chapter 7

electrodynamics—
fields and waves

The electromagnetic field laws, derived thus far from the empirically determined Coulomb–Lorentz forces, are correct on the time scales of our own physical experiences. However, just as Newton's force law must be corrected for material speeds approaching that of light, the field laws must be corrected when fast time variations are on the order of the time it takes light to travel over the length of a system. Unlike the abstractness of relativistic mechanics, the complete electrodynamic equations describe a familiar phenomenon—propagation of electromagnetic waves. Throughout the rest of this text, we will examine when appropriate the low-frequency limits to justify the past quasi-static assumptions.

7-1 MAXWELL'S EQUATIONS

7-1-1 Displacement Current Correction to Ampere's Law

In the historical development of electromagnetic field theory through the nineteenth century, charge and its electric field were studied separately from currents and their magnetic fields. Until Faraday showed that a time varying magnetic field generates an electric field, it was thought that the electric and magnetic fields were distinct and uncoupled. Faraday believed in the duality that a time varying electric field should also generate a magnetic field, but he was not able to prove this supposition.

It remained for James Clerk Maxwell to show that Faraday's hypothesis was correct and that without this correction Ampere's law and conservation of charge were inconsistent:

$$\nabla\times\mathbf{H}=\mathbf{J}_f\Rightarrow\nabla\cdot\mathbf{J}_f=0 \tag{1}$$

$$\nabla\cdot\mathbf{J}_f+\frac{\partial\rho_f}{\partial t}=0 \tag{2}$$

for if we take the divergence of Ampere's law in (1), the current density must have zero divergence because the divergence of the curl of a vector is always zero. This result contradicts (2) if a time varying charge is present. Maxwell realized that adding the displacement current on the right-hand side of Ampere's law would satisfy charge conservation, because of Gauss's law relating **D** to ρ_f ($\nabla \cdot \mathbf{D} = \rho_f$).

This simple correction has far-reaching consequences, because we will be able to show the existence of electromagnetic waves that travel at the speed of light c, thus proving that light is an electromagnetic wave. Because of the significance of Maxwell's correction, the complete set of coupled electromagnetic field laws are called Maxwell's equations:

Faraday's Law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \Rightarrow \oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \tag{3}$$

Ampere's law with Maxwell's displacement current correction

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \Rightarrow \oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J}_f \cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D} \cdot d\mathbf{S} \tag{4}$$

Gauss's laws

$$\nabla \cdot \mathbf{D} = \rho_f \Rightarrow \oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho_f \, dV \tag{5}$$

$$\nabla \cdot \mathbf{B} = 0 \Rightarrow \oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \tag{6}$$

Conservation of charge

$$\nabla \cdot \mathbf{J}_f + \frac{\partial \rho_f}{\partial t} = 0 \Rightarrow \oint_S \mathbf{J}_f \cdot d\mathbf{S} + \frac{d}{dt}\int_V \rho_f \, dV = 0 \tag{7}$$

As we have justified, (7) is derived from the divergence of (4) using (5).

Note that (6) is not independent of (3) for if we take the divergence of Faraday's law, $\nabla \cdot \mathbf{B}$ could at most be a time-independent function. Since we assume that at some point in time $\mathbf{B} = 0$, this function must be zero.

The symmetry in Maxwell's equations would be complete if a magnetic charge density appeared on the right-hand side of Gauss's law in (6) with an associated magnetic current due to the flow of magnetic charge appearing on the right-hand side of (3). Thus far, no one has found a magnetic charge or current, although many people are actively looking. Throughout this text we accept (3)–(7) keeping in mind that if magnetic charge is discovered, we must modify (3) and (6) and add an equation like (7) for conservation of magnetic charge.

7-1-2 Circuit Theory as a Quasi-static Approximation

Circuit theory assumes that the electric and magnetic fields are highly localized within the circuit elements. Although the displacement current is dominant within a capacitor, it is negligible outside so that Ampere's law can neglect time variations of **D** making the current divergence-free. Then we obtain Kirchoff's current law that the algebraic sum of all currents flowing into (or out of) a node is zero:

$$\nabla \cdot \mathbf{J} = 0 \Rightarrow \oint_S \mathbf{J} \cdot d\mathbf{S} = 0 \Rightarrow \Sigma i_k = 0 \tag{8}$$

Similarly, time varying magnetic flux that is dominant within inductors and transformers is assumed negligible outside so that the electric field is curl free. We then have Kirchoff's voltage law that the algebraic sum of voltage drops (or rises) around any closed loop in a circuit is zero:

$$\nabla \times \mathbf{E} = 0 \Rightarrow \mathbf{E} = -\nabla V \Rightarrow \oint_L \mathbf{E} \cdot d\mathbf{l} = 0 \Rightarrow \Sigma v_k = 0 \tag{9}$$

7-2 CONSERVATION OF ENERGY

7-2-1 Poynting's Theorem

We expand the vector quantity

$$\begin{aligned} \nabla \cdot (\mathbf{E} \times \mathbf{H}) &= \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) \\ &= -\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{E} \cdot \mathbf{J}_f \end{aligned} \tag{1}$$

where we change the curl terms using Faraday's and Ampere's laws.

For linear homogeneous media, including free space, the constitutive laws are

$$\mathbf{D} = \varepsilon \mathbf{E}, \qquad \mathbf{B} = \mu \mathbf{H} \tag{2}$$

so that (1) can be rewritten as

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) + \frac{\partial}{\partial t}\left(\tfrac{1}{2}\varepsilon E^2 + \tfrac{1}{2}\mu H^2\right) = -\mathbf{E} \cdot \mathbf{J}_f \tag{3}$$

which is known as Poynting's theorem. We integrate (3) over a closed volume, using the divergence theorem to convert the first term to a surface integral:

$$\underbrace{\oint_S (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{S}}_{\int_V \nabla\cdot(\mathbf{E}\times\mathbf{H})\,dV}+\frac{d}{dt}\int_V (\tfrac{1}{2}\varepsilon E^2+\tfrac{1}{2}\mu H^2)\,dV=-\int_V \mathbf{E}\cdot\mathbf{J}_f\,dV \quad (4)$$

We recognize the time derivative in (4) as operating on the electric and magnetic energy densities, which suggests the interpretation of (4) as

$$P_{out}+\frac{dW}{dt}=-P_d \quad (5)$$

where P_{out} is the total electromagnetic power flowing out of the volume with density

$$\boldsymbol{S}=\mathbf{E}\times\mathbf{H}\ \text{watts/m}^2\ [\text{kg-s}^{-3}] \quad (6)$$

where $\boldsymbol{S}$ is called the Poynting vector, W is the electromagnetic stored energy, and P_d is the power dissipated or generated:

$$P_{out}=\oint_S (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{S}=\oint_S \boldsymbol{S}\cdot d\mathbf{S}$$

$$W=\int_V [\tfrac{1}{2}\varepsilon E^2+\tfrac{1}{2}\mu H^2]\,dV \quad (7)$$

$$P_d=\oint_V \mathbf{E}\cdot\mathbf{J}_f\,dV$$

If $\mathbf{E}$ and $\mathbf{J}_f$ are in the same direction as in an Ohmic conductor ($\mathbf{E}\cdot\mathbf{J}_f=\sigma E^2$), then P_d is positive, representing power dissipation since the right-hand side of (5) is negative. A source that supplies power to the volume has $\mathbf{E}$ and $\mathbf{J}_f$ in opposite directions so that P_d is negative.

7-2-2 A Lossy Capacitor

Poynting's theorem offers a different and to some a paradoxical explanation of power flow to circuit elements. Consider the cylindrical lossy capacitor excited by a time varying voltage source in Figure 7-1. The terminal current has both Ohmic and displacement current contributions:

$$i=\frac{\varepsilon A}{l}\frac{dv}{dt}+\frac{\sigma A v}{l}=C\frac{dv}{dt}+\frac{v}{R},\qquad C=\frac{\varepsilon A}{l},\qquad R=\frac{l}{\sigma A} \quad (8)$$

From a circuit theory point of view we would say that the power flows from the terminal wires, being dissipated in the

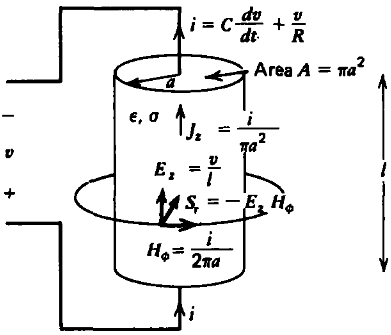

Figure 7-1 The power delivered to a lossy cylindrical capacitor vi is partly dissipated by the Ohmic conduction and partly stored in the electric field. This power can also be thought to flow-in radially from the surrounding electric and magnetic fields via the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$.

resistance and stored as electrical energy in the capacitor:

$$P = vi = \frac{v^2}{R} + \frac{d}{dt}(\tfrac{1}{2}Cv^2) \tag{9}$$

We obtain the same results from a field's viewpoint using Poynting's theorem. Neglecting fringing, the electric field is simply

$$E_z = v/l \tag{10}$$

while the magnetic field at the outside surface of the resistor is generated by the conduction and displacement currents:

$$\oint_L \mathbf{H} \cdot \mathbf{dl} = \int_S \left(J_z + \varepsilon\frac{\partial E_z}{\partial t}\right) dS \Rightarrow H_\phi 2\pi a = \frac{\sigma A v}{l} + \frac{\varepsilon}{l} A \frac{dv}{dt} = i \tag{11}$$

where we recognize the right-hand side as the terminal current in (8),

$$H_\phi = i/(2\pi a) \tag{12}$$

The power flow through the surface at r = a surrounding the resistor is then radially inward,

$$\oint_S (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{dS} = -\int_S \frac{v}{l} \frac{i}{2\pi a} a \, d\phi \, dz = -vi \tag{13}$$

and equals the familiar circuit power formula. The minus sign arises because the left-hand side of (13) is the power out of the volume as the surface area element $\mathbf{dS}$ points radially outwards. From the field point of view, power flows into the lossy capacitor from the electric and magnetic fields outside the resistor via the Poynting vector. Whether the power is thought to flow along the terminal wires or from the surrounding fields is a matter of convenience as the results are identical. The presence of the electric and magnetic fields are directly due to the voltage and current. It is impossible to have the fields without the related circuit variables.

7-2-3 Power in Electric Circuits

We saw in (13) that the flux of $\boldsymbol{S}$ entering the surface surrounding a circuit element just equals vi. We can show this for the general network with N terminals in Figure 7-2 using the quasi-static field laws that describe networks outside the circuit elements:

$$\begin{aligned} \nabla\times\mathbf{E}=0 &\Rightarrow \mathbf{E}=-\nabla V \\ \nabla\times\mathbf{H}=\mathbf{J}_f &\Rightarrow \nabla\cdot\mathbf{J}_f=0 \end{aligned} \tag{14}$$

We then can rewrite the electromagnetic power into a surface as

$$\begin{aligned} P_{\text{in}} &= -\oint_S \mathbf{E}\times\mathbf{H}\cdot d\mathbf{S} \\ &= -\int_V \nabla\cdot(\mathbf{E}\times\mathbf{H})\,dV \\ &= \int_V \nabla\cdot(\nabla V\times\mathbf{H})\,dV \end{aligned} \tag{15}$$

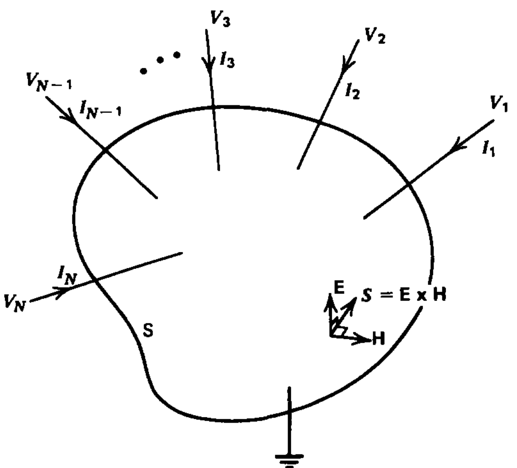

Figure 7-2 The circuit power into an N terminal network $\sum_{k=1}^{N} V_k I_k$ equals the electromagnetic power flow into the surface surrounding the network, $-\oint_S \mathbf{E}\times\mathbf{H}\cdot d\mathbf{S}$.

where the minus is introduced because we want the power in and we use the divergence theorem to convert the surface integral to a volume integral. We expand the divergence term as

$$\begin{aligned}\nabla \cdot (\nabla V \times \mathbf{H}) &= \mathbf{H} \cdot (\nabla \times \nabla V)^{\nearrow 0} - \nabla V \cdot (\nabla \times \mathbf{H}) \\ &= -\mathbf{J}_f \cdot \nabla V = -\nabla \cdot (\mathbf{J}_f V)\end{aligned} \tag{16}$$

where we use (14).

Substituting (16) into (15) yields

$$\begin{aligned}P_{\text{in}} &= -\int_V \nabla \cdot (\mathbf{J}_f V)\, dV \\ &= -\oint_S \mathbf{J}_f V \cdot \mathbf{dS}\end{aligned} \tag{17}$$

where we again use the divergence theorem. On the surface S, the potential just equals the voltages on each terminal wire allowing V to be brought outside the surface integral:

$$\begin{aligned}P_{\text{in}} &= \sum_{k=1}^{N} -V_k \oint_S \mathbf{J}_f \cdot \mathbf{dS} \\ &= \sum_{k=1}^{N} V_k I_k\end{aligned} \tag{18}$$

where we recognize the remaining surface integral as just being the negative (remember **dS** points outward) of each terminal current flowing into the volume. This formula is usually given as a postulate along with Kirchoff's laws in most circuit theory courses. Their correctness follows from the quasi-static field laws that are only an approximation to more general phenomena which we continue to explore.

7-2-4 The Complex Poynting's Theorem

For many situations the electric and magnetic fields vary sinusoidally with time:

$$\begin{aligned}\mathbf{E}(\mathbf{r}, t) &= \text{Re}\,[\hat{\mathbf{E}}(\mathbf{r})\, e^{j\omega t}] \\ \mathbf{H}(\mathbf{r}, t) &= \text{Re}\,[\hat{\mathbf{H}}(\mathbf{r})\, e^{j\omega t}]\end{aligned} \tag{19}$$

where the caret is used to indicate a complex amplitude that can vary with position **r**. The instantaneous power density is obtained by taking the cross product of **E** and **H**. However, it is often useful to calculate the time-average power density $<S>$, where we can avoid the lengthy algebraic and trigonometric manipulations in expanding the real parts in (19).

A simple rule for the time average of products is obtained by realizing that the real part of a complex number is equal to one half the sum of the complex number and its conjugate (denoted by a superscript asterisk). The power density is then

$$\begin{aligned} \boldsymbol{S}(\mathbf{r}, t) &= \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) \\ &= \tfrac{1}{4}[\hat{\mathbf{E}}(\mathbf{r})\, e^{j\omega t} + \hat{\mathbf{E}}^*(\mathbf{r})\, e^{-j\omega t}] \times [\hat{\mathbf{H}}(\mathbf{r})\, e^{j\omega t} + \hat{\mathbf{H}}^*(\mathbf{r})\, e^{-j\omega t}] \\ &= \tfrac{1}{4}[\hat{\mathbf{E}}(\mathbf{r}) \times \hat{\mathbf{H}}(\mathbf{r})\, e^{2j\omega t} + \hat{\mathbf{E}}^*(\mathbf{r}) \times \hat{\mathbf{H}}(\mathbf{r}) + \hat{\mathbf{E}}(\mathbf{r}) \times \hat{\mathbf{H}}^*(\mathbf{r}) \\ &\quad + \hat{\mathbf{E}}^*(\mathbf{r}) \times \hat{\mathbf{H}}^*(\mathbf{r})\, e^{-2j\omega t}] \end{aligned} \tag{20}$$

The time average of (20) is then

$$\begin{aligned} <\boldsymbol{S}> &= \tfrac{1}{4}[\hat{\mathbf{E}}^*(\mathbf{r}) \times \hat{\mathbf{H}}(\mathbf{r}) + \hat{\mathbf{E}}(\mathbf{r}) \times \hat{\mathbf{H}}^*(\mathbf{r})] \\ &= \tfrac{1}{2} \operatorname{Re} [\hat{\mathbf{E}}(\mathbf{r}) \times \hat{\mathbf{H}}^*(\mathbf{r})] \\ &= \tfrac{1}{2} \operatorname{Re} [\hat{\mathbf{E}}^*(\mathbf{r}) \times \hat{\mathbf{H}}(\mathbf{r})] \end{aligned} \tag{21}$$

as the complex exponential terms $e^{\pm 2j\omega t}$ average to zero over a period $T = 2\pi/\omega$ and we again realized that the first bracketed term on the right-hand side of (21) was the sum of a complex function and its conjugate.

Motivated by (21) we define the complex Poynting vector as

$$\hat{\boldsymbol{S}} = \tfrac{1}{2}\hat{\mathbf{E}}(\mathbf{r}) \times \hat{\mathbf{H}}^*(\mathbf{r}) \tag{22}$$

whose real part is just the time-average power density.

We can now derive a complex form of Poynting's theorem by rewriting Maxwell's equations for sinusoidal time variations as

$$\begin{aligned} \nabla \times \hat{\mathbf{E}}(\mathbf{r}) &= -j\omega\mu \hat{\mathbf{H}}(\mathbf{r}) \\ \nabla \times \hat{\mathbf{H}}(\mathbf{r}) &= \hat{\mathbf{J}}_f(\mathbf{r}) + j\omega\varepsilon \hat{\mathbf{E}}(\mathbf{r}) \\ \nabla \cdot \hat{\mathbf{E}}(\mathbf{r}) &= \hat{\rho}_f(\mathbf{r})/\varepsilon \\ \nabla \cdot \hat{\mathbf{B}}(\mathbf{r}) &= 0 \end{aligned} \tag{23}$$

and expanding the product

$$\begin{aligned} \nabla \cdot \hat{\boldsymbol{S}} = \nabla \cdot [\tfrac{1}{2}\hat{\mathbf{E}}(\mathbf{r}) \times \hat{\mathbf{H}}^*(\mathbf{r})] &= \tfrac{1}{2}[\hat{\mathbf{H}}^*(\mathbf{r}) \cdot \nabla \times \hat{\mathbf{E}}(\mathbf{r}) - \hat{\mathbf{E}}(\mathbf{r}) \cdot \nabla \times \hat{\mathbf{H}}^*(\mathbf{r})] \\ &= \tfrac{1}{2}[-j\omega\mu |\hat{\mathbf{H}}(\mathbf{r})|^2 + j\omega\varepsilon |\hat{\mathbf{E}}(\mathbf{r})|^2] - \tfrac{1}{2}\hat{\mathbf{E}}(\mathbf{r}) \cdot \hat{\mathbf{J}}_f^*(\mathbf{r}) \end{aligned} \tag{24}$$

which can be rewritten as

$$\nabla \cdot \hat{\boldsymbol{S}} + 2j\omega[<w_m> - <w_e>] = -\hat{P}_d \tag{25}$$

where

$$\begin{aligned} <w_m> &= \tfrac{1}{4}\mu |\hat{\mathbf{H}}(\mathbf{r})|^2 \\ <w_e> &= \tfrac{1}{4}\varepsilon |\hat{\mathbf{E}}(\mathbf{r})|^2 \\ \hat{P}_d &= \tfrac{1}{2}\hat{\mathbf{E}}(\mathbf{r}) \cdot \hat{\mathbf{J}}_f^*(\mathbf{r}) \end{aligned} \tag{26}$$

We note that $<w_m>$ and $<w_e>$ are the time-average magnetic and electric energy densities and that the complex Poynting's theorem depends on their difference rather than their sum.

