chapter 5

the magnetic field

The ancient Chinese knew that the iron oxide magnetite (Fe_3O_4) attracted small pieces of iron. The first application of this effect was the navigation compass, which was not developed until the thirteenth century. No major advances were made again until the early nineteenth century when precise experiments discovered the properties of the magnetic field.

5-1 FORCES ON MOVING CHARGES

5-1-1 The Lorentz Force Law

It was well known that magnets exert forces on each other, but in 1820 Oersted discovered that a magnet placed near a current carrying wire will align itself perpendicular to the wire. Each charge q in the wire, moving with velocity $\mathbf{v}$ in the magnetic field $\mathbf{B}$ [teslas, (kg-s^{-2}-A^{-1})], felt the empirically determined Lorentz force perpendicular to both $\mathbf{v}$ and $\mathbf{B}$

$$\mathbf{f} = q(\mathbf{v}\times\mathbf{B}) \tag{1}$$

as illustrated in Figure 5-1. A distribution of charge feels a differential force $d\mathbf{f}$ on each moving incremental charge element dq:

$$d\mathbf{f} = dq(\mathbf{v}\times\mathbf{B}) \tag{2}$$

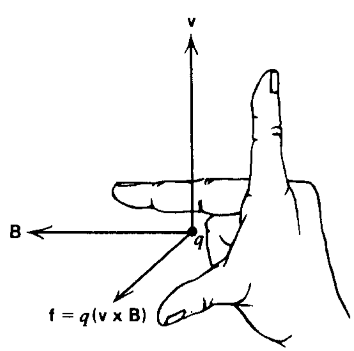


Figure 5-1 A charge moving through a magnetic field experiences the Lorentz force perpendicular to both its motion and the magnetic field.

Moving charges over a line, surface, or volume, respectively constitute line, surface, and volume currents, as in Figure 5-2, where (2) becomes

$$d\mathbf{f}=\begin{cases}\rho_f\mathbf{v}\times\mathbf{B}\,dV=\mathbf{J}\times\mathbf{B}\,dV & (\mathbf{J}=\rho_f\mathbf{v},\ \text{volume current density})\\ \sigma_f\mathbf{v}\times\mathbf{B}\,dS=\mathbf{K}\times\mathbf{B}\,dS & \\ & (\mathbf{K}=\sigma_f\mathbf{v},\ \text{surface current density})\\ \lambda_f\mathbf{v}\times\mathbf{B}\,dl=\mathbf{I}\times\mathbf{B}\,dl & (\mathbf{I}=\lambda_f\mathbf{v},\ \text{line current})\end{cases} \tag{3}$$

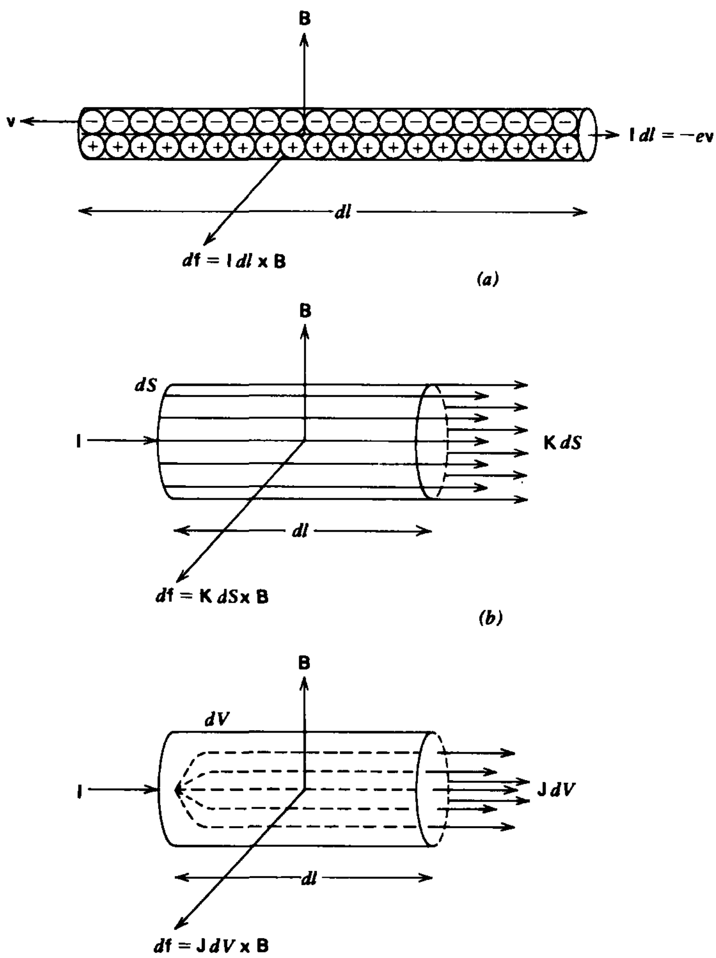


Figure 5-2 Moving line, surface, and volume charge distributions constitute currents. (*a*) In metallic wires the net charge is zero since there are equal amounts of negative and positive charges so that the Coulombic force is zero. Since the positive charge is essentially stationary, only the moving electrons contribute to the line current in the direction opposite to their motion. (*b*) Surface current. (*c*) Volume current.

The total magnetic force on a current distribution is then obtained by integrating (3) over the total volume, surface, or contour containing the current. If there is a net charge with its associated electric field **E**, the total force densities include the Coulombic contribution:

$$\begin{aligned}
\mathbf{f} &= q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \quad \text{Newton} \\
\mathbf{F}_L &= \lambda_f(\mathbf{E}+\mathbf{v}\times\mathbf{B}) = \lambda_f\mathbf{E}+\mathbf{I}\times\mathbf{B} \quad \text{N/m} \\
\mathbf{F}_S &= \sigma_f(\mathbf{E}+\mathbf{v}\times\mathbf{B}) = \sigma_f\mathbf{E}+\mathbf{K}\times\mathbf{B} \quad \text{N/m}^2 \\
\mathbf{F}_V &= \rho_f(\mathbf{E}+\mathbf{v}\times\mathbf{B}) = \rho_f\mathbf{E}+\mathbf{J}\times\mathbf{B} \quad \text{N/m}^3
\end{aligned} \tag{4}$$

In many cases the net charge in a system is very small so that the Coulombic force is negligible. This is often true for conduction in metal wires. A net current still flows because of the difference in velocities of each charge carrier.

Unlike the electric field, the magnetic field cannot change the kinetic energy of a moving charge as the force is perpendicular to the velocity. It can alter the charge's trajectory but not its velocity magnitude.

5-1-2 Charge Motions in a Uniform Magnetic Field

The three components of Newton's law for a charge q of mass m moving through a uniform magnetic field $B_z\mathbf{i}_z$ are

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v}\times\mathbf{B} \Rightarrow \begin{cases} m\dfrac{dv_x}{dt} = qv_yB_z \\ m\dfrac{dv_y}{dt} = -qv_xB_z \\ m\dfrac{dv_z}{dt} = 0 \Rightarrow v_z = \text{const} \end{cases} \tag{5}$$

The velocity component along the magnetic field is unaffected. Solving the first equation for v_y and substituting the result into the second equation gives us a single equation in v_x:

$$\frac{d^2v_x}{dt^2} + \omega_0^2 v_x = 0, \qquad v_y = \frac{1}{\omega_0}\frac{dv_x}{dt}, \qquad \omega_0 = \frac{qB_z}{m} \tag{6}$$

where ω_0 is called the Larmor angular velocity or the cyclotron frequency (see Section 5-1-4). The solutions to (6) are

$$\begin{aligned}
v_x &= A_1 \sin \omega_0 t + A_2 \cos \omega_0 t \\
v_y &= \frac{1}{\omega_0}\frac{dv_x}{dt} = A_1 \cos \omega_0 t - A_2 \sin \omega_0 t
\end{aligned} \tag{7}$$

where A_1 and A_2 are found from initial conditions. If at $t=0$,

$$\mathbf{v}(t=0)=v_0\mathbf{i}_x \tag{8}$$

then (7) and Figure 5-3*a* show that the particle travels in a circle, with constant speed v_0 in the xy plane:

$$v=v_0(\cos\omega_0 t\,\mathbf{i}_x-\sin\omega_0 t\,\mathbf{i}_y) \tag{9}$$

with radius

$$R=v_0/\omega_0 \tag{10}$$

If the particle also has a velocity component along the magnetic field in the z direction, the charge trajectory becomes a helix, as shown in Figure 5-3*b*.

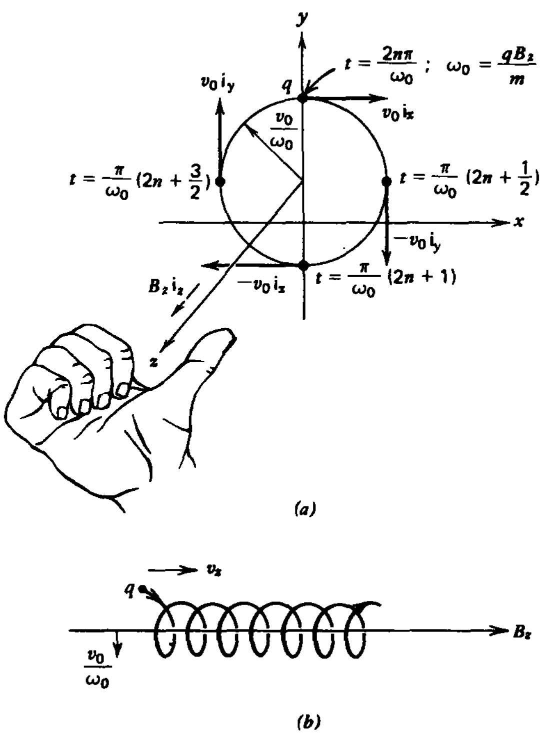


Figure 5-3 (*a*) A positive charge q, initially moving perpendicular to a magnetic field, feels an orthogonal force putting the charge into a circular motion about the magnetic field where the Lorentz force is balanced by the centrifugal force. Note that the charge travels in the direction (in this case clockwise) so that its self-field through the loop [see Section 5-2-1] is opposite in direction to the applied field. (*b*) A velocity component in the direction of the magnetic field is unaffected resulting in a helical trajectory.

5-1-3 The Mass Spectrograph

The mass spectrograph uses the circular motion derived in Section 5-1-2 to determine the masses of ions and to measure the relative proportions of isotopes, as shown in Figure 5-4. Charges enter between parallel plate electrodes with a y-directed velocity distribution. To pick out those charges with a particular magnitude of velocity, perpendicular electric and magnetic fields are imposed so that the net force on a charge is

$$f_x = q(E_x + v_y B_z) \tag{11}$$

For charges to pass through the narrow slit at the end of the channel, they must not be deflected by the fields so that the force in (11) is zero. For a selected velocity $v_y = v_0$ this requires a negatively x directed electric field

$$E_x = \frac{V}{s} = -v_0 B_0 \tag{12}$$

which is adjusted by fixing the applied voltage V. Once the charge passes through the slit, it no longer feels the electric field and is only under the influence of the magnetic field. It thus travels in a circle of radius

$$r = \frac{v_0}{\omega_0} = \frac{v_0 m}{q B_0} \tag{13}$$

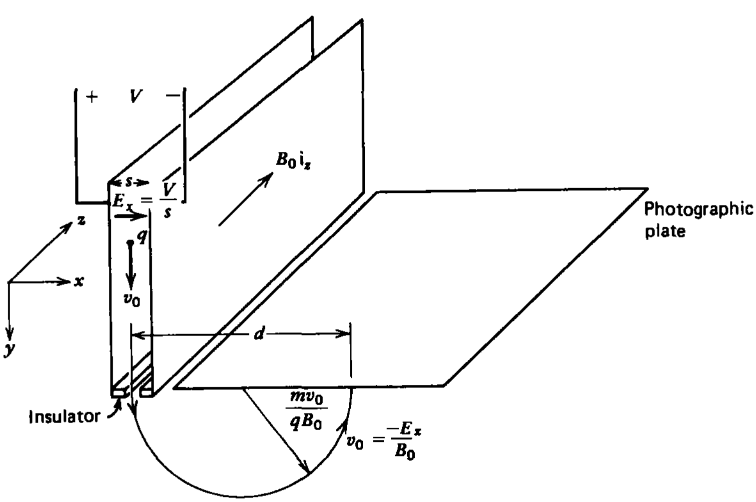


Figure 5-4 The mass spectrograph measures the mass of an ion by the radius of its trajectory when moving perpendicular to a magnetic field. The crossed uniform electric field selects the ion velocity that can pass through the slit.

which is directly proportional to the mass of the ion. By measuring the position of the charge when it hits the photographic plate, the mass of the ion can be calculated. Different isotopes that have the same number of protons but different amounts of neutrons will hit the plate at different positions.

For example, if the mass spectrograph has an applied voltage of $V = -100$ V across a 1-cm gap ($E_x = -10^4$ V/m) with a magnetic field of 1 tesla, only ions with velocity

$$v_y = -E_x/B_0 = 10^4 \text{ m/sec} \tag{14}$$

will pass through. The three isotopes of magnesium, ${}_{12}Mg^{24}$, ${}_{12}Mg^{25}$, ${}_{12}Mg^{26}$, each deficient of one electron, will hit the photographic plate at respective positions:

$$d = 2r = \frac{2\times 10^4 N(1.67\times 10^{-27})}{1.6\times 10^{-19}(1)} \approx 2\times 10^{-4} N$$
$$\Rightarrow 0.48, 0.50, 0.52 \text{ cm} \tag{15}$$

where N is the number of protons and neutrons ($m = 1.67\times 10^{-27}$ kg) in the nucleus.

5-1-4 The Cyclotron

A cyclotron brings charged particles to very high speeds by many small repeated accelerations. Basically it is composed of a split hollow cylinder, as shown in Figure 5-5, where each half is called a "dee" because their shape is similar to the

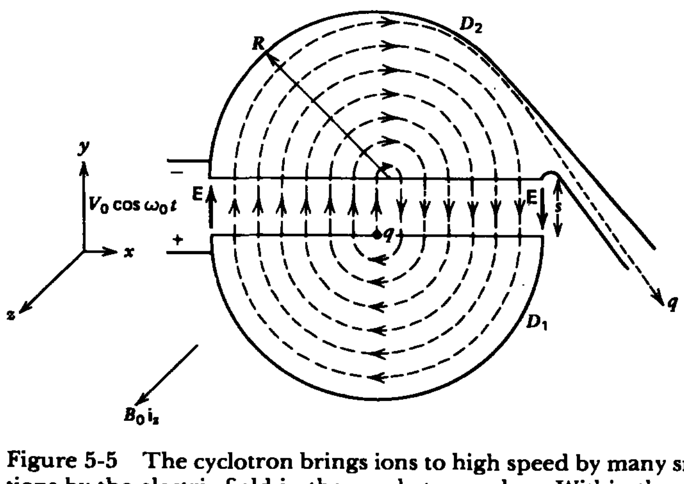


Figure 5-5 The cyclotron brings ions to high speed by many small repeated accelerations by the electric field in the gap between dees. Within the dees the electric field is negligible so that the ions move in increasingly larger circular orbits due to an applied magnetic field perpendicular to their motion.

fourth letter of the alphabet. The dees are put at a sinusoidally varying potential difference. A uniform magnetic field $B_0\mathbf{i}_z$ is applied along the axis of the cylinder. The electric field is essentially zero within the cylindrical volume and assumed uniform $E_y = v(t)/s$ in the small gap between dees. A charge source at the center of D_1 emits a charge q of mass m with zero velocity at the peak of the applied voltage at $t = 0$. The electric field in the gap accelerates the charge towards D_2. Because the gap is so small the voltage remains approximately constant at V_0 while the charge is traveling between dees so that its displacement and velocity are

$$m\frac{dv_y}{dt} = \frac{qV_0}{s} \Rightarrow v_y = \frac{qV_0}{sm}t$$
$$v_y = \frac{dy}{dt} \Rightarrow y = \frac{qV_0t^2}{2ms} \tag{16}$$

The charge thus enters D_2 at time $t = [2ms^2/qV_0]^{1/2}$ later with velocity $v_y = \sqrt{2qV_0/m}$. Within D_2 the electric field is negligible so that the charge travels in a circular orbit of radius $r = v_y/\omega_0 = mv_y/qB_0$ due to the magnetic field alone. The frequency of the voltage is adjusted to just equal the angular velocity $\omega_0 = qB_0/m$ of the charge, so that when the charge re-enters the gap between dees the polarity has reversed accelerating the charge towards D_1 with increased velocity. This process is continually repeated, since every time the charge enters the gap the voltage polarity accelerates the charge towards the opposite dee, resulting in a larger radius of travel. Each time the charge crosses the gap its velocity is increased by the same amount so that after n gap traversals its velocity and orbit radius are

$$v_n = \left(\frac{2qnV_0}{m}\right)^{1/2}, \qquad R_n = \frac{v_n}{\omega_0} = \left(\frac{2nmV_0}{qB_0^2}\right)^{1/2} \tag{17}$$

If the outer radius of the dees is R, the maximum speed of the charge

$$v_{\max} = \omega_0 R = \frac{qB_0}{m}R \tag{18}$$

is reached after $2n = qB_0^2R^2/mV_0$ round trips when $R_n = R$. For a hydrogen ion ($q = 1.6\times 10^{-19}$ coul, $m = 1.67\times 10^{-27}$ kg), within a magnetic field of 1 tesla ($\omega_0 \approx 9.6\times 10^7$ radian/sec) and peak voltage of 100 volts with a cyclotron radius of one meter, we reach $v_{\max} = 9.6\times 10^7$ m/s (which is about 30% of the speed of light) in about $2n \approx 9.6\times 10^5$ round-trips, which takes a time $\tau = 4n\pi/\omega_0 \approx 2\pi/100 \approx 0.06$ sec. To reach this speed with an electrostatic accelerator would require

$$\tfrac{1}{2}mv^2 = qV \Rightarrow V = \frac{mv_{max}^2}{2q} \approx 48 \times 10^6 \text{ Volts} \tag{19}$$

The cyclotron works at much lower voltages because the angular velocity of the ions remains constant for fixed qB_0/m and thus arrives at the gap in phase with the peak of the applied voltage so that it is sequentially accelerated towards the opposite dee. It is not used with electrons because their small mass allows them to reach relativistic velocities close to the speed of light, which then greatly increases their mass, decreasing their angular velocity ω_0, putting them out of phase with the voltage.

5-1-5 Hall Effect

When charges flow perpendicular to a magnetic field, the transverse displacement due to the Lorentz force can give rise to an electric field. The geometry in Figure 5-6 has a uniform magnetic field $B_0\mathbf{i}_z$ applied to a material carrying a current in the y direction. For positive charges as for holes in a p-type semiconductor, the charge velocity is also in the positive y direction, while for negative charges as occur in metals or in n-type semiconductors, the charge velocity is in the negative y direction. In the steady state where the charge velocity does not vary with time, the net force on the charges must be zero,

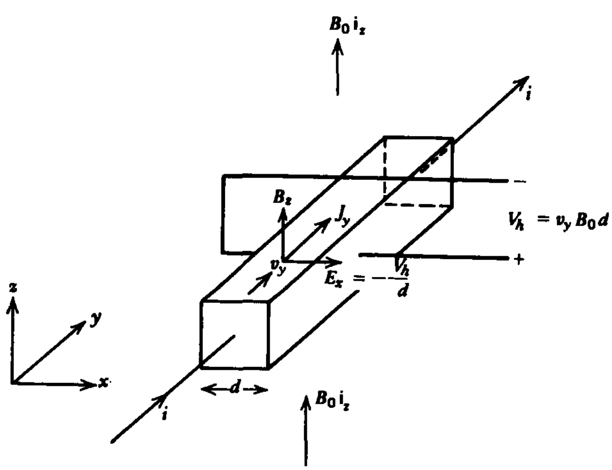


Figure 5-6 A magnetic field perpendicular to a current flow deflects the charges transversely giving rise to an electric field and the Hall voltage. The polarity of the voltage is the same as the sign of the charge carriers.

which requires the presence of an x-directed electric field

$$\mathbf{E}+\mathbf{v}\times\mathbf{B}=0\Rightarrow E_x=-v_yB_0 \tag{20}$$

A transverse potential difference then develops across the material called the Hall voltage:

$$V_h=-\int_0^d E_x\,dx=v_yB_0d \tag{21}$$

The Hall voltage has its polarity given by the sign of v_y; positive voltage for positive charge carriers and negative voltage for negative charges. This measurement provides an easy way to determine the sign of the predominant charge carrier for conduction.

