{Z_{nL} + 1} = \frac{j}{2 + j} \tag{10}$$

which has magnitude

$$|\Gamma_L| = 1/\sqrt{5} \approx 0.447 \tag{11}$$

so that the voltage standing wave ratio is

$$\text{VSWR} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} \approx 2.62 \tag{12}$$

The disadvantage to single-stub tuning is that it is not easy to vary the length l_1. Generally new elements can only be connected at the ends of the line and not inbetween.

8-5-3 Double-Stub Matching

This difficulty of not having a variable length line can be overcome by using two short circuited stubs a fixed length apart, as shown in Figure 8-25*a*. This fixed length is usually $\frac{3}{8}\lambda$. A match is made by adjusting the length of the stubs l_1 and

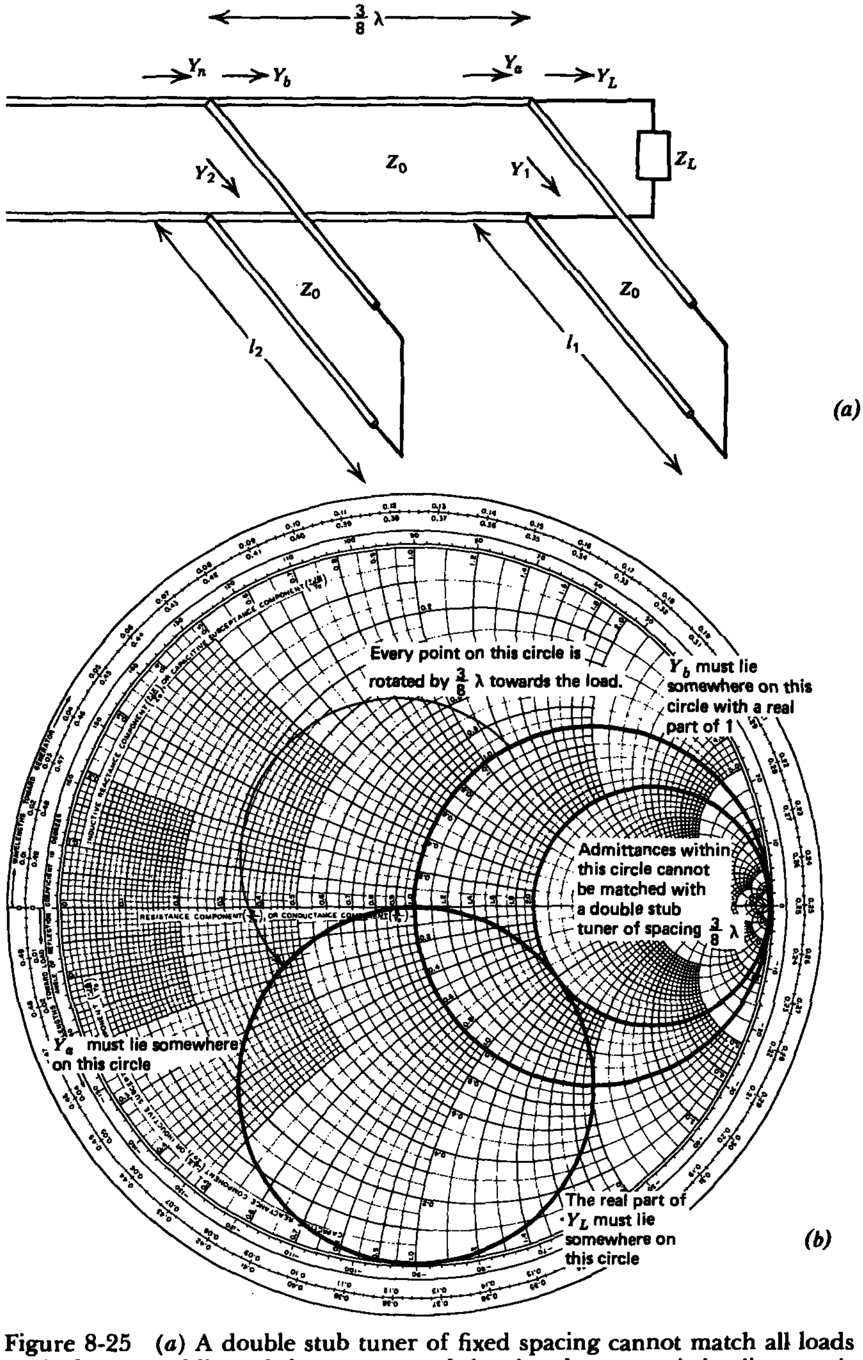


Figure 8-25 (*a*) A double stub tuner of fixed spacing cannot match all loads but is useful because additional elements can only be placed at transmission line terminations and not at any general position along a line as required for a single-stub tuner. (*b*) Smith chart construction. If the stubs are $\frac{3}{8}\lambda$ apart, normalized load admittances whose real part exceeds 2 cannot be matched.

l_2. One problem with the double-stub tuner is that not all loads can be matched for a given stub spacing.

The normalized admittances at each junction are related as

$$\begin{aligned} Y_a &= Y_1 + Y_L \\ Y_n &= Y_2 + Y_b \end{aligned} \tag{13}$$

where Y_1 and Y_2 are the purely reactive admittances of the stubs reflected back to the junctions while Y_b is the admittance of Y_a reflected back towards the load by $\frac{3}{8}\lambda$. For a match we require that Y_n be unity. Since Y_2 is purely imaginary, the real part of Y_b must lie on the circle with a real part of unity. Then Y_a must lie somewhere on this circle when each point on the circle is reflected back by $\frac{3}{8}\lambda$. This generates another circle that is $\frac{3}{2}\pi$ back in the counterclockwise direction as we are moving toward the load, as illustrated in Figure 8-25*b*. To find the conditions for a match, we work from left to right towards the load using the following reasoning:

(i) Since Y_2 is purely imaginary, the real part of Y_b must lie on the circle with a real part of unity, as in Figure 8-25*b*.

(ii) Every possible point on Y_b must be reflected towards the load by $\frac{3}{8}\lambda$ to find the locus of possible match for Y_a. This generates another circle that is $\frac{3}{2}\pi$ back in the counterclockwise direction as we move towards the load, as in Figure 8-25*b*.

Again since Y_1 is purely imaginary, the real part of Y_a must also equal the real part of the load admittance. This yields two possible solutions if the load admittance is outside the forbidden circle enclosing all load admittances with a real part greater than 2. Only loads with normalized admittances whose real part is less than 2 can be matched by the double-stub tuner of $\frac{3}{8}\lambda$ spacing. Of course, if a load is within the forbidden circle, it can be matched by a double-stub tuner if the stub spacing is different than $\frac{3}{8}\lambda$.

EXAMPLE 8-4 DOUBLE-STUB MATCHING

The load impedance $Z_L = 50(1+j)$ on a 50-Ohm line is to be matched by a double-stub tuner of $\frac{3}{8}\lambda$ spacing. What stub lengths l_1 and l_2 are necessary?

SOLUTION

The normalized load impedance $Z_{nL} = 1+j$ corresponds to a normalized load admittance:

$$Y_{nL} = 0.5(1-j)$$

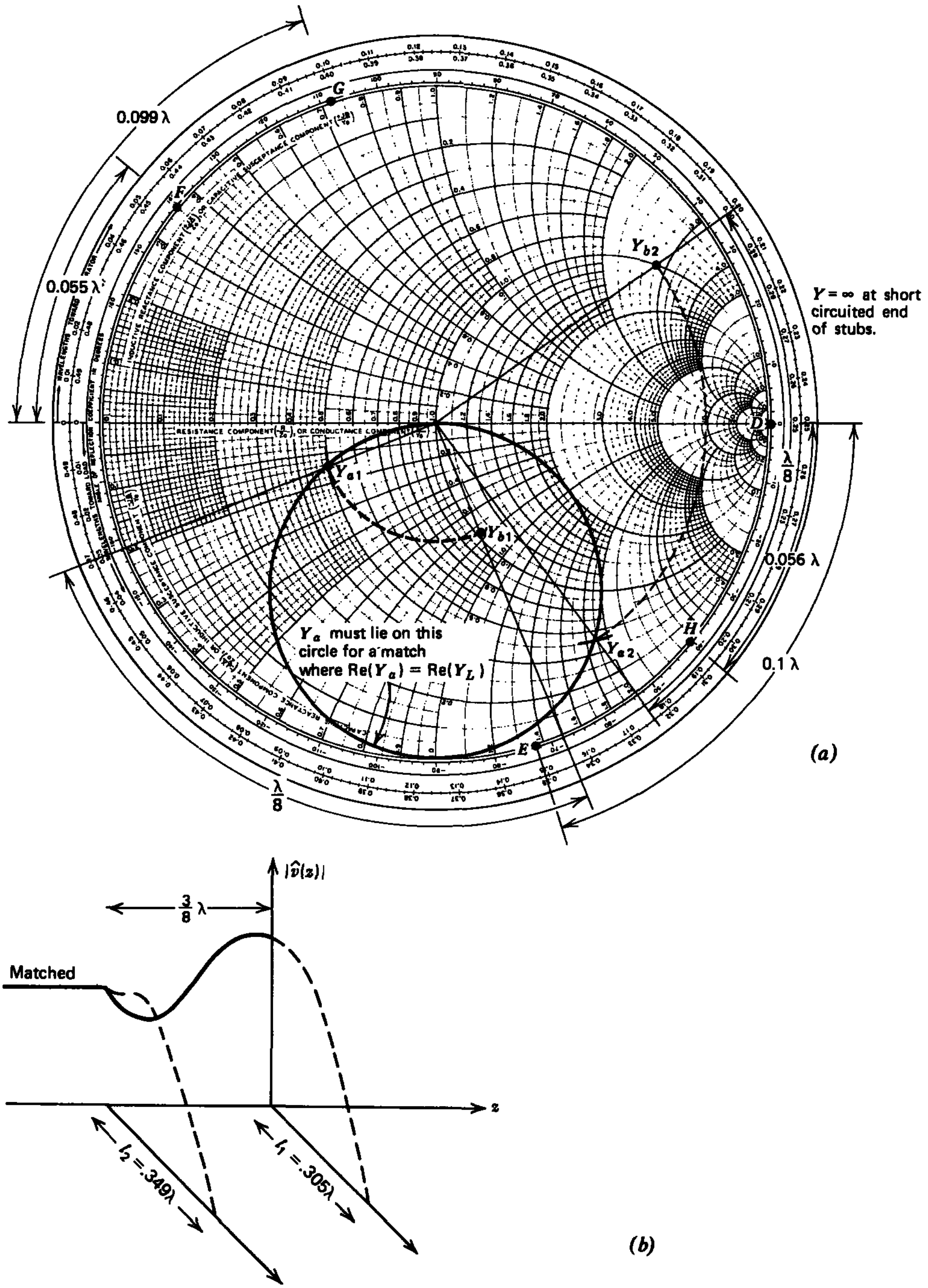


Figure 8-26 (*a*) The Smith chart construction for a double-stub tuner of $\frac{3}{8}\lambda$ spacing with $Z_{nL} = 1 + j$. (*b*) The voltage standing wave pattern.

Then the two solutions for Y_a lie on the intersection of the circle shown in Figure 8-26a with the $r=0.5$ circle:

$$Y_{a1}=0.5-0.14j$$

$$Y_{a2}=0.5-1.85j$$

We then find Y_1 by solving for the imaginary part of the upper equation in (13):

$$Y_1=j\,\mathrm{Im}\,(Y_a-Y_L)=\begin{cases}0.36j\Rightarrow l_1=0.305\lambda & (F)\\ -1.35j\Rightarrow l_1=0.1\lambda & (E)\end{cases}$$

By rotating the Y_a solutions by $\frac{3}{8}\lambda$ back to the generator (270° clockwise, which is equivalent to 90° counterclockwise), their intersection with the $r=1$ circle gives the solutions for Y_b as

$$Y_{b1}=1.0-0.72j$$

$$Y_{b2}=1.0+2.7j$$

This requires Y_2 to be

$$Y_2=-j\,\mathrm{Im}\,(Y_b)=\begin{cases}0.72j\Rightarrow l_2=0.349\lambda & (G)\\ -2.7j\Rightarrow l_2=0.056\lambda & (H)\end{cases}$$

The voltage standing wave pattern along the line and stubs is shown in Figure 8.26b. Note the continuity of voltage at the junctions. The actual stub lengths can be those listed plus any integer multiple of $\lambda/2$.

8-6 THE RECTANGULAR WAVEGUIDE

We showed in Section 8-1-2 that the electric and magnetic fields for TEM waves have the same form of solutions in the plane transverse to the transmission line axis as for statics. The inner conductor within a closed transmission line structure such as a coaxial cable is necessary for TEM waves since it carries a surface current and a surface charge distribution, which are the source for the magnetic and electric fields. A hollow conducting structure, called a waveguide, cannot propagate TEM waves since the static fields inside a conducting structure enclosing no current or charge is zero.

However, new solutions with electric or magnetic fields along the waveguide axis as well as in the transverse plane are allowed. Such solutions can also propagate along transmission lines. Here the axial displacement current can act as a source of the transverse magnetic field giving rise to transverse magnetic (TM) modes as the magnetic field lies entirely within the transverse plane. Similarly, an axial time varying magnetic field generates transverse electric (TE) modes. The most general allowed solutions on a transmission line are TEM, TM, and TE modes. Removing the inner conductor on a closed transmission line leaves a waveguide that can only propagate TM and TE modes.

8-6-1 Governing Equations

To develop these general solutions we return to Maxwell's equations in a linear source-free material:

$$\begin{aligned}\nabla\times\mathbf{E}&=-\mu\frac{\partial\mathbf{H}}{\partial t}\\ \nabla\times\mathbf{H}&=\varepsilon\frac{\partial\mathbf{E}}{\partial t}\\ \varepsilon\nabla\cdot\mathbf{E}&=0\\ \mu\nabla\cdot\mathbf{H}&=0\end{aligned}\tag{1}$$

Taking the curl of Faraday's law, we expand the double cross product and then substitute Ampere's law to obtain a simple vector equation in **E** alone:

$$\begin{aligned}\nabla\times(\nabla\times\mathbf{E})&=\nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E}\\ &=-\mu\frac{\partial}{\partial t}(\nabla\times\mathbf{H})\\ &=-\varepsilon\mu\frac{\partial^2\mathbf{E}}{\partial t^2}\end{aligned}\tag{2}$$

Since $\nabla\cdot\mathbf{E}=0$ from Gauss's law when the charge density is zero, (2) reduces to the vector wave equation in **E**:

$$\nabla^2\mathbf{E}=\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2},\qquad c^2=\frac{1}{\varepsilon\mu}\tag{3}$$

If we take the curl of Ampere's law and perform similar operations, we also obtain the vector wave equation in **H**:

$$\nabla^2\mathbf{H}=\frac{1}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}\tag{4}$$

The solutions for **E** and **H** in (3) and (4) are not independent. If we solve for either **E** or **H**, the other field is obtained from (1). The vector wave equations in (3) and (4) are valid for any shaped waveguide. In particular, we limit ourselves in this text to waveguides whose cross-sectional shape is rectangular, as shown in Figure 8-27.

8-6-2 Transverse Magnetic (TM) Modes

We first consider TM modes where the magnetic field has x and y components but no z component. It is simplest to solve (3) for the z component of electric field and then obtain the other electric and magnetic field components in terms of E_z directly from Maxwell's equations in (1).

We thus assume solutions of the form

$$E_z = \text{Re}\,[\hat{E}_z(x, y)\, e^{j(\omega t - k_z z)}] \tag{5}$$

where an exponential z dependence is assumed because the cross-sectional area of the waveguide is assumed to be uniform in z so that none of the coefficients in (1) depends on z. Then substituting into (3) yields the Helmholtz equation:

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} - \left(k_z^2 - \frac{\omega^2}{c^2}\right)\hat{E}_z = 0 \tag{6}$$

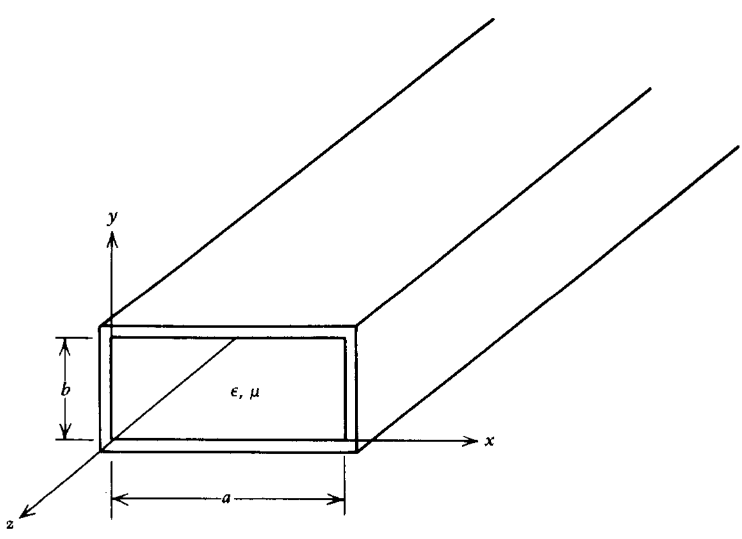


Figure 8-27 A lossless waveguide with rectangular cross section.