7-3 TRANSVERSE ELECTROMAGNETIC WAVES

7-3-1 Plane Waves

Let us try to find solutions to Maxwell's equations that only depend on the z coordinate and time in linear media with permittivity ε and permeability μ. In regions where there are no sources so that $\rho_f=0$, $\mathbf{J}_f=0$, Maxwell's equations then reduce to

$$-\frac{\partial E_y}{\partial z}\mathbf{i}_x+\frac{\partial E_x}{\partial z}\mathbf{i}_y=-\mu\frac{\partial \mathbf{H}}{\partial t} \tag{1}$$

$$-\frac{\partial H_y}{\partial z}\mathbf{i}_x+\frac{\partial H_x}{\partial z}\mathbf{i}_y=\varepsilon\frac{\partial \mathbf{E}}{\partial t} \tag{2}$$

$$\varepsilon\frac{\partial E_z}{\partial z}=0 \tag{3}$$

$$\mu\frac{\partial H_z}{\partial z}=0 \tag{4}$$

These relations tell us that at best E_z and H_z are constant in time and space. Because they are uncoupled, in the absence of sources we take them to be zero. By separating vector components in (1) and (2) we see that E_x is coupled to H_y and E_y is coupled to H_x:

$$\begin{aligned}\frac{\partial E_x}{\partial z}&=-\mu\frac{\partial H_y}{\partial t}, &\quad \frac{\partial E_y}{\partial z}&=\mu\frac{\partial H_x}{\partial t}\\ \frac{\partial H_y}{\partial z}&=-\varepsilon\frac{\partial E_x}{\partial t}, &\quad \frac{\partial H_x}{\partial z}&=\varepsilon\frac{\partial E_y}{\partial t}\end{aligned} \tag{5}$$

forming two sets of independent equations. Each solution has perpendicular electric and magnetic fields. The power flow $\mathbf{S}=\mathbf{E}\times\mathbf{H}$ for each solution is z directed also being perpendicular to $\mathbf{E}$ and $\mathbf{H}$. Since the fields and power flow are mutually perpendicular, such solutions are called transverse electromagnetic waves (TEM). They are waves because if we take $\partial/\partial z$ of the upper equations and $\partial/\partial t$ of the lower equations and solve for the electric fields, we obtain one-dimensional wave equations:

$$\frac{\partial^2 E_x}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2},\qquad \frac{\partial^2 E_y}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2} \tag{6}$$

where c is the speed of the wave,

$$c=\frac{1}{\sqrt{\varepsilon\mu}}=\frac{1}{\sqrt{\varepsilon_0\mu_0}\sqrt{\varepsilon_r\mu_r}}\approx\frac{3\times 10^8}{\sqrt{\varepsilon_r\mu_r}}\text{ m/sec} \tag{7}$$

In free space, where $\varepsilon_r=1$ and $\mu_r=1$, this quantity equals the speed of light in vacuum which demonstrated that light is a transverse electromagnetic wave. If we similarly take $\partial/\partial t$ of the upper and $\partial/\partial z$ of the lower equations in (5), we obtain wave equations in the magnetic fields:

$$\frac{\partial^2 H_y}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 H_y}{\partial t^2},\qquad \frac{\partial^2 H_x}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 H_x}{\partial t^2} \tag{8}$$

7-3-2 The Wave Equation

(a) Solutions

These equations arise in many physical systems, so their solutions are well known. Working with the E_x and H_y equations, the solutions are

$$\begin{aligned} E_x(z,t)&=E_+(t-z/c)+E_-(t+z/c)\\ H_y(z,t)&=H_+(t-z/c)+H_-(t+z/c)\end{aligned} \tag{9}$$

where the functions E_+, E_-, H_+, and H_- depend on initial conditions in time and boundary conditions in space. These solutions can be easily verified by defining the arguments α and β with their resulting partial derivatives as

$$\begin{aligned} \alpha&=t-\frac{z}{c}\Rightarrow\frac{\partial\alpha}{\partial t}=1,\qquad \frac{\partial\alpha}{\partial z}=-\frac{1}{c}\\ \beta&=t+\frac{z}{c}\Rightarrow\frac{\partial\beta}{\partial t}=1,\qquad \frac{\partial\beta}{\partial z}=\frac{1}{c}\end{aligned} \tag{10}$$

and realizing that the first partial derivatives of $E_x(z,t)$ are

$$\begin{aligned} \frac{\partial E_x}{\partial t}&=\frac{dE_+}{d\alpha}\frac{\partial\alpha}{\partial t}+\frac{dE_-}{d\beta}\frac{\partial\beta}{\partial t}\\ &=\frac{dE_+}{d\alpha}+\frac{dE_-}{d\beta}\\ \frac{\partial E_x}{\partial z}&=\frac{dE_+}{d\alpha}\frac{\partial\alpha}{\partial z}+\frac{dE_-}{d\beta}\frac{\partial\beta}{\partial z}\\ &=\frac{1}{c}\left(-\frac{dE_+}{d\alpha}+\frac{dE_-}{d\beta}\right)\end{aligned} \tag{11}$$

The second derivatives are then

$$\begin{aligned}
\frac{\partial^2 E_x}{\partial t^2} &= \frac{d^2 E_+}{d\alpha^2}\frac{\partial\alpha}{\partial t} + \frac{d^2 E_-}{d\beta^2}\frac{\partial\beta}{\partial t} \\
&= \frac{d^2 E_+}{d\alpha^2} + \frac{d^2 E_-}{d\beta^2} \\
\frac{\partial^2 E_x}{\partial z^2} &= \frac{1}{c}\left(-\frac{d^2 E_+}{d\alpha^2}\frac{\partial\alpha}{\partial z} + \frac{d^2 E_-}{d\beta^2}\frac{\partial\beta}{\partial z}\right) \\
&= \frac{1}{c^2}\left(\frac{d^2 E_+}{d\alpha^2} + \frac{d^2 E_-}{d\beta^2}\right) = \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2}
\end{aligned} \tag{12}$$

which satisfies the wave equation of (6). Similar operations apply for H_y, E_y, and H_x.

In (9), the pair H_+ and E_+ as well as the pair H_- and E_- are not independent, as can be seen by substituting the solutions of (9) back into (5) and using (11):

$$\frac{\partial E_x}{\partial z} = -\mu\frac{\partial H_y}{\partial t} \Rightarrow \frac{1}{c}\left(-\frac{dE_+}{d\alpha} + \frac{dE_-}{d\beta}\right) = -\mu\left(\frac{dH_+}{d\alpha} + \frac{dH_-}{d\beta}\right) \tag{13}$$

The functions of α and β must separately be equal,

$$\frac{d}{d\alpha}(E_+ - \mu c H_+) = 0, \qquad \frac{d}{d\beta}(E_- + \mu c H_-) = 0 \tag{14}$$

which requires that

$$E_+ = \mu c H_+ = \sqrt{\frac{\mu}{\varepsilon}}H_+, \qquad E_- = -\mu c H_- = -\sqrt{\frac{\mu}{\varepsilon}}H_- \tag{15}$$

where we use (7). Since $\sqrt{\mu/\varepsilon}$ has units of Ohms, this quantity is known as the wave impedance η,

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \approx 120\pi\sqrt{\frac{\mu_r}{\varepsilon_r}} \tag{16}$$

and has value $120\pi \approx 377$ ohm in free space ($\mu_r = 1$, $\varepsilon_r = 1$).

The power flux density in TEM waves is

$$\begin{aligned}
\boldsymbol{S} = \mathbf{E}\times\mathbf{H} &= [E_+(t - z/c) + E_-(t + z/c)]\mathbf{i}_x \\
&\qquad \times [H_+(t - z/c) + H_-(t + z/c)]\mathbf{i}_y \\
&= (E_+H_+ + E_-H_- + E_-H_+ + E_+H_-)\mathbf{i}_z
\end{aligned} \tag{17}$$

Using (15) and (16) this result can be written as

$$S_z = \frac{1}{\eta}(E_+^2 - E_-^2) \tag{18}$$

where the last two cross terms in (17) cancel because of the minus sign relating E_- to H_- in (15). For TEM waves the total power flux density is due to the difference in power densities between the squares of the positively z-directed and negatively z-directed waves.

(b) Properties

The solutions of (9) are propagating waves at speed c. To see this, let us examine $E_+(t-z/c)$ and consider the case where at $z=0$, $E_+(t)$ is the staircase pulse shown in Figure 7-3*a*. In Figure 7-3*b* we replace the argument t by $t-z/c$. As long as the function E_+ is plotted versus its argument, no matter what its argument is, the plot remains unchanged. However, in Figure 7-3*c* the function $E_+(t-z/c)$ is plotted versus t resulting in the pulse being translated in time by an amount z/c. To help in plotting this translated function, we use the following logic:

(i) The pulse jumps to amplitude E_0 when the argument is zero. When the argument is $t-z/c$, this occurs for $t=z/c$.

(ii) The pulse jumps to amplitude $2E_0$ when the argument is T. When the argument is $t-z/c$, this occurs for $t=T+z/c$.

(iii) The pulse returns to zero when the argument is $2T$. For the argument $t-z/c$, we have $t=2T+z/c$.

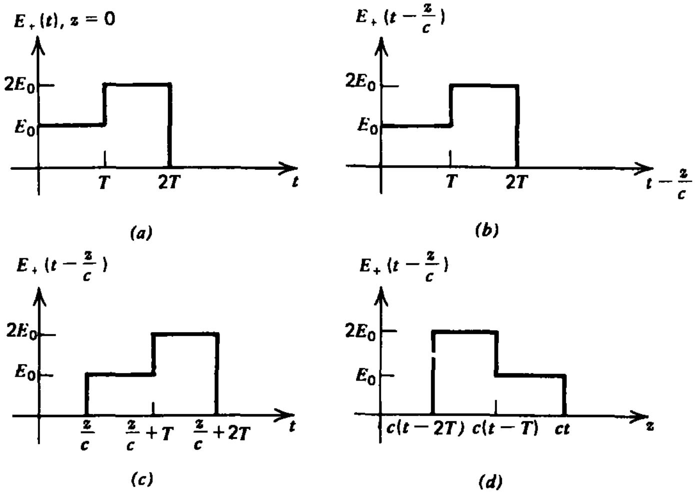

Figure 7-3 (a) $E_+(t)$ at $z=0$ is a staircase pulse. (b) $E_+(\phi)$ always has the same shape as (a) when plotted versus ϕ, no matter what ϕ is. Here $\phi=t-z/c$. (c) When plotted versus t, the pulse is translated in time where z must be positive to keep t positive. (d) When plotted versus z, it is translated and inverted. The pulse propagates at speed c in the positive z direction.

Note that z can only be positive as causality imposes the condition that time can only be increasing. The response at any positive position z to an initial E_+ pulse imposed at $z=0$ has the same shape in time but occurs at a time z/c later. The pulse travels the distance z at the speed c. This is why the function $E_+(t-z/c)$ is called a positively traveling wave.

In Figure 7-3d we plot the same function versus z. Its appearance is inverted as that part of the pulse generated first (step of amplitude E_0) will reach any positive position z first. The second step of amplitude $2E_0$ has not traveled as far since it was generated a time T later. To help in plotting, we use the same criterion on the argument as used in the plot versus time, only we solve for z. The important rule we use is that as long as the argument of a function remains constant, the value of the function is unchanged, no matter how the individual terms in the argument change.

Thus, as long as

$$t-z/c=\text{const} \tag{19}$$

$E_+(t-z/c)$ is unchanged. As time increases, so must z to satisfy (19) at the rate

$$t-\frac{z}{c}=\text{const} \Rightarrow \frac{dz}{dt}=c \tag{20}$$

to keep the E_+ function constant.

For similar reasons $E_-(t+z/c)$ represents a traveling wave at the speed c in the negative z direction as an observer must move to keep the argument $t+z/c$ constant at speed:

$$t+\frac{z}{c}=\text{const} \Rightarrow \frac{dz}{dt}=-c \tag{21}$$

as demonstrated for the same staircase pulse in Figure 7-4. Note in Figure 7-4d that the pulse is not inverted when plotted versus z as it was for the positively traveling wave, because that part of the pulse generated first (step of amplitude E_0) reaches the maximum distance but in the negative z direction. These differences between the positively and negatively traveling waves are functionally due to the difference in signs in the arguments $(t-z/c)$ and $(t+z/c)$.

7-3-3 Sources of Plane Waves

These solutions are called plane waves because at any constant z plane the fields are constant and do not vary with the x and y coordinates.

The idealized source of a plane wave is a time varying current sheet of infinite extent that we take to be x directed,

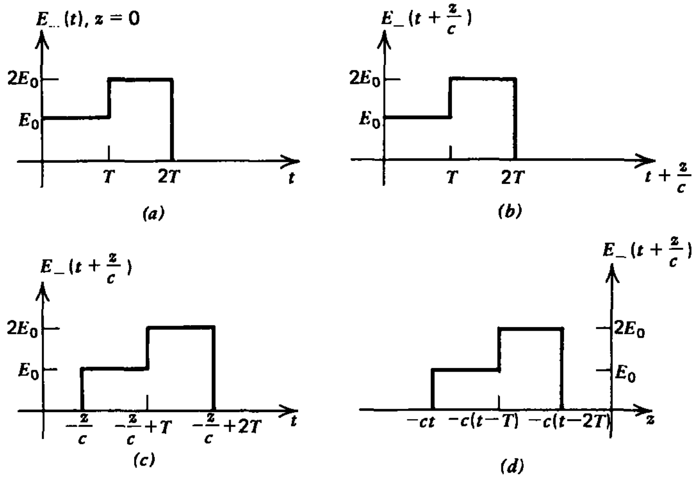

Figure 7-4 (a) $E_-(t)$ at $z=0$ is a staircase pulse. (b) $E_-(\phi)$ always has the same form of (a) when plotted versus ϕ. Here $\phi = t + z/c$. (c) When plotted versus t, the pulse is translated in time where z must be negative to keep t positive. (d) When plotted versus z, it is translated but not inverted.

as shown in Figure 7-5. From the boundary condition on the discontinuity of tangential **H**, we find that the x-directed current sheet gives rise to a y-directed magnetic field:

$$H_y(z=0_+) - H_y(z=0_-) = -K_x(t) \tag{22}$$

In general, a uniform current sheet gives rise to a magnetic field perpendicular to the direction of current flow but in the plane of the sheet. Thus to generate an x-directed magnetic field, a y-directed surface current is required.

Since there are no other sources, the waves must travel away from the sheet so that the solutions on each side of the sheet are of the form

$$H_y(z,t) = \begin{cases} H_+(t - z/c) \\ H_-(t + z/c) \end{cases} \qquad E_x(z,t) = \begin{cases} \eta H_+(t - z/c), & z > 0 \\ -\eta H_-(t + z/c), & z < 0 \end{cases} \tag{23}$$

For $z > 0$, the waves propagate only in the positive z direction. In the absence of any other sources or boundaries, there can be no negatively traveling waves in this region. Similarly for $z < 0$, we only have waves propagating in the $-z$ direction. In addition to the boundary condition of (22), the tangential component of **E** must be continuous across the sheet at $z = 0$

$$\left.\begin{aligned} H_+(t) - H_-(t) &= -K_x(t) \\ \eta[H_+(t) + H_-(t)] &= 0 \end{aligned}\right\} \Rightarrow H_+(t) = -H_-(t) = \frac{-K_x(t)}{2} \tag{24}$$

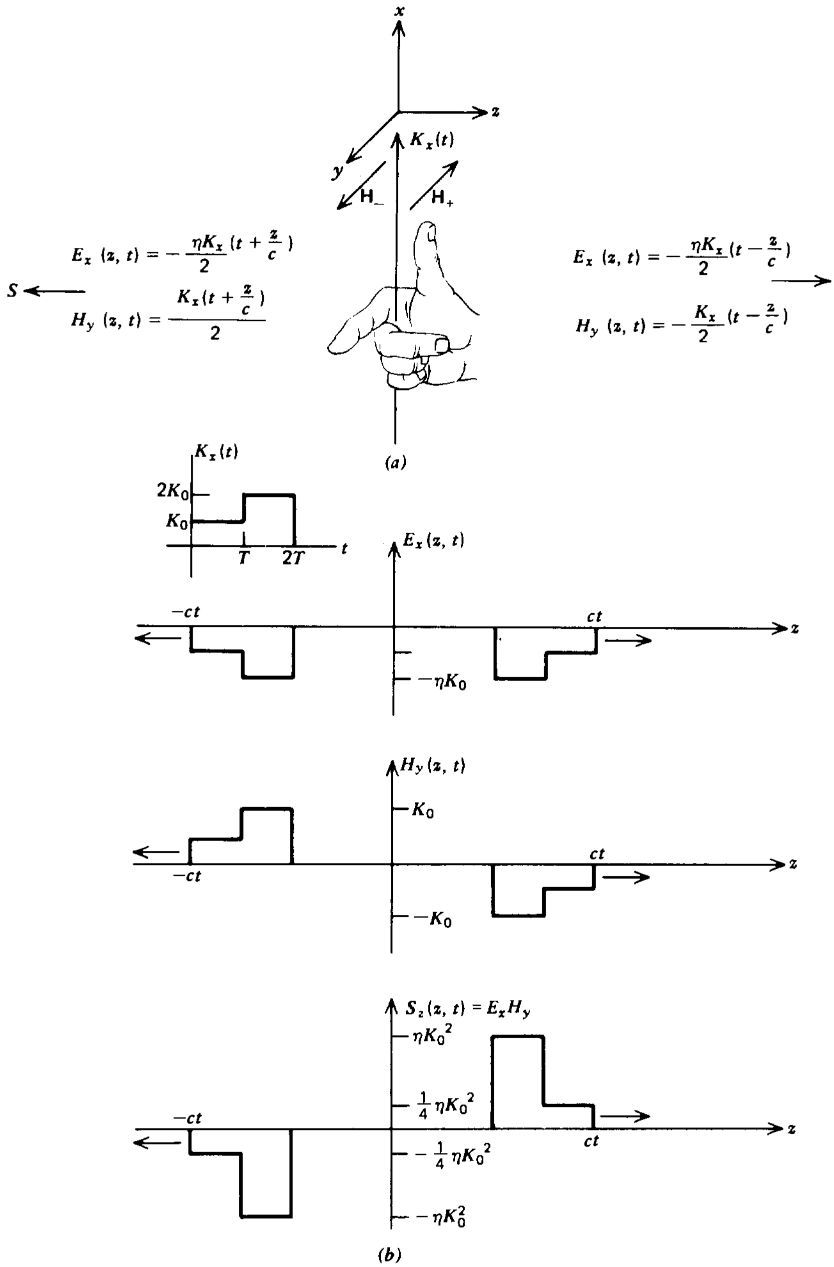

Figure 7-5 (*a*) A linearly polarized plane wave is generated by an infinite current sheet. The electric field is in the direction opposite to the current on either side of the sheet. The magnetic field is perpendicular to the current but in the plane of the current sheet and in opposite directions as given by the right-hand rule on either side of the sheet. The power flow $\mathbf{S}$ is thus perpendicular to the current and to the sheet. (*b*) The field solutions for $t > 2T$ if the current source is a staircase pulse in time.

so that the electric and magnetic fields have the same shape as the current. Because the time and space shape of the fields remains unchanged as the waves propagate, linear dielectric media are said to be nondispersive.

Note that the electric field at $z=0$ is in the opposite direction as the current, so the power per unit area delivered by the current sheet,

$$-\mathbf{E}(z=0,t)\cdot\mathbf{K}_x(t)=\frac{\eta K_x^2(t)}{2} \tag{25}$$

is equally carried away by the Poynting vector on each side of the sheet:

$$\mathbf{S}(z \doteq 0)=\mathbf{E}\times\mathbf{H}=\begin{cases}\dfrac{\eta K_x^2(t)}{4}\mathbf{i}_z, & z>0\\[2ex] \dfrac{-\eta K_x^2(t)}{4}\mathbf{i}_z, & z<0\end{cases} \tag{26}$$

7-3-4 A Brief Introduction to the Theory of Relativity

Maxwell's equations show that electromagnetic waves propagate at the speed $c_0=1/\sqrt{\varepsilon_0\mu_0}$ in vacuum. Our natural intuition would tell us that if we moved at a speed v we would measure a wave speed of c_0-v when moving in the same direction as the wave, and a speed c_0+v when moving in the opposite direction. However, our intuition would be wrong, for nowhere in the free space, source-free Maxwell's equations does the speed of the observer appear. Maxwell's equations predict that the speed of electromagnetic waves is c_0 for all observers no matter their relative velocity. This assumption is a fundamental postulate of the theory of relativity and has been verified by all experiments. The most notable experiment was performed by A. A. Michelson and E. W. Morley in the late nineteenth century, where they showed that the speed of light reflected between mirrors is the same whether it propagated in the direction parallel or perpendicular to the velocity of the earth. This postulate required a revision of the usual notions of time and distance.

If the surface current sheet of Section 7-3-3 is first turned on at $t=0$, the position of the wave front on either side of the sheet at time t later obeys the equality

$$z^2-c_0^2t^2=0 \tag{27}$$

Similarly, an observer in a coordinate system moving with constant velocity $u\mathbf{i}_z$ which is aligned with the current sheet at

$t=0$ finds the wavefront position to obey the equality

$$z'^2 - c_0^2 t'^2 = 0 \tag{28}$$

The two coordinate systems must be related by a linear transformation of the form

$$z' = a_1 z + a_2 t, \qquad t' = b_1 z + b_2 t \tag{29}$$

The position of the origin of the moving frame ($z'=0$) as measured in the stationary frame is $z = vt$, as shown in Figure 7-6, so that a_1 and a_2 are related as

$$0 = a_1 vt + a_2 t \Rightarrow a_1 v + a_2 = 0 \tag{30}$$

We can also equate the two equalities of (27) and (28),

$$z^2 - c_0^2 t^2 = z'^2 - c_0^2 t'^2 = (a_1 z + a_2 t)^2 - c_0^2 (b_1 z + b_2 t)^2 \tag{31}$$

so that combining terms yields

$$z^2(1 - a_1^2 + c_0^2 b_1^2) - c_0^2 t^2\left(1 + \frac{a_2^2}{c_0^2} - b_2^2\right) - 2(a_1 a_2 - c_0^2 b_1 b_2) zt = 0 \tag{32}$$

Since (32) must be true for all z and t, each of the coefficients must be zero, which with (30) gives solutions

$$\begin{aligned} a_1 &= \frac{1}{\sqrt{1-(v/c_0)^2}}, \qquad & b_1 &= \frac{-v/c_0^2}{\sqrt{1-(v/c_0)^2}} \\ a_2 &= \frac{-v}{\sqrt{1-(v/c_0)^2}}, \qquad & b_2 &= \frac{1}{\sqrt{1-(v/c_0)^2}} \end{aligned} \tag{33}$$

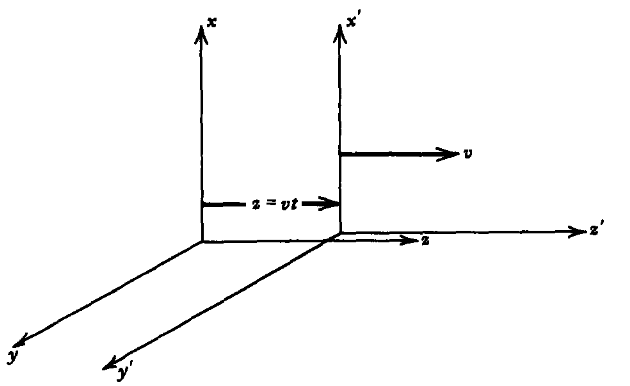

Figure 7-6 The primed coordinate system moves at constant velocity $v\mathbf{i}_z$ with respect to a stationary coordinate system. The free space speed of an electromagnetic wave is c_0 as measured by observers in either coordinate system no matter the velocity v.

The transformations of (29) are then

$$z' = \frac{z - vt}{\sqrt{1 - (v/c_0)^2}}, \qquad t' = \frac{t - vz/c_0^2}{\sqrt{1 - (v/c_0)^2}} \tag{34}$$

and are known as the Lorentz transformations. Measured lengths and time intervals are different for observers moving at different speeds. If the velocity v is much less than the speed of light, (34) reduces to the Galilean transformations,

$$\lim_{v/c \ll 1} z' \approx z - vt, \qquad t' \approx t \tag{35}$$

which describe our usual experiences at nonrelativistic speeds.

The coordinates perpendicular to the motion are unaffected by the relative velocity between reference frames

$$x' = x, \qquad y' = y \tag{36}$$

Continued development of the theory of relativity is beyond the scope of this text and is worth a course unto itself. Applying the Lorentz transformation to Newton's law and Maxwell's equations yield new results that at first appearance seem contrary to our experiences because we live in a world where most material velocities are much less than c_0. However, continued experiments on such disparate time and space scales as between atomic physics and astronomics verify the predictions of relativity theory, in part spawned by Maxwell's equations.