5-2 MAGNETIC FIELD DUE TO CURRENTS

Once it was demonstrated that electric currents exert forces on magnets, Ampere immediately showed that electric currents also exert forces on each other and that a magnet could be replaced by an equivalent current with the same result. Now magnetic fields could be turned on and off at will with their strength easily controlled.

5-2-1 The Biot–Savart Law

Biot and Savart quantified Ampere's measurements by showing that the magnetic field $\mathbf{B}$ at a distance $\mathbf{r}$ from a moving charge is

$$\mathbf{B}=\frac{\mu_0 q\mathbf{v}\times\mathbf{i}_r}{4\pi r^2}\ \text{teslas (kg-s}^{-2}\text{-A}^{-1}\text{)} \tag{1}$$

as in Figure 5-7a, where μ_0 is a constant called the permeability of free space and in SI units is defined as having the exact numerical value

$$\mu_0\equiv 4\pi\times 10^{-7}\ \text{henry/m (kg-m-A}^{-2}\text{-s}^{-2}\text{)} \tag{2}$$

The 4π is introduced in (1) for the same reason it was introduced in Coulomb's law in Section 2-2-1. It will cancel out a 4π contribution in frequently used laws that we will soon derive from (1). As for Coulomb's law, the magnetic field drops off inversely as the square of the distance, but its direction is now perpendicular both to the direction of charge flow and to the line joining the charge to the field point.

In the experiments of Ampere and those of Biot and Savart, the charge flow was constrained as a line current within a wire. If the charge is distributed over a line with

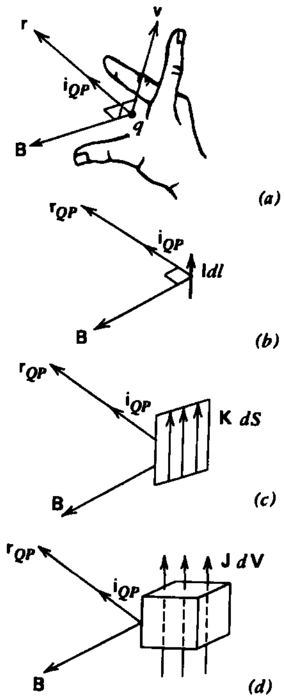


Figure 5-7 The magnetic field generated by a current is perpendicular to the current and the unit vector joining the current element to the field point; (*a*) point charge; (*b*) line current; (*c*) surface current; (*d*) volume current.

current **I**, or a surface with current per unit length **K**, or over a volume with current per unit area **J**, we use the differential-sized current elements, as in Figures 5-7*b*–5-7*d*:

$$dq\,\mathbf{v}=\begin{cases}\mathbf{I}\,dl & \text{(line current)}\\ \mathbf{K}\,dS & \text{(surface current)}\\ \mathbf{J}\,dV & \text{(volume current)}\end{cases} \tag{3}$$

The total magnetic field for a current distribution is then obtained by integrating the contributions from all the incremental elements:

$$\mathbf{B}=\begin{cases}\dfrac{\mu_0}{4\pi}\displaystyle\int_L \frac{\mathbf{I}\,dl\times\mathbf{i}_{QP}}{r_{QP}^2} & \text{(line current)}\\[2ex] \dfrac{\mu_0}{4\pi}\displaystyle\int_S \frac{\mathbf{K}\,dS\times\mathbf{i}_{QP}}{r_{QP}^2} & \text{(surface current)}\\[2ex] \dfrac{\mu_0}{4\pi}\displaystyle\int_V \frac{\mathbf{J}\,dV\times\mathbf{i}_{QP}}{r_{QP}^2} & \text{(volume current)}\end{cases} \tag{4}$$

The direction of the magnetic field due to a current element is found by the right-hand rule, where if the forefinger of the right hand points in the direction of current and the middle finger in the direction of the field point, then the thumb points in the direction of the magnetic field. This magnetic field **B** can then exert a force on other currents, as given in Section 5-1-1.

5-2-2 Line Currents

A constant current I_1 flows in the z direction along a wire of infinite extent, as in Figure 5-8a. Equivalently, the right-hand rule allows us to put our thumb in the direction of current. Then the fingers on the right hand curl in the direction of **B**, as shown in Figure 5-8a. The unit vector in the direction of the line joining an incremental current element $I_1\,dz$ at z to a field point P is

$$\mathbf{i}_{QP} = \mathbf{i}_r \cos\theta - \mathbf{i}_z \sin\theta = \mathbf{i}_r \frac{r}{r_{QP}} - \mathbf{i}_z \frac{z}{r_{QP}} \tag{5}$$

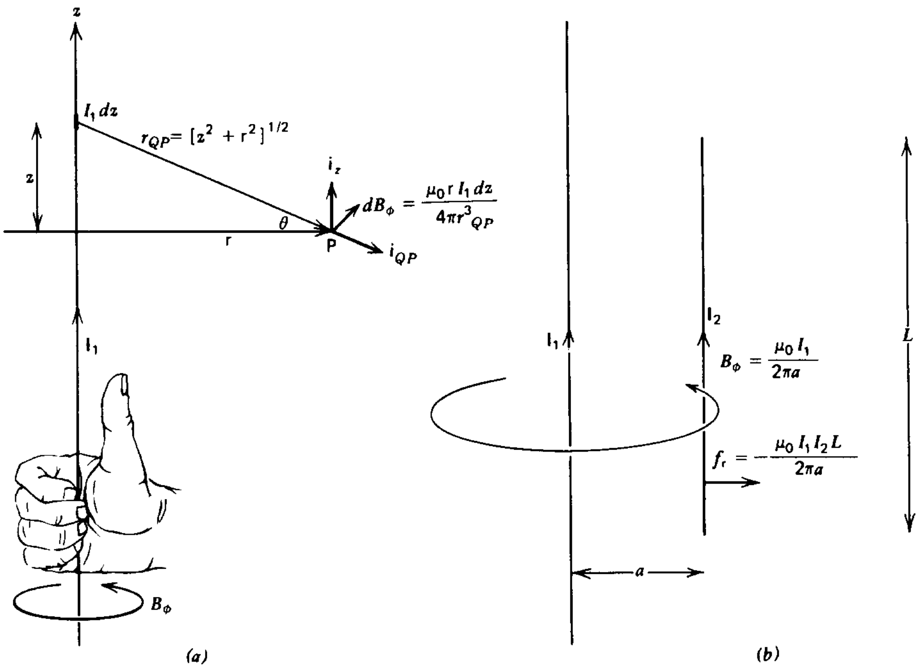


Figure 5-8 (a) The magnetic field due to an infinitely long z-directed line current is in the ϕ direction. (b) Two parallel line currents attract each other if flowing in the same direction and repel if oppositely directed.

with distance

$$r_{QP} = (z^2 + r^2)^{1/2} \tag{6}$$

The magnetic field due to this current element is given by (4) as

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I_1\, dz (\mathbf{i}_z \times \mathbf{i}_{QP})}{r_{QP}^2} = \frac{\mu_0 I_1 r\, dz}{4\pi (z^2 + r^2)^{3/2}} \mathbf{i}_\phi \tag{7}$$

The total magnetic field from the line current is obtained by integrating the contributions from all elements:

$$\begin{aligned} B_\phi &= \frac{\mu_0 I_1 r}{4\pi} \int_{-\infty}^{+\infty} \frac{dz}{(z^2 + r^2)^{3/2}} \\ &= \frac{\mu_0 I_1 r}{4\pi} \frac{z}{r^2 (z^2 + r^2)^{1/2}} \bigg|_{z=-\infty}^{+\infty} \\ &= \frac{\mu_0 I_1}{2\pi r} \end{aligned} \tag{8}$$

If a second line current I_2 of finite length L is placed at a distance a and parallel to I_1, as in Figure 5-8b, the force on I_2 due to the magnetic field of I_1 is

$$\begin{aligned} \mathbf{f} &= \int_{-L/2}^{+L/2} I_2\, dz\, \mathbf{i}_z \times \mathbf{B} \\ &= \int_{-L/2}^{+L/2} I_2\, dz \frac{\mu_0 I_1}{2\pi a} (\mathbf{i}_z \times \mathbf{i}_\phi) \\ &= -\frac{\mu_0 I_1 I_2 L}{2\pi a} \mathbf{i}_r \end{aligned} \tag{9}$$

If both currents flow in the same direction ($I_1 I_2 > 0$), the force is attractive, while if they flow in opposite directions ($I_1 I_2 < 0$), the force is repulsive. This is opposite in sense to the Coulombic force where opposite charges attract and like charges repel.

5-2-3 Current Sheets

(a) Single Sheet of Surface Current

A constant current $K_0 \mathbf{i}_z$ flows in the $y = 0$ plane, as in Figure 5-9a. We break the sheet into incremental line currents $K_0\, dx$, each of which gives rise to a magnetic field as given by (8). From Table 1-2, the unit vector $\mathbf{i}_\phi$ is equivalent to the Cartesian components

$$\mathbf{i}_\phi = -\sin\phi\, \mathbf{i}_x + \cos\phi\, \mathbf{i}_y \tag{10}$$

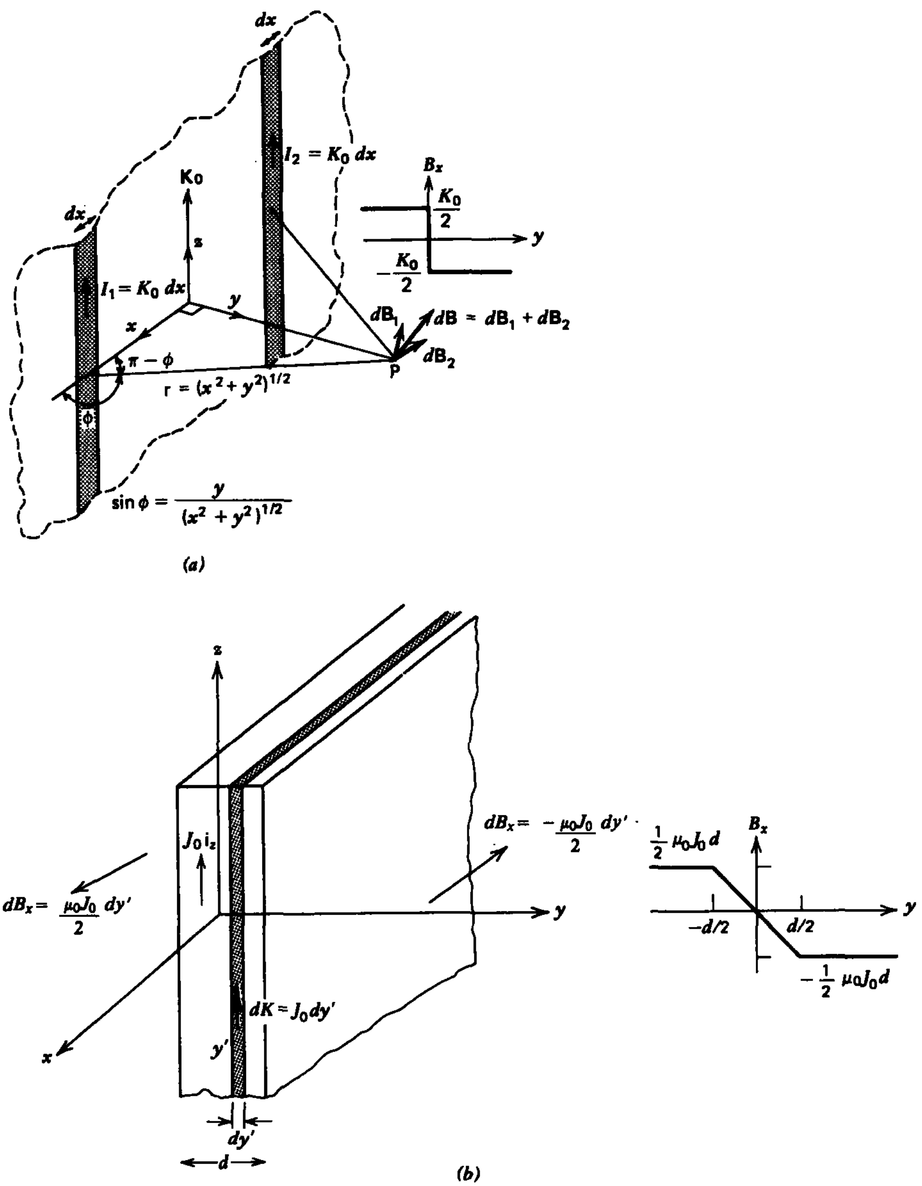


Figure 5-9 (*a*) A uniform surface current of infinite extent generates a uniform magnetic field oppositely directed on each side of the sheet. The magnetic field is perpendicular to the surface current but parallel to the plane of the sheet. (*b*) The magnetic field due to a slab of volume current is found by superimposing the fields due to incremental surface currents. (*c*) Two parallel but oppositely directed surface current sheets have fields that add in the region between the sheets but cancel outside the sheet. (*d*) The force on a current sheet is due to the average field on each side of the sheet as found by modeling the sheet as a uniform volume current distributed over an infinitesimal thickness Δ.

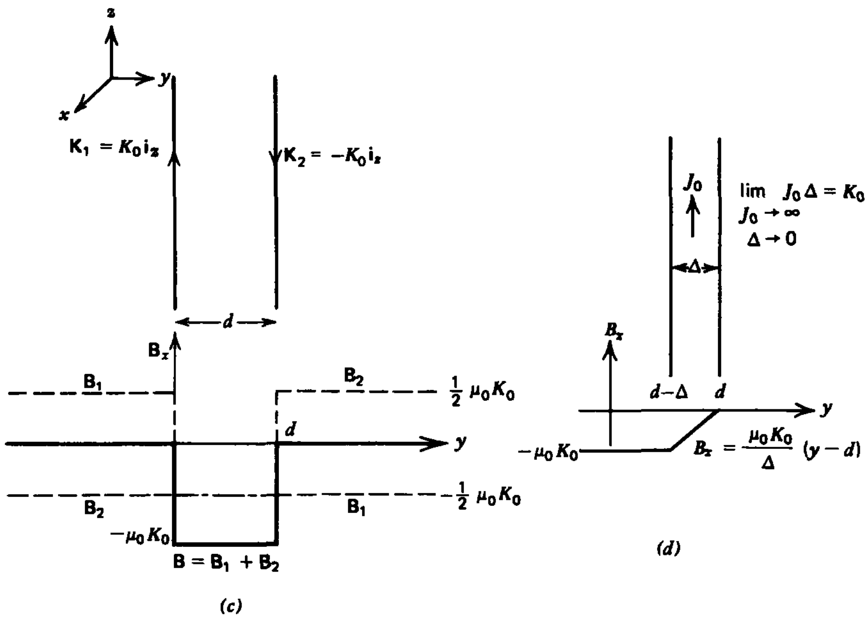


Figure 5-9

The symmetrically located line charge elements a distance x on either side of a point P have y magnetic field components that cancel but x components that add. The total magnetic field is then

$$\begin{aligned} B_x &= -\int_{-\infty}^{+\infty} \frac{\mu_0 K_0 \sin\phi}{2\pi(x^2+y^2)^{1/2}}\,dx \\ &= \frac{-\mu_0 K_0 y}{2\pi}\int_{-\infty}^{+\infty}\frac{dx}{(x^2+y^2)} \\ &= \frac{-\mu_0 K_0}{2\pi}\tan^{-1}\frac{x}{y}\bigg|_{-\infty}^{+\infty} \\ &= \begin{cases} -\mu_0 K_0/2, & y>0 \\ \mu_0 K_0/2, & y<0 \end{cases} \end{aligned} \tag{11}$$

The field is constant and oppositely directed on each side of the sheet.

(b) Slab of Volume Current

If the z-directed current $J_0 \mathbf{i}_z$ is uniform over a thickness d, as in Figure 5-9b, we break the slab into incremental current sheets $J_0\,dy'$. The magnetic field from each current sheet is given by (11). When adding the contributions of all the differential-sized sheets, those to the left of a field point give a negatively x directed magnetic field while those to the right contribute a positively x-directed field:

$$B_x = \begin{cases} \displaystyle\int_{-d/2}^{+d/2} \frac{-\mu_0 J_0\, dy'}{2} = \frac{-\mu_0 J_0 d}{2}, & y > \dfrac{d}{2} \\ \displaystyle\int_{-d/2}^{+d/2} \frac{\mu_0 J_0\, dy'}{2} = \frac{\mu_0 J_0 d}{2}, & y < -\dfrac{d}{2} \\ \displaystyle\int_{-d/2}^{y} \frac{-\mu_0 J_0\, dy'}{2} + \int_{y}^{d/2} \frac{\mu_0 J_0\, dy'}{2} = -\mu_0 J_0 y, & -\dfrac{d}{2} \le y \le \dfrac{d}{2} \end{cases} \tag{12}$$

The total force per unit area on the slab is zero:

$$\begin{aligned} F_{S_y} &= \int_{-d/2}^{+d/2} J_0 B_x\, dy = -\mu_0 J_0^2 \int_{-d/2}^{+d/2} y\, dy \\ &= -\mu_0 J_0^2 \frac{y^2}{2}\bigg|_{-d/2}^{+d/2} = 0 \end{aligned} \tag{13}$$

A current distribution cannot exert a net force on itself.

(c) Two Parallel Current Sheets

If a second current sheet with current flowing in the opposite direction $-K_0\mathbf{i}_z$ is placed at $y = d$ parallel to a current sheet $K_0\mathbf{i}_z$ at $y = 0$, as in Figure 5-9c, the magnetic field due to each sheet alone is

$$\mathbf{B}_1 = \begin{cases} \dfrac{-\mu_0 K_0}{2}\mathbf{i}_x, & y > 0 \\ \dfrac{\mu_0 K_0}{2}\mathbf{i}_x, & y < 0 \end{cases} \qquad \mathbf{B}_2 = \begin{cases} \dfrac{\mu_0 K_0}{2}\mathbf{i}_x, & y > d \\ \dfrac{-\mu_0 K_0}{2}\mathbf{i}_x, & y < d \end{cases} \tag{14}$$

Thus in the region outside the sheets, the fields cancel while they add in the region between:

$$\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 = \begin{cases} -\mu_0 K_0 \mathbf{i}_x, & 0 < y < d \\ 0, & y < 0,\ y > d \end{cases} \tag{15}$$

The force on a surface current element on the second sheet is

$$\mathbf{df} = -K_0 \mathbf{i}_z\, dS \times \mathbf{B} \tag{16}$$

However, since the magnetic field is discontinuous at the current sheet, it is not clear which value of magnetic field to use. To take the limit properly, we model the current sheet at $y = d$ as a thin volume current with density J_0 and thickness Δ, as in Figure 5-9d, where $K_0 = J_0\Delta$.

The results of (12) show that in a slab of uniform volume current, the magnetic field changes linearly to its values at the surfaces

$$B_x(y=d-\Delta)=-\mu_0 K_0$$
$$B_x(y=d)=0 \tag{17}$$

so that the magnetic field within the slab is

$$B_x=\frac{\mu_0 K_0}{\Delta}(y-d) \tag{18}$$

The force per unit area on the slab is then

$$\begin{aligned}\mathbf{F}_S&=-\int_{d-\Delta}^{d}\frac{\mu_0 K_0}{\Delta}J_0(y-d)\mathbf{i}_y\,dy\\&=\frac{-\mu_0 K_0 J_0}{\Delta}\frac{(y-d)^2}{2}\mathbf{i}_y\Big|_{d-\Delta}^{d}\\&=\frac{\mu_0 K_0 J_0\Delta}{2}\mathbf{i}_y=\frac{\mu_0 K_0^2}{2}\mathbf{i}_y\end{aligned} \tag{19}$$

The force acts to separate the sheets because the currents are in opposite directions and thus repel one another.