This equation can be solved by assuming the same product solution as used for solving Laplace's equation in Section 4-2-1, of the form

$$\hat{E}_z(x, y) = X(x)Y(y) \tag{7}$$

where $X(x)$ is only a function of the x coordinate and $Y(y)$ is only a function of y. Substituting this assumed form of solution into (6) and dividing through by $X(x)Y(y)$ yields

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = k_z^2 - \frac{\omega^2}{c^2} \tag{8}$$

When solving Laplace's equation in Section 4-2-1 the right-hand side was zero. Here the reasoning is the same. The first term on the left-hand side in (8) is only a function of x while the second term is only a function of y. The only way a function of x and a function of y can add up to a constant for all x and y is if each function alone is a constant,

$$\begin{aligned} \frac{1}{X}\frac{d^2X}{dx^2} &= -k_x^2 \\ \frac{1}{Y}\frac{d^2Y}{dy^2} &= -k_y^2 \end{aligned} \tag{9}$$

where the separation constants must obey the relation

$$k_x^2 + k_y^2 + k_z^2 = k^2 = \omega^2/c^2 \tag{10}$$

When we solved Laplace's equation in Section 4-2-6, there was no time dependence so that $\omega = 0$. Then we found that at least one of the wavenumbers was imaginary, yielding decaying solutions. For finite frequencies it is possible for all three wavenumbers to be real for pure propagation. The values of these wavenumbers will be determined by the dimensions of the waveguide through the boundary conditions.

The solutions to (9) are sinusoids so that the transverse dependence of the axial electric field $\hat{E}_z(x, y)$ is

$$\hat{E}_z(x, y) = (A_1 \sin k_x x + A_2 \cos k_x x)(B_1 \sin k_y y + B_2 \cos k_y y) \tag{11}$$

Because the rectangular waveguide in Figure 8-27 is composed of perfectly conducting walls, the tangential component of electric field at the walls is zero:

$$\begin{aligned} \hat{E}_z(x, y = 0) &= 0, \qquad \hat{E}_z(x = 0, y) = 0 \\ \hat{E}_z(x, y = b) &= 0, \qquad \hat{E}_z(x = a, y) = 0 \end{aligned} \tag{12}$$

These boundary conditions then require that A_2 and B_2 are zero so that (11) simplifies to

$$\hat{E}_z(x, y) = E_0 \sin k_x x \sin k_y y \tag{13}$$

where E_0 is a field amplitude related to a source strength and the transverse wavenumbers must obey the equalities

$$\begin{aligned} k_x &= m\pi/a, \qquad m = 1, 2, 3, \ldots \\ k_y &= n\pi/b, \qquad n = 1, 2, 3, \ldots \end{aligned} \tag{14}$$

Note that if either m or n is zero in (13), the axial electric field is zero. The waveguide solutions are thus described as TM_{mn} modes where both m and n are integers greater than zero.

The other electric field components are found from the z component of Faraday's law, where $\mathbf{H}_z = 0$ and the charge-free Gauss's law in (1):

$$\begin{aligned} \frac{\partial E_y}{\partial x} &= \frac{\partial E_x}{\partial y} \\ \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} &= 0 \end{aligned} \tag{15}$$

By taking $\partial/\partial x$ of the top equation and $\partial/\partial y$ of the lower equation, we eliminate E_x to obtain

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} = -\frac{\partial^2 E_z}{\partial y\, \partial z} \tag{16}$$

where the right-hand side is known from (13). The general solution for E_y must be of the same form as (11), again requiring the tangential component of electric field to be zero at the waveguide walls,

$$\hat{E}_y(x = 0, y) = 0, \qquad \hat{E}_y(x = a, y) = 0 \tag{17}$$

so that the solution to (16) is

$$\hat{E}_y = -\frac{jk_y k_z E_0}{k_x^2 + k_y^2} \sin k_x x \cos k_y y \tag{18}$$

We then solve for E_x using the upper equation in (15):

$$\hat{E}_x = -\frac{jk_x k_z E_0}{k_x^2 + k_y^2} \cos k_x x \sin k_y y \tag{19}$$

where we see that the boundary conditions

$$\hat{E}_x(x, y = 0) = 0, \qquad \hat{E}_x(x, y = b) = 0 \tag{20}$$

are satisfied.

The magnetic field is most easily found from Faraday's law

$$\hat{\mathbf{H}}(x, y) = -\frac{1}{j\omega\mu}\nabla\times\hat{\mathbf{E}}(x, y) \tag{21}$$

to yield

$$\begin{aligned}
\hat{H}_x &= -\frac{1}{j\omega\mu}\left(\frac{\partial \hat{E}_z}{\partial y} - \frac{\partial \hat{E}_y}{\partial z}\right) \\
&= -\frac{k_y k^2}{j\omega\mu(k_x^2 + k_y^2)} E_0 \sin k_x x \cos k_y y \\
&= \frac{j\omega\varepsilon k_y}{k_x^2 + k_y^2} E_0 \sin k_x x \cos k_y y \\
\hat{H}_y &= -\frac{1}{j\omega\mu}\left(\frac{\partial \hat{E}_x}{\partial z} - \frac{\partial \hat{E}_z}{\partial x}\right) \\
&= \frac{k_x k^2 E_0}{j\omega\mu(k_x^2 + k_y^2)} \cos k_x x \sin k_y y \\
&= -\frac{j\omega\varepsilon k_x}{k_x^2 + k_y^2} E_0 \cos k_x x \sin k_y y \\
\hat{H}_z &= 0
\end{aligned} \tag{22}$$

Note the boundary conditions of the normal component of **H** being zero at the waveguide walls are automatically satisfied:

$$\begin{aligned}
\hat{H}_y(x, y=0) = 0, \qquad \hat{H}_y(x, y=b) = 0 \\
\hat{H}_x(x=0, y) = 0, \qquad \hat{H}_y(x=a, y) = 0
\end{aligned} \tag{23}$$

The surface charge distribution on the waveguide walls is found from the discontinuity of normal **D** fields:

$$\begin{aligned}
\hat{\sigma}_f(x=0, y) &= \varepsilon\hat{E}_x(x=0, y) = -\frac{jk_z k_x \varepsilon}{k_x^2 + k_y^2} E_0 \sin k_y y \\
\hat{\sigma}_f(x=a, y) &= -\varepsilon\hat{E}_x(x=a, y) = \frac{jk_z k_x \varepsilon}{k_x^2 + k_y^2} E_0 \cos m\pi \sin k_y y \\
\hat{\sigma}_f(x, y=0) &= \varepsilon\hat{E}_y(x, y=0) = -\frac{jk_z k_y \varepsilon}{k_x^2 + k_y^2} E_0 \sin k_x x \\
\hat{\sigma}_f(x, y=b) &= -\varepsilon\hat{E}_y(x, y=b) = \frac{jk_z k_y \varepsilon}{k_x^2 + k_y^2} E_0 \cos n\pi \sin k_x x
\end{aligned} \tag{24}$$

Similarly, the surface currents are found by the discontinuity in the tangential components of **H** to be purely z directed:

$$\begin{aligned}\hat{K}_z(x, y=0) &= -\hat{H}_x(x, y=0) = \frac{k_y k^2 E_0 \sin k_x x}{j\omega\mu(k_x^2+k_y^2)}\\ \hat{K}_z(x, y=b) &= \hat{H}_x(x, y=b) = -\frac{k_y k^2 E_0}{j\omega\mu(k_x^2+k_y^2)}\sin k_x x \cos n\pi\\ \hat{K}_z(x=0, y) &= \hat{H}_y(x=0, y) = \frac{k_x k^2 E_0}{j\omega\mu(k_x^2+k_y^2)}\sin k_y y\\ \hat{K}_z(x=a, y) &= -\hat{H}_y(x=a, y) = -\frac{k_x k^2 E_0 \cos m\pi \sin k_y y}{j\omega\mu(k_x^2+k_y^2)}\end{aligned} \tag{25}$$

We see that if m or n are even, the surface charges and surface currents on opposite walls are of opposite sign, while if m or n are odd, they are of the same sign. This helps us in plotting the field lines for the various TM_{mn} modes shown in Figure 8-28. The electric field is always normal and the magnetic field tangential to the waveguide walls. Where the surface charge is positive, the electric field points out of the wall, while it points in where the surface charge is negative. For higher order modes the field patterns shown in Figure 8-28 repeat within the waveguide.