7-4 SINUSOIDAL TIME VARIATIONS

7-4-1 Frequency and Wavenumber

If the current sheet of Section 7-3-3 varies sinusoidally with time as $\text{Re}\,(K_0\, e^{j\omega t})$, the wave solutions require the fields to vary as $e^{j\omega(t-z/c)}$ and $e^{j\omega(t+z/c)}$:

$$H_y(z,t) = \begin{cases} \text{Re}\left(-\dfrac{K_0}{2}\, e^{j\omega(t-z/c)}\right), & z > 0 \\ \text{Re}\left(+\dfrac{K_0}{2}\, e^{j\omega(t+z/c)}\right), & z < 0 \end{cases}$$

$$E_x(z,t) = \begin{cases} \text{Re}\left(-\dfrac{\eta K_0}{2}\, e^{j\omega(t-z/c)}\right), & z > 0 \\ \text{Re}\left(-\dfrac{\eta K_0}{2}\, e^{j\omega(t+z/c)}\right), & z < 0 \end{cases} \tag{1}$$

At a fixed time the fields then also vary sinusoidally with position so that it is convenient to define the wavenumber as

$$k=\frac{2\pi}{\lambda}=\frac{\omega}{c}=\omega\sqrt{\mu\varepsilon} \tag{2}$$

where λ is the fundamental spatial period of the wave. At a fixed position the waveform is also periodic in time with period T:

$$T=\frac{1}{f}=\frac{2\pi}{\omega} \tag{3}$$

where f is the frequency of the source. Using (3) with (2) gives us the familiar frequency-wavelength formula:

$$\omega=kc \Rightarrow f\lambda=c \tag{4}$$

Throughout the electromagnetic spectrum, summarized in Figure 7-7, time varying phenomena differ only in the scaling of time and size. No matter the frequency or wavelength, although easily encompassing 20 orders of magnitude, electromagnetic phenomena are all described by Maxwell's equations. Note that visible light only takes up a tiny fraction of the spectrum.

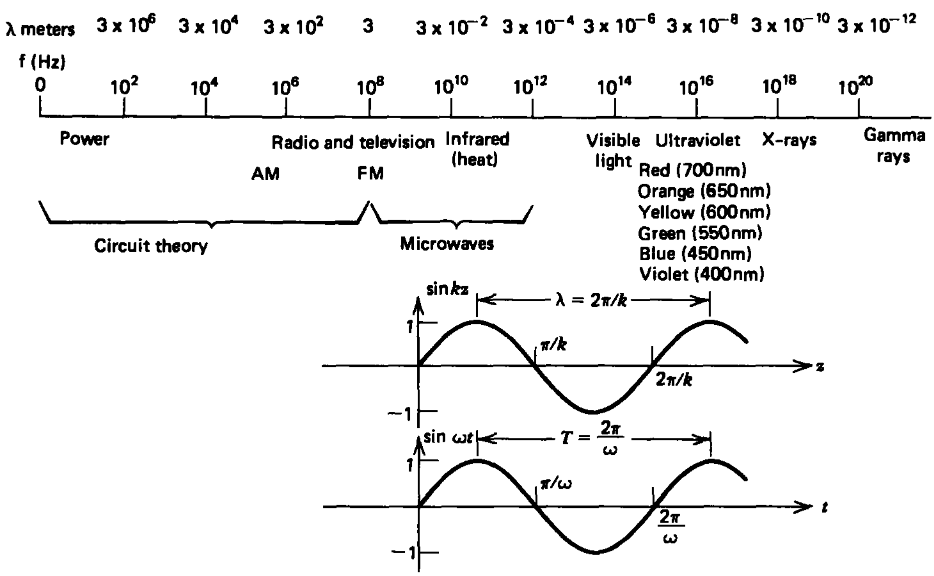

Figure 7-7 Time varying electromagnetic phenomena differ only in the scaling of time (frequency) and size (wavelength). In linear dielectric media the frequency and wavelength are related as $f\lambda=c$ ($\omega=kc$), where $c=1/\sqrt{\varepsilon\mu}$ is the speed of light in the medium.

For a single sinusoidally varying plane wave, the time-average electric and magnetic energy densities are equal because the electric and magnetic field amplitudes are related through the wave impedance η:

$$<w_m> = <w_e> = \tfrac{1}{4}\mu|\mathbf{H}|^2 = \tfrac{1}{4}\varepsilon|\mathbf{E}|^2 = \tfrac{1}{16}\mu K_0^2 \qquad (5)$$

From the complex Poynting theorem derived in Section 7-2-4, we then see that in a lossless region with no sources for $|z|>0$ that $\hat{P}_d = 0$ so that the complex Poynting vector has zero divergence. With only one-dimensional variations with z, this requires the time-average power density to be a constant throughout space on each side of the current sheet:

$$\begin{aligned} <S> &= \tfrac{1}{2}\,\mathrm{Re}\,[\hat{\mathbf{E}}(\mathbf{r})\times\hat{\mathbf{H}}^*(\mathbf{r})] \\ &= \begin{cases} \tfrac{1}{8}\eta K_0^2\mathbf{i}_z, & z>0 \\ -\tfrac{1}{8}\eta K_0^2\mathbf{i}_z, & z<0 \end{cases} \end{aligned} \qquad (6)$$

The discontinuity in $<S>$ at $z=0$ is due to the power output of the source.

7-4-2 Doppler Frequency Shifts

If the sinusoidally varying current sheet $\mathrm{Re}\,(K_0\, e^{j\omega t})$ moves with constant velocity $v\mathbf{i}_z$, as in Figure 7-8, the boundary conditions are no longer at $z=0$ but at $z=vt$. The general form of field solutions are then:

$$\begin{aligned} H_y(z,t) &= \begin{cases} \mathrm{Re}\,(\hat{H}_+\, e^{j\omega_+(t-z/c)}), & z>vt \\ \mathrm{Re}\,(\hat{H}_-\, e^{j\omega_-(t+z/c)}), & z<vt \end{cases} \\ E_x(z,t) &= \begin{cases} \mathrm{Re}\,(\eta\hat{H}_+\, e^{j\omega_+(t-z/c)}), & z>vt \\ \mathrm{Re}\,(-\eta\hat{H}_-\, e^{j\omega_-(t+z/c)}), & z<vt \end{cases} \end{aligned} \qquad (7)$$

where the frequencies of the fields ω_+ and ω_- on each side of the sheet will be different from each other as well as differing from the frequency of the current source ω. We assume $v/c \ll 1$ so that we can neglect relativistic effects discussed in Section 7-3-4. The boundary conditions

$$\begin{aligned} E_{x+}(z=vt) = E_{x-}(z=vt) &\Rightarrow \hat{H}_+\, e^{j\omega_+t(1-v/c)} = -\hat{H}_-\, e^{j\omega_-t(1+v/c)} \\ H_{y+}(z=vt) - H_{y-}(z=vt) &= -K_x \\ &\Rightarrow \hat{H}_+\, e^{j\omega_+t(1-v/c)} - \hat{H}_-\, e^{j\omega_-t(1+v/c)} = -K_0\, e^{j\omega t} \end{aligned} \qquad (8)$$

must be satisfied for all values of t so that the exponential time factors in (8) must all be equal, which gives the shifted

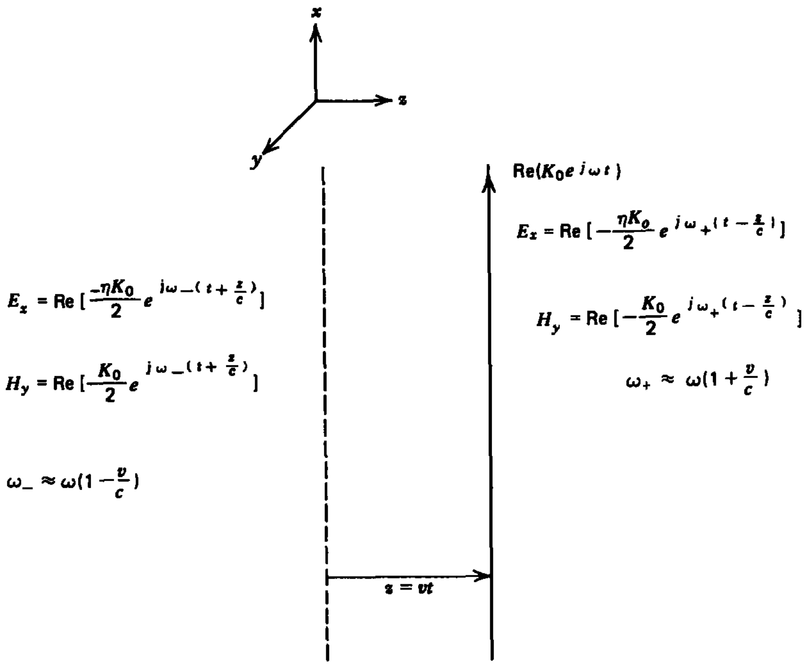

Figure 7-8 When a source of electromagnetic waves moves towards an observer, the frequency is raised while it is lowered when it moves away from an observer.

frequencies on each side of the sheet as

$$\omega_+ = \frac{\omega}{1 - v/c} \approx \omega\left(1 + \frac{v}{c}\right),$$
$$\omega_- = \frac{\omega}{1 + v/c} \approx \omega\left(1 - \frac{v}{c}\right) \Rightarrow \hat{H}_+ = -\hat{H}_- = -\frac{K_0}{2} \tag{9}$$

where $v/c \ll 1$. When the source is moving towards an observer, the frequency is raised while it is lowered when it moves away. Such frequency changes due to the motion of a source or observer are called Doppler shifts and are used to measure the velocities of moving bodies in radar systems. For $v/c \ll 1$, the frequency shifts are a small percentage of the driving frequency, but in absolute terms can be large enough to be easily measured. At a velocity $v = 300$ m/sec with a driving frequency of $f = 10^{10}$ Hz, the frequency is raised and lowered on each side of the sheet by $\Delta f = \pm f(v/c) = \pm 10^4$ Hz.

7-4-3 Ohmic Losses

Thus far we have only considered lossless materials. If the medium also has an Ohmic conductivity σ, the electric field will cause a current flow that must be included in Ampere's law:

$$\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t}$$
$$\frac{\partial H_y}{\partial z} = -J_x - \varepsilon \frac{\partial E_x}{\partial t} = -\sigma E_x - \varepsilon \frac{\partial E_x}{\partial t} \tag{10}$$

where for conciseness we only consider the x-directed electric field solution as the same results hold for the E_y, H_x solution. Our wave solutions of Section 7-3-2 no longer hold with this additional term, but because Maxwell's equations are linear with constant coefficients, for sinusoidal time variations the solutions in space must also be exponential functions, which we write as

$$E_x(z, t) = \text{Re}\,(\hat{E}_0\, e^{j(\omega t - kz)})$$
$$H_y(z, t) = \text{Re}\,(\hat{H}_0\, e^{j(\omega t - kz)}) \tag{11}$$

where $\hat{E}_0$ and $\hat{H}_0$ are complex amplitudes and the wavenumber k is no longer simply related to ω as in (4) but is found by substituting (11) back into (10):

$$-jk\hat{E}_0 = -j\omega\mu\hat{H}_0$$
$$-jk\hat{H}_0 = -j\omega\varepsilon(1 + \sigma/j\omega\varepsilon)\hat{E}_0 \tag{12}$$

This last relation was written in a way that shows that the conductivity enters in the same way as the permittivity so that we can define a complex permittivity $\hat{\varepsilon}$ as

$$\hat{\varepsilon} = \varepsilon(1 + \sigma/j\omega\varepsilon) \tag{13}$$

Then the solutions to (12) are

$$\frac{\hat{E}_0}{\hat{H}_0} = \frac{\omega\mu}{k} = \frac{k}{\omega\hat{\varepsilon}} \Rightarrow k^2 = \omega^2\mu\hat{\varepsilon} = \omega^2\mu\varepsilon\left(1 + \frac{\sigma}{j\omega\varepsilon}\right) \tag{14}$$

which is similar in form to (2) with a complex permittivity.

There are two interesting limits of (14):

(a) Low Loss Limit

If the conductivity is small so that $\sigma/\omega\varepsilon \ll 1$, then the solution of (14) reduces to

$$\lim_{\sigma/\omega\varepsilon \ll 1} k = \pm\omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma}{2j\omega\varepsilon}\right) = \pm\left(\frac{\omega}{c} - \frac{j\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}\right) \tag{15}$$

where c is the speed of the light in the medium if there were no losses, $c = 1/\sqrt{\mu\varepsilon}$. Because of the spatial exponential dependence in (11), the real part of k is the same as for the lossless case and represents the sinusoidal spatial distribution of the fields. The imaginary part of k represents the exponential decay of the fields due to the Ohmic losses with exponential decay length $\frac{1}{2}\sigma\eta$, where $\eta = \sqrt{\mu/\varepsilon}$ is the wave impedance. Note that for waves traveling in the positive z direction we take the upper positive sign in (15) using the lower negative sign for negatively traveling waves so that the solutions all decay and do not grow for distances far from the source. This solution is only valid for small σ so that the wave is only slightly damped as it propagates, as illustrated in Figure 7-9a.

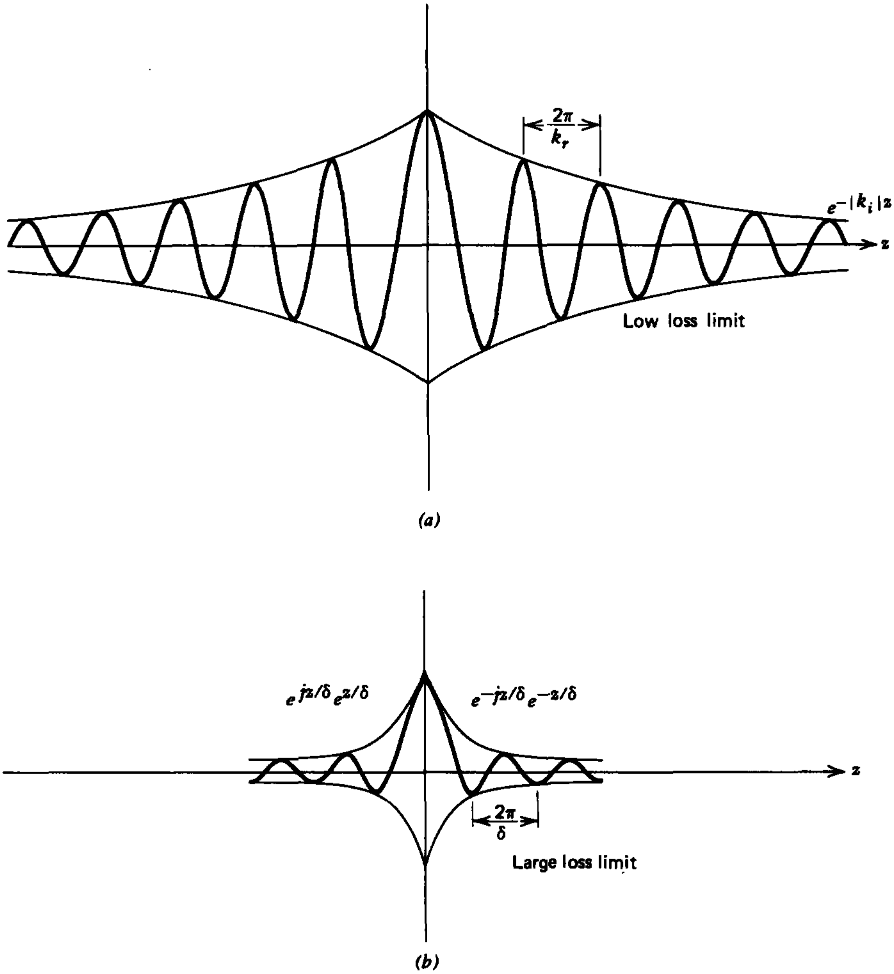

Figure 7-9 (a) In a slightly lossy dielectric, the fields decay away from a source at a slow rate while the wavelength is essentially unchanged. (b) In the large loss limit the spatial decay rate is equal to the skin depth. The wavelength also equals the skin depth.

(b) Large Loss Limit

In the other extreme of a highly conducting material so that $\sigma/\omega\varepsilon \gg 1$, (14) reduces to

$$\lim_{\sigma/\omega\varepsilon \gg 1} k^2 \approx -j\omega\mu\sigma \Rightarrow k = \pm\frac{(1-j)}{\delta}, \qquad \delta = \sqrt{\frac{2}{\omega\mu\sigma}} \tag{16}$$

where δ is just the skin depth found in Section 6-4-3 for magneto-quasi-static fields within a conductor. The skin-depth term also arises for electrodynamic fields because the large loss limit has negligible displacement current compared to the conduction currents.

Because the real and imaginary part of k have equal magnitudes, the spatial decay rate is large so that within a few oscillation intervals the fields are negligibly small, as illustrated in Figure 7-9*b*. For a metal like copper with $\mu = \mu_0 = 4\pi \times 10^{-7}$ henry/m and $\sigma \approx 6 \times 10^7$ siemens/m at a frequency of 1 MHz, the skin depth is $\delta \approx 6.5 \times 10^{-5}$ m.

7-4-4 High-Frequency Wave Propagation in Media

Ohm's law is only valid for frequencies much below the collision frequencies of the charge carriers, which is typically on the order of 10^{13} Hz. In this low-frequency regime the inertia of the particles is negligible. For frequencies much higher than the collision frequency the inertia dominates and the current constitutive law for a single species of charge carrier q with mass m and number density n is as found in Section 3-2-2*d*:

$$\partial \mathbf{J}_f/\partial t = \omega_p^2 \varepsilon \mathbf{E} \tag{17}$$

where $\omega_p = \sqrt{q^2 n/m\varepsilon}$ is the plasma frequency. This constitutive law is accurate for radio waves propagating in the ionosphere, for light waves propagating in many dielectrics, and is also valid for superconductors where the collision frequency is zero.

Using (17) rather than Ohm's law in (10) for sinusoidal time and space variations as given in (11), Maxwell's equations are

$$\begin{aligned} \frac{\partial E_x}{\partial z} &= -\mu\frac{\partial H_y}{\partial t} \Rightarrow -jk\hat{E}_0 = -j\omega\mu\hat{H}_0 \\ \frac{\partial H_y}{\partial z} &= -J_x - \varepsilon\frac{\partial E_x}{\partial t} \Rightarrow -jk\hat{H}_0 = -j\omega\varepsilon\left(1 - \frac{\omega_p^2}{\omega^2}\right)\hat{E}_0 \end{aligned} \tag{18}$$

The effective permittivity is now frequency dependent:

$$\hat{\varepsilon} = \varepsilon(1 - \omega_p^2/\omega^2) \tag{19}$$

The solutions to (18) are

$$\frac{\hat{E}_0}{\hat{H}_0}=\frac{\omega\mu}{k}=\frac{k}{\omega\hat{\varepsilon}}\Rightarrow k^2=\omega^2\mu\hat{\varepsilon}=\frac{\omega^2-\omega_p^2}{c^2} \tag{20}$$

For $\omega>\omega_p$, k is real and we have pure propagation where the wavenumber depends on the frequency. For $\omega<\omega_p$, k is imaginary representing pure exponential decay.

Poynting's theorem for this medium is

$$\begin{aligned}\nabla\cdot\boldsymbol{S}+\frac{\partial}{\partial t}\left(\tfrac{1}{2}\varepsilon|\mathbf{E}|^2+\tfrac{1}{2}\mu|\mathbf{H}|^2\right)&=-\mathbf{E}\cdot\mathbf{J}_f=-\frac{1}{\omega_p^2\varepsilon}\mathbf{J}_f\cdot\frac{\partial\mathbf{J}_f}{\partial t}\\&=-\frac{\partial}{\partial t}\left(\frac{1}{\omega_p^2\varepsilon}\frac{|\mathbf{J}_f|^2}{2}\right)\end{aligned} \tag{21}$$

Because this system is lossless, the right-hand side of (21) can be brought to the left-hand side and lumped with the energy densities:

$$\nabla\cdot\boldsymbol{S}+\frac{\partial}{\partial t}\left[\tfrac{1}{2}\varepsilon|\mathbf{E}|^2+\tfrac{1}{2}\mu|\mathbf{H}|^2+\frac{1}{2}\frac{1}{\omega_p^2\varepsilon}|\mathbf{J}_f|^2\right]=0 \tag{22}$$

This new energy term just represents the kinetic energy density of the charge carriers since their velocity is related to the current density as

$$\mathbf{J}_f=qn\mathbf{v}\Rightarrow\frac{1}{2}\frac{1}{\omega_p^2\varepsilon}|\mathbf{J}_f|^2=\tfrac{1}{2}mn|\mathbf{v}|^2 \tag{23}$$

7-4-5 Dispersive Media

When the wavenumber is not proportional to the frequency of the wave, the medium is said to be dispersive. A nonsinusoidal time signal (such as a square wave) will change shape and become distorted as the wave propagates because each Fourier component of the signal travels at a different speed.

To be specific, consider a stationary current sheet source at $z=0$ composed of two signals with slightly different frequencies:

$$\begin{aligned}K(t)&=K_0[\cos(\omega_0+\Delta\omega)t+\cos(\omega_0-\Delta\omega)t]\\&=2K_0\cos\Delta\omega t\cos\omega_0 t\end{aligned} \tag{24}$$

With $\Delta\omega\ll\omega$ the fast oscillations at frequency ω_0 are modulated by the slow envelope function at frequency $\Delta\omega$. In a linear dielectric medium this wave packet would propagate away from the current sheet at the speed of light, $c=1/\sqrt{\varepsilon\mu}$).

If the medium is dispersive with the wavenumber $k(\omega)$ being a function of ω, each frequency component in (24) travels at a slightly different speed. Since each frequency is very close to ω_0 we expand $k(\omega)$ as

$$\begin{aligned} k(\omega_0+\Delta\omega) &\approx k(\omega_0)+\frac{dk}{d\omega}\bigg|_{\omega_0}\Delta\omega \\ k(\omega_0-\Delta\omega) &\approx k(\omega_0)-\frac{dk}{d\omega}\bigg|_{\omega_0}\Delta\omega \end{aligned} \tag{25}$$

where for propagation $k(\omega_0)$ must be real.