Just as we found for the electric field on either side of a sheet of surface charge in Section 3-9-1, when the magnetic field is discontinuous on either side of a current sheet $\mathbf{K}$, being $\mathbf{B}_1$ on one side and $\mathbf{B}_2$ on the other, the average magnetic field is used to compute the force on the sheet:

$$\mathbf{df}=\mathbf{K}\,dS\times\frac{(\mathbf{B}_1+\mathbf{B}_2)}{2} \tag{20}$$

In our case

$$\mathbf{B}_1=-\mu_0 K_0\mathbf{i}_x,\qquad \mathbf{B}_2=0 \tag{21}$$

5-2-4 Hoops of Line Current

(a) Single hoop

A circular hoop of radius a centered about the origin in the xy plane carries a constant current I, as in Figure 5-10a. The distance from any point on the hoop to a point at z along the z axis is

$$r_{QP}=(z^2+a^2)^{1/2} \tag{22}$$

in the direction

$$\mathbf{i}_{QP}=\frac{(-a\mathbf{i}_r+z\mathbf{i}_z)}{(z^2+a^2)^{1/2}} \tag{23}$$

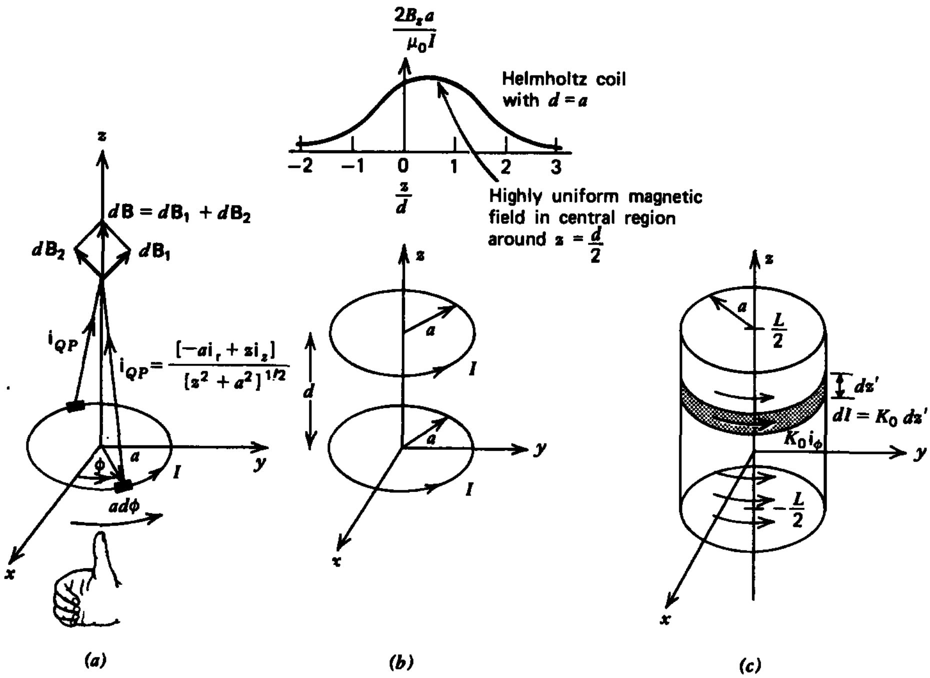


Figure 5-10 (*a*) The magnetic field due to a circular current loop is z directed along the axis of the hoop. (*b*) A Helmholtz coil, formed by two such hoops at a distance apart d equal to their radius, has an essentially uniform field near the center at $z = d/2$. (*c*) The magnetic field on the axis of a cylinder with a ϕ-directed surface current is found by integrating the fields due to incremental current loops.

so that the incremental magnetic field due to a current element of differential size is

$$\begin{aligned} d\mathbf{B} &= \frac{\mu_0}{4\pi r_{QP}^2} Ia\, d\phi\, \mathbf{i}_\phi \times \mathbf{i}_{QP} \\ &= \frac{\mu_0 Ia\, d\phi}{4\pi(z^2+a^2)^{3/2}}(a\mathbf{i}_z + z\mathbf{i}_r) \end{aligned} \tag{24}$$

The radial unit vector changes direction as a function of ϕ, being oppositely directed at $-\phi$, so that the total magnetic field due to the whole hoop is purely z directed:

$$\begin{aligned} B_z &= \frac{\mu_0 Ia^2}{4\pi(z^2+a^2)^{3/2}} \int_0^{2\pi} d\phi \\ &= \frac{\mu_0 Ia^2}{2(z^2+a^2)^{3/2}} \end{aligned} \tag{25}$$

The direction of the magnetic field can be checked using the right-hand rule. Curling the fingers on the right hand in the direction of the current puts the thumb in the direction of the magnetic field. Note that the magnetic field along the z axis is positively z directed both above and below the hoop.

(b) Two Hoops (Helmholtz Coil)

Often it is desired to have an accessible region in space with an essentially uniform magnetic field. This can be arranged by placing another coil at $z = d$, as in Figure 5-10*b*. Then the total magnetic field along the z axis is found by superposing the field of (25) for each hoop:

$$B_z = \frac{\mu_0 I a^2}{2}\left(\frac{1}{(z^2+a^2)^{3/2}} + \frac{1}{((z-d)^2+a^2)^{3/2}}\right) \tag{26}$$

We see then that the slope of B_z,

$$\frac{\partial B_z}{\partial z} = \frac{3\mu_0 I a^2}{2}\left(\frac{-z}{(z^2+a^2)^{5/2}} - \frac{(z-d)}{((z-d)^2+a^2)^{5/2}}\right) \tag{27}$$

is zero at $z = d/2$. The second derivative,

$$\frac{\partial^2 B_z}{\partial z^2} = \frac{3\mu_0 I a^2}{2}\left(\frac{5z^2}{(z^2+a^2)^{7/2}} - \frac{1}{(z^2+a^2)^{5/2}} + \frac{5(z-d)^2}{((z-d)^2+a^2)^{7/2}} - \frac{1}{((z-d)^2+a^2)^{5/2}}\right) \tag{28}$$

can also be set to zero at $z = d/2$, if $d = a$, giving a highly uniform field around the center of the system, as plotted in Figure 5-10*b*. Such a configuration is called a Helmholtz coil.

(c) Hollow Cylinder of Surface Current

A hollow cylinder of length L and radius a has a uniform surface current $K_0\mathbf{i}_\phi$ as in Figure 5-10*c*. Such a configuration is arranged in practice by tightly winding N turns of a wire around a cylinder and imposing a current I through the wire. Then the current per unit length is

$$K_0 = NI/L \tag{29}$$

The magnetic field along the z axis at the position z due to each incremental hoop at z' is found from (25) by replacing z by $(z - z')$ and I by $K_0\,dz'$:

$$dB_z = \frac{\mu_0 a^2 K_0\,dz'}{2[(z-z')^2+a^2]^{3/2}} \tag{30}$$

The total axial magnetic field is then

$$B_z=\int_{z'=-L/2}^{+L/2}\frac{\mu_0 a^2 K_0}{2}\frac{dz'}{[(z-z')^2+a^2]^{3/2}}$$

$$=\frac{\mu_0 a^2 K_0}{2}\frac{(z'-z)}{a^2[(z-z')^2+a^2]^{1/2}}\bigg|_{z'=-L/2}^{+L/2}$$

$$=\frac{\mu_0 K_0}{2}\left(\frac{-z+L/2}{[(z-L/2)^2+a^2]^{1/2}}+\frac{z+L/2}{[(z+L/2)^2+a^2]^{1/2}}\right) \tag{31}$$

As the cylinder becomes very long, the magnetic field far from the ends becomes approximately constant

$$\lim_{L\to\infty} B_z=\mu_0 K_0 \tag{32}$$

5-3 DIVERGENCE AND CURL OF THE MAGNETIC FIELD

Because of our success in examining various vector operations on the electric field, it is worthwhile to perform similar operations on the magnetic field. We will need to use the following vector identities from Section 1-5-4, Problem 1-24 and Sections 2-4-1 and 2-4-2:

$$\nabla\cdot(\nabla\times\mathbf{A})=0 \tag{1}$$

$$\nabla\times(\nabla f)=0 \tag{2}$$

$$\nabla\left(\frac{1}{r_{QP}}\right)=-\frac{\mathbf{i}_{QP}}{r_{QP}^2} \tag{3}$$

$$\int_V \nabla^2\left(\frac{1}{r_{QP}}\right)dV=\begin{cases}0, & r_{QP}\neq 0\\ -4\pi, & r_{QP}=0\end{cases} \tag{4}$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot\nabla\times\mathbf{B} \tag{5}$$

$$\nabla\times(\mathbf{A}\times\mathbf{B})=(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}+(\nabla\cdot\mathbf{B})\mathbf{A}-(\nabla\cdot\mathbf{A})\mathbf{B} \tag{6}$$

$$\nabla(\mathbf{A}\cdot\mathbf{B})=(\mathbf{A}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{A}+\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A}) \tag{7}$$

5-3-1 Gauss's Law for the Magnetic Field

Using (3) the magnetic field due to a volume distribution of current $\mathbf{J}$ is rewritten as

$$\mathbf{B}=\frac{\mu_0}{4\pi}\int_V\frac{\mathbf{J}\times\mathbf{i}_{QP}}{r_{QP}^2}dV$$

$$=\frac{-\mu_0}{4\pi}\int_V \mathbf{J}\times\nabla\left(\frac{1}{r_{QP}}\right)dV \tag{8}$$

If we take the divergence of the magnetic field with respect to field coordinates, the del operator can be brought inside the integral as the integral is only over the source coordinates:

$$\nabla \cdot \mathbf{B} = \frac{-\mu_0}{4\pi} \int_V \nabla \cdot \left[\mathbf{J} \times \nabla\left(\frac{1}{r_{QP}}\right)\right] dV \tag{9}$$

The integrand can be expanded using (5)

$$\nabla \cdot \left[\mathbf{J} \times \nabla\left(\frac{1}{r_{QP}}\right)\right] = \nabla\left(\frac{1}{r_{QP}}\right) \cdot \underbrace{(\nabla \times \mathbf{J})}_{0} - \mathbf{J} \cdot \underbrace{\nabla \times \left[\nabla\left(\frac{1}{r_{QP}}\right)\right]}_{0} = 0 \tag{10}$$

The first term on the right-hand side in (10) is zero because $\mathbf{J}$ is not a function of field coordinates, while the second term is zero from (2), the curl of the gradient is always zero. Then (9) reduces to

$$\nabla \cdot \mathbf{B} = 0 \tag{11}$$

This contrasts with Gauss's law for the displacement field where the right-hand side is equal to the electric charge density. Since nobody has yet discovered any net magnetic charge, there is no source term on the right-hand side of (11).

The divergence theorem gives us the equivalent integral representation

$$\int_V \nabla \cdot \mathbf{B}\, dV = \oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \tag{12}$$

which tells us that the net magnetic flux through a closed surface is always zero. As much flux enters a surface as leaves it. Since there are no magnetic charges to terminate the magnetic field, the field lines are always closed.

5-3-2 Ampere's Circuital Law

We similarly take the curl of (8) to obtain

$$\nabla \times \mathbf{B} = \frac{-\mu_0}{4\pi} \int_V \nabla \times \left[\mathbf{J} \times \nabla\left(\frac{1}{r_{QP}}\right)\right] dV \tag{13}$$

where again the del operator can be brought inside the integral and only operates on r_{QP}.

We expand the integrand using (6):

$$\nabla \times \left[\mathbf{J} \times \nabla\left(\frac{1}{r_{QP}}\right)\right] = \left[\nabla\left(\frac{1}{r_{QP}}\right) \cdot \underbrace{\nabla\right]\mathbf{J}}_{0} - (\mathbf{J} \cdot \nabla)\nabla\left(\frac{1}{r_{QP}}\right)$$
$$+ \left[\nabla^2\left(\frac{1}{r_{QP}}\right)\right]\mathbf{J} - \underbrace{(\nabla \cdot \mathbf{J})}_{0}\nabla\left(\frac{1}{r_{QP}}\right) \tag{14}$$

where two terms on the right-hand side are zero because **J** is not a function of the field coordinates. Using the identity of (7),

$$\nabla\left[\mathbf{J}\cdot\nabla\left(\frac{1}{r_{QP}}\right)\right]=\underbrace{\left[\nabla\left(\frac{1}{r_{QP}}\right)\cdot\nabla\right]\mathbf{J}}_{0}+(\mathbf{J}\cdot\nabla)\nabla\left(\frac{1}{r_{QP}}\right)$$
$$+\nabla\left(\frac{1}{r_{QP}}\right)\times\underbrace{(\nabla\times\mathbf{J})}_{0}+\mathbf{J}\times\underbrace{\left[\nabla\times\nabla\left(\frac{1}{r_{QP}}\right)\right]}_{0}\tag{15}$$

the second term on the right-hand side of (14) can be related to a pure gradient of a quantity because the first and third terms on the right of (15) are zero since **J** is not a function of field coordinates. The last term in (15) is zero because the curl of a gradient is always zero. Using (14) and (15), (13) can be rewritten as

$$\nabla\times\mathbf{B}=\frac{\mu_0}{4\pi}\int_V\left\{\nabla\left[\mathbf{J}\cdot\nabla\left(\frac{1}{r_{QP}}\right)\right]-\mathbf{J}\nabla^2\left(\frac{1}{r_{QP}}\right)\right\}dV\tag{16}$$

Using the gradient theorem, a corollary to the divergence theorem, (see Problem 1-15*a*), the first volume integral is converted to a surface integral

$$\nabla\times\mathbf{B}=\frac{\mu_0}{4\pi}\left[\underbrace{\int_S\mathbf{J}\cdot\nabla\left(\frac{1}{r_{QP}}\right)d\mathbf{S}}_{0}-\int_V\mathbf{J}\nabla^2\left(\frac{1}{r_{QP}}\right)dV\right]\tag{17}$$

This surface completely surrounds the current distribution so that S is outside in a zero current region where $\mathbf{J}=0$ so that the surface integral is zero. The remaining volume integral is nonzero only when $r_{QP}=0$, so that using (4) we finally obtain

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}\tag{18}$$

which is known as Ampere's law.

Stokes' theorem applied to (18) results in Ampere's circuital law:

$$\int_S\nabla\times\frac{\mathbf{B}}{\mu_0}\cdot d\mathbf{S}=\oint_L\frac{\mathbf{B}}{\mu_0}\cdot d\mathbf{l}=\int_S\mathbf{J}\cdot d\mathbf{S}\tag{19}$$

Like Gauss's law, choosing the right contour based on symmetry arguments often allows easy solutions for **B**.

If we take the divergence of both sides of (18), the left-hand side is zero because the divergence of the curl of a vector is always zero. This requires that magnetic field systems have divergence-free currents so that charge cannot accumulate. Currents must always flow in closed loops.

5-3-3 Currents With Cylindrical Symmetry

(a) Surface Current

A surface current $K_0\mathbf{i}_z$ flows on the surface of an infinitely long hollow cylinder of radius a. Consider the two symmetrically located line charge elements $dI = K_0\, a d\phi$ and their effective fields at a point P in Figure 5-11a. The magnetic field due to both current elements cancel in the radial direction but add in the ϕ direction. The total magnetic field can be found by doing a difficult integration over ϕ. However,

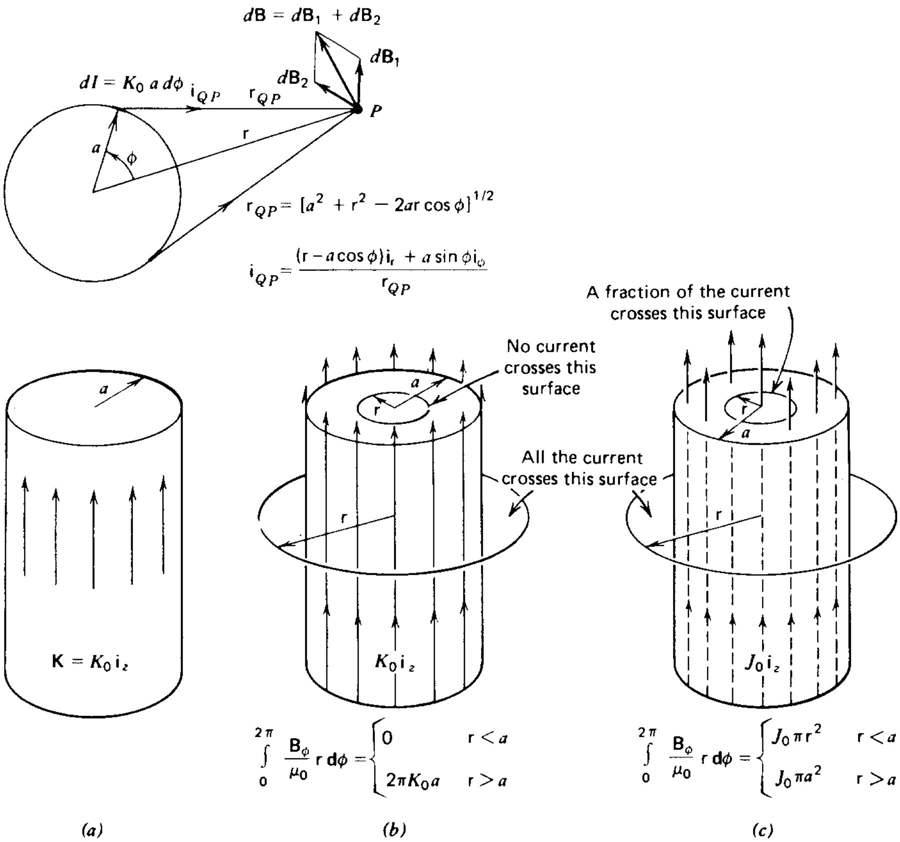


Figure 5-11 (a) The magnetic field of an infinitely long cylinder carrying a surface current parallel to its axis can be found using the Biot–Savart law for each incremental line current element. Symmetrically located elements have radial field components that cancel but ϕ field components that add. (b) Now that we know that the field is purely ϕ directed, it is easier to use Ampere's circuital law for a circular contour concentric with the cylinder. For $r < a$ no current passes through the contour while for $r > a$ all the current passes through the contour. (c) If the current is uniformly distributed over the cylinder the smaller contour now encloses a fraction of the current.

using Ampere's circuital law of (19) is much easier. Since we know the magnetic field is ϕ directed and by symmetry can only depend on r and not ϕ or z, we pick a circular contour of constant radius r as in Figure 5-11*b*. Since $\mathbf{dl} = r\, d\phi\, \mathbf{i}_\phi$ is in the same direction as $\mathbf{B}$, the dot product between the magnetic field and $\mathbf{dl}$ becomes a pure multiplication. For $r < a$ no current passes through the surface enclosed by the contour, while for $r > a$ all the current is purely perpendicular to the normal to the surface of the contour:

$$\oint_L \frac{\mathbf{B}}{\mu_0}\cdot \mathbf{dl} = \int_0^{2\pi} \frac{B_\phi}{\mu_0} r\, d\phi = \frac{2\pi r B_\phi}{\mu_0} = \begin{cases} K_0 2\pi a = I, & r > a \\ 0, & r < a \end{cases} \tag{20}$$

where I is the total current on the cylinder.

The magnetic field is thus

$$B_\phi = \begin{cases} \mu_0 K_0 a/r = \mu_0 I/(2\pi r), & r > a \\ 0, & r < a \end{cases} \tag{21}$$

Outside the cylinder, the magnetic field is the same as if all the current was concentrated along the axis as a line current.

(b) Volume Current

If the cylinder has the current uniformly distributed over the volume as $J_0\mathbf{i}_z$, the contour surrounding the whole cylinder still has the total current $I = J_0\pi a^2$ passing through it. If the contour has a radius smaller than that of the cylinder, only the fraction of current proportional to the enclosed area passes through the surface as shown in Figure 5-11c:

$$\oint_L \frac{B_\phi}{\mu_0} r\, d\phi = \frac{2\pi r B_\phi}{\mu_0} = \begin{cases} J_0\pi a^2 = I, & r > a \\ J_0 \pi r^2 = I r^2/a^2, & r < a \end{cases} \tag{22}$$

so that the magnetic field is

$$B_\phi = \begin{cases} \dfrac{\mu_0 J_0 a^2}{2r} = \dfrac{\mu_0 I}{2\pi r}, & r > a \\[2ex] \dfrac{\mu_0 J_0 r}{2} = \dfrac{\mu_0 I r}{2\pi a^2}, & r < a \end{cases} \tag{23}$$

5-4 THE VECTOR POTENTIAL

5-4-1 Uniqueness

Since the divergence of the magnetic field is zero, we may write the magnetic field as the curl of a vector,

$$\nabla \cdot \mathbf{B} = 0 \Rightarrow \mathbf{B} = \nabla \times \mathbf{A} \tag{1}$$

where **A** is called the vector potential, as the divergence of the curl of any vector is always zero. Often it is easier to calculate **A** and then obtain the magnetic field from (1).