Slots are often cut in waveguide walls to allow the insertion of a small sliding probe that measures the electric field. These slots must be placed at positions of zero surface current so that the field distributions of a particular mode are only negligibly disturbed. If a slot is cut along the z direction on the $y=b$ surface at $x=a/2$, the surface current given in (25) is zero for TM modes if $\sin(k_x a/2)=0$, which is true for the $m=$ even modes.

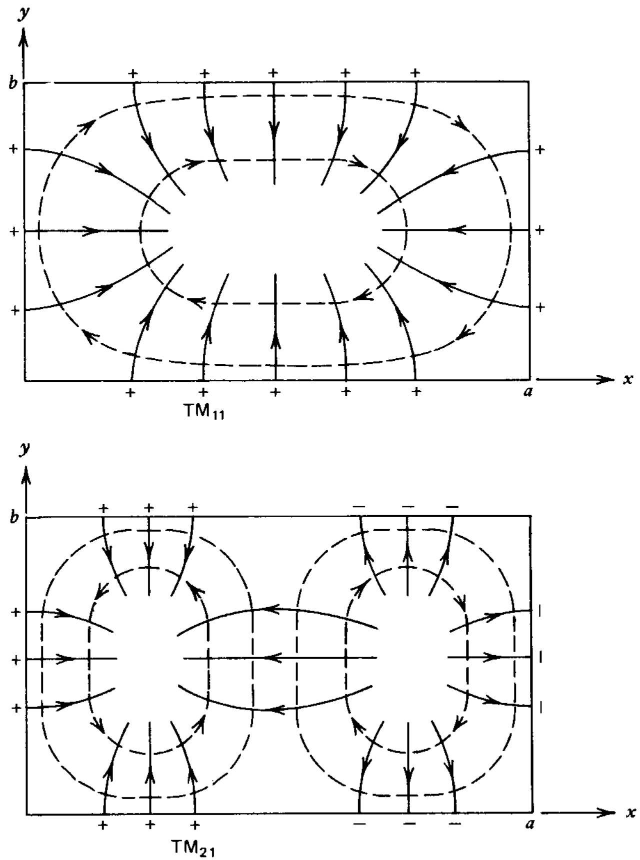
8-6-3 Transverse Electric (TE) Modes

When the electric field lies entirely in the xy plane, it is most convenient to first solve (4) for H_z. Then as for TM modes we assume a solution of the form

$$H_z = \mathrm{Re}\,[\hat{H}_z(x, y)\, e^{j(\omega t - k_z z)}] \tag{26}$$

which when substituted into (4) yields

$$\frac{\partial^2 \hat{H}_z}{\partial x^2}+\frac{\partial^2 \hat{H}_z}{\partial y^2}-\left(k_z^2-\frac{\omega^2}{c^2}\right)\hat{H}_z=0 \tag{27}$$



Electric field (———)

$$\hat{E}_x = \frac{-jk_xk_zE_0}{k_x^2+k_y^2}\cos k_xx\,\sin k_yy$$

$$\hat{E}_y = \frac{-jk_yk_zE_0}{k_x^2+k_y^2}\sin k_xx\,\cos k_yy$$

$$\hat{E}_z = E_0\sin k_xx\,\sin k_yy$$

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{k_y}{k_x}\frac{\tan k_xx}{\tan k_yy}$$

$$\Rightarrow \frac{[\cos k_xx]^{(k_y/k_x)^2}}{\cos k_yy} = \text{const}$$

Magnetic field (- - - -)

$$\hat{H}_x = \frac{j\omega\varepsilon k_y}{k_x^2+k_y^2}E_0\sin k_xx\,\cos k_yy$$

$$\hat{H}_y = \frac{-j\omega\varepsilon k_x}{k_x^2+k_y^2}E_0\cos k_xx\,\sin k_yy$$

$$\frac{dy}{dx} = \frac{H_y}{H_x} = \frac{-k_x\cot k_xx}{k_y\cot k_yy}$$

$$\Rightarrow \sin k_xx\,\sin k_yy = \text{const}$$

$$k_x = \frac{m\pi}{a}, \quad k_y = \frac{n\pi}{b}, \quad k_z = \left[\frac{\omega^2}{c^2} - k_x^2 - k_y^2\right]^{1/2}$$

Figure 8-28 The transverse electric and magnetic field lines for the TM_{11} and TM_{21} modes. The electric field is purely z directed where the field lines converge.

Again this equation is solved by assuming a product solution and separating to yield a solution of the same form as (11):

$$\hat{H}_z(x, y) = (A_1 \sin k_x x + A_2 \cos k_x x)(B_1 \sin k_y y + B_2 \cos k_y y) \quad (28)$$

The boundary conditions of zero normal components of **H** at the waveguide walls require that

$$\begin{aligned} \hat{H}_x(x=0, y) &= 0, \qquad \hat{H}_x(x=a, y) = 0 \\ \hat{H}_y(x, y=0) &= 0, \qquad \hat{H}_y(x, y=b) = 0 \end{aligned} \quad (29)$$

Using identical operations as in (15)–(20) for the TM modes the magnetic field solutions are

$$\begin{aligned} \hat{H}_x &= \frac{jk_z k_x H_0}{k_x^2 + k_y^2} \sin k_x x \cos k_y y, \qquad k_x = \frac{m\pi}{a}, \quad k_y = \frac{n\pi}{b} \\ \hat{H}_y &= \frac{jk_z k_y H_0}{k_x^2 + k_y^2} \cos k_x x \sin k_y y \\ \hat{H}_z &= H_0 \cos k_x x \cos k_y y \end{aligned} \quad (30)$$

The electric field is then most easily obtained from Ampere's law in (1),

$$\hat{\mathbf{E}} = \frac{1}{j\omega\varepsilon} \nabla \times \hat{\mathbf{H}} \quad (31)$$

to yield

$$\begin{aligned} \hat{E}_x &= \frac{1}{j\omega\varepsilon}\left(\frac{\partial}{\partial y}\hat{H}_z - \frac{\partial}{\partial z}\hat{H}_y\right) \\ &= -\frac{k_y k^2 H_0}{j\omega\varepsilon(k_x^2 + k_y^2)} \cos k_x x \sin k_y y \\ &= \frac{j\omega\mu k_y}{k_x^2 + k_y^2} H_0 \cos k_x x \sin k_y y \\ \hat{E}_y &= \frac{1}{j\omega\varepsilon}\left(\frac{\partial \hat{H}_x}{\partial z} - \frac{\partial \hat{H}_z}{\partial x}\right) \\ &= \frac{k_x k^2 H_0}{j\omega\varepsilon(k_x^2 + k_y^2)} \sin k_x x \cos k_y y \\ &= -\frac{j\omega\mu k_x}{k_x^2 + k_y^2} H_0 \sin k_x x \cos k_y y \\ \hat{E}_z &= 0 \end{aligned} \quad (32)$$

We see in (32) that as required the tangential components of the electric field at the waveguide walls are zero. The surface charge densities on each of the walls are:

$$\begin{aligned}
\hat{\sigma}_f(x=0,y) &= \varepsilon\hat{E}_x(x=0,y) = \frac{-k_y k^2 H_0}{j\omega(k_x^2+k_y^2)}\sin k_y y\\
\hat{\sigma}_f(x=a,y) &= -\varepsilon\hat{E}_x(x=a,y) = \frac{k_y k^2 H_0}{j\omega(k_x^2+k_y^2)}\cos m\pi \sin k_y y\\
\hat{\sigma}_f(x,y=0) &= \varepsilon\hat{E}_y(x,y=0) = \frac{k_x k^2 H_0}{j\omega(k_x^2+k_y^2)}\sin k_x x\\
\hat{\sigma}_f(x,y=b) &= -\varepsilon\hat{E}_y(x,y=b) = -\frac{k_x k^2 H_0}{j\omega(k_x^2+k_y^2)}\cos n\pi \sin k_x x
\end{aligned} \tag{33}$$

