

# chapter 3

*polarization and  
conduction*



electrons symmetrically so that the net positive and negative charges are separated. An applied electric field then exerts a torque on the molecule that tends to align it with the field. The ions in a molecule can also undergo slight relative displacements that gives rise to ionic polarizability.

The slightly separated charges for these cases form electric dipoles. Dielectric materials have a distribution of such dipoles. Even though these materials are charge neutral because each dipole contains an equal amount of positive and negative charges, a net charge can accumulate in a region if there is a local imbalance of positive or negative dipole ends. The net polarization charge in such a region is also a source of the electric field in addition to any other free charges.

### 3-1-1 The Electric Dipole

The simplest model of an electric dipole, shown in Figure 3-2a, has a positive and negative charge of equal magnitude  $q$  separated by a small vector displacement  $\mathbf{d}$  directed from the negative to positive charge along the  $z$  axis. The electric potential is easily found at any point  $P$  as the superposition of potentials from each point charge alone:

$$V = \frac{q}{4\pi\epsilon_0 r_+} - \frac{q}{4\pi\epsilon_0 r_-} \quad (1)$$

The general potential and electric field distribution for any displacement  $\mathbf{d}$  can be easily obtained from the geometry relating the distances  $r_+$  and  $r_-$  to the spherical coordinates  $r$  and  $\theta$ . By symmetry, these distances are independent of the angle  $\phi$ . However, in dielectric materials the separation between charges are of atomic dimensions and so are very small compared to distances of interest far from the dipole. So, with  $r_+$  and  $r_-$  much greater than the dipole spacing  $d$ , we approximate them as

$$\begin{aligned} r_+ &\approx r - \frac{d}{2} \cos \theta \\ \lim_{r \gg d} r_- &\approx r + \frac{d}{2} \cos \theta \end{aligned} \quad (2)$$

Then the potential of (1) is approximately

$$V \approx \frac{qd \cos \theta}{4\pi\epsilon_0 r^2} = \frac{\mathbf{p} \cdot \mathbf{i}_r}{4\pi\epsilon_0 r^2} \quad (3)$$