From Ampere's law, the vector potential is related to the current density as

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} \tag{2}$$

We see that (1) does not uniquely define **A**, as we can add the gradient of any term to **A** and not change the value of the magnetic field, since the curl of the gradient of any function is always zero:

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla f \Rightarrow \mathbf{B} = \nabla \times (\mathbf{A} + \nabla f) = \nabla \times \mathbf{A} \tag{3}$$

Helmholtz's theorem states that to uniquely specify a vector, both its curl and divergence must be specified and that far from the sources, the fields must approach zero. To prove this theorem, let's say that we are given the curl and divergence of **A** and we are to determine what **A** is. Is there any other vector **C**, different from **A** that has the same curl and divergence? We try **C** of the form

$$\mathbf{C} = \mathbf{A} + \mathbf{a} \tag{4}$$

and we will prove that **a** is zero.

By definition, the curl of **C** must equal the curl of **A** so that the curl of **a** must be zero:

$$\nabla \times \mathbf{C} = \nabla \times (\mathbf{A} + \mathbf{a}) = \nabla \times \mathbf{A} \Rightarrow \nabla \times \mathbf{a} = 0 \tag{5}$$

This requires that **a** be derivable from the gradient of a scalar function f:

$$\nabla \times \mathbf{a} = 0 \Rightarrow \mathbf{a} = \nabla f \tag{6}$$

Similarly, the divergence condition requires that the divergence of **a** be zero,

$$\nabla \cdot \mathbf{C} = \nabla \cdot (\mathbf{A} + \mathbf{a}) = \nabla \cdot \mathbf{A} \Rightarrow \nabla \cdot \mathbf{a} = 0 \tag{7}$$

so that the Laplacian of f must be zero,

$$\nabla \cdot \mathbf{a} = \nabla^2 f = 0 \tag{8}$$

In Chapter 2 we obtained a similar equation and solution for the electric potential that goes to zero far from the charge distribution:

$$\nabla^2 V = -\frac{\rho}{\varepsilon} \Rightarrow V = \int_V \frac{\rho\, dV}{4\pi\varepsilon r_{QP}} \tag{9}$$

If we equate f to V, then ρ must be zero giving us that the scalar function f is also zero. That is, the solution to Laplace's equation of (8) for zero sources everywhere is zero, even though Laplace's equation in a region does have nonzero solutions if there are sources in other regions of space. With f zero, from (6) we have that the vector $\mathbf{a}$ is also zero and then $\mathbf{C} = \mathbf{A}$, thereby proving Helmholtz's theorem.

5-4-2 The Vector Potential of a Current Distribution

Since we are free to specify the divergence of the vector potential, we take the simplest case and set it to zero:

$$\nabla \cdot \mathbf{A} = 0 \tag{10}$$

Then (2) reduces to

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \tag{11}$$

Each vector component of (11) is just Poisson's equation so that the solution is also analogous to (9)

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}\, dV}{r_{QP}} \tag{12}$$

The vector potential is often easier to use since it is in the same direction as the current, and we can avoid the often complicated cross product in the Biot–Savart law. For moving point charges, as well as for surface and line currents, we use (12) with the appropriate current elements:

$$\mathbf{J}\, dV \rightarrow \mathbf{K}\, dS \rightarrow \mathbf{I}\, dL \rightarrow q\mathbf{v} \tag{13}$$

5-4-3 The Vector Potential and Magnetic Flux

Using Stokes' theorem, the magnetic flux through a surface can be expressed in terms of a line integral of the vector potential:

$$\Phi = \int_S \mathbf{B} \cdot \mathbf{dS} = \int_S \nabla \times \mathbf{A} \cdot \mathbf{dS} = \oint_L \mathbf{A} \cdot \mathbf{dl} \tag{14}$$

(a) Finite Length Line Current

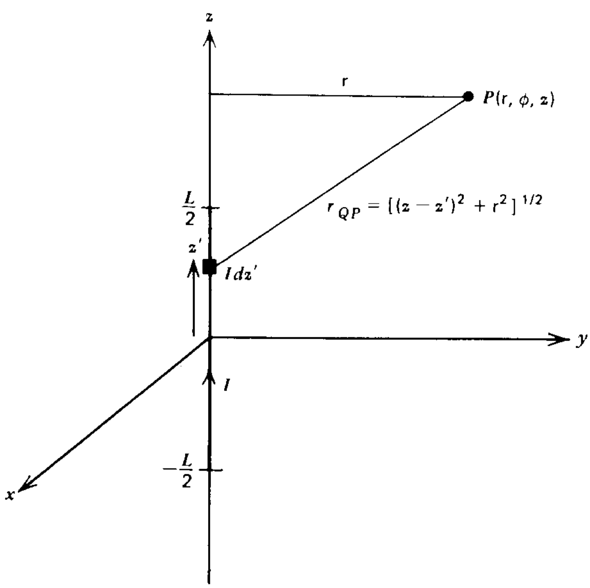
The problem of a line current I of length L, as in Figure 5-12a appears to be nonphysical as the current must be continuous. However, we can imagine this line current to be part of a closed loop and we calculate the vector potential and magnetic field from this part of the loop.

The distance r_{QP} from the current element $I\,dz'$ to the field point at coordinate (r, ϕ, z) is

$$r_{QP} = [(z - z')^2 + r^2]^{1/2} \tag{15}$$

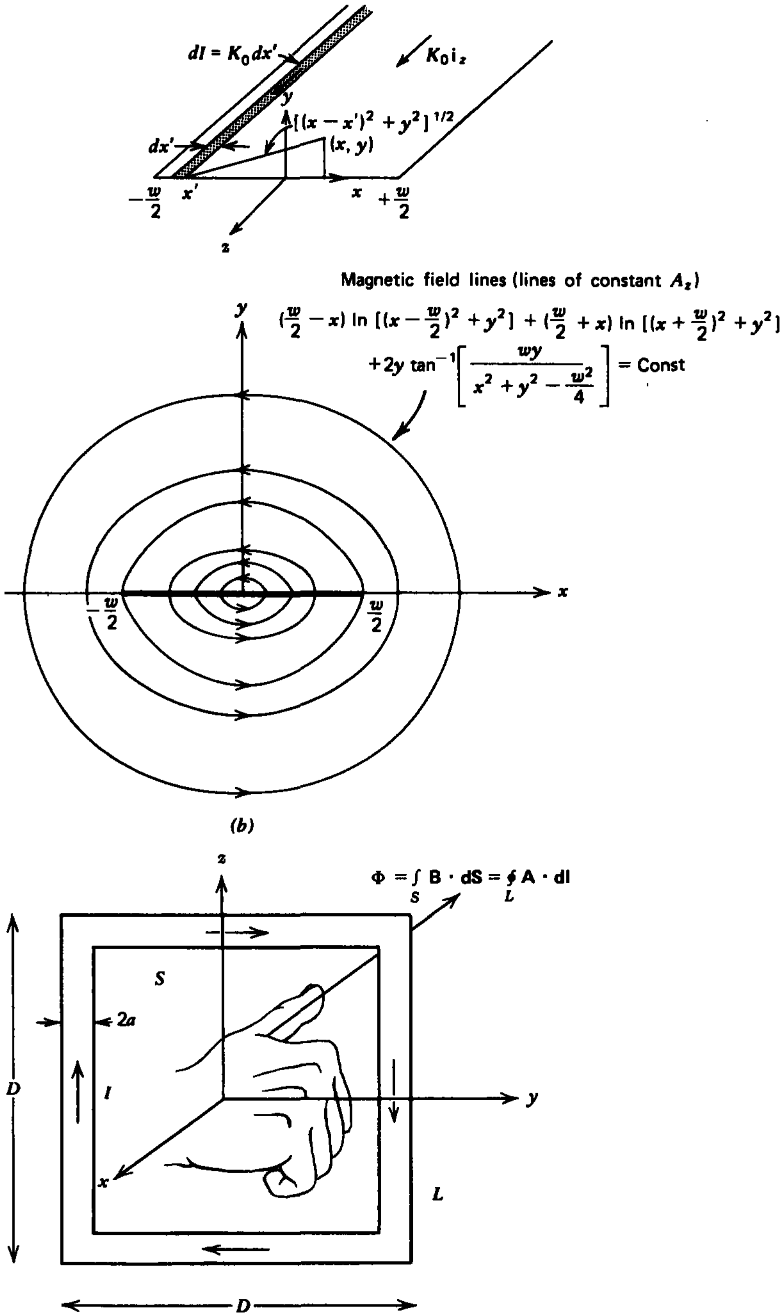
The vector potential is then

$$\begin{aligned} A_z &= \frac{\mu_0 I}{4\pi} \int_{-L/2}^{L/2} \frac{dz'}{[(z-z')^2 + r^2]^{1/2}} \\ &= \frac{\mu_0 I}{4\pi} \ln\left(\frac{-z + L/2 + [(z - L/2)^2 + r^2]^{1/2}}{-(z + L/2) + [(z + L/2)^2 + r^2]^{1/2}}\right) \\ &= \frac{\mu_0 I}{4\pi}\left(\sinh^{-1}\frac{-z + L/2}{r} + \sinh^{-1}\frac{z + L/2}{r}\right) \end{aligned} \tag{16}$$



(a)

Figure 5-12 (a) The magnetic field due to a finite length line current is most easily found using the vector potential, which is in the direction of the current. This problem is physical only if the line current is considered to be part of a closed loop. (b) The magnetic field from a length w of surface current is found by superposing the vector potential of (a) with $L \to \infty$. The field lines are lines of constant A_z. (c) The magnetic flux through a square current loop is in the $-x$ direction by the right-hand rule.


$dI = K_0 dx'$
$K_0 \mathbf{i}_z$
$[(x-x')^2+y^2]^{1/2}$
dx'
(x, y)
$-\frac{w}{2}$ x' x $+\frac{w}{2}$
y
z
Magnetic field lines (lines of constant A_z)
$(\frac{w}{2}-x)\ln[(x-\frac{w}{2})^2+y^2]+(\frac{w}{2}+x)\ln[(x+\frac{w}{2})^2+y^2]$
$+2y\tan^{-1}\left[\frac{wy}{x^2+y^2-\frac{w^2}{4}}\right]=\text{Const}$
$-\frac{w}{2}$
$\frac{w}{2}$
x
(b)
$\Phi=\int_S \mathbf{B}\cdot d\mathbf{S}=\oint_L \mathbf{A}\cdot d\mathbf{l}$
S
2a
D
I
L
z
y
x
D

(c)

Figure 5-12

with associated magnetic field

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$= \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\mathbf{i}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\mathbf{i}_\phi + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right)\mathbf{i}_z$$

$$= -\frac{\partial A_z}{\partial r}\mathbf{i}_\phi$$

$$= \frac{-\mu_0 I r}{4\pi}\left(\frac{1}{[(z-L/2)^2 + r^2]^{1/2}\{-z + L/2 + [(z-L/2)^2 + r^2]^{1/2}\}}\right.$$

$$\left. - \frac{1}{[(z+L/2)^2 + r^2]^{1/2}\{-(z+L/2) + [(z+L/2)^2 + r^2]^{1/2}\}}\right)\mathbf{i}_\phi$$

$$= \frac{\mu_0 I}{4\pi r}\left(\frac{-z + L/2}{[r^2 + (z-L/2)^2]^{1/2}} + \frac{z + L/2}{[r^2 + (z+L/2)^2]^{1/2}}\right)\mathbf{i}_\phi \qquad (17)$$

For large L, (17) approaches the field of an infinitely long line current as given in Section 5-2-2:

$$\lim_{L\to\infty}\begin{cases} A_z = \dfrac{-\mu_0 I}{2\pi}\ln r + \text{const} \\ B_\phi = -\dfrac{\partial A_z}{\partial r} = \dfrac{\mu_0 I}{2\pi r}\end{cases} \qquad (18)$$

Note that the vector potential constant in (18) is infinite, but this is unimportant as this constant has no contribution to the magnetic field.

(b) Finite Width Surface Current

If a surface current $K_0\mathbf{i}_z$, of width w, is formed by laying together many line current elements, as in Figure 5-12*b*, the vector potential at (x, y) from the line current element $K_0\,dx'$ at position x' is given by (18):

$$dA_z = \frac{-\mu_0 K_0\,dx'}{4\pi}\ln[(x-x')^2 + y^2] \qquad (19)$$

The total vector potential is found by integrating over all elements:

$$A_z = \frac{-\mu_0 K_0}{4\pi}\int_{-w/2}^{+w/2} \ln[(x-x')^2+y^2]\,dx'$$

$$= \frac{-\mu_0 K_0}{4\pi}\left((x'-x)\ln[(x-x')^2+y^2]-2(x'-x) + 2y\tan^{-1}\frac{(x'-x)}{y}\right)\Bigg|_{-w/2}^{+w/2}$$

$$= \frac{-\mu_0 K_0}{4\pi}\left\{\left(\frac{w}{2}-x\right)\ln\left[\left(x-\frac{w}{2}\right)^2+y^2\right] + \left(\frac{w}{2}+x\right)\ln\left[\left(x+\frac{w}{2}\right)^2+y^2\right] - 2w + 2y\tan^{-1}\frac{wy}{y^2+x^2-w^2/4}\right\} \qquad (20)^*$$

The magnetic field is then

$$\mathbf{B} = \mathbf{i}_x\frac{\partial A_z}{\partial y} - \mathbf{i}_y\frac{\partial A_z}{\partial x}$$
$$= \frac{-\mu_0 K_0}{4\pi}\left(2\tan^{-1}\frac{wy}{y^2+x^2-w^2/4}\mathbf{i}_x + \ln\frac{(x+w/2)^2+y^2}{(x-w/2)^2+y^2}\mathbf{i}_y\right) \qquad (21)$$

The vector potential in two-dimensional geometries is also useful in plotting field lines,

$$\frac{dy}{dx} = \frac{B_y}{B_x} = \frac{-\partial A_z/\partial x}{\partial A_z/\partial y} \qquad (22)$$

for if we cross multiply (22),

$$\frac{\partial A_z}{\partial x}dx + \frac{\partial A_z}{\partial y}dy = dA_z = 0 \Rightarrow A_z = \text{const} \qquad (23)$$

we see that it is constant on a field line. The field lines in Figure 5-12*b* are just lines of constant A_z. The vector potential thus plays the same role as the electric stream function in Sections 4.3.2*b* and 4.4.3*b*.

(c) Flux Through a Square Loop

The vector potential for the square loop in Figure 5-12*c* with very small radius *a* is found by superposing (16) for each side with each component of **A** in the same direction as the current in each leg. The resulting magnetic field is then given by four

$^*\tan^{-1}(a-b)+\tan^{-1}(a+b)=\tan^{-1}\dfrac{2a}{1-a^2+b^2}$

terms like that in (17) so that the flux can be directly computed by integrating the normal component of **B** over the loop area. This method is straightforward but the algebra is cumbersome.

An easier method is to use (14) since we already know the vector potential along each leg. We pick a contour that runs along the inside wire boundary at small radius a. Since each leg is identical, we only have to integrate over one leg, then multiply the result by 4:

$$\begin{aligned}\Phi&=4\int_{\substack{a-D/2\\ r=a}}^{-a+D/2} A_z\,dz\\
&=\frac{\mu_0 I}{\pi}\int_{a-D/2}^{-a+D/2}\left(\sinh^{-1}\frac{-z+D/2}{a}+\sinh^{-1}\frac{z+D/2}{a}\right)dz\\
&=\frac{\mu_0 I}{\pi}\left\{-\left(\frac{D}{2}-z\right)\sinh^{-1}\frac{-z+D/2}{a}+\left[\left(\frac{D}{2}-z\right)^2+a^2\right]^{1/2}\right.\\
&\quad\left.+\left(\frac{D}{2}+z\right)\sinh^{-1}\frac{z+D/2}{a}-\left[\left(\frac{D}{2}+z\right)^2+a^2\right]^{1/2}\right\}\Bigg|_{a-D/2}^{-a+D/2}\\
&=2\frac{\mu_0 I}{\pi}\left(-a\sinh^{-1}1+a\sqrt{2}+(D-a)\sinh^{-1}\frac{D-a}{a}\right.\\
&\quad\left.-[(D-a)^2+a^2]^{1/2}\right)\end{aligned}\tag{24}$$

As a becomes very small, (24) reduces to

$$\lim_{a\to 0}\Phi=2\frac{\mu_0 I}{\pi}D\left(\sinh^{-1}\left(\frac{D}{a}\right)-1\right)\tag{25}$$

We see that the flux through the loop is proportional to the current. This proportionality constant is called the self-inductance and is only a function of the geometry:

$$L=\frac{\Phi}{I}=2\frac{\mu_0 D}{\pi}\left(\sinh^{-1}\left(\frac{D}{a}\right)-1\right)\tag{26}$$

Inductance is more fully developed in Chapter 6.

5-5 MAGNETIZATION

Our development thus far has been restricted to magnetic fields in free space arising from imposed current distributions. Just as small charge displacements in dielectric materials contributed to the electric field, atomic motions constitute microscopic currents, which also contribute to the magnetic field. There is a direct analogy between polarization and magnetization, so our development will parallel that of Section 3-1.

5-5-1 The Magnetic Dipole

Classical atomic models describe an atom as orbiting electrons about a positively charged nucleus, as in Figure 5-13.

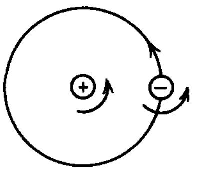


Figure 5-13 Atomic currents arise from orbiting electrons in addition to the spin contributions from the electron and nucleus.

The nucleus and electron can also be imagined to be spinning. The simplest model for these atomic currents is analogous to the electric dipole and consists of a small current loop of area dS carrying a current I, as in Figure 5-14. Because atomic dimensions are so small, we are only interested in the magnetic field far from this magnetic dipole. Then the shape of the loop is not important, thus for simplicity we take it to be rectangular.

The vector potential for this loop is then

$$\mathbf{A}=\frac{\mu_0 I}{4\pi}\left[dx\left(\frac{1}{r_3}-\frac{1}{r_1}\right)\mathbf{i}_x+dy\left(\frac{1}{r_4}-\frac{1}{r_2}\right)\mathbf{i}_y\right] \qquad (1)$$

where we assume that the distance from any point on each side of the loop to the field point P is approximately constant.

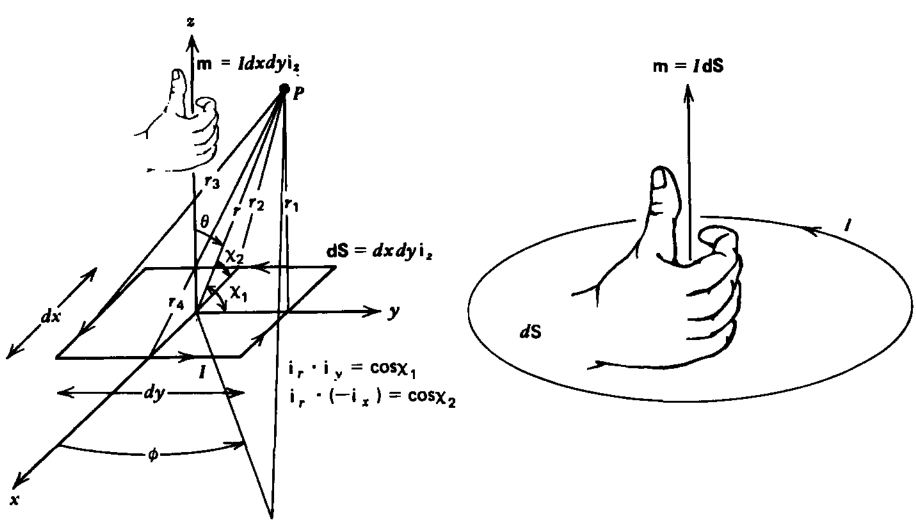


Figure 5-14 A magnetic dipole consists of a small circulating current loop. The magnetic moment is in the direction normal to the loop by the right-hand rule.