For TE modes, the surface currents determined from the discontinuity of tangential **H** now flow in closed paths on the waveguide walls:

$$\begin{aligned}
\hat{\mathbf{K}}(x=0,y) &= \mathbf{i}_x\times\hat{\mathbf{H}}(x=0,y)\\
&= \mathbf{i}_z\hat{H}_y(x=0,y)-\mathbf{i}_y\hat{H}_z(x=0,y)\\
\hat{\mathbf{K}}(x=a,y) &= -\mathbf{i}_x\times\hat{\mathbf{H}}(x=a,y)\\
&= -\mathbf{i}_z\hat{H}_y(x=a,y)+\mathbf{i}_y\hat{H}_z(x=a,y)\\
\hat{\mathbf{K}}(x,y=0) &= \mathbf{i}_y\times\hat{\mathbf{H}}(x,y=0)\\
&= -\mathbf{i}_z\hat{H}_x(x,y=0)+\mathbf{i}_x\hat{H}_z(x,y=0)\\
\hat{\mathbf{K}}(x,y=b) &= -\mathbf{i}_y\times\hat{\mathbf{H}}(x,y=b)\\
&= \mathbf{i}_z\hat{H}_x(x,y=b)-\mathbf{i}_x\hat{H}_z(x,y=b)
\end{aligned} \tag{34}$$

Note that for TE modes either n or m (but not both) can be zero and still yield a nontrivial set of solutions. As shown in Figure 8-29, when n is zero there is no variation in the fields in the y direction and the electric field is purely y directed while the magnetic field has no y component. The TE_{11} and TE_{21} field patterns are representative of the higher order modes.

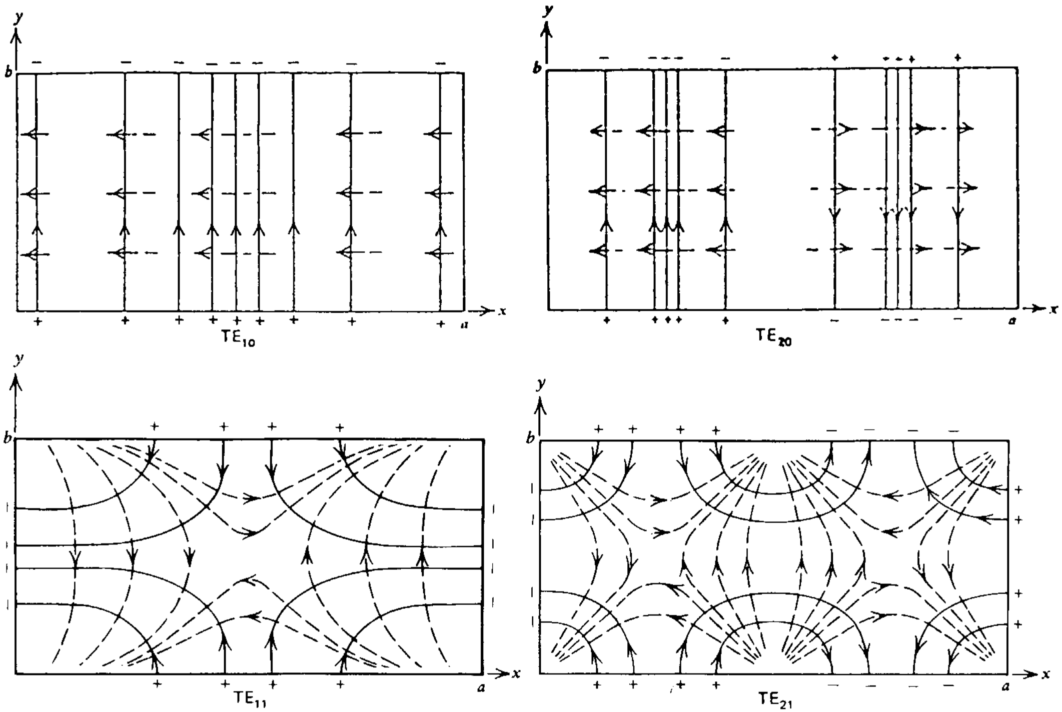
8-6-4 Cut-Off

The transverse wavenumbers are

$$k_x=\frac{m\pi}{a},\qquad k_y=\frac{n\pi}{b} \tag{35}$$

so that the axial variation of the fields is obtained from (10) as

$$k_z=\left[\frac{\omega_2}{c^2}-k_x^2-k_y^2\right]^{1/2}=\left[\frac{\omega^2}{c^2}-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2\right]^{1/2} \tag{36}$$



Electric field (———)

$$\hat{E}_x = \frac{j\omega\mu k_y}{k_x^2 + k_y^2} H_0 \cos k_x x \sin k_y y$$

$$\hat{E}_y = \frac{-j\omega\mu k_x}{k_x^2 + k_y^2} H_0 \sin k_x x \cos k_y y$$

$$k_x = \frac{m\pi}{a}, \quad k_y = \frac{n\pi}{b}, \quad k_z = \left[\frac{\omega^2}{c^2} - k_x^2 - k_y^2\right]^{1/2}$$

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{-k_x}{k_y} \frac{\tan k_x x}{\tan k_y y}$$

$$\Rightarrow \cos k_x x \cos k_y y = \text{const}$$

Magnetic field (- - - -)

$$\hat{H}_x = \frac{j k_z k_x H_0}{k_x^2 + k_y^2} \sin k_x x \cos k_y y$$

$$\hat{H}_y = \frac{j k_z k_y H_0}{k_x^2 + k_y^2} \cos k_x x \sin k_y y$$

$$\hat{H}_z = H_0 \cos k_x x \cos k_y y$$

$$\frac{dy}{dx} = \frac{H_y}{H_x} = \frac{k_y}{k_x} \frac{\cot k_x x}{\cot k_y y}$$

$$\Rightarrow \frac{[\sin k_x x]^{(k_y/k_x)^2}}{\sin k_y y} = \text{const}$$

Figure 8-29 (*a*) The transverse electric and magnetic field lines for various TE modes. The magnetic field is purely z directed where the field lines converge. The TE_{10} mode is called the dominant mode since it has the lowest cut-off frequency. (*b*) Surface current lines for the TE_{10} mode.

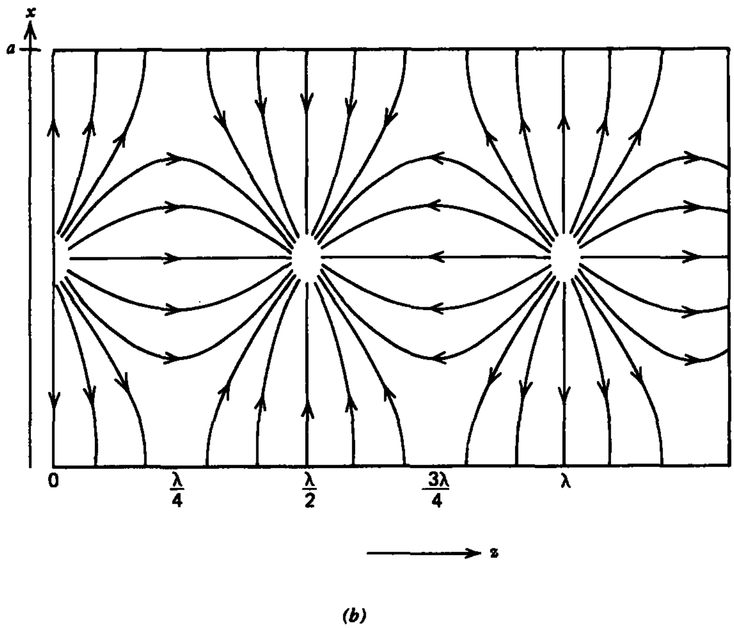


Figure 8-29

Thus, although k_x and k_y are real, k_z can be either pure real or pure imaginary. A real value of k_z represents power flow down the waveguide in the z direction. An imaginary value of k_z means exponential decay with no time-average power flow. The transition from propagating waves (k_z real) to evanescence (k_z imaginary) occurs for $k_z = 0$. The frequency when k_z is zero is called the cut-off frequency ω_c:

$$\omega_c = c\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{1/2} \tag{37}$$

This frequency varies for each mode with the mode parameters m and n. If we assume that a is greater than b, the lowest cut-off frequency occurs for the TE_{10} mode, which is called the dominant or fundamental mode. No modes can propagate below this lowest critical frequency ω_{c0}:

$$\omega_{c0} = \frac{\pi c}{a} \Rightarrow f_{c0} = \frac{\omega_{c0}}{2\pi} = \frac{c}{2a}\ \text{Hz} \tag{38}$$

If an air-filled waveguide has $a = 1$ cm, then $f_{c0} = 1.5\times 10^{10}$ Hz, while if $a = 10$ m, then $f_{c0} = 15$ MHz. This explains why we usually cannot hear the radio when driving through a tunnel. As the frequency is raised above ω_{c0}, further modes can propagate.