The fields for waves propagating in the $+z$ direction are then of the following form:

$$\begin{aligned} E_x(z,t) &= \text{Re}\,\hat{E}_0\left(\exp\left\{j\left[(\omega_0+\Delta\omega)t-\left(k(\omega_0)+\frac{dk}{d\omega}\bigg|_{\omega_0}\Delta\omega\right)z\right]\right\}\right. \\ &\quad \left.+\exp\left\{j\left[(\omega_0-\Delta\omega)t-\left(k(\omega_0)-\frac{dk}{d\omega}\bigg|_{\omega_0}\Delta\omega\right)z\right]\right\}\right) \\ &= \text{Re}\left(\hat{E}_0\exp\{j[\omega_0 t-k(\omega_0)z]\}\left\{\exp\left[j\,\Delta\omega\left(t-\frac{dk}{d\omega}\bigg|_{\omega_0}z\right)\right]\right.\right. \\ &\quad \left.\left.+\exp\left[-j\,\Delta\omega\left(t-\frac{dk}{d\omega}\bigg|_{\omega_0}z\right)\right]\right\}\right) \\ &= 2E_0\cos(\omega_0 t-k(\omega_0)z)\cos\Delta\omega\left(t-\frac{dk}{d\omega}\bigg|_{\omega_0}z\right) \end{aligned} \tag{26}$$

where without loss of generality we assume in the last relation that $\hat{E}_0=E_0$ is real. This result is plotted in Figure 7-10 as a function of z for fixed time. The fast waves with argument $\omega_0 t-k(\omega_0)z$ travel at the phase speed $v_p=\omega_0/k(\omega_0)$ through the modulating envelope with argument $\Delta\omega(t-dk/d\omega|_{\omega_0}z)$. This envelope itself travels at the slow speed

$$t-\frac{dk}{d\omega}\bigg|_{\omega_0}z=\text{const}\Rightarrow\frac{dz}{dt}=v_g=\frac{d\omega}{dk}\bigg|_{\omega_0} \tag{27}$$

known as the group velocity, for it is the velocity at which a packet of waves within a narrow frequency band around ω_0 will travel.

For linear media the group and phase velocities are equal:

$$\begin{aligned} \omega = kc &\Rightarrow v_p=\frac{\omega}{k}=c \\ v_g &= \frac{d\omega}{dk}=c \end{aligned} \tag{28}$$

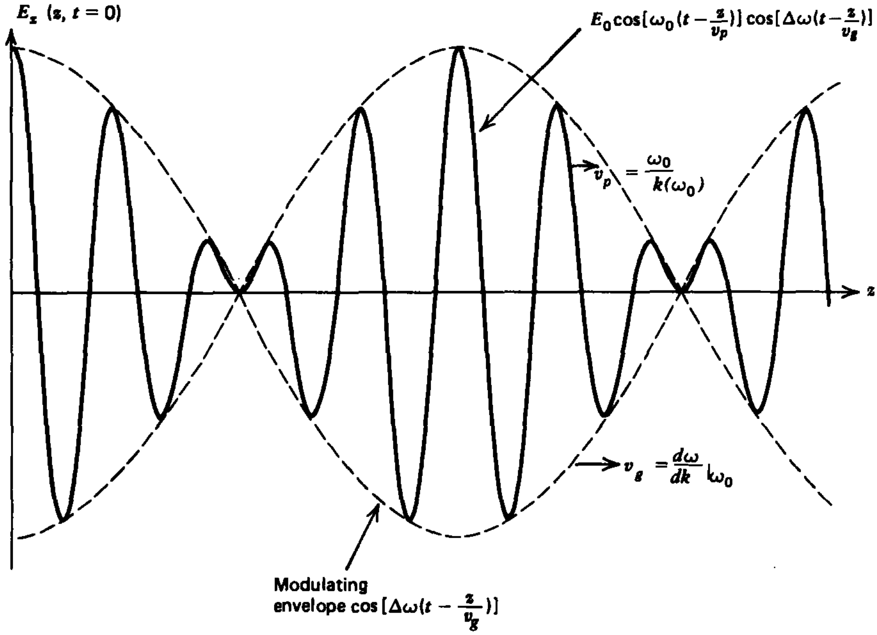

Figure 7-10 In a dispersive medium the shape of the waves becomes distorted so the velocity of a wave is not uniquely defined. For a group of signals within a narrow frequency band the modulating envelope travels at the group velocity v_g. The signal within the envelope propagates through at the phase velocity v_p.

while from Section 7-4-4 in the high-frequency limit for conductors, we see that

$$\omega^2 = k^2c^2 + \omega_p^2 \Rightarrow v_p = \frac{\omega}{k}$$
$$v_g = \frac{d\omega}{dk} = \frac{k}{\omega}c^2 \tag{29}$$

where the velocities only make sense when k is real so that $\omega > \omega_p$. Note that in this limit

$$v_g v_p = c^2 \tag{30}$$

Group velocity only has meaning in a dispersive medium when the signals of interest are clustered over a narrow frequency range so that the slope defined by (27) is approximately constant and real.

7-4-6 Polarization

The two independent sets of solutions of Section 7-3-1 both have their power flow $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ in the z direction. One solution is said to have its electric field polarized in the x direction while the second has its electric field polarized in the y direction. Each solution alone is said to be linearly polarized because the electric field always points in the same direction for all time. If both field solutions are present, the direction of net electric field varies with time. In particular, let us say that the x and y components of electric field at any value of z differ in phase by angle ϕ:

$$\mathbf{E} = \text{Re}\,[E_{x_0}\mathbf{i}_x + E_{y_0}\,e^{j\phi}\mathbf{i}_y]\,e^{j\omega t} = E_{x_0}\cos\omega t\,\mathbf{i}_x + E_{y_0}\cos(\omega t+\phi)\mathbf{i}_y \tag{31}$$

We can eliminate time as a parameter, realizing from (31) that

$$\cos\omega t = E_x/E_{x_0}$$
$$\sin\omega t = \frac{\cos\omega t\cos\phi - E_y/E_{y_0}}{\sin\phi} = \frac{(E_x/E_{x_0})\cos\phi - E_y/E_{y_0}}{\sin\phi} \tag{32}$$

and using the identity that

$$\sin^2\omega t + \cos^2\omega t = 1 = \left(\frac{E_x}{E_{x_0}}\right)^2 + \frac{(E_x/E_{x_0})^2\cos^2\phi + (E_y/E_{y_0})^2 - (2E_xE_y/E_{x_0}E_{y_0})\cos\phi}{\sin^2\phi} \tag{33}$$

to give us the equation of an ellipse relating E_x to E_y:

$$\left(\frac{E_x}{E_{x_0}}\right)^2 + \left(\frac{E_y}{E_{y_0}}\right)^2 - \frac{2E_xE_y}{E_{x_0}E_{y_0}}\cos\phi = \sin^2\phi \tag{34}$$

as plotted in Figure 7-11*a*. As time increases the electric field vector traces out an ellipse each period so this general case of the superposition of two linear polarizations with arbitrary phase ϕ is known as elliptical polarization. There are two important special cases:

(a) Linear Polarization

If E_x and E_y are in phase so that $\phi = 0$, (34) reduces to

$$\left(\frac{E_x}{E_{x_0}} - \frac{E_y}{E_{y_0}}\right)^2 = 0 \Rightarrow \tan\theta = \frac{E_y}{E_x} = \frac{E_{y_0}}{E_{x_0}} \tag{35}$$

The electric field at all times is at a constant angle θ to the x axis. The electric field amplitude oscillates with time along this line, as in Figure 7-11*b*. Because its direction is always along the same line, the electric field is linearly polarized.

(b) Circular Polarization

If both components have equal amplitudes but are 90° out of phase,

$$E_{x_0} = E_{y_0} \equiv E_0, \qquad \phi = \pm\pi/2 \tag{36}$$

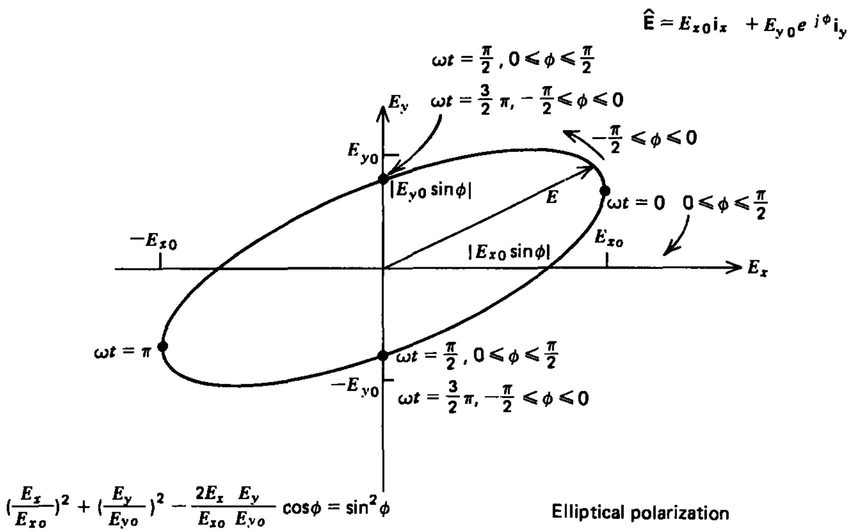

Figure 7-11 (*a*) Two perpendicular field components with phase difference ϕ have the tip of the net electric field vector tracing out an ellipse each period. (*b*) If both field components are in phase, the ellipse reduces to a straight line. (*c*) If the field components have the same magnitude but are 90° out of phase, the ellipse becomes a circle. The polarization is left circularly polarized to z-directed power flow if the electric field rotates clockwise and is (*d*) right circularly polarized if it rotates counterclockwise.

(34) reduces to the equation of a circle:

$$E_x^2 + E_y^2 = E_0^2 \tag{37}$$

The tip of the electric field vector traces out a circle as time evolves over a period, as in Figure 7-11*c*. For the upper (+) sign for ϕ in (36), the electric field rotates clockwise while the negative sign has the electric field rotating counterclockwise. These cases are, respectively, called left and right circular polarization for waves propagating in the $+z$ direction as found by placing the thumb of either hand in the direction of power flow. The fingers on the left hand curl in the direction of the rotating field for left circular polarization, while the fingers of the right hand curl in the direction of the rotating field for right circular polarization. Left and right circular polarizations reverse for waves traveling in the $-z$ direction.

7-4-7 Wave Propagation in Anisotropic Media

Many properties of plane waves have particular applications to optics. Because visible light has a wavelength on the order of 500 nm, even a pencil beam of light 1 mm wide is 2000 wavelengths wide and thus approximates a plane wave.

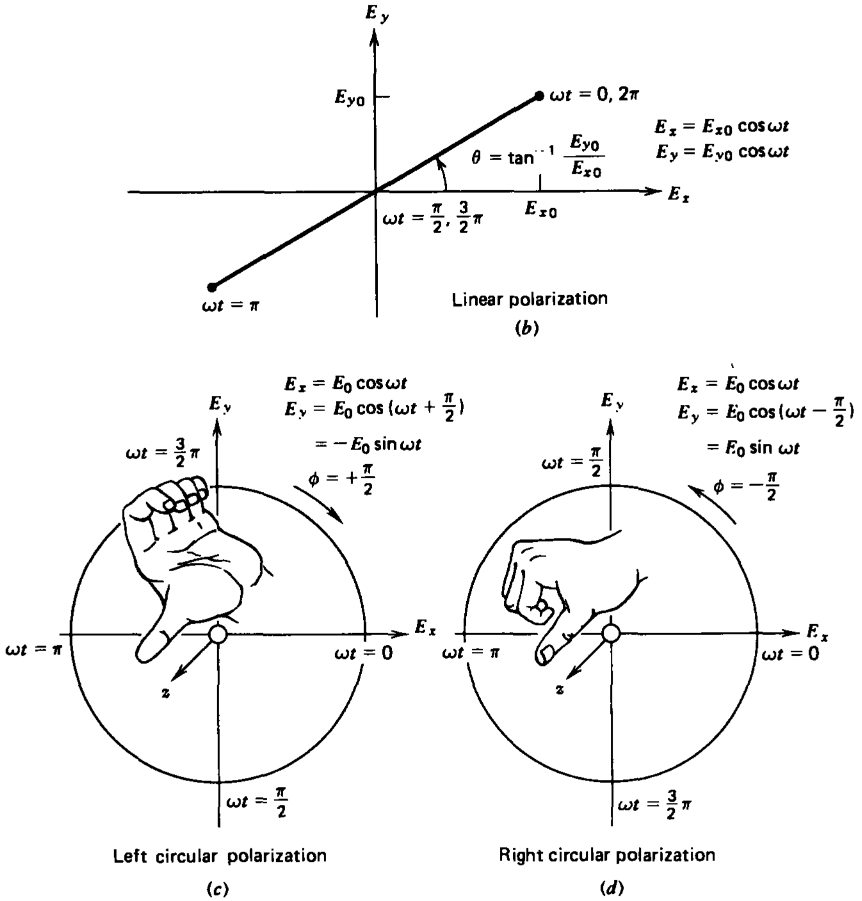

Figure 7-11

(a) Polarizers

Light is produced by oscillating molecules whether in a light bulb or by the sun. This natural light is usually unpolarized as each molecule oscillates in time and direction independent of its neighbors so that even though the power flow may be in a single direction the electric field phase changes randomly with time and the source is said to be incoherent. Lasers, an acronym for "light amplification by stimulated emission of radiation," emits coherent light by having all the oscillating molecules emit in time phase.

A polarizer will only pass those electric field components aligned with the polarizer's transmission axis so that the transmitted light is linearly polarized. Polarizers are made of such crystals as tourmaline, which exhibit dichroism—the selective absorption of the polarization along a crystal axis.

The polarization perpendicular to this axis is transmitted.

Because tourmaline polarizers are expensive, fragile, and of small size, improved low cost and sturdy sheet polarizers were developed by embedding long needlelike crystals or chainlike molecules in a plastic sheet. The electric field component in the long direction of the molecules or crystals is strongly absorbed while the perpendicular component of the electric field is passed.

For an electric field of magnitude E_0 at angle ϕ to the transmission axis of a polarizer, the magnitude of the transmitted field is

$$E_t = E_0 \cos \phi \tag{38}$$

so that the time-average power flux density is

$$\begin{aligned} <S> &= |\tfrac{1}{2}\mathrm{Re}\,[\hat{\mathbf{E}}(\mathbf{r}) \times \hat{\mathbf{H}}^*(\mathbf{r})]| \\ &= \frac{1}{2}\frac{E_0^2}{\eta}\cos^2\phi \end{aligned} \tag{39}$$

which is known as the law of Malus.

(b) Double Refraction (Birefringence)

If a second polarizer, now called the analyzer, is placed parallel to the first but with its transmission axis at right angles, as in Figure 7-12, no light is transmitted. The combination is called a polariscope. However, if an anisotropic crystal is inserted between the polarizer and analyzer, light is transmitted through the analyzer. In these doubly refracting crystals, light polarized along the optic axis travels at speed $c_\parallel$ while light polarized perpendicular to the axis travels at a slightly different speed $c_\perp$. The crystal is said to be birefringent. If linearly polarized light is incident at 45° to the axis,

$$\mathbf{E}(z=0, t) = E_0(\mathbf{i}_x + \mathbf{i}_y)\,\mathrm{Re}\,(e^{j\omega t}) \tag{40}$$

the components of electric field along and perpendicular to the axis travel at different speeds:

$$\begin{aligned} E_x(z, t) &= E_0\,\mathrm{Re}\,(e^{j(\omega t - k_\parallel z)}), \qquad k_\parallel = \omega/c_\parallel \\ E_y(z, t) &= E_0\,\mathrm{Re}\,(e^{j(\omega t - k_\perp z)}), \qquad k_\perp = \omega/c_\perp \end{aligned} \tag{41}$$

After exiting the crystal at $z = l$, the total electric field is

$$\begin{aligned} \mathbf{E}(z=l, t) &= E_0\,\mathrm{Re}\,[e^{j\omega t}(e^{-jk_\parallel l}\mathbf{i}_x + e^{-jk_\perp l}\mathbf{i}_y)] \\ &= E_0\,\mathrm{Re}\,[e^{j(\omega t - k_\parallel l)}(\mathbf{i}_x + e^{j(k_\parallel - k_\perp)l}\mathbf{i}_y)] \end{aligned} \tag{42}$$

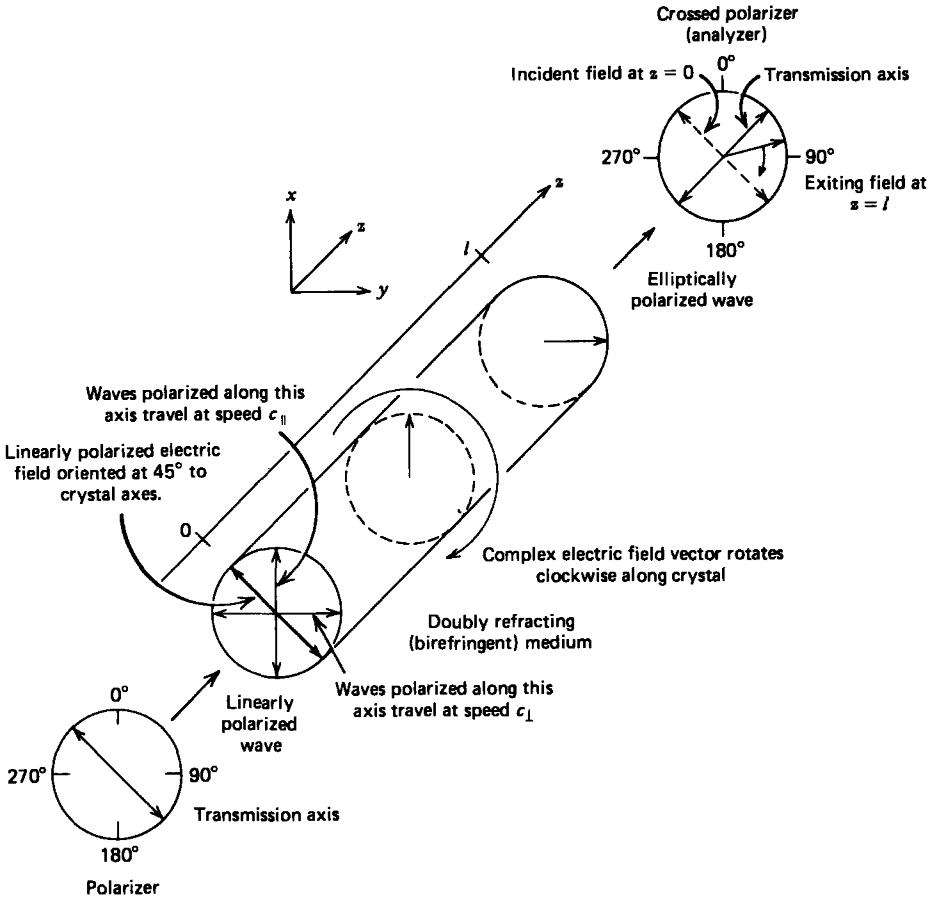

Figure 7-12 When a linearly polarized wave passes through a doubly refracting (birefringent) medium at an angle to the crystal axes, the transmitted light is elliptically polarized.

which is of the form of (31) for an elliptically polarized wave where the phase difference is

$$\phi = (k_\parallel - k_\perp)l = \omega l\left(\frac{1}{c_\parallel} - \frac{1}{c_\perp}\right) \tag{43}$$

When ϕ is an integer multiple of 2π, the light exiting the crystal is the same as if the crystal were not there so that it is not transmitted through the analyzer. If ϕ is an odd integer multiple of π, the exiting light is also linearly polarized but perpendicularly to the incident light so that it is polarized in the same direction as the transmission axis of the analyzer, and thus is transmitted. Such elements are called half-wave plates at the frequency of operation. When ϕ is an odd integer multiple of $\pi/2$, the exiting light is circularly polarized and the crystal serves as a quarter-wave plate. However, only that polarization of light along the transmission axis of the analyzer is transmitted.

Double refraction occurs naturally in many crystals due to their anisotropic molecular structure. Many plastics and glasses that are generally isotropic have induced birefringence when mechanically stressed. When placed within a polariscope the photoelastic stress patterns can be seen. Some liquids, notably nitrobenzene, also become birefringent when stressed by large electric fields. This phenomena is called the Kerr effect. Electro-optical measurements allow electric field mapping in the dielectric between high voltage stressed electrodes, useful in the study of high voltage conduction and breakdown phenomena. The Kerr effect is also used as a light switch in high-speed shutters. A parallel plate capacitor is placed within a polariscope so that in the absence of voltage no light is transmitted. When the voltage is increased the light is transmitted, being a maximum when $\phi = \pi$. (See problem 17.)