Using the law of cosines, these distances are related as

$$r_1^2 = r^2 + \left(\frac{dy}{2}\right)^2 - r\,dy\cos\chi_1; \qquad r_2^2 = r^2 + \left(\frac{dx}{2}\right)^2 - r\,dx\cos\chi_2$$
$$r_3^2 = r^2 + \left(\frac{dy}{2}\right)^2 + r\,dy\cos\chi_1, \qquad r_4^2 = r^2 + \left(\frac{dx}{2}\right)^2 + r\,dx\cos\chi_2 \tag{2}$$

where the angles χ_1 and χ_2 are related to the spherical coordinates from Table 1-2 as

$$\mathbf{i}_r \cdot \mathbf{i}_y = \cos\chi_1 = \sin\theta\sin\phi$$
$$-\mathbf{i}_r \cdot \mathbf{i}_x = \cos\chi_2 = -\sin\theta\cos\phi \tag{3}$$

In the far field limit (1) becomes

$$\lim_{\substack{r\gg dx\\ r\gg dy}} \mathbf{A} = \frac{\mu_0 I}{4\pi}\left[\frac{dx}{r}\left(\frac{1}{\left[1+\frac{dy}{2r}\left(\frac{dy}{2r}+2\cos\chi_1\right)\right]^{1/2}} - \frac{1}{\left[1+\frac{dy}{2r}\left(\frac{dy}{2r}-2\cos\chi_1\right)\right]^{1/2}}\right)\right.$$
$$\left. + \frac{dy}{r}\left(\frac{1}{\left[1+\frac{dx}{2r}\left(\frac{dx}{2r}+2\cos\chi_2\right)\right]^{1/2}} - \frac{1}{\left[1+\frac{dx}{2r}\left(\frac{dx}{2r}-2\cos\chi_2\right)\right]^{1/2}}\right)\right]$$
$$\approx \frac{-\mu_0 I}{4\pi r^2}\,dx\,dy[\cos\chi_1\mathbf{i}_x + \cos\chi_2\mathbf{i}_y] \tag{4}$$

Using (3), (4) further reduces to

$$\mathbf{A} = \frac{\mu_0 I\,dS}{4\pi r^2}\sin\theta[-\sin\phi\,\mathbf{i}_x + \cos\phi\,\mathbf{i}_y]$$
$$= \frac{\mu_0 I\,dS}{4\pi r^2}\sin\theta\,\mathbf{i}_\phi \tag{5}$$

where we again used Table 1-2 to write the bracketed Cartesian unit vector term as $\mathbf{i}_\phi$. The magnetic dipole moment $\mathbf{m}$ is defined as the vector in the direction perpendicular to the loop (in this case $\mathbf{i}_z$) by the right-hand rule with magnitude equal to the product of the current and loop area:

$$\mathbf{m} = I\,dS\,\mathbf{i}_z = I\,\mathbf{dS} \tag{6}$$

Then the vector potential can be more generally written as

$$\mathbf{A}=\frac{\mu_0 m}{4\pi r^2}\sin\theta\,\mathbf{i}_\phi=\frac{\mu_0\mathbf{m}}{4\pi r^2}\times\mathbf{i}_r \tag{7}$$

with associated magnetic field

$$\begin{aligned}\mathbf{B}=\nabla\times\mathbf{A}&=\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\phi\sin\theta)\mathbf{i}_r-\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\mathbf{i}_\theta\\&=\frac{\mu_0 m}{4\pi r^3}[2\cos\theta\,\mathbf{i}_r+\sin\theta\,\mathbf{i}_\theta]\end{aligned} \tag{8}$$

This field is identical in form to the electric dipole field of Section 3-1-1 if we replace p/ε_0 by $\mu_0 m$.

5-5-2 Magnetization Currents

Ampere modeled magnetic materials as having the volume filled with such infinitesimal circulating current loops with number density N, as illustrated in Figure 5-15. The magnetization vector $\mathbf{M}$ is then defined as the magnetic dipole density:

$$\mathbf{M}=N\mathbf{m}=NI\,\mathbf{dS}\text{ amp/m} \tag{9}$$

For the differential sized contour in the xy plane shown in Figure 5-15, only those dipoles with moments in the x or y directions (thus z components of currents) will give rise to currents crossing perpendicularly through the surface bounded by the contour. Those dipoles completely within the contour give no net current as the current passes through the contour twice, once in the positive z direction and on its return in the negative z direction. Only those dipoles on either side of the edges—so that the current only passes through the contour once, with the return outside the contour—give a net current through the loop.

Because the length of the contour sides Δx and Δy are of differential size, we assume that the dipoles along each edge do not change magnitude or direction. Then the net total current linked by the contour near each side is equal to the product of the current per dipole I and the number of dipoles that just pass through the contour once. If the normal vector to the dipole loop (in the direction of $\mathbf{m}$) makes an angle θ with respect to the direction of the contour side at position x, the net current linked along the line at x is

$$-IN\,dS\,\Delta y\cos\theta|_x=-M_y(x)\,\Delta y \tag{10}$$

The minus sign arises because the current within the contour adjacent to the line at coordinate x flows in the $-z$ direction.

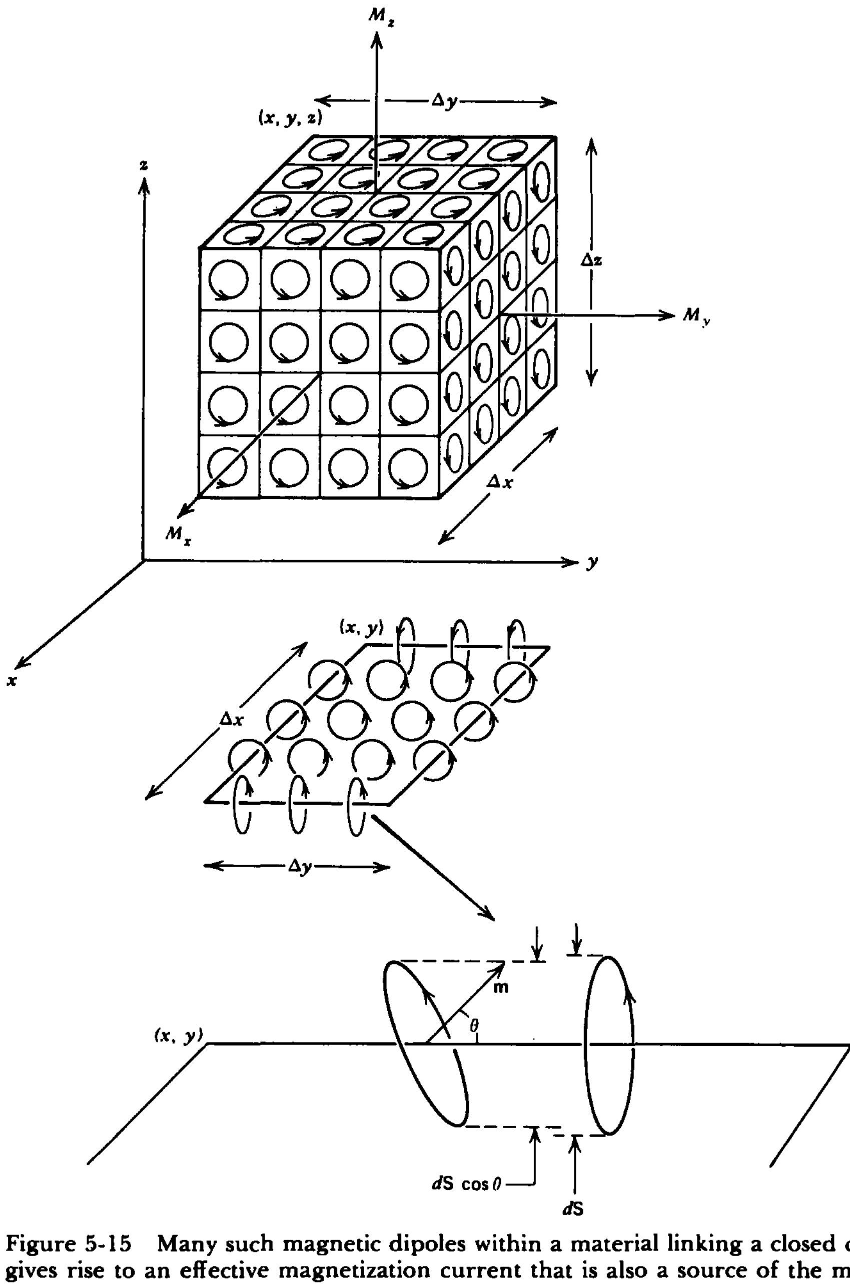


Figure 5-15 Many such magnetic dipoles within a material linking a closed contour gives rise to an effective magnetization current that is also a source of the magnetic field.

Similarly, near the edge at coordinate $x+\Delta x$, the net current linked perpendicular to the contour is

$$IN\,dS\,\Delta y\cos\theta|_{x+\Delta x}=M_y(x+\Delta x)\,\Delta y \qquad (11)$$

Along the edges at y and $y+\Delta y$, the current contributions are

$$IN\,dS\,\Delta x\cos\theta|_y = M_x(y)\,\Delta x$$
$$-IN\,dS\,\Delta x\cos\theta|_{y+\Delta y} = -M_x(y+\Delta y)\,\Delta x \qquad (12)$$

The total current in the z direction linked by this contour is thus the sum of contributions in (10)–(12):

$$I_{z\,\text{tot}} = \Delta x\,\Delta y\left(\frac{M_y(x+\Delta x)-M_y(x)}{\Delta x}-\frac{M_x(y+\Delta y)-M_x(y)}{\Delta y}\right) \qquad (13)$$

If the magnetization is uniform, the net total current is zero as the current passing through the loop at one side is canceled by the current flowing in the opposite direction at the other side. Only if the magnetization changes with position can there be a net current through the loop's surface. This can be accomplished if either the current per dipole, area per dipole, density of dipoles, or angle of orientation of the dipoles is a function of position.

In the limit as Δx and Δy become small, terms on the right-hand side in (13) define partial derivatives so that the current per unit area in the z direction is

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} J_z = \frac{I_{z\,\text{tot}}}{\Delta x\,\Delta y} = \left(\frac{\partial M_y}{\partial x}-\frac{\partial M_x}{\partial y}\right) = (\nabla\times\mathbf{M})_z \qquad (14)$$

which we recognize as the z component of the curl of the magnetization. If we had orientated our loop in the xz or yz planes, the current density components would similarly obey the relations

$$J_y = \left(\frac{\partial M_x}{\partial z}-\frac{\partial M_z}{\partial x}\right) = (\nabla\times\mathbf{M})_y$$
$$J_x = \left(\frac{\partial M_z}{\partial y}-\frac{\partial M_y}{\partial z}\right) = (\nabla\times\mathbf{M})_x \qquad (15)$$

so that in general

$$\mathbf{J}_m = \nabla\times\mathbf{M} \qquad (16)$$

where we subscript the current density with an m to represent the magnetization current density, often called the Amperian current density.

These currents are also sources of the magnetic field and can be used in Ampere's law as

$$\nabla\times\frac{\mathbf{B}}{\mu_0} = \mathbf{J}_m + \mathbf{J}_f = \mathbf{J}_f + \nabla\times\mathbf{M} \qquad (17)$$

where $\mathbf{J}_f$ is the free current due to the motion of free charges as contrasted to the magnetization current $\mathbf{J}_m$, which is due to the motion of bound charges in materials.

As we can only impose free currents, it is convenient to define the vector **H** as the magnetic field intensity to be distinguished from **B**, which we will now call the magnetic flux density:

$$\mathbf{H}=\frac{\mathbf{B}}{\mu_0}-\mathbf{M}\Rightarrow\mathbf{B}=\mu_0(\mathbf{H}+\mathbf{M}) \tag{18}$$

Then (17) can be recast as

$$\nabla\times\left(\frac{\mathbf{B}}{\mu_0}-\mathbf{M}\right)=\nabla\times\mathbf{H}=\mathbf{J}_f \tag{19}$$

The divergence and flux relations of Section 5-3-1 are unchanged and are in terms of the magnetic flux density **B**. In free space, where $\mathbf{M}=0$, the relation of (19) between **B** and **H** reduces to

$$\mathbf{B}=\mu_0\mathbf{H} \tag{20}$$

This is analogous to the development of the polarization with the relationships of **D**, **E**, and **P**. Note that in (18), the constant parameter μ_0 multiplies both **H** and **M**, unlike the permittivity ε_0 which only multiplies **E**.

Equation (19) can be put into an equivalent integral form using Stokes' theorem:

$$\int_S(\nabla\times\mathbf{H})\cdot d\mathbf{S}=\oint_L\mathbf{H}\cdot d\mathbf{l}=\int_S\mathbf{J}_f\cdot d\mathbf{S} \tag{21}$$

The free current density $\mathbf{J}_f$ is the source of the **H** field, the magnetization current density $\mathbf{J}_m$ is the source of the **M** field, while the total current, $\mathbf{J}_f+\mathbf{J}_m$, is the source of the **B** field.

5-5-3 Magnetic Materials

There are direct analogies between the polarization processes found in dielectrics and magnetic effects. The constitutive law relating the magnetization **M** to an applied magnetic field **H** is found by applying the Lorentz force to our atomic models.

(a) Diamagnetism

The orbiting electrons as atomic current loops is analogous to electronic polarization, with the current in the direction opposite to their velocity. If the electron ($e=1.6\times10^{-19}$ coul) rotates at angular speed ω at radius R, as in Figure 5-16, the current and dipole moment are

$$I=\frac{e\omega}{2\pi},\qquad m=I\pi R^2=\frac{e\omega}{2}R^2 \tag{22}$$

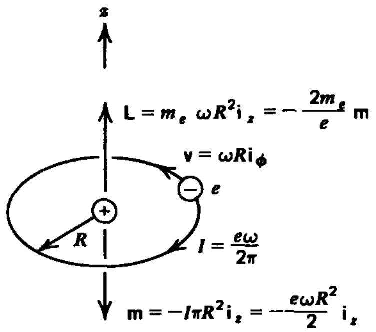


Figure 5-16 The orbiting electron has its magnetic moment **m** in the direction opposite to its angular momentum **L** because the current is opposite to the electron's velocity.

Note that the angular momentum **L** and magnetic moment **m** are oppositely directed and are related as

$$\mathbf{L} = m_e R \mathbf{i}_r \times \mathbf{v} = m_e \omega R^2 \mathbf{i}_z = -\frac{2m_e}{e}\mathbf{m} \tag{23}$$

where $m_e = 9.1 \times 10^{-31}$ kg is the electron mass.

Since quantum theory requires the angular momentum to be quantized in units of $h/2\pi$, where Planck's constant is $h = 6.62 \times 10^{-34}$ joule-sec, the smallest unit of magnetic moment, known as the Bohr magneton, is

$$m_B = \frac{eh}{4\pi m_e} \approx 9.3 \times 10^{-24} \text{ amp-m}^2 \tag{24}$$

Within a homogeneous material these dipoles are randomly distributed so that for every electron orbiting in one direction, another electron nearby is orbiting in the opposite direction so that in the absence of an applied magnetic field there is no net magnetization.

The Coulombic attractive force on the orbiting electron towards the nucleus with atomic number Z is balanced by the centrifugal force:

$$m_e \omega^2 R = \frac{Ze^2}{4\pi\varepsilon_0 R^2} \tag{25}$$

Since the left-hand side is just proportional to the square of the quantized angular momentum, the orbit radius R is also quantized for which the smallest value is

$$R = \frac{4\pi\varepsilon_0}{m_e Z e^2}\left(\frac{h}{2\pi}\right)^2 \approx \frac{5 \times 10^{-11}}{Z} \text{ m} \tag{26}$$

with resulting angular speed

$$\omega = \frac{Z^2 e^4 m_e}{(4\pi\varepsilon_0)^2 (h/2\pi)^3} \approx 1.3 \times 10^{16} Z^2 \tag{27}$$

When a magnetic field $H_0\mathbf{i}_z$ is applied, as in Figure 5-17, electron loops with magnetic moment opposite to the field feel an additional radial force inwards, while loops with colinear moment and field feel a radial force outwards. Since the orbital radius R cannot change because it is quantized, this magnetic force results in a change of orbital speed $\Delta\omega$:

$$m_e(\omega + \Delta\omega_1)^2 R = e\left(\frac{Ze}{4\pi\varepsilon_0 R^2} + (\omega + \Delta\omega_1) R \mu_0 H_0\right)$$

$$m_e(\omega + \Delta\omega_2)^2 R = e\left(\frac{Ze}{4\pi\varepsilon_0 R^2} - (\omega + \Delta\omega_2) R \mu_0 H_0\right) \tag{28}$$

where the first electron speeds up while the second one slows down.

Because the change in speed $\Delta\omega$ is much less than the natural speed ω, we solve (28) approximately as

$$\Delta\omega_1 = \frac{e\omega\mu_0 H_0}{2m_e\omega - e\mu_0 H_0}$$

$$\Delta\omega_2 = \frac{-e\omega\mu_0 H_0}{2m_e\omega + e\mu_0 H_0} \tag{29}$$

where we neglect quantities of order $(\Delta\omega)^2$. However, even with very high magnetic field strengths of $H_0 = 10^6$ amp/m we see that usually

$$e\mu_0 H_0 \ll 2m_e\omega_0$$
$$(1.6\times 10^{-19})(4\pi \times 10^{-7})10^6 \ll 2(9.1\times 10^{-31})(1.3\times 10^{16}) \tag{30}$$

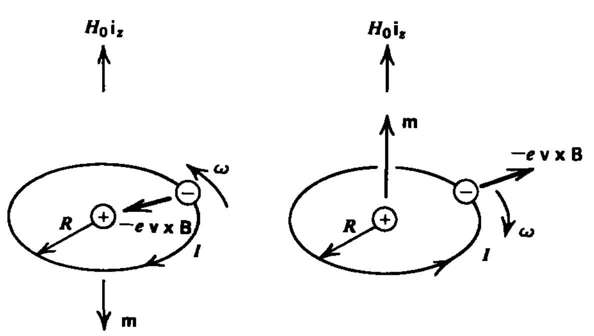


Figure 5-17 Diamagnetic effects, although usually small, arise in all materials because dipoles with moments parallel to the magnetic field have an increase in the orbiting electron speed while those dipoles with moments opposite to the field have a decrease in speed. The loop radius remains constant because it is quantized.

so that (29) further reduces to

$$\Delta\omega_1 \approx -\Delta\omega_2 \approx \frac{e\mu_0 H_0}{2m_e} \approx 1.1\times 10^5 H_0 \tag{31}$$

The net magnetic moment for this pair of loops,

$$m = \frac{eR^2}{2}(\omega_2-\omega_1) = -eR^2\,\Delta\omega_1 = \frac{-e^2\mu_0 R^2}{2m_e}H_0 \tag{32}$$

is opposite in direction to the applied magnetic field.

If we have N such loop pairs per unit volume, the magnetization field is

$$\mathbf{M} = N\mathbf{m} = -\frac{Ne^2\mu_0 R^2}{2m_e}H_0\mathbf{i}_z \tag{33}$$

which is also oppositely directed to the applied magnetic field.

Since the magnetization is linearly related to the field, we define the magnetic susceptibility χ_m as

$$\mathbf{M} = \chi_m\mathbf{H}, \qquad \chi_m = -\frac{Ne^2\mu_0 R^2}{2m_e} \tag{34}$$

where χ_m is negative. The magnetic flux density is then

$$\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M}) = \mu_0(1+\chi_m)\mathbf{H} = \mu_0\mu_r\mathbf{H} = \mu\mathbf{H} \tag{35}$$

where $\mu_r = 1+\chi_m$ is called the relative permeability and μ is the permeability. In free space $\chi_m = 0$ so that $\mu_r = 1$ and $\mu = \mu_0$. The last relation in (35) is usually convenient to use, as all the results in free space are still correct within linear permeable material if we replace μ_0 by μ. In diamagnetic materials, where the susceptibility is negative, we have that $\mu_r < 1$, $\mu < \mu_0$. However, substituting in our typical values

$$\chi_m = -\frac{Ne^2\mu_0 R^2}{2m\cdot} \approx \frac{4.4\times 10^{-35}}{Z^2}N \tag{36}$$

we see that even with $N \approx 10^{30}$ atoms/m^3, χ_m is much less than unity so that diamagnetic effects are very small.