The phase and group velocity of the waves are

$$v_p=\frac{\omega}{k_z}=\frac{\omega}{\left[\frac{\omega^2}{c^2}-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2\right]^{1/2}}$$
$$v_g=\frac{d\omega}{dk_z}=\frac{k_z c^2}{\omega}=\frac{c^2}{v_p}\Rightarrow v_g v_p=c^2 \tag{39}$$

At cut-off, $v_g=0$ and $v_p=\infty$ with their product always a constant.

8-6-5 Waveguide Power Flow

The time-averaged power flow per unit area through the waveguide is found from the Poynting vector:

$$<S>=\tfrac{1}{2}\operatorname{Re}(\hat{\mathbf{E}}\times\hat{\mathbf{H}}^*) \tag{40}$$

(a) Power Flow for the TM Modes

Substituting the field solutions found in Section 8-6-2 into (40) yields

$$\begin{aligned}<S>&=\tfrac{1}{2}\operatorname{Re}[(\hat{E}_x\mathbf{i}_x+\hat{E}_y\mathbf{i}_y+\hat{E}_z\mathbf{i}_z)\,e^{-jk_z z}\times(\hat{H}_x^*\mathbf{i}_x+\hat{H}_y^*\mathbf{i}_y)\,e^{+jk_z^* z}]\\&=\tfrac{1}{2}\operatorname{Re}[(\hat{E}_x\hat{H}_y^*-\hat{E}_y\hat{H}_x^*)\mathbf{i}_z+\hat{E}_z(\hat{H}_x^*\mathbf{i}_y-\hat{H}_y^*\mathbf{i}_x)]\,e^{-j(k_z-k_z^*)z}\end{aligned} \tag{41}$$

where we remember that k_z may be imaginary for a particular mode if the frequency is below cut-off. For propagating modes where k_z is real so that $k_z=k_z^*$, there is no z dependence in (41). For evanescent modes where k_z is pure imaginary, the z dependence of the Poynting vector is a real decaying exponential of the form $e^{-2|k_z|z}$. For either case we see from (13) and (22) that the product of $\hat{E}_z$ with $\hat{H}_x$ and $\hat{H}_y$ is pure imaginary so that the real parts of the x- and y-directed time average power flow are zero in (41). Only the z-directed power flow can have a time average:

$$\begin{aligned}<S>=\frac{\omega\varepsilon|E_0|^2}{2(k_x^2+k_y^2)}\operatorname{Re}[k_z\,e^{-j(k_z-k_z^*)z}(k_x^2\cos^2 k_x x\sin^2 k_y y\\+k_y^2\sin^2 k_x x\cos^2 k_y y)]\mathbf{i}_z\end{aligned} \tag{42}$$

If k_z is imaginary, we have that $<S>=0$ while a real k_z results in a nonzero time-average power flow. The total z-directed power flow is found by integrating (42) over the cross-sectional area of the waveguide:

$$<P> = \int_{x=0}^{a}\int_{y=0}^{b} <S_z>\, dx\, dy$$
$$= \frac{\omega\varepsilon k_z ab E_0^2}{8(k_x^2+k_y^2)} \tag{43}$$

where it is assumed that k_z is real, and we used the following identities:

$$\int_0^a \sin^2\frac{m\pi x}{a}\,dx = \frac{a}{m\pi}\left(\frac{1}{2}\frac{m\pi x}{a} - \frac{1}{4}\sin\frac{2m\pi x}{a}\right)\Big|_0^a$$
$$= \begin{cases} a/2, & m \neq 0 \\ 0, & m = 0 \end{cases}$$
$$\int_0^a \cos^2\frac{m\pi x}{a}\,dx = \frac{a}{m\pi}\left(\frac{1}{2}\frac{m\pi x}{a} + \frac{1}{4}\sin\frac{2m\pi x}{a}\right)\Big|_0^a \tag{44}$$
$$= \begin{cases} a/2, & m \neq 0 \\ a, & m = 0 \end{cases}$$

For the TM modes, both m and n must be nonzero.

(b) Power Flow for the TE Modes

The same reasoning is used for the electromagnetic fields found in Section 8-6-3 substituted into (40):

$$<\mathbf{S}> = \tfrac{1}{2}\,\text{Re}\,[(\hat{E}_x\mathbf{i}_x + \hat{E}_y\mathbf{i}_y)\,e^{-jk_z z}\times(\hat{H}_x^*\mathbf{i}_x + \hat{H}_y^*\mathbf{i}_y + \hat{H}_z^*\mathbf{i}_z)\,e^{+jk_z^* z}]$$
$$= \tfrac{1}{2}\,\text{Re}\,[(\hat{E}_x\hat{H}_y^* - \hat{E}_y\hat{H}_x^*)\mathbf{i}_z - \hat{H}_z^*(\hat{E}_x\mathbf{i}_y - \hat{E}_y\mathbf{i}_x)]\,e^{-j(k_z-k_z^*)z} \tag{45}$$

Similarly, again we have that the product of H_z^* with $\hat{E}_x$ and $\hat{E}_y$ is pure imaginary so that there are no x- and y-directed time average power flows. The z-directed power flow reduces to

$$<S_z> = \frac{1}{2}\frac{\omega\mu H_0^2}{(k_x^2+k_y^2)}(k_y^2\cos^2 k_x x\sin^2 k_y y$$
$$+ k_x^2\sin^2 k_x x\cos^2 k_y y)\,\text{Re}\,(k_z\,e^{-j(k_z-k_z^*)z}) \tag{46}$$

Again we have nonzero z-directed time average power flow only if k_z is real. Then the total z-directed power is

$$<P> = \int_{x=0}^{a}\int_{y=0}^{b} <S_z>\,dx\,dy = \begin{cases} \dfrac{\omega\mu k_z ab H_0^2}{8(k_x^2+k_y^2)}, & m, n \neq 0 \\[2ex] \dfrac{\omega\mu k_z ab H_0^2}{4(k_x^2+k_y^2)}, & m \text{ or } n = 0 \end{cases} \tag{47}$$

where we again used the identities of (44). Note the factor of 2 differences in (47) for either the TE_{10} or TE_{01} modes. Both m and n cannot be zero as the TE_{00} mode reduces to the trivial spatially constant uncoupled z-directed magnetic field.

8-6-6 Wall Losses

If the waveguide walls have a high but noninfinite Ohmic conductivity σ_w, we can calculate the spatial attenuation rate using the approximate perturbation approach described in Section 8-3-4*b*. The fields decay as $e^{-\alpha z}$, where

$$\alpha = \frac{1}{2}\frac{<P_{dL}>}{<P>} \tag{48}$$

where $<P_{dL}>$ is the time-average dissipated power per unit length and $<P>$ is the electromagnetic power flow in the lossless waveguide derived in Section 8-6-5 for each of the modes.