7-5 NORMAL INCIDENCE ONTO A PERFECT CONDUCTOR

A uniform plane wave with x-directed electric field is normally incident upon a perfectly conducting plane at $z = 0$, as shown in Figure 7-13. The presence of the boundary gives rise to a reflected wave that propagates in the $-z$ direction. There are no fields within the perfect conductor. The known incident fields traveling in the $+z$ direction can be written as

$$\begin{aligned} \mathbf{E}_i(z,t) &= \text{Re}\,(\hat{E}_i\, e^{j(\omega t - kz)}\mathbf{i}_x) \\ \mathbf{H}_i(z,t) &= \text{Re}\left(\frac{\hat{E}_i}{\eta_0}\, e^{j(\omega t - kz)}\mathbf{i}_y\right) \end{aligned} \tag{1}$$

while the reflected fields propagating in the $-z$ direction are similarly

$$\begin{aligned} \mathbf{E}_r(z,t) &= \text{Re}\,(\hat{E}_r\, e^{j(\omega t + kz)}\mathbf{i}_x) \\ \mathbf{H}_r(z,t) &= \text{Re}\left(\frac{-\hat{E}_r}{\eta_0}\, e^{j(\omega t + kz)}\mathbf{i}_y\right) \end{aligned} \tag{2}$$

where in the lossless free space

$$\eta_0 = \sqrt{\mu_0/\varepsilon_0}, \qquad k = \omega\sqrt{\varepsilon_0\mu_0} \tag{3}$$

Note the minus sign difference in the spatial exponential phase factors of (1) and (2) as the waves are traveling in opposite directions. The amplitude of incident and reflected magnetic fields are given by the ratio of electric field amplitude to the wave impedance, as derived in Eq. (15) of Section

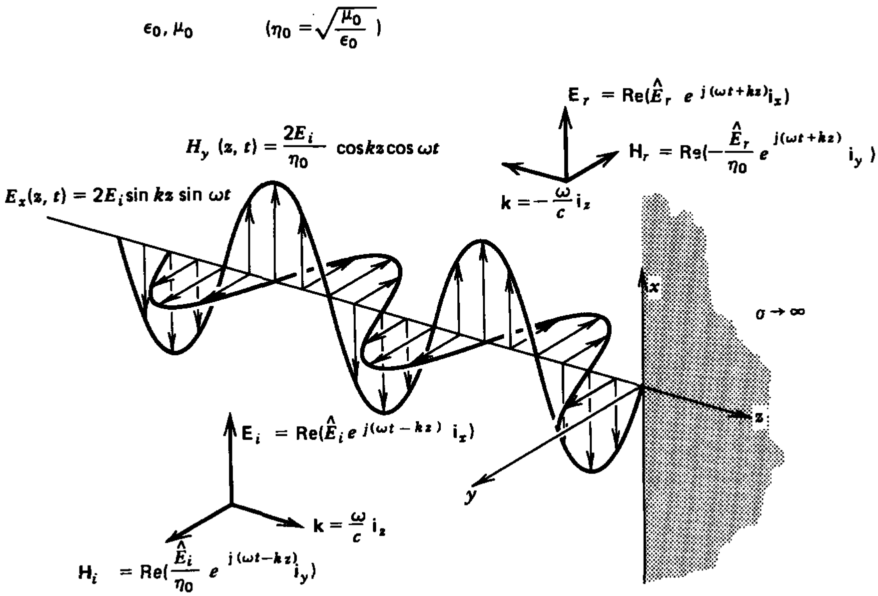

Figure 7-13 A uniform plane wave normally incident upon a perfect conductor has zero electric field at the conducting surface thus requiring a reflected wave. The source of this reflected wave is the surface current at $z=0$, which equals the magnetic field there. The total electric and magnetic fields are 90° out of phase in time and space.

7-3-2. The negative sign in front of the reflected magnetic field for the wave in the $-z$ direction arises because the power flow $\mathbf{S}_r = \mathbf{E}_r \times \mathbf{H}_r$ in the reflected wave must also be in the $-z$ direction.

The total electric and magnetic fields are just the sum of the incident and reflected fields. The only unknown parameter E_r can be evaluated from the boundary condition at $z=0$ where the tangential component of $\mathbf{E}$ must be continuous and thus zero along the perfect conductor:

$$\hat{E}_i + \hat{E}_r = 0 \Rightarrow \hat{E}_r = -\hat{E}_i \tag{4}$$

The total fields are then the sum of the incident and reflected fields

$$\begin{aligned}
\mathbf{E}_x(z,t) &= \mathbf{E}_i(z,t) + \mathbf{E}_r(z,t) \\
&= \mathrm{Re}\,[\hat{E}_i(e^{-jkz} - e^{+jkz})\, e^{j\omega t}] \\
&= 2E_i \sin kz \sin \omega t \\
\mathbf{H}_y(z,t) &= \mathbf{H}_i(z,t) + \mathbf{H}_r(z,t) \\
&= \mathrm{Re}\left(\frac{\hat{E}_i}{\eta_0}(e^{-jkz} + e^{+jkz})\, e^{j\omega t}\right) \\
&= \frac{2E_i}{\eta_0} \cos kz \cos \omega t
\end{aligned} \tag{5}$$

where we take $\hat{E}_i = E_i$ to be real. The electric and magnetic fields are 90° out of phase with each other both in time and space. We note that the two oppositely traveling wave solutions combined for a standing wave solution. The total solution does not propagate but is a standing sinusoidal solution in space whose amplitude varies sinusoidally in time.

A surface current flows on the perfect conductor at $z=0$ due to the discontinuity in tangential component of $\mathbf{H}$,

$$K_x = H_y(z=0) = \frac{2E_i}{\eta_0}\cos\omega t \tag{6}$$

giving rise to a force per unit area on the conductor,

$$\mathbf{F} = \tfrac{1}{2}\mathbf{K}\times\mu_0\mathbf{H} = \tfrac{1}{2}\mu_0 H_y^2(z=0)\mathbf{i}_z = 2\varepsilon_0 E_i^2\cos^2\omega t\,\mathbf{i}_z \tag{7}$$

known as the radiation pressure. The factor of $\frac{1}{2}$ arises in (7) because the force on a surface current is proportional to the average value of magnetic field on each side of the interface, here being zero for $z=0_+$.

7-6 NORMAL INCIDENCE ONTO A DIELECTRIC

7-6-1 Lossless Dielectric

We replace the perfect conductor with a lossless dielectric of permittivity ε_2 and permeability μ_2, as in Figure 7-14, with a uniform plane wave normally incident from a medium with permittivity ε_1 and permeability μ_1. In addition to the incident and reflected fields for $z<0$, there are transmitted fields which propagate in the $+z$ direction within the medium for $z>0$:

$$\left.\begin{aligned}
\mathbf{E}_i(z,t) &= \operatorname{Re}[\hat{E}_i e^{j(\omega t-k_1 z)}\mathbf{i}_x], & k_1 &= \omega\sqrt{\varepsilon_1\mu_1}\\
\mathbf{H}_i(z,t) &= \operatorname{Re}\left[\frac{\hat{E}_i}{\eta_1}e^{j(\omega t-k_1 z)}\mathbf{i}_y\right], & \eta_1 &= \sqrt{\frac{\mu_1}{\varepsilon_1}}\\
\mathbf{E}_r(z,t) &= \operatorname{Re}[\hat{E}_r e^{j(\omega t+k_1 z)}\mathbf{i}_x] & &\\
\mathbf{H}_r(z,t) &= \operatorname{Re}\left[-\frac{\hat{E}_r}{\eta_1}e^{j(\omega t+k_1 z)}\mathbf{i}_y\right] & &
\end{aligned}\right\}\quad z<0$$

$$\left.\begin{aligned}
\mathbf{E}_t(z,t) &= \operatorname{Re}[\hat{E}_t e^{j(\omega t-k_2 z)}\mathbf{i}_x], & k_2 &= \omega\sqrt{\varepsilon_2\mu_2}\\
\mathbf{H}_t(z,t) &= \operatorname{Re}\left[\frac{\hat{E}_t}{\eta_2}e^{j(\omega t-k_2 z)}\mathbf{i}_y\right], & \eta_2 &= \sqrt{\frac{\mu_2}{\varepsilon_2}}.
\end{aligned}\right\}\quad z>0 \tag{1}$$

It is necessary in (1) to use the appropriate wavenumber and impedance within each region. There is no wave traveling in the $-z$ direction in the second region as we assume no boundaries or sources for $z>0$.

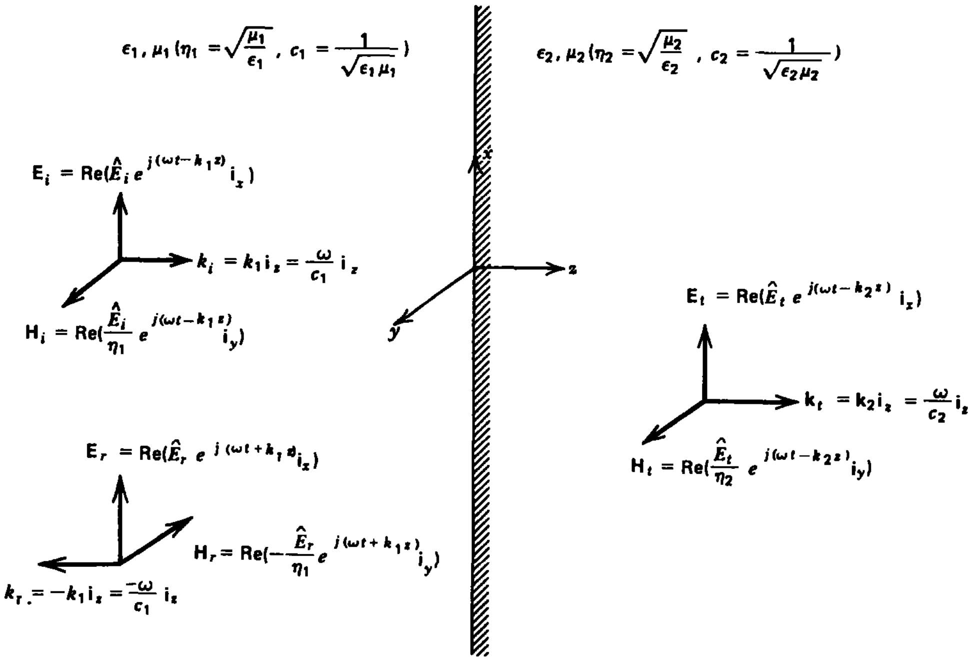

Figure 7-14 A uniform plane wave normally incident upon a dielectric interface separating two different materials has part of its power reflected and part transmitted.

The unknown quantities $\hat{E}_r$ and $\hat{E}_t$ can be found from the boundary conditions of continuity of tangential **E** and **H** at $z=0$,

$$\hat{E}_i + \hat{E}_r = \hat{E}_t$$
$$\frac{\hat{E}_i - \hat{E}_r}{\eta_1} = \frac{\hat{E}_t}{\eta_2} \tag{2}$$

from which we find the reflection R and transmission T field coefficients as

$$R = \frac{\hat{E}_r}{\hat{E}_i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$
$$T = \frac{\hat{E}_t}{\hat{E}_i} = \frac{2\eta_2}{\eta_2 + \eta_1} \tag{3}$$

where from (2)

$$1 + R = T \tag{4}$$

If both mediums have the same wave impedance, $\eta_1 = \eta_2$, there is no reflected wave.

7-6-2 Time-Average Power Flow

The time-average power flow in the region $z<0$ is

$$\begin{aligned}<S_{zi}> &= \tfrac{1}{2}\,\mathrm{Re}\,[\hat{E}_x(z)\hat{H}_y^*(z)]\\ &= \frac{1}{2\eta_1}\,\mathrm{Re}\,[\hat{E}_i\,e^{-jk_1z}+\hat{E}_r\,e^{+jk_1z}][\hat{E}_i^*\,e^{+jk_1z}-\hat{E}_r^*\,e^{-jk_1z}]\\ &= \frac{1}{2\eta_1}[|\hat{E}_i|^2-|\hat{E}_r|^2]\\ &\quad+\frac{1}{2\eta_1}\underbrace{\mathrm{Re}\,[\hat{E}_r\hat{E}_i^*\,e^{+2jk_1z}-\hat{E}_r^*\hat{E}_i\,e^{-2jk_1z}]}_{0}\end{aligned}\tag{5}$$

The last term on the right-hand side of (5) is zero as it is the difference between a number and its complex conjugate, which is pure imaginary and equals $2j$ times its imaginary part. Being pure imaginary, its real part is zero. Thus the time-average power flow just equals the difference in the power flows in the incident and reflected waves as found more generally in Section 7-3-2. The coupling terms between oppositely traveling waves have no time-average yielding the simple superposition of time-average powers:

$$\begin{aligned}<S_{zi}> &= \frac{1}{2\eta_1}[|\hat{E}_i|^2-|\hat{E}_r|^2]\\ &= \frac{|\hat{E}_i|^2}{2\eta_1}[1-R^2]\end{aligned}\tag{6}$$

This net time-average power flows into the dielectric medium, as it also equals the transmitted power;

$$<S_{zt}> = \frac{1}{2\eta_2}|\hat{E}_t|^2 = \frac{|\hat{E}_i|^2T^2}{2\eta_2} = \frac{|\hat{E}_i|^2}{2\eta_1}[1-R^2]\tag{7}$$

7-6-3 Lossy Dielectric

If medium 2 is lossy with Ohmic conductivity σ, the solutions of (3) are still correct if we replace the permittivity ε_2 by the complex permittivity $\hat{\varepsilon}_2$,

$$\hat{\varepsilon}_2=\varepsilon_2\left(1+\frac{\sigma}{j\omega\varepsilon_2}\right)\tag{8}$$

so that the wave impedance in region 2 is complex:

$$\eta_2=\sqrt{\mu_2/\hat{\varepsilon}_2}\tag{9}$$

We can easily explore the effect of losses in the low and large loss limits.

(a) Low Losses

If the Ohmic conductivity is small, we can neglect it in all terms except in the wavenumber k_2:

$$\lim_{\sigma/\omega\varepsilon_2 \ll 1} k_2 \approx \omega\sqrt{\varepsilon_2\mu_2} - \frac{j}{2}\sigma\sqrt{\frac{\mu_2}{\varepsilon_2}} \tag{10}$$

The imaginary part of k_2 gives rise to a small rate of exponential decay in medium 2 as the wave propagates away from the $z=0$ boundary.

(b) Large Losses

For large conductivities so that the displacement current is negligible in medium 2, the wavenumber and impedance in region 2 are complex:

$$\lim_{\sigma/\omega\varepsilon_2 \gg 1} \begin{cases} k_2 = \dfrac{1-j}{\delta}, & \delta = \sqrt{\dfrac{2}{\omega\mu_2\sigma}} \\ \eta_2 = \sqrt{\dfrac{j\omega\mu_2}{\sigma}} = \dfrac{1+j}{\sigma\delta} \end{cases} \tag{11}$$

The fields decay within a characteristic distance equal to the skin depth δ. This is why communications to submerged submarines are difficult. For seawater, $\mu_2 = \mu_0 = 4\pi \times 10^{-7}$ henry/m and $\sigma \approx 4$ siemens/m so that for 1 MHz signals, $\delta \approx 0.25$ m. However, at 100 Hz the skin depth increases to 25 meters. If a submarine is within this distance from the surface, it can receive the signals. However, it is difficult to transmit these low frequencies because of the large free space wavelength, $\lambda \approx 3 \times 10^6$ m. Note that as the conductivity approaches infinity,

$$\lim_{\sigma\to\infty} \begin{cases} k_2 = \infty \\ \eta_2 = 0 \end{cases} \Rightarrow \begin{cases} R = -1 \\ T = 0 \end{cases} \tag{12}$$

so that the field solution approaches that of normal incidence upon a perfect conductor found in Section 7-5.

EXAMPLE 7-1 DIELECTRIC COATING

A thin lossless dielectric with permittivity ε and permeability μ is coated onto the interface between two infinite half-spaces of lossless media with respective properties (ε_1, μ_1) and (ε_2, μ_2), as shown in Figure 7-15. What coating parameters ε and μ and thickness d will allow all the time-average power

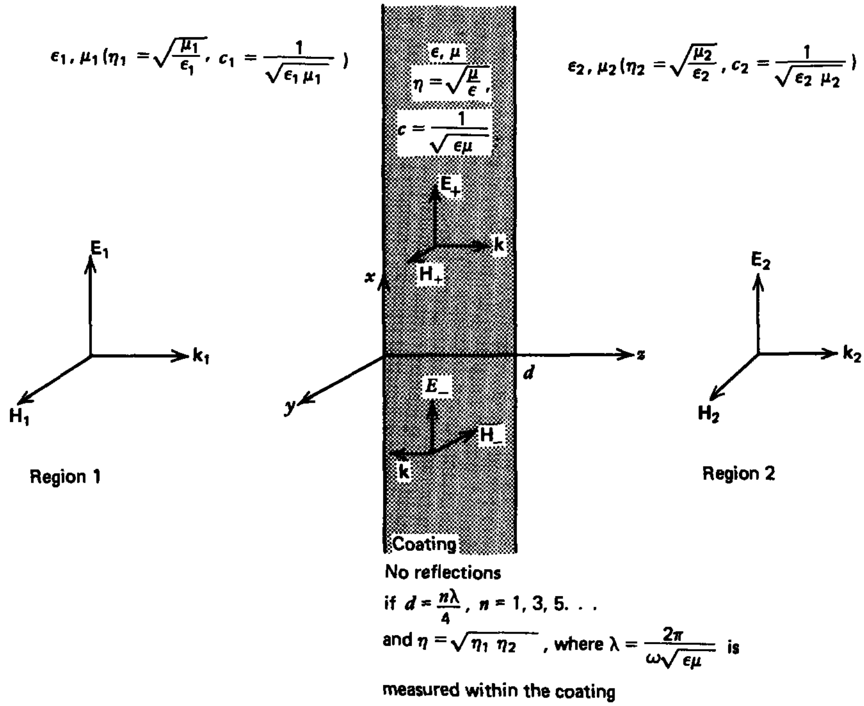

Figure 7-15 A suitable dielectric coating applied on the interface of discontinuity between differing media can eliminate reflections at a given frequency.

from region 1 to be transmitted through the coating to region 2? Such coatings are applied to optical components such as lenses to minimize unwanted reflections and to maximize the transmitted light intensity.

SOLUTION

For all the incident power to be transmitted into region 2, there can be no reflected field in region 1, although we do have oppositely traveling waves in the coating due to the reflection at the second interface. Region 2 only has positively z-directed power flow. The fields in each region are thus of the following form:

Region 1

$$\mathbf{E}_1 = \text{Re}\,[\hat{E}_1\, e^{j(\omega t - k_1 z)}\mathbf{i}_x], \qquad k_1 = \omega/c_1 = \omega\sqrt{\varepsilon_1\mu_1}$$

$$\mathbf{H}_1 = \text{Re}\left[\frac{\hat{E}_1}{\eta_1}\, e^{j(\omega t - k_1 z)}\mathbf{i}_y\right], \qquad \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$

Coating

$$\mathbf{E}_+ = \text{Re}\,[\hat{E}_+\, e^{j(\omega t - kz)}\mathbf{i}_x], \qquad k = \omega/c = \omega\sqrt{\varepsilon\mu}$$

$$\mathbf{H}_+ = \text{Re}\left[\frac{\hat{E}_+}{\eta}\, e^{j(\omega t - kz)}\mathbf{i}_y\right], \qquad \eta = \sqrt{\frac{\mu}{\varepsilon}}$$

$$\mathbf{E}_- = \text{Re}\,[\hat{E}_-\, e^{j(\omega t + kz)}\mathbf{i}_x]$$

$$\mathbf{H}_- = \text{Re}\left[-\frac{\hat{E}_-}{\eta}\, e^{j(\omega t + kz)}\mathbf{i}_y\right]$$

Region 2

$$\mathbf{E}_2 = \text{Re}\,[\hat{E}_2\, e^{j(\omega t - k_2 z)}\mathbf{i}_x], \qquad k_2 = \omega/c_2 = \omega\sqrt{\varepsilon_2\mu_2}$$

$$\mathbf{H}_2 = \text{Re}\left[\frac{\hat{E}_2}{\eta_2}\, e^{j(\omega t - k_2 z)}\mathbf{i}_y\right], \qquad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$$

Continuity of tangential **E** and **H** at $z = 0$ and $z = d$ requires

$$\hat{E}_1 = \hat{E}_+ + \hat{E}_-, \qquad \frac{\hat{E}_1}{\eta_1} = \frac{\hat{E}_+ - \hat{E}_-}{\eta}$$

$$\hat{E}_+\, e^{-jkd} + \hat{E}_-\, e^{+jkd} = \hat{E}_2\, e^{-jk_2 d}$$

$$\frac{\hat{E}_+\, e^{-jkd} - \hat{E}_-\, e^{+jkd}}{\eta} = \frac{\hat{E}_2\, e^{-jk_2 d}}{\eta_2}$$

Each of these amplitudes in terms of $\hat{E}_1$ is then

$$\hat{E}_+ = \frac{1}{2}\left(1 + \frac{\eta}{\eta_1}\right)\hat{E}_1$$

$$\hat{E}_- = \frac{1}{2}\left(1 - \frac{\eta}{\eta_1}\right)\hat{E}_1$$

$$\hat{E}_2 = e^{jk_2 d}[\hat{E}_+\, e^{-jkd} + \hat{E}_-\, e^{+jkd}]$$

$$= \frac{\eta_2}{\eta}\, e^{jk_2 d}[\hat{E}_+\, e^{-jkd} - \hat{E}_-\, e^{+jkd}]$$

Solving this last relation self-consistently requires that

$$\hat{E}_+\, e^{-jkd}\left(1 - \frac{\eta_2}{\eta}\right) + \hat{E}_-\, e^{jkd}\left(1 + \frac{\eta_2}{\eta}\right) = 0$$

Writing $\hat{E}_+$ and $\hat{E}_-$ in terms of $\hat{E}_1$ yields

$$\left(1 + \frac{\eta}{\eta_1}\right)\left(1 - \frac{\eta_2}{\eta}\right) + e^{2jkd}\left(1 + \frac{\eta_2}{\eta}\right)\left(1 - \frac{\eta}{\eta_1}\right) = 0$$

Since this relation is complex, the real and imaginary parts must separately be satisfied. For the imaginary part to be zero requires that the coating thickness d be an integral number of quarter wavelengths as measured within the coating,

$$2kd = n\pi \Rightarrow d = n\lambda/4, \qquad n = 1, 2, 3, \dots$$

The real part then requires

$$\left(1+\frac{\eta}{\eta_1}\right)\left(1-\frac{\eta_2}{\eta}\right) \pm \left(1+\frac{\eta_2}{\eta}\right)\left(1-\frac{\eta}{\eta_1}\right) = 0\begin{cases} n \text{ even} \\ n \text{ odd}\end{cases}$$

For the upper sign where d is a multiple of half-wavelengths the only solution is

$$\eta_2 = \eta_1 \quad (d = n\lambda/4, \quad n = 2, 4, 6, \dots)$$

which requires that media 1 and 2 be the same so that the coating serves no purpose. If regions 1 and 2 have differing wave impedances, we must use the lower sign where d is an odd integer number of quarter wavelengths so that

$$\eta^2 = \eta_1\eta_2 \Rightarrow \eta = \sqrt{\eta_1\eta_2} \quad (d = n\lambda/4, \quad n = 1, 3, 5, \dots)$$

Thus, if the coating is a quarter wavelength thick as measured within the coating, or any odd integer multiple of this thickness with its wave impedance equal to the geometrical average of the impedances in each adjacent region, all the time-average power flow in region 1 passes through the coating into region 2:

$$\begin{aligned} \langle S_z \rangle &= \frac{1}{2}\frac{|\hat{E}_1|^2}{\eta_1} = \frac{1}{2}\frac{|\hat{E}_2|^2}{\eta_2} \\ &= \tfrac{1}{2}\,\mathrm{Re}\left[(\hat{E}_+ e^{-jkz} + \hat{E}_- e^{+jkz})\frac{(\hat{E}_+^* e^{+jkz} - \hat{E}_-^* e^{-jkz})}{\eta}\right] \\ &= \frac{1}{2\eta}(|\hat{E}_+|^2 - |\hat{E}_-|^2) \end{aligned}$$

Note that for a given coating thickness d, there is no reflection only at select frequencies corresponding to wavelengths $d = n\lambda/4$, $n = 1, 3, 5, \dots$. For a narrow band of wavelengths about these select wavelengths, reflections are small. The magnetic permeability of coatings and of the glass used in optical components are usually that of free space while the permittivities differ. The permittivity of the coating ε is then picked so that

$$\varepsilon = \sqrt{\varepsilon_2\varepsilon_0}$$

and with a thickness corresponding to the central range of the wavelengths of interest (often in the visible).