(b) Paramagnetism

As for orientation polarization, an applied magnetic field exerts a torque on each dipole tending to align its moment with the field, as illustrated for the rectangular magnetic dipole with moment at an angle θ to a uniform magnetic field **B** in Figure 5-18*a*. The force on each leg is

$$\begin{aligned} \mathbf{df}_1 &= -\mathbf{df}_2 = I\,\Delta x\,\mathbf{i}_x\times\mathbf{B} = I\,\Delta x[B_y\mathbf{i}_z - B_z\mathbf{i}_y] \\ \mathbf{df}_3 &= -\mathbf{df}_4 = I\,\Delta y\,\mathbf{i}_y\times\mathbf{B} = I\,\Delta y(-B_x\mathbf{i}_z + B_z\mathbf{i}_x) \end{aligned} \tag{37}$$

In a uniform magnetic field, the forces on opposite legs are equal in magnitude but opposite in direction so that the net

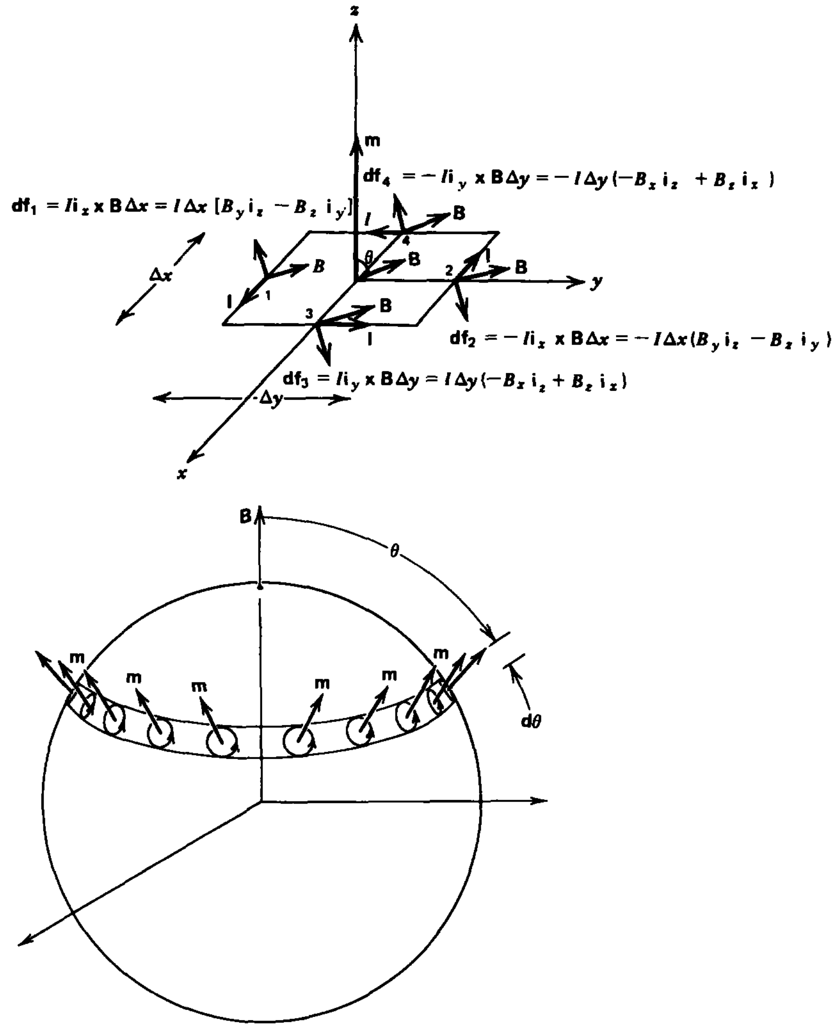


Figure 5-18 (*a*) A torque is exerted on a magnetic dipole with moment at an angle θ to an applied magnetic field. (*b*) From Boltzmann statistics, thermal agitation opposes the alignment of magnetic dipoles. All the dipoles at an angle θ, together have a net magnetization in the direction of the applied field.

force on the loop is zero. However, there is a torque:

$$\begin{aligned}\mathbf{T} &= \sum_{n=1}^{4} \mathbf{r} \times \mathbf{df}_n \\ &= \frac{\Delta y}{2}(-\mathbf{i}_y \times \mathbf{df}_1 + \mathbf{i}_y \times \mathbf{df}_2) + \frac{\Delta x}{2}(\mathbf{i}_x \times \mathbf{df}_3 - \mathbf{i}_x \times \mathbf{df}_4) \\ &= I\,\Delta x\,\Delta y(B_x \mathbf{i}_y - B_y \mathbf{i}_x) = \mathbf{m} \times \mathbf{B} \end{aligned} \qquad (38)$$

The incremental amount of work necessary to turn the dipole by a small angle $d\theta$ is

$$dW = T\,d\theta = m\mu_0 H_0 \sin\theta\, d\theta \tag{39}$$

so that the total amount of work necessary to turn the dipole from $\theta = 0$ to any value of θ is

$$W = \int_0^\theta T\,d\theta = -m\mu_0 H_0 \cos\theta\Big|_0^\theta = m\mu_0 H_0(1-\cos\theta) \tag{40}$$

This work is stored as potential energy, for if the dipole is released it will try to orient itself with its moment parallel to the field. Thermal agitation opposes this alignment where Boltzmann statistics describes the number density of dipoles having energy W as

$$n = n_1 e^{-W/kT} = n_1 e^{-m\mu_0 H_0(1-\cos\theta)/kT} = n_0 e^{m\mu_0 H_0 \cos\theta/kT} \tag{41}$$

where we lump the constant energy contribution in (40) within the amplitude n_0, which is found by specifying the average number density of dipoles N within a sphere of radius R:

$$\begin{aligned} N &= \frac{1}{\frac{4}{3}\pi R^3}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\int_{r=0}^{R} n_0\, e^{a\cos\theta}\, r^2 \sin\theta\, dr\, d\theta\, d\phi \\ &= \frac{n_0}{2}\int_{\theta=0}^{\pi} \sin\theta\, e^{a\cos\theta}\, d\theta \end{aligned} \tag{42}$$

where we let

$$a = m\mu_0 H_0/kT \tag{43}$$

With the change of variable

$$u = a\cos\theta, \qquad du = -a\sin\theta\, d\theta \tag{44}$$

the integration in (42) becomes

$$N = \frac{-n_0}{2a}\int_a^{-a} e^u\, du = \frac{n_0}{a}\sinh a \tag{45}$$

so that (41) becomes

$$n = \frac{Na}{\sinh a} e^{a\cos\theta} \tag{46}$$

From Figure 5-18*b* we see that all the dipoles in the shell over the interval θ to $\theta + d\theta$ contribute to a net magnetization. which is in the direction of the applied magnetic field:

$$dM = \frac{mn}{\frac{4}{3}\pi R^3}\cos\theta\, r^2 \sin\theta\, dr\, d\theta\, d\phi \tag{47}$$

so that the total magnetization due to all the dipoles within the sphere is

$$M = \frac{maN}{2\sinh a}\int_{\theta=0}^{\pi} \sin\theta\cos\theta\, e^{a\cos\theta}\, d\theta \tag{48}$$

Again using the change of variable in (44), (48) integrates to

$$\begin{aligned} M &= \frac{-mN}{2a\sinh a}\int_{a}^{-a} ue^{u}\,du \\ &= \frac{-mN}{2a\sinh a}\, e^{u}(u-1)\Big|_{a}^{-a} \\ &= \frac{-mN}{2a\sinh a}\,[e^{-a}(-a-1)-e^{a}(a-1)] \\ &= \frac{-mN}{a\sinh a}\,[-a\cosh a+\sinh a] \\ &= mN[\coth a - 1/a] \end{aligned} \tag{49}$$

which is known as the Langevin equation and is plotted as a function of reciprocal temperature in Figure 5-19. At low temperatures (high a) the magnetization saturates at $M = mN$ as all the dipoles have their moments aligned with the field. At room temperature, a is typically very small. Using the parameters in (26) and (27) in a strong magnetic field of $H_0 = 10^6$ amps/m, a is much less than unity:

$$a = \frac{m\mu_0 H_0}{kT} = \frac{e\omega}{2}R^2\frac{\mu_0 H_0}{kT} \approx 8\times 10^{-4} \tag{50}$$

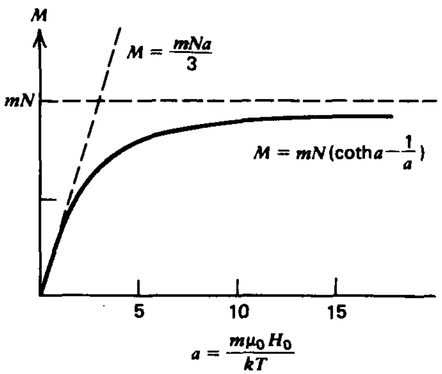


Figure 5-19 The Langevin equation describes the net magnetization. At low temperatures (high a) all the dipoles align with the field causing saturation. At high temperatures ($a \ll 1$) the magnetization increases linearly with field.

In this limit, Langevin's equation simplifies to

$$\lim_{a\ll 1} M \approx mN\left[\frac{1+a^2/2}{a+a^3/6}-\frac{1}{a}\right]$$
$$\approx mN\left(\frac{(1+a^2/2)(1-a^3/6)}{a}-\frac{1}{a}\right]$$
$$\approx \frac{mNa}{3} \approx \frac{\mu_0 m^2 N}{3kT} H_0 \tag{51}$$

In this limit the magnetic susceptibility χ_m is positive:

$$\mathbf{M} = \chi_m \mathbf{H}, \qquad \chi_m = \frac{\mu_0 m^2 N}{3kT} \tag{52}$$

but even with $N \approx 10^{30}$ atoms/m^3, it is still very small:

$$\chi_m \approx 7 \times 10^{-4} \tag{53}$$

(c) Ferromagnetism

As for ferroelectrics (see Section 3-1-5), sufficiently high coupling between adjacent magnetic dipoles in some iron alloys causes them to spontaneously align even in the absence of an applied magnetic field. Each of these microscopic domains act like a permanent magnet, but they are randomly distributed throughout the material so that the macroscopic magnetization is zero. When a magnetic field is applied, the dipoles tend to align with the field so that domains with a magnetization along the field grow at the expense of non-aligned domains.

The friction-like behavior of domain wall motion is a lossy process so that the magnetization varies with the magnetic field in a nonlinear way, as described by the hysteresis loop in Figure 5-20. A strong field aligns all the domains to saturation. Upon decreasing **H**, the magnetization lags behind so that a remanent magnetization M_r exists even with zero field. In this condition we have a permanent magnet. To bring the magnetization to zero requires a negative coercive field $-H_c$.

Although nonlinear, the main engineering importance of ferromagnetic materials is that the relative permeability μ_r is often in the thousands:

$$\mu = \mu_r \mu_0 = \mathbf{B}/\mathbf{H} \tag{54}$$

This value is often so high that in engineering applications we idealize it to be infinity. In this limit

$$\lim_{\mu\to\infty} \mathbf{B} = \mu \mathbf{H} \Rightarrow \mathbf{H} = 0, \qquad \mathbf{B} \text{ finite} \tag{55}$$

the **H** field becomes zero to keep the **B** field finite.

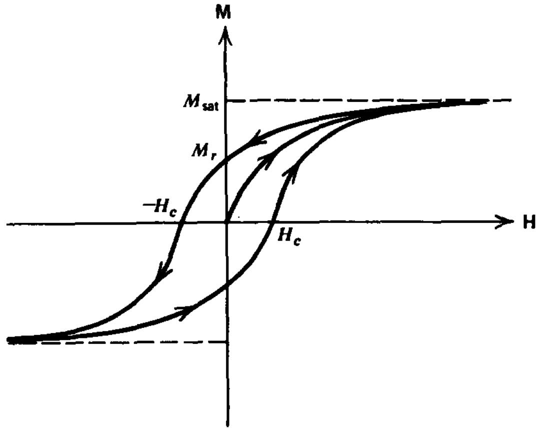


Figure 5-20 Ferromagnetic materials exhibit hysteresis where the magnetization saturates at high field strengths and retains a net remanent magnetization $\mathbf{M}_r$ even when **H** is zero. A coercive field $-\mathbf{H}_c$ is required to bring the magnetization back to zero.

EXAMPLE 5-1 INFINITE LINE CURRENT WITHIN A MAGNETICALLY PERMEABLE CYLINDER

A line current I of infinite extent is within a cylinder of radius a that has permeability μ, as in Figure 5-21. The cylinder is surrounded by free space. What are the **B**, **H**, and **M** fields everywhere? What is the magnetization current?

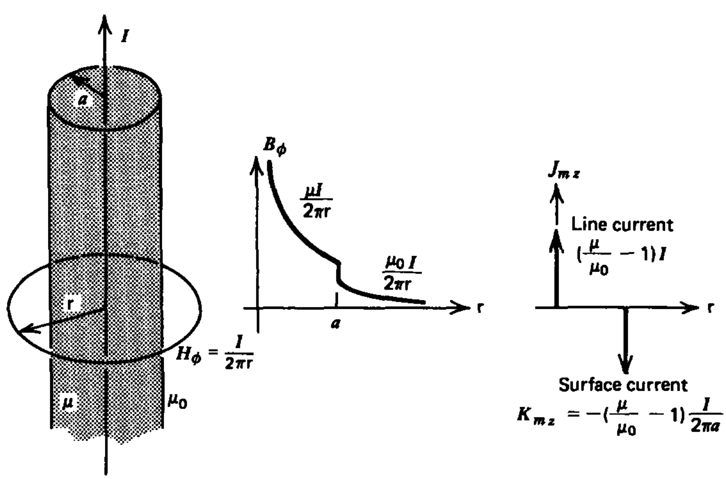


Figure 5-21 A free line current of infinite extent placed within a permeable cylinder gives rise to a line magnetization current along the axis and an oppositely directed surface magnetization current on the cylinder surface.

SOLUTION

Pick a circular contour of radius r around the current. Using the integral form of Ampere's law, (21), the **H** field is of the same form whether inside or outside the cylinder:

$$\oint_L \mathbf{H}\cdot d\mathbf{l} = H_\phi 2\pi r = I \Rightarrow H_\phi = \frac{I}{2\pi r}$$

The magnetic flux density differs in each region because the permeability differs:

$$B_\phi = \begin{cases} \mu H_\phi = \dfrac{\mu I}{2\pi r}, & 0<r<a \\ \mu_0 H_\phi = \dfrac{\mu_0 I}{2\pi r}, & r>a \end{cases}$$

The magnetization is obtained from the relation

$$\mathbf{M} = \frac{\mathbf{B}}{\mu_0} - \mathbf{H}$$

as

$$M_\phi = \begin{cases} \left(\dfrac{\mu}{\mu_0} - 1\right) H_\phi = \dfrac{\mu - \mu_0}{\mu_0}\dfrac{I}{2\pi r}, & 0<r<a \\ 0, & r>a \end{cases}$$

The volume magnetization current can be found using (16):

$$\mathbf{J}_m = \nabla \times \mathbf{M} = -\frac{\partial M_\phi}{\partial z}\mathbf{i}_r + \frac{1}{r}\frac{\partial}{\partial r}(rM_\phi)\mathbf{i}_z = 0, \qquad 0<r<a$$

There is no bulk magnetization current because there are no bulk free currents. However, there is a line magnetization current at $r=0$ and a surface magnetization current at $r=a$. They are easily found using the integral form of (16) from Stokes' theorem:

$$\int_S \nabla \times \mathbf{M}\cdot d\mathbf{S} = \oint_L \mathbf{M}\cdot d\mathbf{l} = \int_S \mathbf{J}_m \cdot d\mathbf{S}$$

Pick a contour around the center of the cylinder with $r<a$:

$$M_\phi 2\pi r = \left(\frac{\mu - \mu_0}{\mu_0}\right) I = I_m$$

where I_m is the magnetization line current. The result remains unchanged for any radius $r<a$ as no more current is enclosed since $\mathbf{J}_m = 0$ for $0<r<a$. As soon as $r>a$, M_ϕ becomes zero so that the total magnetization current becomes zero. Therefore, at $r=a$ a surface magnetization current must flow whose total current is equal in magnitude but opposite in sign to the line magnetization current:

$$K_{zm}=\frac{-I_m}{2\pi a}=-\frac{(\mu-\mu_0)I}{\mu_0 2\pi a}$$

5-6 BOUNDARY CONDITIONS

At interfacial boundaries separating materials of differing properties, the magnetic fields on either side of the boundary must obey certain conditions. The procedure is to use the integral form of the field laws for differential sized contours, surfaces, and volumes in the same way as was performed for electric fields in Section 3-3.

To summarize our development thus far, the field laws for magnetic fields in differential and integral form are

$$\nabla\times\mathbf{H}=\mathbf{J}_f, \qquad \oint_L \mathbf{H}\cdot d\mathbf{l}=\int_S \mathbf{J}_f\cdot d\mathbf{S} \tag{1}$$

$$\nabla\times\mathbf{M}=\mathbf{J}_m, \qquad \oint_L \mathbf{M}\cdot d\mathbf{l}=\int_S \mathbf{J}_m\cdot d\mathbf{S} \tag{2}$$

$$\nabla\cdot\mathbf{B}=0, \qquad \oint_S \mathbf{B}\cdot d\mathbf{S}=0 \tag{3}$$

5-6-1 Tangential Component of H

We apply Ampere's circuital law of (1) to the contour of differential size enclosing the interface, as shown in Figure 5-22*a*. Because the interface is assumed to be infinitely thin, the short sides labelled *c* and *d* are of zero length and so offer

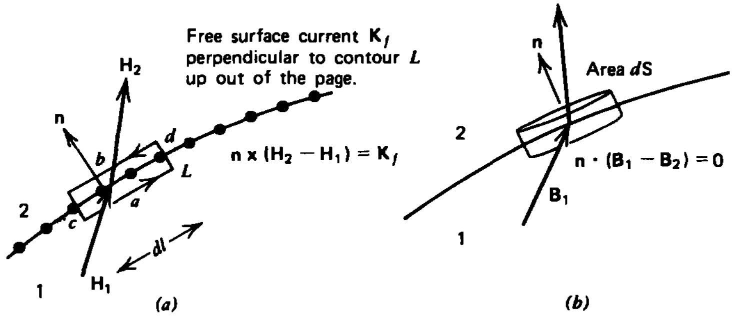


Figure 5-22 (*a*) The tangential component of **H** can be discontinuous in a free surface current across a boundary. (*b*) The normal component of **B** is always continuous across an interface.

no contribution to the line integral. The remaining two sides yield

$$\oint_L \mathbf{H}\cdot d\mathbf{l}=(H_{1t}-H_{2t})\,dl=K_{fn}\,dl \tag{4}$$

where K_{fn} is the component of free surface current perpendicular to the contour by the right-hand rule in this case up out of the page. Thus, the tangential component of magnetic field can be discontinuous by a free surface current,

$$(H_{1t}-H_{2t})=K_{fn}\Rightarrow \mathbf{n}\times(\mathbf{H}_2-\mathbf{H}_1)=\mathbf{K}_f \tag{5}$$

where the unit normal points from region 1 towards region 2. If there is no surface current, the tangential component of **H** is continuous.

5-6-2 Tangential Component of M

Equation (2) is of the same form as (6) so we may use the results of (5) replacing **H** by **M** and $\mathbf{K}_f$ by $\mathbf{K}_m$, the surface magnetization current:

$$(M_{1t}-M_{2t})=K_{mn},\qquad \mathbf{n}\times(\mathbf{M}_2-\mathbf{M}_1)=\mathbf{K}_m \tag{6}$$

This boundary condition confirms the result for surface magnetization current found in Example 5-1.

5-6-3 Normal Component of B

Figure 5-22*b* shows a small volume whose upper and lower surfaces are parallel and are on either side of the interface. The short cylindrical side, being of zero length, offers no contribution to (3), which thus reduces to

$$\oint_S \mathbf{B}\cdot d\mathbf{S}=(B_{2n}-B_{1n})\,dS=0 \tag{7}$$

yielding the boundary condition that the component of **B** normal to an interface of discontinuity is always continuous:

$$B_{1n}-B_{2n}=0\Rightarrow \mathbf{n}\cdot(\mathbf{B}_1-\mathbf{B}_2)=0 \tag{8}$$

EXAMPLE 5-2 MAGNETIC SLAB WITHIN A UNIFORM MAGNETIC FIELD

A slab of infinite extent in the x and y directions is placed within a uniform magnetic field $H_0\mathbf{i}_z$ as shown in Figure 5-23.

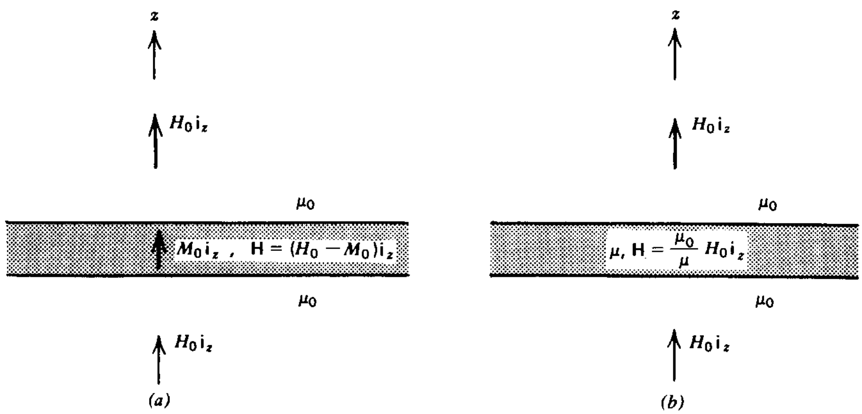


Figure 5-23 A (*a*) permanently magnetized or (*b*) linear magnetizable material is placed within a uniform magnetic field.