In particular, we calculate α for the TE_{10} mode ($k_x = \pi/a$, $k_y = 0$). The waveguide fields are then

$$\begin{aligned}\hat{\mathbf{H}} &= H_0\left(\mathbf{i}_x \frac{jk_z a}{\pi}\sin\frac{\pi x}{a} + \cos\frac{\pi x}{a}\mathbf{i}_z\right)\\ \hat{\mathbf{E}} &= -\frac{j\omega\mu a}{\pi}H_0 \sin\frac{\pi x}{a}\mathbf{i}_y\end{aligned} \tag{49}$$

The surface current on each wall is found from (34) as

$$\begin{aligned}\hat{\mathbf{K}}(x=0,y) &= \hat{\mathbf{K}}(x=a,y) = -H_0\mathbf{i}_y\\ \hat{\mathbf{K}}(x,y=0) &= -\hat{\mathbf{K}}(x,y=b) = H_0\left(-\mathbf{i}_z \frac{jk_z a}{\pi}\sin\frac{\pi x}{a} + \mathbf{i}_x\cos\frac{\pi x}{a}\right)\end{aligned} \tag{50}$$

With lossy walls the electric field component $\mathbf{E}_w$ within the walls is in the same direction as the surface current proportional by a surface conductivity $\sigma_w\delta$, where δ is the skin depth as found in Section 8-3-4*b*. The time-average dissipated power density per unit area in the walls is then:

$$\begin{aligned}<P_d(x=0,y)> &= <P_d(x=a,y)>\\ &= \tfrac{1}{2}\mathrm{Re}\,(\hat{\mathbf{E}}_w\cdot\hat{\mathbf{K}}^*) = \frac{1}{2}\frac{H_0^2}{\sigma_w\delta}\\ <P_d(x,y=0)> &= <P_d(x,y=b)>\\ &= \frac{1}{2}\frac{H_0^2}{\sigma_w\delta}\left[\left(\frac{k_z a}{\pi}\right)^2\sin^2\frac{\pi x}{a} + \cos^2\frac{\pi x}{a}\right]\end{aligned} \tag{51}$$

The total time average dissipated power per unit length $<P_{dL}>$ required in (48) is obtained by integrating each of the terms in (51) along the waveguide walls:

$$<P_{dL}> = \int_0^b [<P_d(x=0, y)> + <P_d(x=a, y)>]\, dy$$
$$+\int_0^a [<P_d(x, y=0)> + <P_d(x, y=b)>]\, dx$$
$$= \frac{H_0^2 b}{\sigma_w \delta} + \frac{H_0^2}{\sigma_w \delta}\int_0^a \left[\left(\frac{k_z a}{\pi}\right)^2 \sin^2\frac{\pi x}{a} + \cos^2\frac{\pi x}{a}\right] dx$$
$$= \frac{H_0^2}{\sigma_w \delta}\left\{b + \frac{a}{2}\left[\left(\frac{k_z a}{\pi}\right)^2 + 1\right]\right\} = \frac{H_0^2}{\sigma_w \delta}\left[b + \frac{a}{2}\left(\frac{\omega^2 a^2}{\pi^2 c^2}\right)\right] \tag{52}$$

while the electromagnetic power above cut-off for the TE_{10} mode is given by (47),

$$<P> = \frac{\omega \mu k_z a b H_0^2}{4(\pi/a)^2} \tag{53}$$

so that

$$\alpha = \frac{1}{2}\frac{<P_{dL}>}{<P>} = \frac{2\left(\frac{\pi}{a}\right)^2\left[b + \frac{a}{2}\left(\frac{\omega^2 a^2}{\pi^2 c^2}\right)\right]}{\omega \mu a b k_z \sigma_w \delta} \tag{54}$$

where

$$k_z = \left[\frac{\omega^2}{c^2} - \left(\frac{\pi}{a}\right)^2\right]^{1/2}; \qquad \frac{\omega}{c} > \frac{\pi}{a} \tag{55}$$

8-7 DIELECTRIC WAVEGUIDE

We found in Section 7-10-6 for fiber optics that electromagnetic waves can also be guided by dielectric structures if the wave travels from the dielectric to free space at an angle of incidence greater than the critical angle. Waves propagating along the dielectric of thickness $2d$ in Figure 8-30 are still described by the vector wave equations derived in Section 8-6-1.

8-7-1 TM Solutions

We wish to find solutions where the fields are essentially confined within the dielectric. We neglect variations with y so that for TM waves propagating in the z direction the z component of electric field is given in Section 8-6-2 as

$$E_z(x, t) = \begin{cases} \text{Re}\,[A_2\, e^{-\alpha(x-d)}\, e^{j(\omega t - k_z z)}], & x \geq d \\ \text{Re}\,[(A_1 \sin k_x x + B_1 \cos k_x x)\, e^{j(\omega t - k_z z)}], & |x| \leq d \\ \text{Re}\,[A_3\, e^{\alpha(x+d)}\, e^{j(\omega t - k_z z)}], & x \leq -d \end{cases} \tag{1}$$

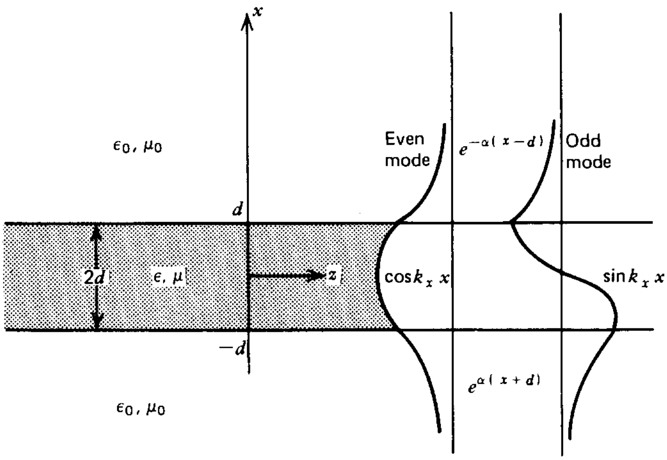


Figure 8-30 TE and TM modes can also propagate along dielectric structures. The fields can be essentially confined to the dielectric over a frequency range if the speed of the wave in the dielectric is less than that outside. It is convenient to separate the solutions into even and odd modes.

where we choose to write the solution outside the dielectric in the decaying wave form so that the fields are predominantly localized around the dielectric.

The wavenumbers and decay rate obey the relations

$$\begin{aligned} k_x^2 + k_z^2 &= \omega^2 \varepsilon\mu \\ -\alpha^2 + k_z^2 &= \omega^2 \varepsilon_0 \mu_0 \end{aligned} \tag{2}$$

The z component of the wavenumber must be the same in all regions so that the boundary conditions can be met at each interface. For propagation in the dielectric and evanescence in free space, we must have that

$$\omega^2 \varepsilon_0 \mu_0 < k_z < \omega^2 \varepsilon\mu \tag{3}$$

All the other electric and magnetic field components can be found from (1) in the same fashion as for metal waveguides in Section 8-6-2. However, it is convenient to separately consider each of the solutions for E_z within the dielectric.

(a) Odd Solutions

If E_z in each half-plane above and below the centerline are oppositely directed, the field within the dielectric must vary solely as $\sin k_x x$:

$$\hat{E}_z = \begin{cases} A_2\, e^{-\alpha(x-d)}, & x \geq d \\ A_1 \sin k_x x, & |x| \leq d \\ A_3\, e^{\alpha(x+d)}, & x \leq -d \end{cases} \tag{4}$$

Then because in the absence of volume charge the electric field has no divergence,

$$\frac{\partial \hat{E}_x}{\partial x} - jk_z\hat{E}_z \Rightarrow \hat{E}_x = \begin{cases} -\frac{jk_z}{\alpha} A_2\, e^{-\alpha(x-d)}, & x \geq d \\ -\frac{jk_z}{k_x} A_1 \cos k_x x, & |x| \leq d \\ \frac{jk_z}{\alpha} A_3\, e^{\alpha(x+d)}, & x \leq -d \end{cases} \tag{5}$$

while from Faraday's law the magnetic field is

$$\hat{H}_y = -\frac{1}{j\omega\mu}\left(-jk_z\hat{E}_x - \frac{\partial \hat{E}_z}{\partial x}\right)$$

$$\Rightarrow \hat{H}_y = \begin{cases} -\frac{j\omega\varepsilon_0 A_2}{\alpha} e^{-\alpha(x-d)}, & x \geq d \\ -\frac{j\omega\varepsilon A_1}{k_x} \cos k_x x, & |x| \leq d \\ \frac{j\omega\varepsilon_0 A_3}{\alpha} e^{\alpha(x+d)}, & x \leq -d \end{cases} \tag{6}$$