7-7 UNIFORM AND NONUNIFORM PLANE WAVES

Our analysis thus far has been limited to waves propagating in the z direction normally incident upon plane interfaces. Although our examples had the electric field polarized in the x direction, the solution procedure is the same for the y-directed electric field polarization as both polarizations are parallel to the interfaces of discontinuity.

7-7-1 Propagation at an Arbitrary Angle

We now consider a uniform plane wave with power flow at an angle θ to the z axis, as shown in Figure 7-16. The electric field is assumed to be y directed, but the magnetic field that is perpendicular to both $\mathbf{E}$ and $\boldsymbol{S}$ now has components in the x and z directions.

The direction of the power flow, which we can call z', is related to the Cartesian coordinates as

$$z' = x \sin\theta + z\cos\theta \tag{1}$$

so that the phase factor kz' can be written as

$$kz' = k_x x + k_z z, \qquad \begin{aligned} k_x &= k\sin\theta \\ k_z &= k\cos\theta \end{aligned} \tag{2}$$

where the wavenumber magnitude is

$$k = \omega\sqrt{\varepsilon\mu} \tag{3}$$

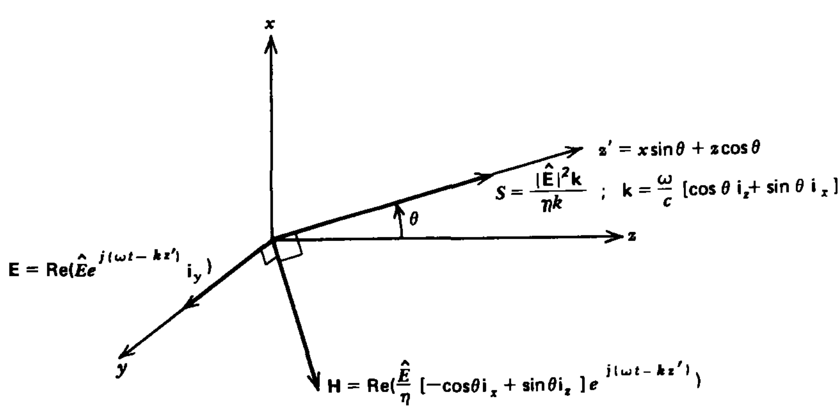

Figure 7-16 The spatial dependence of a uniform plane wave at an arbitrary angle θ can be expressed in terms of a vector wavenumber $\mathbf{k}$ as $e^{-j\mathbf{k}\cdot\mathbf{r}}$, where $\mathbf{k}$ is in the direction of power flow and has magnitude ω/c.

This allows us to write the fields as

$$\mathbf{E}=\text{Re}\,[\hat{E}\,e^{j(\omega t-k_x x-k_z z)}\mathbf{i}_y]$$
$$\mathbf{H}=\text{Re}\left[\frac{\hat{E}}{\eta}(-\cos\theta\mathbf{i}_x+\sin\theta\mathbf{i}_z)\,e^{j(\omega t-k_x x-k_z z)}\right] \tag{4}$$

We note that the spatial dependence of the fields can be written as $e^{-j\mathbf{k}\cdot\mathbf{r}}$, where the wavenumber is treated as a vector:

$$\mathbf{k}=k_x\mathbf{i}_x+k_y\mathbf{i}_y+k_z\mathbf{i}_z \tag{5}$$

with

$$\mathbf{r}=x\mathbf{i}_x+y\mathbf{i}_y+z\mathbf{i}_z \tag{6}$$

so that

$$\mathbf{k}\cdot\mathbf{r}=k_x x+k_y y+k_z z \tag{7}$$

The magnitude of $\mathbf{k}$ is as given in (3) and its direction is the same as the power flow $\boldsymbol{S}$:

$$\boldsymbol{S}=\mathbf{E}\times\mathbf{H}=\frac{|\hat{E}|^2}{\eta}(\cos\theta\mathbf{i}_z+\sin\theta\mathbf{i}_x)\cos^2(\omega t-\mathbf{k}\cdot\mathbf{r})$$
$$=\frac{|\hat{E}|^2\mathbf{k}}{\omega\mu}\cos^2(\omega t-\mathbf{k}\cdot\mathbf{r}) \tag{8}$$

where without loss of generality we picked the phase of $\hat{E}$ to be zero so that it is real.

7-7-2 The Complex Propagation Constant

Let us generalize further by considering fields of the form

$$\mathbf{E}=\text{Re}\,[\hat{\mathbf{E}}\,e^{j\omega t}\,e^{-\boldsymbol{\gamma}\cdot\mathbf{r}}]=\text{Re}\,[\hat{\mathbf{E}}\,e^{j(\omega t-\mathbf{k}\cdot\mathbf{r})}\,e^{-\boldsymbol{\alpha}\cdot\mathbf{r}}]$$
$$\mathbf{H}=\text{Re}\,[\hat{\mathbf{H}}\,e^{j\omega t}\,e^{-\boldsymbol{\gamma}\cdot\mathbf{r}}]=\text{Re}\,[\hat{\mathbf{H}}\,e^{j(\omega t-\mathbf{k}\cdot\mathbf{r})}\,e^{-\boldsymbol{\alpha}\cdot\mathbf{r}}] \tag{9}$$

where $\boldsymbol{\gamma}$ is the complex propagation vector and $\mathbf{r}$ is the position vector of (6):

$$\boldsymbol{\gamma}=\boldsymbol{\alpha}+j\mathbf{k}=\gamma_x\mathbf{i}_x+\gamma_y\mathbf{i}_y+\gamma_z\mathbf{i}_z$$
$$\boldsymbol{\gamma}\cdot\mathbf{r}=\gamma_x x+\gamma_y y+\gamma_z z \tag{10}$$

We have previously considered uniform plane waves in lossless media where the wavenumber $\mathbf{k}$ is pure real and z directed with $\alpha=0$ so that $\boldsymbol{\gamma}$ is pure imaginary. The parameter $\boldsymbol{\alpha}$ represents the decay rate of the fields even though the medium is lossless. If $\boldsymbol{\alpha}$ is nonzero, the solutions are called nonuniform plane waves. We saw this decay in our quasi-static solutions of Laplace's equation where solutions had oscillations in one direction but decay in the perpendicular direction. We would expect this evanescence to remain at low frequencies.

The value of the assumed form of solutions in (9) is that the del (∇) operator in Maxwell's equations can be replaced by the vector operator $-\boldsymbol{\gamma}$:

$$\begin{aligned}\nabla &= \frac{\partial}{\partial x}\mathbf{i}_x + \frac{\partial}{\partial y}\mathbf{i}_y + \frac{\partial}{\partial z}\mathbf{i}_z \\ &= -\gamma_x\mathbf{i}_x - \gamma_y\mathbf{i}_y - \gamma_z\mathbf{i}_z \\ &= -\boldsymbol{\gamma} \end{aligned} \tag{11}$$

This is true because any spatial derivatives only operate on the exponential term in (9). Then the source free Maxwell's equations can be written in terms of the complex amplitudes as

$$\begin{aligned} -\boldsymbol{\gamma}\times\hat{\mathbf{E}} &= -j\omega\mu\hat{\mathbf{H}} \\ -\boldsymbol{\gamma}\times\hat{\mathbf{H}} &= j\omega\varepsilon\hat{\mathbf{E}} \\ -\boldsymbol{\gamma}\cdot\varepsilon\hat{\mathbf{E}} &= 0 \\ -\boldsymbol{\gamma}\cdot\mu\hat{\mathbf{H}} &= 0 \end{aligned} \tag{12}$$

The last two relations tell us that $\boldsymbol{\gamma}$ is perpendicular to both $\mathbf{E}$ and $\mathbf{H}$. If we take $\boldsymbol{\gamma}\times$ the top equation and use the second equation, we have

$$\begin{aligned} -\boldsymbol{\gamma}\times(\boldsymbol{\gamma}\times\hat{\mathbf{E}}) &= -j\omega\mu(\boldsymbol{\gamma}\times\hat{\mathbf{H}}) = -j\omega\mu(-j\omega\varepsilon\hat{\mathbf{E}}) \\ &= -\omega^2\mu\varepsilon\hat{\mathbf{E}} \end{aligned} \tag{13}$$

The double cross product can be expanded as

$$\begin{aligned} -\boldsymbol{\gamma}\times(\boldsymbol{\gamma}\times\hat{\mathbf{E}}) &= -\boldsymbol{\gamma}(\boldsymbol{\gamma}\cdot\hat{\mathbf{E}}) + (\boldsymbol{\gamma}\cdot\boldsymbol{\gamma})\hat{\mathbf{E}} \\ &= (\boldsymbol{\gamma}\cdot\boldsymbol{\gamma})\hat{\mathbf{E}} = -\omega^2\mu\varepsilon\hat{\mathbf{E}} \end{aligned} \tag{14}$$

The $\boldsymbol{\gamma}\cdot\hat{\mathbf{E}}$ term is zero from the third relation in (12). The dispersion relation is then

$$\boldsymbol{\gamma}\cdot\boldsymbol{\gamma} = (\alpha^2 - k^2 + 2j\boldsymbol{\alpha}\cdot\mathbf{k}) = -\omega^2\mu\varepsilon \tag{15}$$

For solution, the real and imaginary parts of (15) must be separately equal:

$$\begin{aligned} \alpha^2 - k^2 &= -\omega^2\mu\varepsilon \\ \boldsymbol{\alpha}\cdot\mathbf{k} &= 0 \end{aligned} \tag{16}$$

When $\boldsymbol{\alpha} = 0$, (16) reduces to the familiar frequency-wavenumber relation of Section 7-3-4.

The last relation now tells us that evanescence (decay) in space as represented by $\boldsymbol{\alpha}$ is allowed by Maxwell's equations, but must be perpendicular to propagation represented by $\mathbf{k}$.

We can compute the time-average power flow for fields of the form of (9) using (12) in terms of either $\hat{\mathbf{E}}$ or $\hat{\mathbf{H}}$ as follows:

$$\begin{aligned}
\langle S\rangle &= \tfrac{1}{2}\,\mathrm{Re}\,(\hat{\mathbf{E}}\times\hat{\mathbf{H}}^*),\\
&= -\tfrac{1}{2}\,\mathrm{Re}\left(\hat{\mathbf{E}}\times\frac{(\boldsymbol{\gamma}^*\times\hat{\mathbf{E}}^*)}{j\omega\mu}\right),\\
&= -\tfrac{1}{2}\,\mathrm{Re}\left(\frac{\boldsymbol{\gamma}^*|\hat{\mathbf{E}}|^2-\hat{\mathbf{E}}^*(\boldsymbol{\gamma}^*\cdot\hat{\mathbf{E}})}{j\omega\mu}\right),\\
&= \frac{1}{2}\frac{\mathbf{k}}{\omega\mu}|\hat{\mathbf{E}}|^2+\tfrac{1}{2}\,\mathrm{Re}\left(\frac{\hat{\mathbf{E}}^*(\boldsymbol{\gamma}^*\cdot\hat{\mathbf{E}})}{j\omega\mu}\right),\\
\langle S\rangle &= \tfrac{1}{2}\,\mathrm{Re}\,(\hat{\mathbf{E}}\times\hat{\mathbf{H}}^*)\\
&= -\tfrac{1}{2}\,\mathrm{Re}\left(\frac{(\boldsymbol{\gamma}\times\hat{\mathbf{H}})}{j\omega\varepsilon}\times\hat{\mathbf{H}}^*\right)\\
&= \tfrac{1}{2}\,\mathrm{Re}\left(\frac{\boldsymbol{\gamma}|\hat{\mathbf{H}}|^2-\hat{\mathbf{H}}(\boldsymbol{\gamma}\cdot\hat{\mathbf{H}}^*)}{j\omega\varepsilon}\right)\\
&= \frac{1}{2}\frac{\mathbf{k}}{\omega\varepsilon}|\hat{\mathbf{H}}|^2-\tfrac{1}{2}\,\mathrm{Re}\left(\frac{\hat{\mathbf{H}}(\boldsymbol{\gamma}\cdot\hat{\mathbf{H}}^*)}{j\omega\varepsilon}\right)
\end{aligned} \tag{17}$$

Although both final expressions in (17) are equivalent, it is convenient to write the power flow in terms of either $\hat{\mathbf{E}}$ or $\hat{\mathbf{H}}$. When $\hat{\mathbf{E}}$ is perpendicular to both the real vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, defined in (10) and (16), the dot product $\boldsymbol{\gamma}^*\cdot\hat{\mathbf{E}}$ is zero. Such a mode is called transverse electric (TE), and we see in (17) that the time-average power flow is still in the direction of the wavenumber $\mathbf{k}$. Similarly, when $\hat{\mathbf{H}}$ is perpendicular to $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, the dot product $\boldsymbol{\gamma}\cdot\hat{\mathbf{H}}^*$ is zero and we have a transverse magnetic (TM) mode. Again, the time-average power flow in (17) is in the direction of $\mathbf{k}$. The magnitude of $\mathbf{k}$ is related to ω in (16).

Note that our discussion has been limited to lossless systems. We can include Ohmic losses if we replace ε by the complex permittivity $\hat{\varepsilon}$ of Section 7-4-3 in (15) and (17). Then, there is always decay ($\alpha\neq 0$) because of Ohmic dissipation (see Problem 22).

7-7-3 Nonuniform Plane Waves

We can examine nonuniform plane wave solutions with nonzero $\boldsymbol{\alpha}$ by considering a current sheet in the $z=0$ plane, which is a traveling wave in the x direction:

$$K_x(z=0)=K_0\cos(\omega t-k_x x)=\mathrm{Re}\,(K_0\,e^{j(\omega t-k_x x)}) \tag{18}$$

The x-directed surface current gives rise to a y-directed magnetic field. Because the system does not depend on the y coordinate, solutions are thus of the following form:

$$H_y = \begin{cases} \mathrm{Re}\,(\hat{H}_1\, e^{j\omega t}\, e^{-\boldsymbol{\gamma}_1\cdot\mathbf{r}}), & z>0 \\ \mathrm{Re}\,(\hat{H}_2\, e^{j\omega t}\, e^{-\boldsymbol{\gamma}_2\cdot\mathbf{r}}), & z<0 \end{cases}$$
$$\mathbf{E} = \begin{cases} \mathrm{Re}\left[-\dfrac{\boldsymbol{\gamma}_1\times\hat{H}_1}{j\omega\varepsilon}\mathbf{i}_y\, e^{j\omega t}\, e^{-\boldsymbol{\gamma}_1\cdot\mathbf{r}}\right], & z>0 \\ \mathrm{Re}\left[-\dfrac{\boldsymbol{\gamma}_2\times\hat{H}_2}{j\omega\varepsilon}\mathbf{i}_y\, e^{j\omega t}\, e^{-\boldsymbol{\gamma}_2\cdot\mathbf{r}}\right], & z<0 \end{cases} \tag{19}$$

where $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ are the complex propagation vectors on each side of the current sheet:

$$\begin{aligned} \boldsymbol{\gamma}_1 &= \gamma_{1x}\mathbf{i}_x + \gamma_{1z}\mathbf{i}_z \\ \boldsymbol{\gamma}_2 &= \gamma_{2x}\mathbf{i}_x + \gamma_{2z}\mathbf{i}_z \end{aligned} \tag{20}$$

The boundary condition of the discontinuity of tangential $\mathbf{H}$ at $z=0$ equaling the surface current yields

$$-\hat{H}_1\, e^{-\gamma_{1x}x} + \hat{H}_2\, e^{-\gamma_{2x}x} = K_0\, e^{-jk_x x} \tag{21}$$

which tells us that the x components of the complex propagation vectors equal the trigonometric spatial dependence of the surface current:

$$\gamma_{1x} = \gamma_{2x} = jk_x \tag{22}$$

The z components of $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ are then determined from (15) as

$$\gamma_x^2 + \gamma_z^2 = -\omega^2\varepsilon\mu \Rightarrow \gamma_z = \pm(k_x^2 - \omega^2\varepsilon\mu)^{1/2} \tag{23}$$

If $k_x^2 < \omega^2\varepsilon\mu$, γ_z is pure imaginary representing propagation and we have uniform plane waves. If $k_x^2 > \omega^2\varepsilon\mu$, then γ_z is pure real representing evanescence in the z direction so that we generate nonuniform plane waves. When $\omega = 0$, (23) corresponds to Laplacian solutions that oscillate in the x direction but decay in the z direction.

The z component of $\boldsymbol{\gamma}$ is of opposite sign in each region,

$$\gamma_{1z} = -\gamma_{2z} = +(k_x^2 - \omega^2\varepsilon\mu)^{1/2} \tag{24}$$

as the waves propagate or decay away from the sheet. Continuity of the tangential component of $\mathbf{E}$ requires

$$\gamma_{1z}\hat{H}_1 = \gamma_{2z}\hat{H}_2 \Rightarrow \hat{H}_2 = -\hat{H}_1 = K_0/2 \tag{25}$$

If $k_x = 0$, we re-obtain the solution of Section 7-4-1. Increasing k_x generates propagating waves with power flow in the $k_x\mathbf{i}_x \pm k_z\mathbf{i}_z$ directions. At $k_x^2 = \omega^2\varepsilon\mu$, $k_z = 0$ so that the power flow is purely x directed with no spatial dependence on z. Further increasing k_x converts k_z to α_z as γ_z becomes real and the fields decay with z.

7-8 OBLIQUE INCIDENCE ONTO A PERFECT CONDUCTOR

7-8-1 E Field Parallel to the Interface

In Figure 7-17*a* we show a uniform plane wave incident upon a perfect conductor with power flow at an angle θ_i to the normal. The electric field is parallel to the surface with the magnetic field having both x and z components:

$$\begin{aligned}\mathbf{E}_i &= \text{Re}\,[\hat{E}_i\, e^{j(\omega t - k_{xi}x - k_{zi}z)}\,\mathbf{i}_y]\\ \mathbf{H}_i &= \text{Re}\left[\frac{\hat{E}_i}{\eta}(-\cos\theta_i\,\mathbf{i}_x + \sin\theta_i\,\mathbf{i}_z)\, e^{j(\omega t - k_{xi}x - k_{zi}z)}\right]\end{aligned} \tag{1}$$

where

$$\left.\begin{aligned}k_{xi} &= k\ \sin\theta_i\\ k_{zi} &= k\ \cos\theta_i\end{aligned}\right\}\quad k = \omega\sqrt{\varepsilon\mu},\quad \eta = \sqrt{\frac{\mu}{\epsilon}} \tag{2}$$

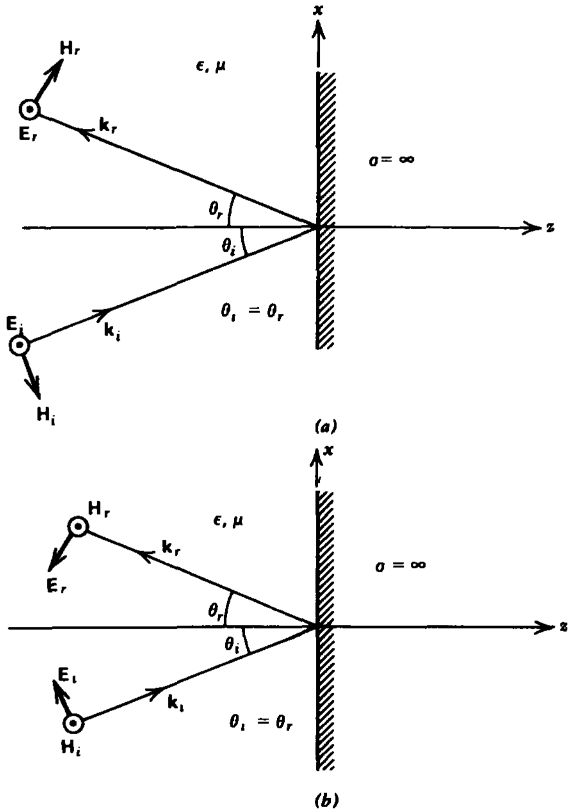

Figure 7-17 A uniform plane wave obliquely incident upon a perfect conductor has its angle of incidence equal to the angle of reflection. (*a*) Electric field polarized parallel to the interface. (*b*) Magnetic field parallel to the interface.