Find the **H** field within the slab when it is

(a) permanently magnetized with magnetization $M_0\mathbf{i}_z$,

(b) a linear permeable material with permeability μ.

SOLUTION

For both cases, (8) requires that the **B** field across the boundaries be continuous as it is normally incident.

(a) For the permanently magnetized slab, this requires that

$$\mu_0 H_0 = \mu_0(H + M_0) \Rightarrow H = H_0 - M_0$$

Note that when there is no externally applied field ($H_0 = 0$), the resulting field within the slab is oppositely directed to the magnetization so that $\mathbf{B} = 0$.

(b) For a linear permeable medium (8) requires

$$\mu_0 H_0 = \mu H \Rightarrow H = \frac{\mu_0}{\mu} H_0$$

For $\mu > \mu_0$ the internal magnetic field is reduced. If H_0 is set to zero, the magnetic field within the slab is also zero.

5-7 MAGNETIC FIELD BOUNDARY VALUE PROBLEMS

5-7-1 The Method of Images

A line current I of infinite extent in the z direction is a distance d above a plane that is either perfectly conducting or infinitely permeable, as shown in Figure 5-24. For both cases

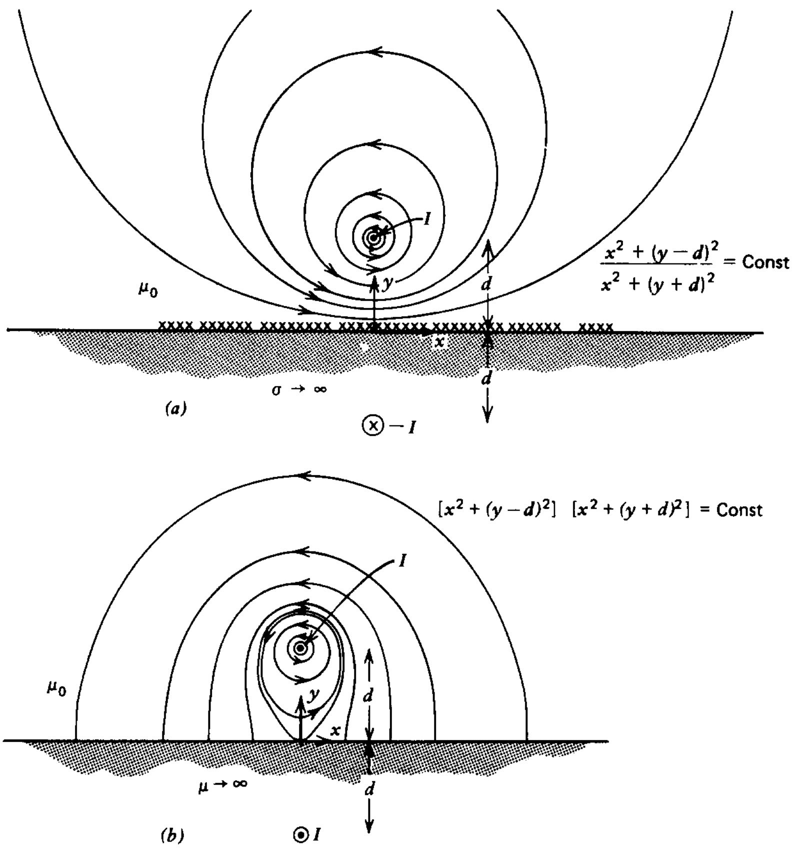


Figure 5-24 (*a*) A line current above a perfect conductor induces an oppositely directed surface current that is equivalent to a symmetrically located image line current. (*b*) The field due to a line current above an infinitely permeable medium is the same as if the medium were replaced by an image current now in the same direction as the original line current.

the **H** field within the material must be zero but the boundary conditions at the interface are different. In the perfect conductor both **B** and **H** must be zero, so that at the interface the normal component of **B** and thus **H** must be continuous and thus zero. The tangential component of **H** is discontinuous in a surface current.

In the infinitely permeable material **H** is zero but **B** is finite. No surface current can flow because the material is not a conductor, so the tangential component of **H** is continuous and thus zero. The **B** field must be normally incident.

Both sets of boundary conditions can be met by placing an image current I at $y=-d$ flowing in the opposite direction for the conductor and in the same direction for the permeable material.

Using the upper sign for the conductor and the lower sign for the infinitely permeable material, the vector potential due to both currents is found by superposing the vector potential found in Section 5-4-3*a*, Eq. (18), for each infinitely long line current:

$$A_z = \frac{-\mu_0 I}{2\pi}\{\ln[x^2+(y-d)^2]^{1/2} \mp \ln[x^2+(y+d)^2]^{1/2}\}$$

$$= \frac{-\mu_0 I}{4\pi}\{\ln[x^2+(y-d)^2] \mp \ln[x^2+(y+d)^2]\} \qquad (1)$$

with resultant magnetic field

$$\mathbf{H} = \frac{1}{\mu_0}\nabla\times\mathbf{A} = \frac{1}{\mu_0}\left(\mathbf{i}_x\frac{\partial A_z}{\partial y} - \mathbf{i}_y\frac{\partial A_z}{\partial x}\right)$$

$$= \frac{-I}{2\pi}\left\{\frac{(y-d)\mathbf{i}_x - x\mathbf{i}_y}{[x^2+(y-d)^2]} \mp \frac{(y+d)\mathbf{i}_x - x\mathbf{i}_y}{[x^2+(y+d)^2]}\right\} \qquad (2)$$

The surface current distribution for the conducting case is given by the discontinuity in tangential **H**,

$$K_z = -H_x(y=0) = -\frac{Id}{\pi[d^2+x^2]} \qquad (3)$$

which has total current

$$I_T = \int_{-\infty}^{+\infty} K_z\,dx = -\frac{Id}{\pi}\int_{-\infty}^{+\infty}\frac{dx}{(x^2+d^2)}$$

$$= -\frac{Id}{\pi}\frac{1}{d}\tan^{-1}\frac{x}{d}\Big|_{-\infty}^{+\infty} = -I \qquad (4)$$

just equal to the image current.

The force per unit length on the current for each case is just due to the magnetic field from its image:

$$\mathbf{f} = \pm\frac{\mu_0 I^2}{4\pi d}\mathbf{i}_y \qquad (5)$$

being repulsive for the conductor and attractive for the permeable material.

The magnetic field lines plotted in Figure 5-24 are just lines of constant A_z as derived in Section 5-4-3*b*. Right next to the line current the self-field term dominates and the field lines are circles. The far field in Figure 5-24*b*, when the line and image current are in the same direction, is the same as if we had a single line current of $2I$.

5-7-2 Sphere in a Uniform Magnetic Field

A sphere of radius R is placed within a uniform magnetic field $H_0\mathbf{i}_z$. The sphere and surrounding medium may have any of the following properties illustrated in Figure 5-25:

(i) Sphere has permeability μ_2 and surrounding medium has permeability μ_1.
(ii) Perfectly conducting sphere in free space.
(iii) Uniformly magnetized sphere $M_2\mathbf{i}_z$ in a uniformly magnetized medium $M_1\mathbf{i}_z$.

For each of these three cases, there are no free currents in either region so that the governing equations in each region are

$$\nabla\cdot\mathbf{B}=0$$
$$\nabla\times\mathbf{H}=0 \tag{5}$$

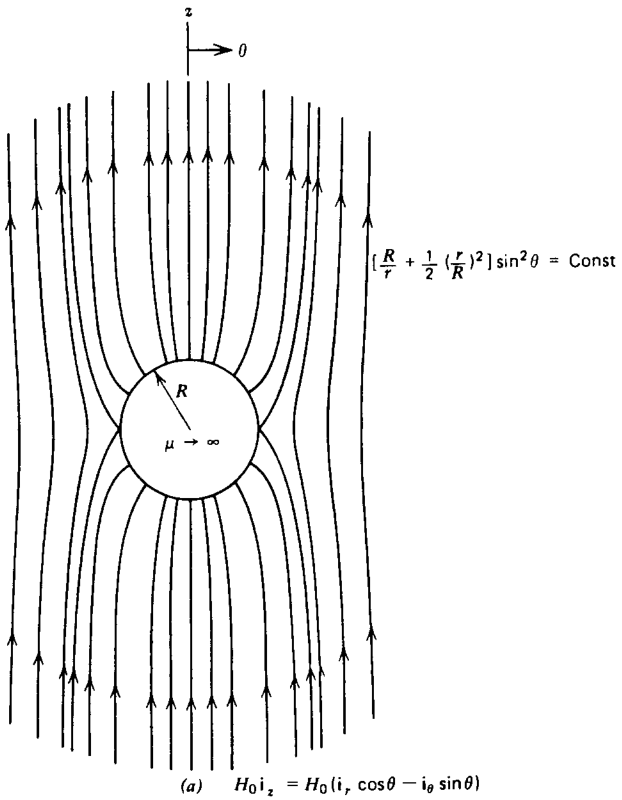


Figure 5-25 Magnetic field lines about an (*a*) infinitely permeable and (*b*) perfectly conducting sphere in a uniform magnetic field.

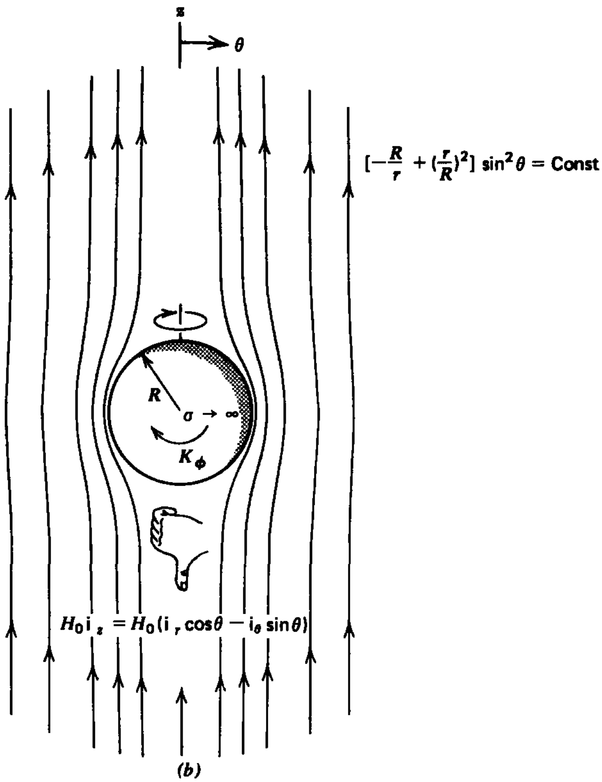


Figure 5-25

Because the curl of **H** is zero, we can define a scalar magnetic potential

$$\mathbf{H} = \nabla\chi \tag{6}$$

where we avoid the use of a negative sign as is used with the electric field since the potential χ is only introduced as a mathematical convenience and has no physical significance. With **B** proportional to **H** or for uniform magnetization, the divergence of **H** is also zero so that the scalar magnetic potential obeys Laplace's equation in each region:

$$\nabla^2\chi = 0 \tag{7}$$

We can then use the same techniques developed for the electric field in Section 4-4 by trying a scalar potential in each region as

$$\chi = \begin{cases} Ar\cos\theta, & r<R \\ (Dr + C/r^2)\cos\theta & r>R \end{cases} \tag{8}$$

The associated magnetic field is then

$$\mathbf{H}=\nabla\chi=\frac{\partial\chi}{\partial r}\mathbf{i}_r+\frac{1}{r}\frac{\partial\chi}{\partial\theta}\mathbf{i}_\theta+\frac{1}{r\sin\theta}\frac{\partial\chi}{\partial\phi}\mathbf{i}_\phi$$

$$=\begin{cases}A(\mathbf{i}_r\cos\theta-\mathbf{i}_\theta\sin\theta)=A\mathbf{i}_z, & r<R\\ (D-2C/r^3)\cos\theta\mathbf{i}_r-(D+C/r^3)\sin\theta\mathbf{i}_\theta, & r>R\end{cases}\tag{9}$$

For the three cases, the magnetic field far from the sphere must approach the uniform applied field:

$$\mathbf{H}(r=\infty)=H_0\mathbf{i}_z=H_0(\mathbf{i}_r\cos\theta-\mathbf{i}_\theta\sin\theta)\Rightarrow D=H_0\tag{10}$$

The other constants, A and C, are found from the boundary conditions at $r=R$. The field within the sphere is uniform, in the same direction as the applied field. The solution outside the sphere is the imposed field plus a contribution as if there were a magnetic dipole at the center of the sphere with moment $m_z=4\pi C$.

(i) If the sphere has a different permeability from the surrounding region, both the tangential components of **H** and the normal components of **B** are continuous across the spherical surface:

$$\begin{aligned}&H_\theta(r=R_+)=H_\theta(r=R_-)\Rightarrow A=D+C/R^3\\ &B_r(r=R_+)=B_r(r=R_-)\Rightarrow\mu_1H_r(r=R_+)=\mu_2H_r(r=R_-)\end{aligned}\tag{11}$$

which yields solutions

$$A=\frac{3\mu_1H_0}{\mu_2+2\mu_1},\qquad C=-\frac{\mu_2-\mu_1}{\mu_2+2\mu_1}R^3H_0\tag{12}$$

The magnetic field distribution is then

$$\mathbf{H}=\begin{cases}\dfrac{3\mu_1H_0}{\mu_2+2\mu_1}(\mathbf{i}_r\cos\theta-\mathbf{i}_\theta\sin\theta)=\dfrac{3\mu_1H_0\mathbf{i}_z}{\mu_2+2\mu_1}, & r<R\\ H_0\left\{\left[1+\dfrac{2R^3}{r^3}\left(\dfrac{\mu_2-\mu_1}{\mu_2+2\mu_1}\right)\right]\cos\theta\mathbf{i}_r\right.\\ \qquad\left.-\left[1-\dfrac{R^3}{r^3}\left(\dfrac{\mu_2-\mu_1}{\mu_2+2\mu_1}\right)\right]\sin\theta\mathbf{i}_\theta\right\}, & r>R\end{cases}\tag{13}$$

The magnetic field lines are plotted in Figure 5-25a when $\mu_2\to\infty$. In this limit, **H** within the sphere is zero, so that the field lines incident on the sphere are purely radial. The field lines plotted are just lines of constant stream function Σ, found in the same way as for the analogous electric field problem in Section 4-4-3b.

(ii) If the sphere is perfectly conducting, the internal magnetic field is zero so that $A=0$. The normal component of **B** right outside the sphere is then also zero:

$$H_r(r=R_+)=0 \Rightarrow C=H_0R^3/2 \tag{14}$$

yielding the solution

$$\mathbf{H}=H_0\left[\left(1-\frac{R^3}{r^3}\right)\cos\theta\,\mathbf{i}_r-\left(1+\frac{R^3}{2r^3}\right)\sin\theta\,\mathbf{i}_\theta\right],\quad r>R \tag{15}$$

The interfacial surface current at $r=R$ is obtained from the discontinuity in the tangential component of **H**:

$$K_\phi=H_\theta(r=R)=-\tfrac{3}{2}H_0\sin\theta \tag{16}$$

The current flows in the negative ϕ direction around the sphere. The right-hand rule, illustrated in Figure 5-25*b*, shows that the resulting field from the induced current acts in the direction opposite to the imposed field. This opposition results in the zero magnetic field inside the sphere.

The field lines plotted in Figure 5-25*b* are purely tangential to the perfectly conducting sphere as required by (14).

(iii) If both regions are uniformly magnetized, the boundary conditions are

$$\begin{aligned} H_\theta(r=R_+)&=H_\theta(r=R_-)\Rightarrow A=D+C/R^3\\ B_r(r=R_+)&=B_r(r=R_-)\Rightarrow H_r(r=R_+)+M_1\cos\theta\\ &\qquad=H_r(r=R_-)+M_2\cos\theta \end{aligned} \tag{17}$$

with solutions

$$\begin{aligned} A&=H_0+\tfrac{1}{3}(M_1-M_2)\\ C&=\frac{R^3}{3}(M_1-M_2) \end{aligned} \tag{18}$$

so that the magnetic field is

$$\mathbf{H}=\begin{cases}\left[H_0+\dfrac{1}{3}(M_1-M_2)\right][\cos\theta\,\mathbf{i}_r-\sin\theta\,\mathbf{i}_\theta]\\ \qquad=\left[H_0+\dfrac{1}{3}(M_1-M_2)\right]\mathbf{i}_z \quad r<R\\ \left(H_0-\dfrac{2R^3}{3r^3}(M_1-M_2)\right)\cos\theta\,\mathbf{i}_r\\ \qquad-\left(H_0+\dfrac{R^3}{3r^3}(M_1-M_2)\right)\sin\theta\,\mathbf{i}_\theta, \quad r>R\end{cases} \tag{19}$$

Because the magnetization is uniform in each region, the curl of **M** is zero everywhere but at the surface of the sphere, so that the volume magnetization current is zero with a surface magnetization current at $r=R$ given by

$$\begin{aligned}\mathbf{K}_m &= \mathbf{n}\times(\mathbf{M}_1-\mathbf{M}_2)\\ &= \mathbf{i}_r\times(M_1-M_2)\mathbf{i}_z\\ &= \mathbf{i}_r\times(M_1-M_2)(\mathbf{i}_r\cos\theta-\sin\theta\mathbf{i}_\theta)\\ &= -(M_1-M_2)\sin\theta\mathbf{i}_\phi \end{aligned} \tag{20}$$

5-8 MAGNETIC FIELDS AND FORCES

5-8-1 Magnetizable Media

A magnetizable medium carrying a free current $\mathbf{J}_f$ is placed within a magnetic field $\mathbf{B}$, which is a function of position. In addition to the Lorentz force, the medium feels the forces on all its magnetic dipoles. Focus attention on the rectangular magnetic dipole shown in Figure 5-26. The force on each current carrying leg is

$$\begin{aligned} \mathbf{f} &= \mathbf{i}\,dl\times(B_x\mathbf{i}_x+B_y\mathbf{i}_y+B_z\mathbf{i}_z)\\ \Rightarrow \mathbf{f}(x) &= -i\,\Delta y[-B_x\mathbf{i}_z+B_z\mathbf{i}_x]|_x\\ \mathbf{f}(x+\Delta x) &= i\,\Delta y[-B_x\mathbf{i}_z+B_z\mathbf{i}_x]|_{x+\Delta x}\\ \mathbf{f}(y) &= i\,\Delta x[B_y\mathbf{i}_z-B_z\mathbf{i}_y]|_y\\ \mathbf{f}(y+\Delta y) &= -i\,\Delta x[B_y\mathbf{i}_z-B_z\mathbf{i}_y]|_{y+\Delta y} \end{aligned} \tag{1}$$

so that the total force on the dipole is

$$\begin{aligned} \mathbf{f} &= \mathbf{f}(x)+\mathbf{f}(x+\Delta x)+\mathbf{f}(y)+\mathbf{f}(y+\Delta y)\\ &= i\,\Delta x\,\Delta y\left[\frac{B_z(x+\Delta x)-B_z(x)}{\Delta x}\mathbf{i}_x-\frac{B_x(x+\Delta x)-B_x(x)}{\Delta x}\mathbf{i}_z\right.\\ &\left.\quad+\frac{B_z(y+\Delta y)-B_z(y)}{\Delta y}\mathbf{i}_y-\frac{B_y(y+\Delta y)-B_y(y)}{\Delta y}\mathbf{i}_z\right] \end{aligned} \tag{2}$$

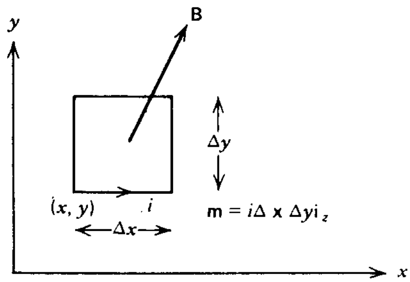


Figure 5-26 A magnetic dipole in a magnetic field **B**.