At the boundaries where $x = \pm d$ the tangential electric and magnetic fields are continuous:

$$E_z(x = \pm d_-) = E_z(x = \pm d_+) \Rightarrow A_1 \sin k_x d = A_2$$

$$-A_1 \sin k_x d = A_3$$

$$H_y(x = \pm d_-) = H_y(x = \pm d_+) \Rightarrow \frac{-j\omega\varepsilon A_1}{k_x} \cos k_x d = \frac{-j\omega\varepsilon_0 A_2}{\alpha} \tag{7}$$

$$\frac{-j\omega\varepsilon A_1}{k_x} \cos k_x d = \frac{j\omega\varepsilon_0 A_3}{\alpha}$$

which when simultaneously solved yields

$$\left.\begin{aligned} \frac{A_2}{A_1} &= \sin k_x d = \frac{\varepsilon\alpha}{\varepsilon_0 k_x} \cos k_x d \\ \frac{A_3}{A_1} &= -\sin k_x d = -\frac{\varepsilon\alpha}{\varepsilon_0 k_x} \cos k_x d \end{aligned}\right\} \Rightarrow \alpha = \frac{\varepsilon_0}{\varepsilon} k_x \tan k_x d \tag{8}$$

The allowed values of α and k_x are obtained by self-consistently solving (8) and (2), which in general requires a numerical method. The critical condition for a guided wave occurs when $\alpha = 0$, which requires that $k_x d = n\pi$ and $k_z^2 = \omega^2\varepsilon_0\mu_0$. The critical frequency is then obtained from (2) as

$$\omega^2 = \frac{k_x^2}{\varepsilon\mu - \varepsilon_0\mu_0} = \frac{(n\pi/d)^2}{\varepsilon\mu - \varepsilon_0\mu_0} \tag{9}$$

Note that this occurs for real frequencies only if $\varepsilon\mu > \varepsilon_0\mu_0$.

(b) Even Solutions

If E_z is in the same direction above and below the dielectric, solutions are similarly

$$\hat{E}_z = \begin{cases} B_2\, e^{-\alpha(x-d)}, & x \geq d \\ B_1 \cos k_x x, & |x| \leq d \\ B_3\, e^{\alpha(x+d)}, & x \leq -d \end{cases} \tag{10}$$

$$\hat{E}_x = \begin{cases} -\dfrac{jk_z}{\alpha} B_2\, e^{-\alpha(x-d)}, & x \geq d \\ \dfrac{jk_z}{k_x} B_1 \sin k_x x, & |x| \leq d \\ \dfrac{jk_z}{\alpha} B_3\, e^{\alpha(x+d)}, & x \leq -d \end{cases} \tag{11}$$

$$\hat{H}_y = \begin{cases} -\dfrac{j\omega\varepsilon_0}{\alpha} B_2\, e^{-\alpha(x-d)}, & x \geq d \\ \dfrac{j\omega\varepsilon}{k_x} B_1 \sin k_x x, & |x| \leq d \\ \dfrac{j\omega\varepsilon_0}{\alpha} B_3\, e^{\alpha(x+d)}, & x \leq -d \end{cases} \tag{12}$$

Continuity of tangential electric and magnetic fields at $x = \pm d$ requires

$$B_1 \cos k_x d = B_2, \qquad B_1 \cos k_x d = B_3$$

$$\frac{j\omega\varepsilon}{k_x} B_1 \sin k_x d = -\frac{j\omega\varepsilon_0}{\alpha} B_2, \qquad -\frac{j\omega\varepsilon B_1}{k_x} \sin k_x d = \frac{j\omega\varepsilon_0 B_3}{\alpha} \tag{13}$$

or

$$\left.\begin{aligned} \frac{B_2}{B_1} &= \cos k_x d = -\frac{\varepsilon\alpha}{\varepsilon_0 k_x} \sin k_x d \\ \frac{B_3}{B_1} &= \cos k_x d = -\frac{\varepsilon\alpha}{\varepsilon_0 k_x} \sin k_x d \end{aligned}\right\} \Rightarrow \alpha = -\frac{\varepsilon_0 k_x}{\varepsilon} \cot k_x d \tag{14}$$

8-7-2 TE Solutions

The same procedure is performed for the TE solutions by first solving for H_z.

(a) Odd Solutions

$$\hat{H}_z = \begin{cases} A_2\, e^{-\alpha(x-d)}, & x \geq d \\ A_1 \sin k_x x, & |x| \leq d \\ A_3\, e^{\alpha(x+d)}, & x \leq -d \end{cases} \tag{15}$$

$$\hat{H}_x = \begin{cases} -\dfrac{jk_z}{\alpha} A_2\, e^{-\alpha(x-d)}, & x \geq d \\ -\dfrac{jk_z}{k_x} A_1 \cos k_x x, & |x| \leq d \\ \dfrac{jk_z}{\alpha} A_3\, e^{\alpha(x+d)}, & x \leq -d \end{cases} \tag{16}$$

$$\hat{E}_y = \begin{cases} \dfrac{j\omega\mu_0}{\alpha} A_2\, e^{-\alpha(x-d)}, & x \geq d \\ \dfrac{j\omega\mu}{k_x} A_1 \cos k_x x, & |x| \leq d \\ -\dfrac{j\omega\mu_0}{\alpha} A_3\, e^{\alpha(x+d)}, & x \leq -d \end{cases} \tag{17}$$

where continuity of tangential **E** and **H** across the boundaries requires

$$\alpha = \frac{\mu_0}{\mu} k_x \tan k_x d \tag{18}$$

(b) Even Solutions

$$\hat{H}_z = \begin{cases} B_2\, e^{-\alpha(x-d)}, & x \geq d \\ B_1 \cos k_x x, & |x| \leq d \\ B_3\, e^{\alpha(x+d)}, & x \leq -d \end{cases} \tag{19}$$

$$\hat{H}_x = \begin{cases} -\dfrac{jk_z}{\alpha} B_2\, e^{-\alpha(x-d)}, & x \geq d \\ \dfrac{jk_z}{k_x} B_1 \sin k_x x, & |x| \leq d \\ \dfrac{jk_z}{\alpha} B_3\, e^{\alpha(x+d)}, & x \leq -d \end{cases} \tag{20}$$

$$\hat{E}_y = \begin{cases} \dfrac{j\omega\mu_0}{\alpha} B_2\, e^{-\alpha(x-d)}, & x \geq d \\ -\dfrac{j\omega\mu}{k_x} B_1 \sin k_x x, & |x| \leq d \\ -\dfrac{j\omega\mu_0}{\alpha} B_3\, e^{\alpha(x+d)}, & x \leq -d \end{cases} \tag{21}$$

where α and k_x are related as

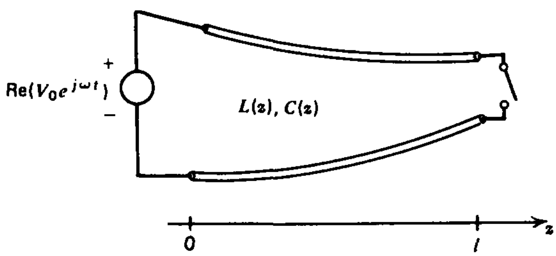
$$\alpha = -\frac{\mu_0}{\mu} k_x \cot k_x d \tag{22}$$

PROBLEMS

Section 8-1

1. Find the inductance and capacitance per unit length and the characteristic impedance for the wire above plane and two wire line shown in Figure 8-3. (**Hint:** See Section 2-6-4*c*.)

2. The inductance and capacitance per unit length on a lossless transmission line is a weak function of z as the distance between electrodes changes slowly with z.



(a) For this case write the transmission line equations as single equations in voltage and current.

(b) Consider an exponential line, where

$$L(z) = L_0\, e^{\alpha z}, \qquad C(z) = C_0\, e^{-\alpha z}$$

If the voltage and current vary sinusoidally with time as

$$v(z, t) = \mathrm{Re}\,[\hat{v}(z)\, e^{j\omega t}], \qquad i(z, t) = \mathrm{Re}\,[\hat{i}(z)\, e^{j\omega t}]$$

find the general form of solution for the spatial distributions of $\hat{v}(z)$ and $\hat{i}(z)$.

(c) The transmission line is excited by a voltage source $V_0 \cos \omega t$ at $z = 0$. What are the voltage and current distributions if the line is short or open circuited at $z = l$?

(d) For what range of frequency do the waves strictly decay with distance? What is the cut-off frequency for wave propagation?

(e) What are the resonant frequencies of the short circuited line?

(f) What condition determines the resonant frequencies of the open circuited line.

3. Two conductors of length l extending over the radial distance $a \leq r \leq b$ are at a constant angle α apart.

(a) What are the electric and magnetic fields in terms of the voltage and current?

(b) Find the inductance and capacitance per unit length. What is the characteristic impedance?