There are no transmitted fields within the perfect conductor, but there is a reflected field with power flow at angle θ_r from the interface normal. The reflected electric field is also in the y direction so the magnetic field, which must be perpendicular to both $\mathbf{E}$ and $\mathbf{S}=\mathbf{E}\times\mathbf{H}$, is in the direction shown in Figure 7-17a:

$$\begin{aligned}\mathbf{E}_r &= \text{Re}\,[\hat{E}_r e^{j(\omega t-k_{xr}x+k_{zr}z)}\mathbf{i}_y]\\ \mathbf{H}_r &= \text{Re}\left[\frac{\hat{E}_r}{\eta}(\cos\theta_r\mathbf{i}_x+\sin\theta_r\mathbf{i}_z)\,e^{j(\omega t-k_{xr}x+k_{zr}z)}\right]\end{aligned} \tag{3}$$

where the reflected wavenumbers are

$$\begin{aligned}k_{xr}&=k\sin\theta_r\\ k_{zr}&=k\cos\theta_r\end{aligned} \tag{4}$$

At this point we do not know the angle of reflection θ_r or the reflected amplitude $\hat{E}_r$. They will be determined from the boundary conditions at $z=0$ of continuity of tangential $\mathbf{E}$ and normal $\mathbf{B}$. Because there are no fields within the perfect conductor these boundary conditions at $z=0$ are

$$\begin{aligned}\hat{E}_i e^{-jk_{xi}x}+\hat{E}_r e^{-jk_{xr}x}&=0\\ \frac{\mu}{\eta}(\hat{E}_i\sin\theta_i e^{-jk_{xi}x}+\hat{E}_r\sin\theta_r e^{-jk_{xr}x})&=0\end{aligned} \tag{5}$$

These conditions must be true for every value of x along $z=0$ so that the phase factors given in (2) and (4) must be equal,

$$k_{xi}=k_{xr}\Rightarrow\theta_i=\theta_r\equiv\theta \tag{6}$$

giving the well-known rule ***that the angle of incidence equals the angle of reflection.*** The reflected field amplitude is then

$$\hat{E}_r=-\hat{E}_i \tag{7}$$

with the boundary conditions in (5) being redundant as they both yield (7). The total fields are then:

$$\begin{aligned}E_y &= \text{Re}\,[\hat{E}_i(e^{-jk_zz}-e^{+jk_zz})\,e^{j(\omega t-k_xx)}]\\ &= 2E_i\sin k_zz\sin(\omega t-k_xx)\\ \mathbf{H} &= \text{Re}\left[\frac{\hat{E}_i}{\eta}[\cos\theta(-e^{-jk_zz}-e^{+jk_zz})\mathbf{i}_x+\sin\theta(e^{-jk_zz}\right.\\ &\qquad \left.-e^{+jk_zz})\mathbf{i}_z]\,e^{j(\omega t-k_xx)}\right]\\ &= \frac{2E_i}{\eta}\left[-\cos\theta\cos k_zz\cos(\omega t-k_xx)\mathbf{i}_x\right.\\ &\qquad \left.+\sin\theta\sin k_zz\sin(\omega t-k_xx)\mathbf{i}_z\right]\end{aligned} \tag{8}$$

where without loss of generality we take $\hat{E}_i$ to be real.

We drop the i and r subscripts on the wavenumbers and angles because they are equal. The fields travel in the x direction parallel to the interface, but are stationary in the z direction. Note that another perfectly conducting plane can be placed at distances d to the left of the interface at

$$k_z d = n\pi \tag{9}$$

where the electric field is already zero without disturbing the solutions of (8). The boundary conditions at the second conductor are automatically satisfied. Such a structure is called a waveguide and is discussed in Section 8-6.

Because the tangential component of $\mathbf{H}$ is discontinuous at $z=0$, a traveling wave surface current flows along the interface,

$$K_y = -H_x(z=0) = \frac{2E_i}{\eta}\cos\theta\cos(\omega t - k_x x) \tag{10}$$

From (8) we compute the time-average power flow as

$$\begin{aligned} <S> &= \tfrac{1}{2}\,\mathrm{Re}\,[\hat{\mathbf{E}}(x,z)\times\hat{\mathbf{H}}^*(x,z)] \\ &= \frac{2E_i^2}{\eta}\sin\theta\sin^2 k_z z\,\mathbf{i}_x \end{aligned} \tag{11}$$

We see that the only nonzero power flow is in the direction parallel to the interfacial boundary and it varies as a function of z.

7-8-2 H Field Parallel to the Interface

If the $\mathbf{H}$ field is parallel to the conducting boundary, as in Figure 7-17b, the incident and reflected fields are as follows:

$$\begin{aligned} \mathbf{E}_i &= \mathrm{Re}\,[\hat{E}_i(\cos\theta_i\mathbf{i}_x - \sin\theta_i\mathbf{i}_z)\,e^{j(\omega t - k_{xi}x - k_{zi}z)}] \\ \mathbf{H}_i &= \mathrm{Re}\left[\frac{\hat{E}_i}{\eta}e^{j(\omega t - k_{xi}x - k_{zi}z)}\mathbf{i}_y\right] \\ \mathbf{E}_r &= \mathrm{Re}\,[\hat{E}_r(-\cos\theta_r\mathbf{i}_x - \sin\theta_r\mathbf{i}_z)\,e^{j(\omega t - k_{xr}x + k_{zr}z)}] \\ \mathbf{H}_r &= \mathrm{Re}\left[\frac{\hat{E}_r}{\eta}e^{j(\omega t - k_{xr}x + k_{zr}z)}\mathbf{i}_y\right] \end{aligned} \tag{12}$$

The tangential component of $\mathbf{E}$ is continuous and thus zero at $z=0$:

$$\hat{E}_i\cos\theta_i\,e^{-jk_{xi}x} - \hat{E}_r\cos\theta_r\,e^{-jk_{xr}x} = 0 \tag{13}$$

There is no normal component of $\mathbf{B}$. This boundary condition must be satisfied for all values of x so again the angle of incidence must equal the angle of reflection ($\theta_i = \theta_r$) so that

$$\hat{E}_i = \hat{E}_r \tag{14}$$

The total **E** and **H** fields can be obtained from (12) by adding the incident and reflected fields and taking the real part;

$$\begin{aligned}\mathbf{E} &= \mathrm{Re}\,\{\hat{E}_i\,[\cos\theta(e^{-jk_z z} - e^{+jk_z z})\mathbf{i}_x \\ &\quad -\sin\theta(e^{-jk_z z} + e^{+jk_z z})\mathbf{i}_z]\,e^{j(\omega t - k_x x)}\} \\ &= 2E_i\{\cos\theta\sin k_z z\sin(\omega t - k_x x)\mathbf{i}_x \\ &\quad -\sin\theta\cos k_z z\cos(\omega t - k_x x)\mathbf{i}_z\} \\ \mathbf{H} &= \mathrm{Re}\left[\frac{\hat{E}_i}{\eta}(e^{-jk_z z} + e^{+jk_z z})\,e^{j(\omega t - k_x x)}\mathbf{i}_y\right] \\ &= \frac{2E_i}{\eta}\cos k_z z\cos(\omega t - k_x x)\mathbf{i}_y\end{aligned} \tag{15}$$

The surface current on the conducting surface at $z = 0$ is given by the tangential component of **H**

$$K_x(z=0) = H_y(z=0) = \frac{2E_i}{\eta}\cos(\omega t - k_x x) \tag{16}$$

while the surface charge at $z = 0$ is proportional to the normal component of electric field,

$$\sigma_f(z=0) = -\varepsilon E_z(z=0) = 2\varepsilon E_i\sin\theta\cos(\omega t - k_x x) \tag{17}$$

Note that (16) and (17) satisfy conservation of current on the conducting surface,

$$\nabla_\Sigma\cdot\mathbf{K} + \frac{\partial\sigma_f}{\partial t} = 0 \Rightarrow \frac{\partial K_x}{\partial x} + \frac{\partial\sigma_f}{\partial t} = 0 \tag{18}$$

where

$$\nabla_\Sigma = \frac{\partial}{\partial x}\mathbf{i}_x + \frac{\partial}{\partial y}\mathbf{i}_y$$

is the surface divergence operator. The time-average power flow for this polarization is also x directed:

$$\begin{aligned}\langle S\rangle &= \tfrac{1}{2}\mathrm{Re}\,(\hat{\mathbf{E}}\times\hat{\mathbf{H}}^*) \\ &= \frac{2E_i^2}{\eta}\sin\theta\cos^2 k_z z\,\mathbf{i}_x\end{aligned} \tag{19}$$

7-9 OBLIQUE INCIDENCE ONTO A DIELECTRIC

7-9-1 E Parallel to the Interface

A plane wave incident upon a dielectric interface, as in Figure 7-18*a*, now has transmitted fields as well as reflected fields. For the electric field polarized parallel to the interface, the fields in each region can be expressed as

$$\begin{aligned}
\mathbf{E}_i &= \text{Re}\,[\hat{E}_i\, e^{j(\omega t - k_{xi}x - k_{zi}z)}\mathbf{i}_y] \\
\mathbf{H}_i &= \text{Re}\left[\frac{\hat{E}_i}{\eta_1}(-\cos\theta_i\mathbf{i}_x + \sin\theta_i\mathbf{i}_z)\, e^{j(\omega t - k_{xi}x - k_{zi}z)}\right] \\
\mathbf{E}_r &= \text{Re}\,[\hat{E}_r\, e^{j(\omega t - k_{xr}x + k_{zr}z)}\mathbf{i}_y] \\
\mathbf{H}_r &= \text{Re}\left[\frac{\hat{E}_r}{\eta_1}(\cos\theta_r\mathbf{i}_x + \sin\theta_r\mathbf{i}_z)\, e^{j(\omega t - k_{xr}x + k_{zr}z)}\right] \\
\mathbf{E}_t &= \text{Re}\,[\hat{E}_t\, e^{j(\omega t - k_{xt}x - k_{zt}z)}\mathbf{i}_y] \\
\mathbf{H}_t &= \text{Re}\left[\frac{\hat{E}_t}{\eta_2}(-\cos\theta_t\mathbf{i}_x + \sin\theta_t\mathbf{i}_z)\, e^{j(\omega t - k_{xt}x - k_{zt}z)}\right]
\end{aligned} \tag{1}$$

where θ_i, θ_r, and θ_t are the angles from the normal of the incident, reflected, and transmitted power flows. The wavenumbers in each region are

$$\begin{aligned}
k_{xi} &= k_1 \sin\theta_i, \qquad k_{xr} = k_1\sin\theta_r, \qquad k_{xt} = k_2\sin\theta_t \\
k_{zi} &= k_1 \cos\theta_i, \qquad k_{zr} = k_1\cos\theta_r, \qquad k_{zt} = k_2\cos\theta_t
\end{aligned} \tag{2}$$

where the wavenumber magnitudes, wave speeds, and wave impedances are

$$\begin{aligned}
k_1 &= \frac{\omega}{c_1}, \qquad k_2 = \frac{\omega}{c_2}, \qquad c_1 = \frac{1}{\sqrt{\varepsilon_1\mu_1}} \\
\eta_1 &= \sqrt{\frac{\mu_1}{\varepsilon_1}}, \qquad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}, \qquad c_2 = \frac{1}{\sqrt{\varepsilon_2\mu_2}}
\end{aligned} \tag{3}$$

The unknown angles and amplitudes in (1) are found from the boundary conditions of continuity of tangential **E** and **H** at the $z = 0$ interface.

$$\begin{aligned}
&\hat{E}_i\, e^{-jk_{xi}x} + \hat{E}_r\, e^{-jk_{xr}x} = \hat{E}_t\, e^{-jk_{xt}x} \\
&\frac{-\hat{E}_i\cos\theta_i\, e^{-jk_{xi}x} + \hat{E}_r\cos\theta_r\, e^{-jk_{xr}x}}{\eta_1} = -\frac{\hat{E}_t\cos\theta_t\, e^{-jk_{xt}x}}{\eta_2}
\end{aligned} \tag{4}$$

These boundary conditions must be satisfied point by point for all x. This requires that the exponential factors also be

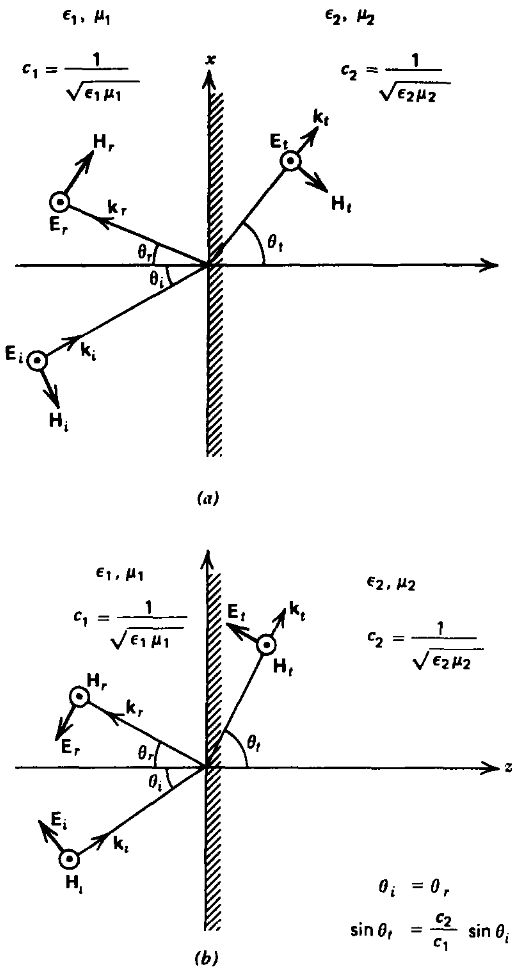

Figure 7-18 A uniform plane wave obliquely incident upon a dielectric interface also has its angle of incidence equal to the angle of reflection while the transmitted angle is given by Snell's law. (*a*) Electric field polarized parallel to the interface. (*b*) Magnetic field parallel to the interface.

equal so that the x components of all wavenumbers must be equal,

$$k_{xi} = k_{xr} = k_{xt} \Rightarrow k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t \tag{5}$$

which relates the angles as

$$\theta_r = \theta_i \tag{6}$$

$$\sin\theta_t = (c_2/c_1)\sin\theta_i \tag{7}$$

As before, the angle of incidence equals the angle of reflection. The transmission angle obeys a more complicated relation called Snell's law relating the sines of the angles. The angle from the normal is largest in that region which has the faster speed of electromagnetic waves.

In optics, the ratio of the speed of light in vacuum, $c_0 = 1/\sqrt{\varepsilon_0\mu_0}$, to the speed of light in the medium is defined as the index of refraction,

$$n_1 = c_0/c_1, \qquad n_2 = c_0/c_2 \tag{8}$$

which is never less than unity. Then Snell's law is written as

$$\sin\theta_t = (n_1/n_2)\sin\theta_i \tag{9}$$

With the angles related as in (6), the reflected and transmitted field amplitudes can be expressed in the same way as for normal incidence (see Section 7-6-1) if we replace the wave impedances by $\eta \to \eta/\cos\theta$ to yield

$$\begin{aligned} R &= \frac{\hat{E}_r}{\hat{E}_i} = \frac{\dfrac{\eta_2}{\cos\theta_t} - \dfrac{\eta_1}{\cos\theta_i}}{\dfrac{\eta_2}{\cos\theta_t} + \dfrac{\eta_1}{\cos\theta_i}} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} \\ T &= \frac{\hat{E}_t}{\hat{E}_i} = \frac{2\eta_2}{\cos\theta_t\left(\dfrac{\eta_2}{\cos\theta_t} + \dfrac{\eta_1}{\cos\theta_i}\right)} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} \end{aligned} \tag{10}$$

In (4) we did not consider the boundary condition of continuity of normal $\mathbf{B}$ at $z = 0$. This boundary condition is redundant as it is the same condition as the upper equation in (4):

$$\frac{\mu_1}{\eta_1}(\hat{E}_i + \hat{E}_r)\sin\theta_i = \frac{\mu_2}{\eta_2}\hat{E}_t\sin\theta_t \Rightarrow (\hat{E}_i + \hat{E}_r) = \hat{E}_t \tag{11}$$

where we use the relation between angles in (6). Since

$$\frac{\mu_1}{\eta_1} = \sqrt{\mu_1\varepsilon_1} = \frac{1}{c_1}, \qquad \frac{\mu_2}{\eta_2} = \sqrt{\mu_2\varepsilon_2} = \frac{1}{c_2} \tag{12}$$

the trigonometric terms in (11) cancel due to Snell's law. There is no normal component of $\mathbf{D}$ so it is automatically continuous across the interface.

7-9-2 Brewster's Angle of No Reflection

We see from (10) that at a certain angle of incidence, there is no reflected field as $R = 0$. This angle is called Brewster's angle:

$$R = 0 \Rightarrow \eta_2\cos\theta_i = \eta_1\cos\theta_t \tag{13}$$

By squaring (13), replacing the cosine terms with sine terms ($\cos^2\theta = 1-\sin^2\theta$), and using Snell's law of (6), the Brewster angle θ_B is found as

$$\sin^2\theta_B = \frac{1-\varepsilon_2\mu_1/(\varepsilon_1\mu_2)}{1-(\mu_1/\mu_2)^2} \tag{14}$$

There is not always a real solution to (14) as it depends on the material constants. The common dielectric case, where $\mu_1 = \mu_2 \equiv \mu$ but $\varepsilon_1 \neq \varepsilon_2$, does not have a solution as the right-hand side of (14) becomes infinite. Real solutions to (14) require the right-hand side to be between zero and one. A Brewster's angle does exist for the uncommon situation where $\varepsilon_1 = \varepsilon_2$ and $\mu_1 \neq \mu_2$:

$$\sin^2\theta_B = \frac{1}{1+\mu_1/\mu_2} \Rightarrow \tan\theta_B = \sqrt{\frac{\mu_2}{\mu_1}} \tag{15}$$

At this Brewster's angle, the reflected and transmitted power flows are at right angles ($\theta_B + \theta_t = \pi/2$) as can be seen by using (6), (13), and (14):

$$\begin{aligned}\cos(\theta_B+\theta_t) &= \cos\theta_B\cos\theta_t - \sin\theta_B\sin\theta_t \\ &= \cos^2\theta_B\sqrt{\frac{\mu_2}{\mu_1}} - \sin^2\theta_B\sqrt{\frac{\mu_1}{\mu_2}} \\ &= \sqrt{\frac{\mu_2}{\mu_1}} - \sin^2\theta_B\left(\sqrt{\frac{\mu_1}{\mu_2}}+\sqrt{\frac{\mu_2}{\mu_1}}\right) = 0\end{aligned} \tag{16}$$

7-9-3 Critical Angle of Transmission

Snell's law in (6) shows us that if $c_2 > c_1$, large angles of incident angle θ_i could result in $\sin\theta_t$ being greater than unity. There is no real angle θ_t that satisfies this condition. The critical incident angle θ_c is defined as that value of θ_i that makes $\theta_t = \pi/2$,

$$\sin\theta_c = c_1/c_2 \tag{17}$$

which has a real solution only if $c_1 < c_2$. At the critical angle, the wavenumber k_{zt} is zero. Lesser incident angles have real values of k_{zt}. For larger incident angles there is no real angle θ_t that satisfies (6). Snell's law must always be obeyed in order to satisfy the boundary conditions at $z=0$ for all x. What happens is that θ_t becomes a complex number that satisfies (6). Although $\sin\theta_t$ is still real, $\cos\theta_t$ is imaginary when $\sin\theta_t$ exceeds unity:

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} \tag{18}$$

This then makes k_{zt} imaginary, which we can write as

$$k_{zt} = k_2 \cos\theta_t = -j\alpha \tag{19}$$

The negative sign of the square root is taken so that waves now decay with z:

$$\begin{aligned} \mathbf{E}_t &= \mathrm{Re}\,[\hat{E}_t e^{j(\omega t - k_{xt}x)} e^{-\alpha z}\mathbf{i}_y] \\ \mathbf{H}_t &= \mathrm{Re}\left[\frac{\hat{E}_t}{\eta_2}(-\cos\theta_t \mathbf{i}_x + \sin\theta_t \mathbf{i}_z)\, e^{j(\omega t - k_{xt}x)} e^{-\alpha z}\right] \end{aligned} \tag{20}$$

The solutions are now nonuniform plane waves, as discussed in Section 7-7.

Complex angles of transmission are a valid mathematical concept. What has happened is that in (1) we wrote our assumed solutions for the transmitted fields in terms of pure propagating waves. Maxwell's equations for an incident angle greater than the critical angle require spatially decaying waves with z in region 2 so that the mathematics forced k_{zt} to be imaginary.

There is no power dissipation since the z-directed time-average power flow is zero,

$$\begin{aligned} \langle S_z \rangle &= -\tfrac{1}{2}\,\mathrm{Re}\,[E_y H_x^*] \\ &= -\tfrac{1}{2}\,\mathrm{Re}\left[\frac{\hat{E}_t \hat{E}_t^*}{\eta_2}(-\cos\theta_t)^* e^{-2\alpha z}\right] = 0 \end{aligned} \tag{21}$$

because $\cos\theta_t$ is pure imaginary so that the bracketed term in (21) is pure imaginary. The incident z-directed time-average power is totally reflected. Even though the time-averaged z-directed transmitted power is zero, there are nonzero but exponentially decaying fields in region 2.

7-9-4 H Field Parallel to the Boundary

For this polarization, illustrated in Figure 7-18b, the fields are

$$\begin{aligned} \mathbf{E}_i &= \mathrm{Re}\,[\hat{E}_i(\cos\theta_i \mathbf{i}_x - \sin\theta_i \mathbf{i}_z)\, e^{j(\omega t - k_{xi}x - k_{zi}z)}] \\ \mathbf{H}_i &= \mathrm{Re}\left[\frac{\hat{E}_i}{\eta_1} e^{j(\omega t - k_{xi}x - k_{zi}z)}\mathbf{i}_y\right] \\ \mathbf{E}_r &= \mathrm{Re}\,[\hat{E}_r(-\cos\theta_r \mathbf{i}_x - \sin\theta_r \mathbf{i}_z)\, e^{j(\omega t - k_{xr}x + k_{zr}z)}] \\ \mathbf{H}_r &= \mathrm{Re}\left[\frac{\hat{E}_r}{\eta_1} e^{j(\omega t - k_{xr}x + k_{zr}z)}\mathbf{i}_y\right] \\ \mathbf{E}_t &= \mathrm{Re}\,[\hat{E}_t(\cos\theta_t \mathbf{i}_x - \sin\theta_t \mathbf{i}_z)\, e^{j(\omega t - k_{xt}x - k_{zt}z)}] \\ \mathbf{H}_t &= \mathrm{Re}\left[\frac{\hat{E}_t}{\eta_2} e^{j(\omega t - k_{xt}x - k_{zt}z)}\mathbf{i}_y\right] \end{aligned} \tag{22}$$

where the wavenumbers and impedances are the same as in (2) and (3).