In the limit of infinitesimal Δx and Δy the bracketed terms define partial derivatives while the coefficient is just the magnetic dipole moment $\mathbf{m} = i\,\Delta x\,\Delta y\,\mathbf{i}_z$:

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \mathbf{f} = m_z\left[\frac{\partial B_z}{\partial x}\mathbf{i}_x - \left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}\right)\mathbf{i}_z + \frac{\partial B_z}{\partial y}\mathbf{i}_y\right] \tag{3}$$

Ampere's and Gauss's law for the magnetic field relate the field components as

$$\nabla\cdot\mathbf{B} = 0 \Rightarrow \frac{\partial B_z}{\partial z} = -\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}\right) \tag{4}$$

$$\nabla\times\mathbf{B} = \mu_0(\mathbf{J}_f+\nabla\times\mathbf{M}) = \mu_0\mathbf{J}_T \Rightarrow \frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z} = \mu_0 J_{Tx}$$

$$\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x} = \mu_0 J_{Ty}$$

$$\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y} = \mu_0 J_{Tz} \tag{5}$$

which puts (3) in the form

$$\mathbf{f} = m_z\left(\frac{\partial B_x}{\partial z}\mathbf{i}_x + \frac{\partial B_y}{\partial z}\mathbf{i}_y + \frac{\partial B_z}{\partial z}\mathbf{i}_z - \mu_0(J_{Ty}\mathbf{i}_x - J_{Tx}\mathbf{i}_y)\right)$$

$$= (\mathbf{m}\cdot\nabla)\mathbf{B} + \mu_0\mathbf{m}\times\mathbf{J}_T \tag{6}$$

where $\mathbf{J}_T$ is the sum of free and magnetization currents.

If there are N such dipoles per unit volume, the force density on the dipoles and on the free current is

$$\begin{aligned}\mathbf{F} = N\mathbf{f} &= (\mathbf{M}\cdot\nabla)\mathbf{B} + \mu_0\mathbf{M}\times\mathbf{J}_T + \mathbf{J}_f\times\mathbf{B}\\ &= \mu_0(\mathbf{M}\cdot\nabla)(\mathbf{H}+\mathbf{M}) + \mu_0\mathbf{M}\times(\mathbf{J}_f+\nabla\times\mathbf{M}) + \mu_0\mathbf{J}_f\times(\mathbf{H}+\mathbf{M})\\ &= \mu_0(\mathbf{M}\cdot\nabla)(\mathbf{H}+\mathbf{M}) + \mu_0\mathbf{M}\times(\nabla\times\mathbf{M}) + \mu_0\mathbf{J}_f\times\mathbf{H}\end{aligned} \tag{7}$$

Using the vector identity

$$\mathbf{M}\times(\nabla\times\mathbf{M}) = -(\mathbf{M}\cdot\nabla)\mathbf{M} + \tfrac{1}{2}\nabla(\mathbf{M}\cdot\mathbf{M}) \tag{8}$$

(7) can be reduced to

$$\mathbf{F} = \mu_0(\mathbf{M}\cdot\nabla)\mathbf{H} + \mu_0\mathbf{J}_f\times\mathbf{H} + \nabla\left(\frac{\mu_0}{2}\mathbf{M}\cdot\mathbf{M}\right) \tag{9}$$

The total force on the body is just the volume integral of $\mathbf{F}$:

$$\mathbf{f} = \int_V \mathbf{F}\,dV \tag{10}$$

In particular, the last contribution in (9) can be converted to a surface integral using the gradient theorem, a corollary to the divergence theorem (see Problem 1-15*a*):

$$\int_V \nabla\left(\frac{\mu_0}{2}\mathbf{M}\cdot\mathbf{M}\right) dV = \oint_S \frac{\mu_0}{2}\mathbf{M}\cdot\mathbf{M}\, d\mathbf{S} \qquad (11)$$

Since this surface S surrounds the magnetizable medium, it is in a region where $\mathbf{M}=0$ so that the integrals in (11) are zero. For this reason the force density of (9) is written as

$$\mathbf{F} = \mu_0(\mathbf{M}\cdot\nabla)\mathbf{H} + \mu_0\mathbf{J}_f\times\mathbf{H} \qquad (12)$$

It is the first term on the right-hand side in (12) that accounts for an iron object to be drawn towards a magnet. Magnetizable materials are attracted towards regions of higher $\mathbf{H}$.

5-8-2 Force on a Current Loop

(a) Lorentz Force Only

Two parallel wires are connected together by a wire that is free to move, as shown in Figure 5-27*a*. A current I is imposed and the whole loop is placed in a uniform magnetic field $B_0\mathbf{i}_x$. The Lorentz force on the moveable wire is

$$f_y = IB_0 l \qquad (13)$$

where we neglect the magnetic field generated by the current, assuming it to be much smaller than the imposed field B_0.

(b) Magnetization Force Only

The sliding wire is now surrounded by an infinitely permeable hollow cylinder of inner radius a and outer radius b, both being small compared to the wire's length l, as in Figure 5-27*b*. For distances near the cylinder, the solution is approximately the same as if the wire were infinitely long. For $r>0$ there is no current, thus the magnetic field is curl and divergence free within each medium so that the magnetic scalar potential obeys Laplace's equation as in Section 5-7-2. In cylindrical geometry we use the results of Section 4-3 and try a scalar potential of the form

$$\chi = \left(Ar + \frac{C}{r}\right)\cos\phi \qquad (14)$$

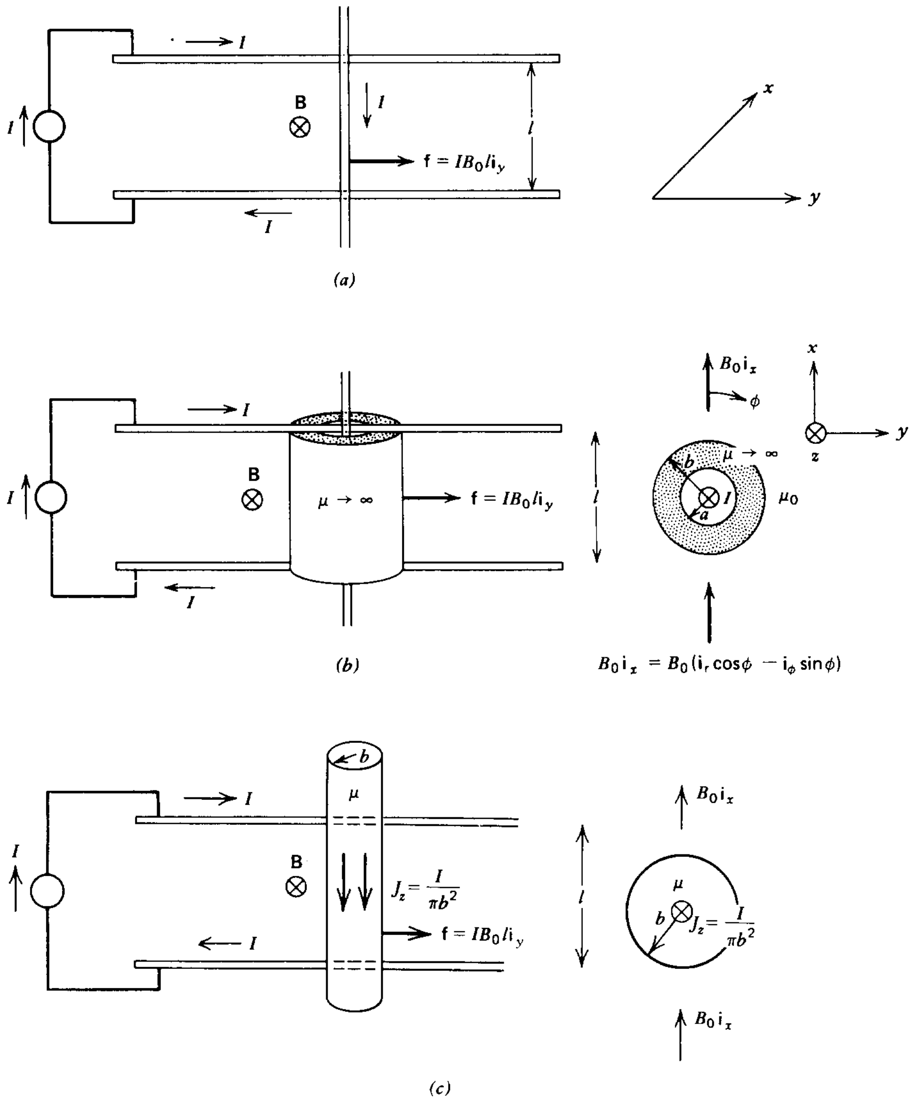


Figure 5-27 (*a*) The Lorentz force on a current carrying wire in a magnetic field. (*b*) If the current-carrying wire is surrounded by an infinitely permeable hollow cylinder, there is no Lorentz force as the imposed magnetic field is zero where the current is. However, the magnetization force on the cylinder is the same as in (*a*). (*c*) The total force on a current-carrying magnetically permeable wire is also unchanged.

in each region, where $\mathbf{B} = \nabla\chi$ because $\nabla \times \mathbf{B} = 0$. The constants are evaluated by requiring that the magnetic field approach the imposed field $B_0\mathbf{i}_x$ at $r = \infty$ and be normally incident onto the infinitely permeable cylinder at $r = a$ and $r = b$. In addition, we must add the magnetic field generated by the line current. The magnetic field in each region is then (see Problem 32*a*):

$$\mathbf{B}=\begin{cases}\dfrac{\mu_0 I}{2\pi r}\mathbf{i}_\phi, & 0<r<a\\ \dfrac{2B_0b^2}{b^2-a^2}\left[\left(1-\dfrac{a^2}{r^2}\right)\cos\phi\,\mathbf{i}_r-\left(1+\dfrac{a^2}{r^2}\right)\sin\phi\,\mathbf{i}_\phi\right]+\dfrac{\mu I}{2\pi r}\mathbf{i}_\phi, & a<r<b\\ B_0\left[\left(1+\dfrac{b^2}{r^2}\right)\cos\phi\,\mathbf{i}_r-\left(1-\dfrac{b^2}{r^2}\right)\sin\phi\,\mathbf{i}_\phi\right]+\dfrac{\mu_0 I}{2\pi r}\mathbf{i}_\phi, & r>b\end{cases} \quad (15)$$

Note the infinite flux density in the iron ($\mu\to\infty$) due to the line current that sets up the finite **H** field. However, we see that none of the imposed magnetic field is incident upon the current carrying wire because it is shielded by the infinitely permeable cylindrical shell so that the Lorentz force contribution on the wire is zero. There is, however, a magnetization force on the cylindrical shell where the internal magnetic field **H** is entirely due to the line current, $H_\phi = I/2\pi r$ because with $\mu\to\infty$, the contribution due to B_0 is negligibly small:

$$\begin{aligned}\mathbf{F}&=\mu_0(\mathbf{M}\cdot\nabla)\mathbf{H}\\ &=\mu_0\left(M_r\frac{\partial}{\partial r}(H_\phi\mathbf{i}_\phi)+\frac{M_\phi}{r}\frac{\partial}{\partial\phi}(H_\phi\mathbf{i}_\phi)\right)\end{aligned} \quad (16)$$

Within the infinitely permeable shell the magnetization and **H** fields are

$$\begin{aligned}H_\phi&=\frac{I}{2\pi r}\\ \mu_0M_r&=B_r-\mu_0\overset{0}{\cancel{H_r}}=\frac{2B_0b^2}{b^2-a^2}\left(1-\frac{a^2}{r^2}\right)\cos\phi\\ \mu_0M_\phi&=B_\phi-\mu_0H_\phi=-\frac{2B_0b^2}{(b^2-a^2)}\left(1+\frac{a^2}{r^2}\right)\sin\phi+\frac{(\mu-\mu_0)I}{2\pi r}\end{aligned} \quad (17)$$

Although H_ϕ only depends on r, the unit vector $\mathbf{i}_\phi$ depends on ϕ:

$$\mathbf{i}_\phi=(-\sin\phi\,\mathbf{i}_x+\cos\phi\,\mathbf{i}_y) \quad (18)$$

so that the force density of (16) becomes

$$\begin{aligned}\mathbf{F}&=-\frac{B_rI}{2\pi r^2}\mathbf{i}_\phi+\frac{(B_\phi-\mu_0H_\phi)I}{2\pi r^2}\frac{d}{d\phi}(\mathbf{i}_\phi)\\ &=\frac{I}{2\pi r^2}[-B_r(-\sin\phi\,\mathbf{i}_x+\cos\phi\,\mathbf{i}_y)\\ &\quad+(B_\phi-\mu_0H_\phi)(-\cos\phi\,\mathbf{i}_x-\sin\phi\,\mathbf{i}_y)]\end{aligned}$$

$$= \frac{I}{2\pi r^2}\left\{-\frac{2B_0b^2}{b^2-a^2}\left[\left(1-\frac{a^2}{r^2}\right)\cos\phi(-\sin\phi\,\mathbf{i}_x+\cos\phi\,\mathbf{i}_y)\right.\right.$$
$$\left.-\left(1+\frac{a^2}{r^2}\right)\sin\phi(\cos\phi\,\mathbf{i}_x+\sin\phi\,\mathbf{i}_y)\right]$$
$$\left.+\frac{(\mu-\mu_0)I}{2\pi r}(\cos\phi\,\mathbf{i}_x+\sin\phi\,\mathbf{i}_y)\right\}$$
$$= \frac{I}{2\pi r^2}\left[-\frac{2B_0b^2}{b^2-a^2}\left(-2\sin\phi\cos\phi\,\mathbf{i}_x-\frac{2a^2}{r^2}\mathbf{i}_y\right)\right.$$
$$\left.+\frac{(\mu-\mu_0)I}{2\pi r}(\cos\phi\,\mathbf{i}_x+\sin\phi\,\mathbf{i}_y)\right] \tag{19}$$

The total force on the cylinder is obtained by integrating (19) over r and ϕ:

$$\mathbf{f} = \int_{\phi=0}^{2\pi}\int_{r=a}^{b} \mathbf{F} l\, r\, dr\, d\phi \tag{20}$$

All the trigonometric terms in (19) integrate to zero over ϕ so that the total force is

$$\begin{aligned} f_y &= \frac{2B_0b^2 Il}{(b^2-a^2)}\int_{r=a}^{b}\frac{a^2}{r^3}\,dr \\ &= -\frac{B_0b^2Il}{(b^2-a^2)}\frac{a^2}{r^2}\bigg|_a^b \\ &= IB_0l \end{aligned} \tag{21}$$

The force on the cylinder is the same as that of an unshielded current-carrying wire given by (13). If the iron core has a finite permeability, the total force on the wire (Lorentz force) and on the cylinder (magnetization force) is again equal to (13). This fact is used in rotating machinery where current-carrying wires are placed in slots surrounded by highly permeable iron material. Most of the force on the whole assembly is on the iron and not on the wire so that very little restraining force is necessary to hold the wire in place. The force on a current-carrying wire surrounded by iron is often calculated using only the Lorentz force, neglecting the presence of the iron. The correct answer is obtained but for the wrong reasons. Actually there is very little **B** field near the wire as it is almost surrounded by the high permeability iron so that the Lorentz force on the wire is very small. The force is actually on the iron core.

(c) Lorentz and Magnetization Forces

If the wire itself is highly permeable with a uniformly distributed current, as in Figure 5-27*c*, the magnetic field is (see Problem 32*a*)

$$\mathbf{H}=\begin{cases}\dfrac{2B_0}{\mu+\mu_0}(\mathbf{i}_r\cos\phi-\mathbf{i}_\phi\sin\phi)+\dfrac{Ir}{2\pi b^2}\mathbf{i}_\phi \\ \qquad =\dfrac{2B_0}{\mu+\mu_0}\mathbf{i}_x+\dfrac{I}{2\pi b^2}(-y\mathbf{i}_x+x\mathbf{i}_y), \quad r<b \\ \dfrac{B_0}{\mu_0}\left[\left(1+\dfrac{b^2}{r^2}\dfrac{\mu-\mu_0}{\mu+\mu_0}\right)\cos\phi\,\mathbf{i}_r\right. \\ \qquad \left.-\left(1-\dfrac{b^2}{r^2}\dfrac{\mu-\mu_0}{\mu+\mu_0}\right)\sin\phi\,\mathbf{i}_\phi\right]+\dfrac{I}{2\pi r}\mathbf{i}_\phi, \quad r>b\end{cases} \tag{22}$$

It is convenient to write the fields within the cylinder in Cartesian coordinates using (18) as then the force density given by (12) is

$$\begin{aligned}\mathbf{F}&=\mu_0(\mathbf{M}\cdot\nabla)\mathbf{H}+\mu_0\mathbf{J}_f\times\mathbf{H}\\ &=(\mu-\mu_0)(\mathbf{H}\cdot\nabla)\mathbf{H}+\frac{\mu_0 I}{\pi b^2}\mathbf{i}_z\times\mathbf{H}\\ &=(\mu-\mu_0)\left(H_x\frac{\partial}{\partial x}+H_y\frac{\partial}{\partial y}\right)(H_x\mathbf{i}_x+H_y\mathbf{i}_y)+\frac{\mu_0 I}{\pi b^2}(H_x\mathbf{i}_y-H_y\mathbf{i}_x)\end{aligned} \tag{23}$$

Since within the cylinder ($r<b$) the partial derivatives of $\mathbf{H}$ are

$$\begin{aligned}\frac{\partial H_x}{\partial x}&=\frac{\partial H_y}{\partial y}=0\\ \frac{\partial H_x}{\partial y}&=-\frac{\partial H_y}{\partial x}=-\frac{I}{2\pi b^2}\end{aligned} \tag{24}$$

(23) reduces to

$$\begin{aligned}\mathbf{F}&=(\mu-\mu_0)\left(H_x\frac{\partial H_y}{\partial x}\mathbf{i}_y+H_y\frac{\partial H_x}{\partial y}\mathbf{i}_x\right)+\frac{\mu_0 I}{\pi b^2}(H_x\mathbf{i}_y-H_y\mathbf{i}_x)\\ &=\frac{I}{2\pi b^2}(\mu+\mu_0)(H_x\mathbf{i}_y-H_y\mathbf{i}_x)\\ &=\frac{I(\mu+\mu_0)}{2\pi b^2}\left[\left(\frac{2B_0}{\mu+\mu_0}-\frac{Iy}{2\pi b^2}\right)\mathbf{i}_y-\frac{Ix}{2\pi b^2}\mathbf{i}_x\right]\end{aligned} \tag{25}$$

Realizing from Table 1-2 that

$$y\mathbf{i}_y+x\mathbf{i}_x=r[\sin\phi\,\mathbf{i}_y+\cos\phi\,\mathbf{i}_x]=r\mathbf{i}_r \tag{26}$$

the force density can be written as

$$\mathbf{F}=\frac{IB_0}{\pi b^2}\mathbf{i}_y-\frac{I^2(\mu+\mu_0)}{(2\pi b^2)^2}\mathrm{r}\,(\sin\phi\,\mathbf{i}_y+\cos\phi\,\mathbf{i}_x) \tag{27}$$

The total force on the permeable wire is

$$\mathbf{f}=\int_{\phi=0}^{2\pi}\int_{\mathrm{r}=0}^{b}\mathbf{F}l\mathrm{r}\,d\mathrm{r}\,d\phi \tag{28}$$

We see that the trigonometric terms in (27) integrate to zero so that only the first term contributes:

$$\begin{aligned} f_y&=\frac{IB_0l}{\pi b^2}\int_{\phi=0}^{2\pi}\int_{\mathrm{r}=0}^{b}\mathrm{r}\,d\mathrm{r}\,d\phi\\ &=IB_0l \end{aligned} \tag{29}$$

The total force on the wire is independent of its magnetic permeability.

PROBLEMS

Section 5-1

1. A charge q of mass m moves through a uniform magnetic field $B_o\mathbf{i}_z$. At $t=0$ its velocity and displacement are

$$\mathbf{v}(t=0)=v_{x0}\mathbf{i}_x+v_{y0}\mathbf{i}_y+v_{z0}\mathbf{i}_z$$

$$\mathbf{r}(t=0)=x_0\mathbf{i}_x+y_0\mathbf{i}_y+z_0\mathbf{i}_z$$

(a) What is the subsequent velocity and displacement?

(b) Show that its motion projected onto the xy plane is a circle. What is the radius of this circle and where is its center?

(c) What is the time dependence of the kinetic energy of the charge $\frac{1}{2}m|\mathbf{v}|^2$?

2. A magnetron is essentially a parallel plate capacitor stressed by constant voltage V_0 where electrons of charge $-e$ are emitted at $x=0$, $y=0$ with zero initial velocity. A transverse magnetic field $B_0\mathbf{i}_z$ is applied. Neglect the electric and magnetic fields due to the electrons in comparison to the applied field.

(a) What is the velocity and displacement of an electron, injected with zero initial velocity at $t=0$?

(b) What value of magnetic field will just prevent the electrons from reaching the other electrode? This is the cut-off magnetic field.