Continuity of tangential **E** and **H** at $z=0$ requires

$$\hat{E}_i \cos\theta_i e^{-jk_{xi}x} - \hat{E}_r \cos\theta_r e^{-jk_{xr}x} = \hat{E}_t \cos\theta_t e^{-jk_{xt}x}$$
$$\frac{\hat{E}_i e^{-jk_{xi}x} + \hat{E}_r e^{-jk_{xr}x}}{\eta_1} = \frac{\hat{E}_t e^{-jk_{xt}x}}{\eta_2} \tag{23}$$

Again the phase factors must be equal so that (5) and (6) are again true. Snell's law and the angle of incidence equalling the angle of reflection are independent of polarization.

We solve (23) for the field reflection and transmission coefficients as

$$R = \frac{\hat{E}_r}{\hat{E}_i} = \frac{\eta_1 \cos\theta_i - \eta_2 \cos\theta_t}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} \tag{24}$$

$$T = \frac{\hat{E}_t}{\hat{E}_i} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} \tag{25}$$

Now we note that the boundary condition of continuity of normal **D** at $z=0$ is redundant to the lower relation in (23),

$$\varepsilon_1 \hat{E}_1 \sin\theta_i + \varepsilon_1 \hat{E}_r \sin\theta_r = \varepsilon_2 \hat{E}_t \sin\theta_t \tag{26}$$

using Snell's law to relate the angles.

For this polarization the condition for no reflected waves is

$$R = 0 \Rightarrow \eta_2 \cos\theta_t = \eta_1 \cos\theta_i \tag{27}$$

which from Snell's law gives the Brewster angle:

$$\sin^2\theta_B = \frac{1 - \varepsilon_1\mu_2/(\varepsilon_2\mu_1)}{1 - (\varepsilon_1/\varepsilon_2)^2} \tag{28}$$

There is now a solution for the usual case where $\mu_1 = \mu_2$ but $\varepsilon_1 \neq \varepsilon_2$:

$$\sin^2\theta_B = \frac{1}{1+\varepsilon_1/\varepsilon_2} \Rightarrow \tan\theta_B = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \tag{29}$$

At this Brewster's angle the reflected and transmitted power flows are at right angles $(\theta_B + \theta_t) = \pi/2$ as can be seen by using (6), (27), and (29)

$$\begin{aligned}\cos(\theta_B + \theta_t) &= \cos\theta_B \cos\theta_t - \sin\theta_B \sin\theta_t \\ &= \cos^2\theta_B \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} - \sin^2\theta_B \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} \\ &= \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} - \sin^2\theta_B \left(\sqrt{\frac{\varepsilon_1}{\varepsilon_2}} + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \right) = 0\end{aligned} \tag{30}$$

Because Snell's law is independent of polarization, the critical angle of (17) is the same for both polarizations. Note that the Brewster's angle for either polarization, if it exists, is always less than the critical angle of (17), as can be particularly seen when $\mu_1 = \mu_2$ for the magnetic field polarized parallel to the interface or when $\varepsilon_1 = \varepsilon_2$ for the electric field polarized parallel to the interface, as then

$$\frac{1}{\sin^2 \theta_B} = \frac{1}{\sin^2 \theta_c} + 1 \tag{31}$$

7-10 APPLICATIONS TO OPTICS

Reflection and refraction of electromagnetic waves obliquely incident upon the interface between dissimilar linear lossless media are governed by the two rules illustrated in Figure 7-19:

(i) The angle of incidence equals the angle of reflection.
(ii) Waves incident from a medium of high light velocity (low index of refraction) to one of low velocity (high index of refraction) are bent towards the normal. If the wave is incident from a low velocity (high index) to high velocity (low index) medium, the light is bent away from the normal. The incident and refracted angles are related by Snell's law.

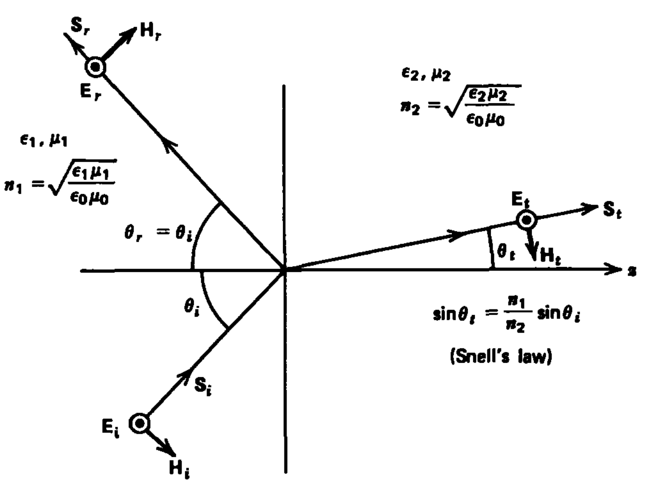

Figure 7-19 A summary of reflection and refraction phenomena across the interface separating two linear media. When $\theta_i = \theta_B$ (Brewster's angle), there is no reflected ray. When $\theta_i > \theta_c$ (critical angle), the transmitted fields decay with z.

Most optical materials, like glass, have a permeability of free space μ_0. Therefore, a Brewster's angle of no reflection only exists if the **H** field is parallel to the boundary.

At the critical angle, which can only exist if light travels from a high index of refraction material (low light velocity) to one of low index (high light velocity), there is a transmitted field that decays with distance as a nonuniform plane wave. However, there is no time-average power carried by this evanescent wave so that all the time-average power is reflected. This section briefly describes various applications of these special angles and the rules governing reflection and refraction.

7-10-1 Reflections from a Mirror

A person has their eyes at height h above their feet and a height Δh below the top of their head, as in Figure 7-20. A mirror in front extends a distance Δy above the eyes and a distance y below. How large must y and Δy be so that the person sees their entire image? The light reflected off the person into the mirror must be reflected again into the person's eyes. Since the angle of incidence equals the angle of reflection, Figure 7-20 shows that $\Delta y = \Delta h/2$ and $y = h/2$.

7-10-2 Lateral Displacement of a Light Ray

A light ray is incident from free space upon a transparent medium with index of refraction n at angle θ_i, as shown in Figure 7-21. The angle of the transmitted light is given by Snell's law:

$$\sin \theta_t = (1/n) \sin \theta_i \qquad (1)$$

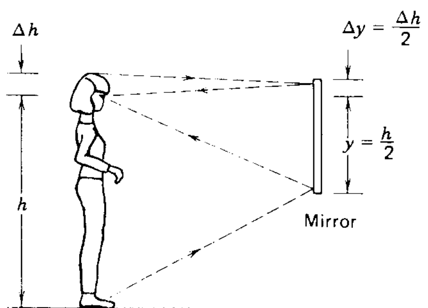

Figure 7-20 Because the angle of incidence equals the angle of reflection, a person can see their entire image if the mirror extends half the distance of extent above and below the eyes.

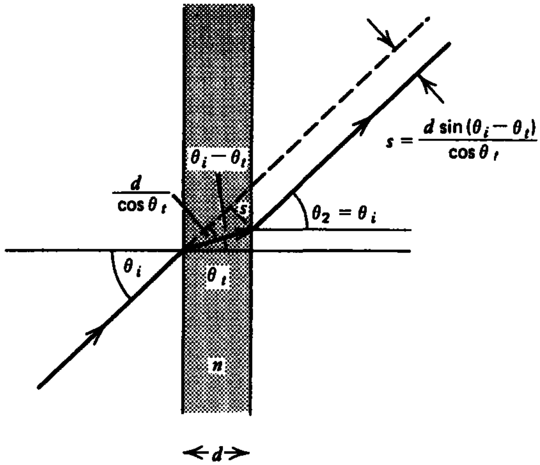

Figure 7-21 A light ray incident upon a glass plate exits the plate into the original medium parallel to its original trajectory but laterally displaced.

When this light hits the second interface, the angle θ_t is now the incident angle so that the transmitted angle θ_2 is again given by Snell's law:

$$\sin\theta_2 = n \sin\theta_t = \sin\theta_i \tag{2}$$

so that the light exits at the original incident angle θ_i. However, it is now shifted by the amount:

$$s = \frac{d \sin(\theta_i - \theta_t)}{\cos\theta_t} \tag{3}$$

If the plate is glass with refractive index $n = 1.5$ and thickness $d = 1$ mm with incident angle $\theta_i = 30°$, the angle θ_t in the glass is

$$\sin\theta_t = 0.33 \Rightarrow \theta_t = 19.5° \tag{4}$$

so that the lateral displacement is $s = 0.19$ mm.

7-10-3 Polarization By Reflection

Unpolarized light is incident upon the piece of glass in Section 7-10-2 with index of refraction $n = 1.5$. Unpolarized light has both **E** and **H** parallel to the interface. We assume that the permeability of the glass equals that of free space and that the light is incident at the Brewster's angle θ_B for light polarized with **H** parallel to the interface. The incident and transmitted angles are then

$$\begin{aligned} \tan\theta_B &= \sqrt{\varepsilon/\varepsilon_0} = n \Rightarrow \theta_B = 56.3^\circ \\ \tan\theta_t &= \sqrt{\varepsilon_0/\varepsilon} = 1/n \Rightarrow \theta_t = 33.7^\circ \end{aligned} \qquad (5)$$

The Brewster's angle is also called the polarizing angle because it can be used to separate the two orthogonal polarizations. The polarization, whose **H** field is parallel to the interface, is entirely transmitted at the first interface with no reflection. The other polarization with electric field parallel to the interface is partially transmitted and reflected. At the second (glass-free space) interface the light is incident at angle θ_t. From (5) we see that this angle is the Brewster's angle with **H** parallel to the interface for light incident from the glass side onto the glass-free space interface. Then again, the **H** parallel to the interface polarization is entirely transmitted while the **E** parallel to the interface polarization is partially reflected and partially transmitted. Thus, the reflected wave is entirely polarized with electric field parallel to the interface. The transmitted waves, although composed of both polarizations, have the larger amplitude with **H**

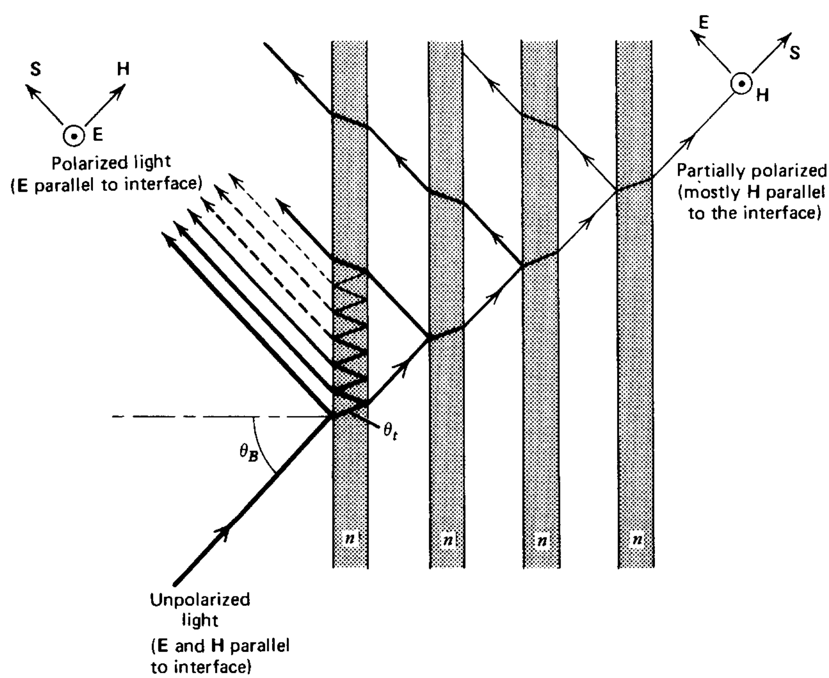

Figure 7-22 Unpolarized light incident upon glass with $\mu = \mu_0$ can be polarized by reflection if it is incident at the Brewster's angle for the polarization with **H** parallel to the interface. The transmitted light becomes more polarized with **H** parallel to the interface by adding more parallel glass plates.

parallel to the interface because it was entirely transmitted with no reflection at both interfaces.

By passing the transmitted light through another parallel piece of glass, the polarization with electric field parallel to the interface becomes further diminished because it is partially reflected, while the other polarization is completely transmitted. With more glass elements, as in Figure 7-22, the transmitted light can be made essentially completely polarized with **H** field parallel to the interface.

7-10-4 Light Propagation In Water

(a) Submerged Source

A light source is a distance d below the surface of water with refractive index $n = 1.33$, as in Figure 7-23. The rays emanate from the source as a cone. Those rays at an angle from the normal greater than the critical angle,

$$\sin\theta_c = 1/n \Rightarrow \theta_c = 48.8° \tag{6}$$

are not transmitted into the air but undergo total internal reflection. A circle of light with diameter

$$D = 2d\tan\theta_c \approx 2.28d \tag{7}$$

then forms on the water's surface due to the exiting light.

(b) Fish Below a Boat

A fish swims below a circular boat of diameter D, as in Figure 7-24. As we try to view the fish from the air above, the incident light ray is bent towards the normal. The region below the boat that we view from above is demarcated by the light rays at grazing incidence to the surface ($\theta_i = \pi/2$) just entering the water ($n = 1.33$) at the sides of the boat. The transmitted angle of these light rays is given from Snell's law as

$$\sin\theta_t = \frac{\sin\theta_i}{n} = \frac{1}{n} \Rightarrow \theta_t = 48.8° \tag{8}$$

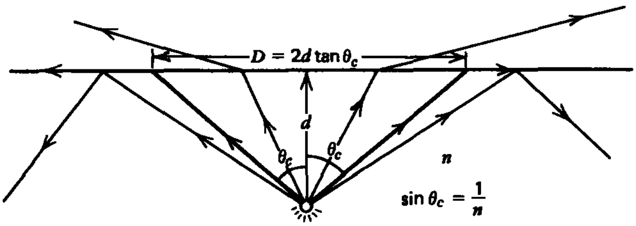

Figure 7-23 Light rays emanating from a source within a high index of refraction medium are totally internally reflected from the surface for angles greater than the critical angle. Lesser angles of incidence are transmitted.

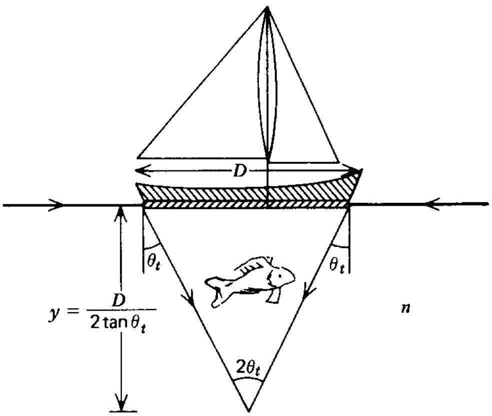

Figure 7-24 A fish cannot be seen from above if it swims below a circular boat within the cone bounded by light rays at grazing incidence entering the water at the side of the boat.

These rays from all sides of the boat intersect at the point a distance y below the boat, where

$$\tan\theta_t = \frac{D}{2y} \Rightarrow y = \frac{D}{2\tan\theta_t} \approx 0.44D \tag{9}$$

If the fish swims within the cone, with vertex at the point y below the boat, it cannot be viewed from above.

7-10-5 Totally Reflecting Prisms

The glass isoceles right triangle in Figure 7-25 has an index of refraction of $n = 1.5$ so that the critical angle for total

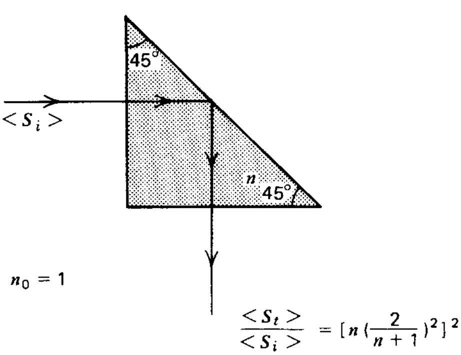

Figure 7-25 A totally reflecting prism. The index of refraction n must exceed $\sqrt{2}$ so that the light incident on the hypotenuse at 45° exceeds the critical angle.

internal reflection is

$$\sin\theta_c=\frac{1}{n}=\frac{1}{1.5}\Rightarrow\theta_c=41.8^\circ \tag{10}$$

The light is normally incident on the vertical face of the prism. The transmission coefficient is then given in Section 7-6-1 as

$$T_1=\frac{\hat{E}_t}{\hat{E}_i}=\frac{2\eta}{\eta+\eta_0}=\frac{2/n}{1+1/n}=\frac{2}{n+1}=0.8 \tag{11}$$

where because the permeability of the prism equals that of free space $n=\sqrt{\varepsilon/\varepsilon_0}$ while $\eta/\eta_0=\sqrt{\varepsilon_0/\varepsilon}=1/n$. The transmitted light is then incident upon the hypotenuse of the prism at an angle of 45°, which exceeds the critical angle so that no power is transmitted and the light is totally reflected being turned through a right angle. The light is then normally incident upon the horizontal face with transmission coefficient:

$$T_2=\frac{\hat{E}_2}{0.8\hat{E}_i}=\frac{2\eta_0}{\eta+\eta_0}=\frac{2}{1/n+1}=\frac{2n}{n+1}=1.2 \tag{12}$$

The resulting electric field amplitude is then

$$\hat{E}_2=T_1T_2\hat{E}_i=0.96\hat{E}_i \tag{13}$$

The ratio of transmitted to incident power density is

$$\frac{<S>}{<S_i>}=\frac{\frac{1}{2}|\hat{E}_2|^2/\eta_0}{\frac{1}{2}|\hat{E}_i|^2/\eta_0}=\frac{|\hat{E}_2|^2}{|\hat{E}_i|^2}=\left(\frac{24}{25}\right)^2\approx 0.92 \tag{14}$$

This ratio can be increased to unity by applying a quarter-wavelength-thick dielectric coating with index of refraction $n_{\text{coating}}=\sqrt{n}$, as developed in Example 7-1. This is not usually done because the ratio in (14) is already large without the expense of a coating.

7-10-6 Fiber Optics

(a) Straight Light Pipe

Long thin fibers of transparent material can guide light along a straight path if the light within the pipe is incident upon the wall at an angle greater than the critical angle ($\sin\theta_c=1/n$):

$$\sin\theta_2=\cos\theta_t\geq\sin\theta_c \tag{15}$$

The light rays are then totally internally reflected being confined to the pipe until they exit, as in Figure 7-26. The

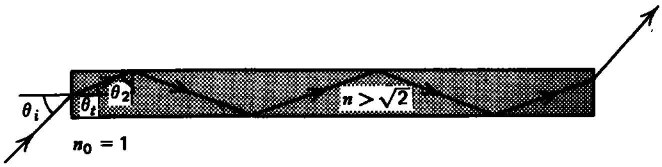

Figure 7-26 The index of refraction of a straight light pipe must be greater than $\sqrt{2}$ for total internal reflections of incident light at any angle.

incident angle is related to the transmitted angle from Snell's law,

$$\sin \theta_t = (1/n) \sin \theta_i \tag{16}$$

so that (15) becomes

$$\cos \theta_t = \sqrt{1 - \sin^2 \theta_t} = \sqrt{1 - (1/n^2) \sin^2 \theta_i} \geq 1/n \tag{17}$$

which when solved for n yields

$$n^2 \geq 1 + \sin^2 \theta_i \tag{18}$$

If this condition is met for grazing incidence ($\theta_i = \pi/2$), all incident light will be passed by the pipe, which requires that

$$n^2 \geq 2 \Rightarrow n \geq \sqrt{2} \tag{19}$$

Most types of glass have $n \approx 1.5$ so that this condition is easily met.

(b) Bent Fibers

Light can also be guided along a tortuous path if the fiber is bent, as in the semi-circular pipe shown in Figure 7-27. The minimum angle to the radial normal for the incident light shown is at the point A. This angle in terms of the radius of the bend and the light pipe width must exceed the critical angle

$$\sin \theta_A = \frac{R}{R+d} \geq \sin \theta_c \tag{20}$$

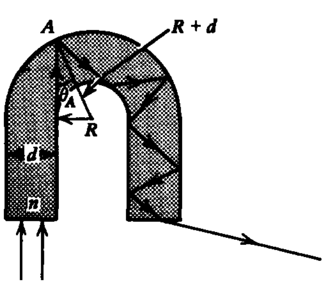

Figure 7-27 Light can be guided along a circularly bent fiber if $R/d > 1/(n-1)$ as then there is always total internal reflection each time the light is incident on the walls.

so that

$$\frac{R/d}{R/d+1} \geq \frac{1}{n} \tag{21}$$

which when solved for R/d requires

$$\frac{R}{d} \geq \frac{1}{n-1} \tag{22}$$

PROBLEMS

Section 7-1

1. For the following electric fields in a linear media of permittivity ε and permeability μ find the charge density, magnetic field, and current density.

(a) $\mathbf{E} = E_0(x\mathbf{i}_x + y\mathbf{i}_y)\sin\omega t$

(b) $\mathbf{E} = E_0(y\mathbf{i}_x - x\mathbf{i}_y)\cos\omega t$

(c) $\mathbf{E} = \text{Re}\,[E_0\, e^{j(\omega t - k_x x - k_z z)}\mathbf{i}_y]$. How must k_x, k_z, and ω be related so that $\mathbf{J} = 0$?

2. An Ohmic conductor of arbitrary shape has an initial charge distribution $\rho_0(\mathbf{r})$ at $t = 0$.

(a) What is the charge distribution for all time?

(b) The initial charge distribution is uniform and is confined between parallel plate electrodes of spacing d. What are the electric and magnetic fields when the electrodes are opened or short circuited?

(c) Repeat (b) for coaxial cylindrical electrodes of inner radius a and outer radius b.

(d) When does a time varying electric field not generate a magnetic field?

3. (a) For linear media of permittivity ε and permeability μ, use the magnetic vector potential $\mathbf{A}$ to rewrite Faraday's law as the curl of a function.

(b) Can a scalar potential function V be defined? What is the electric field in terms of V and $\mathbf{A}$? The choice of V is not unique so pick V so that under static conditions $\mathbf{E} = -\nabla V$.

(c) Use the results of (a) and (b) in Ampere's law with Maxwell's displacement current correction to obtain a single equation in $\mathbf{A}$ and V. (**Hint:** $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$.)

(d) Since we are free to specify $\nabla\cdot\mathbf{A}$, what value should we pick to make (c) an equation just in $\mathbf{A}$? This is called setting the gauge.

(e) Use the results of (a)–(d) in Gauss's law for $\mathbf{D}$ to obtain a single equation in V.