where the vector  $\mathbf{p}$  is called the dipole moment and is defined as

$$\mathbf{p} = q\mathbf{d} \text{ (coul-m)} \quad (4)$$

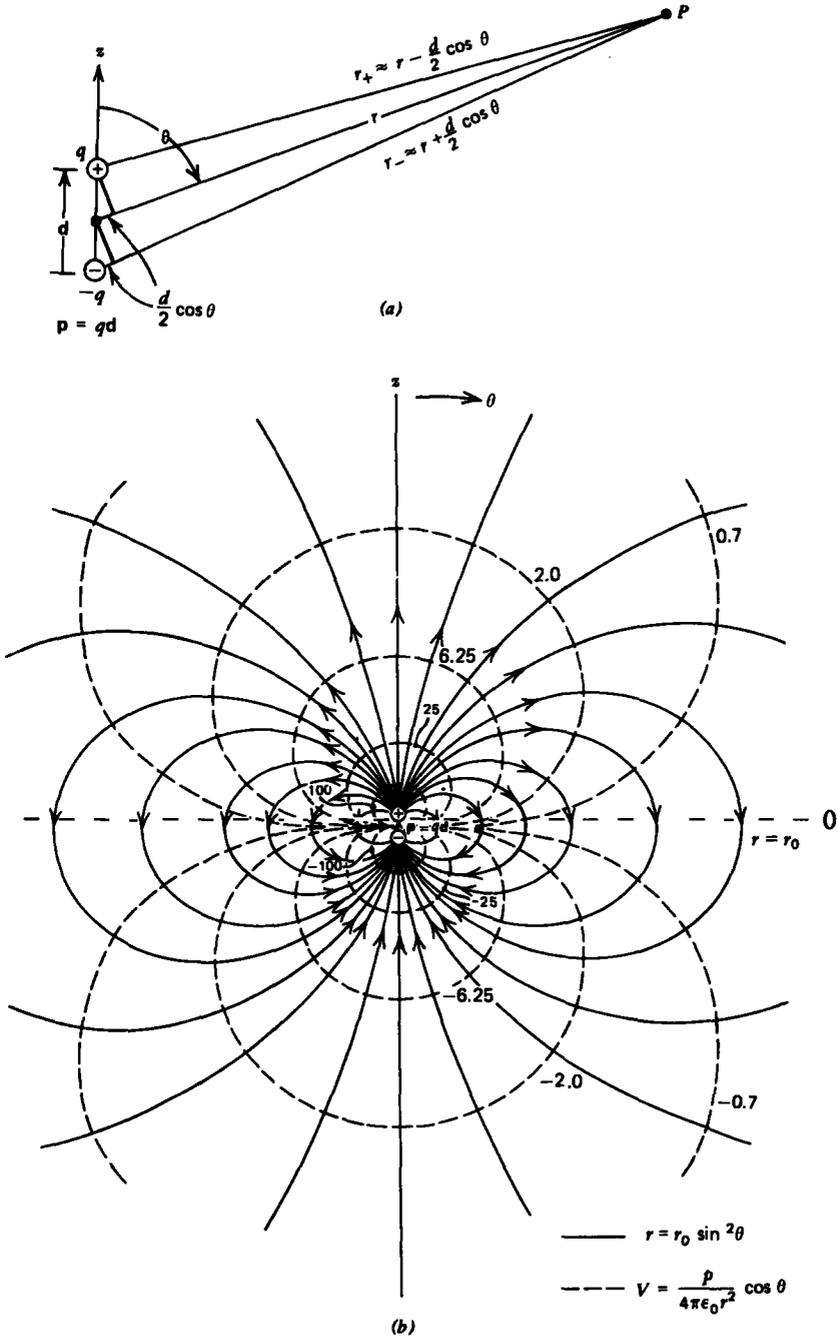


Figure 3-2 (a) The potential at any point  $P$  due to the electric dipole is equal to the sum of potentials of each charge alone. (b) The equi-potential (dashed) and field lines (solid) for a point electric dipole calibrated for  $4\pi\epsilon_0/p = 100$ .

Because the separation of atomic charges is on the order of  $1 \text{ \AA} (10^{-10} \text{ m})$  with a charge magnitude equal to an integer multiple of the electron charge ( $q = 1.6 \times 10^{-19} \text{ coul}$ ), it is convenient to express dipole moments in units of debyes defined as  $1 \text{ debye} = 3.33 \times 10^{-30} \text{ coul}\cdot\text{m}$  so that dipole moments are of order  $p = 1.6 \times 10^{-29} \text{ coul}\cdot\text{m} \approx 4.8 \text{ debyes}$ . The electric field for the point dipole is then

$$\mathbf{E} = -\nabla V = \frac{p}{4\pi\epsilon_0 r^3} [2 \cos \theta \mathbf{i}_r + \sin \theta \mathbf{i}_\theta] = \frac{3(\mathbf{p} \cdot \mathbf{i}_r)\mathbf{i}_r - \mathbf{p}}{4\pi\epsilon_0 r^3} \quad (5)$$

the last expressions in (3) and (5) being coordinate independent. The potential and electric field drop off as a single higher power in  $r$  over that of a point charge because the net charge of the dipole is zero. As one gets far away from the dipole, the fields due to each charge tend to cancel. The point dipole equipotential and field lines are sketched in Figure 3-2*b*. The lines tangent to the electric field are

$$\frac{dr}{r d\theta} = \frac{E_r}{E_\theta} = 2 \cot \theta \Rightarrow r = r_0 \sin^2 \theta \quad (6)$$

where  $r_0$  is the position of the field line when  $\theta = \pi/2$ . All field lines start on the positive charge and terminate on the negative charge.

If there is more than one pair of charges, the definition of dipole moment in (4) is generalized to a sum over all charges,

$$\mathbf{p} = \sum_{\text{all charges}} q_i \mathbf{r}_i \quad (7)$$

where  $\mathbf{r}_i$  is the vector distance from an origin to the charge  $q_i$  as in Figure 3-3. When the net charge in the system is zero ( $\sum q_i = 0$ ), the dipole moment is independent of the choice of origins for if we replace  $\mathbf{r}_i$  in (7) by  $\mathbf{r}_i + \mathbf{r}_0$ , where  $\mathbf{r}_0$  is the constant vector distance between two origins:

$$\begin{aligned} \mathbf{p} &= \sum q_i (\mathbf{r}_i + \mathbf{r}_0) \\ &= \sum q_i \mathbf{r}_i + \mathbf{r}_0 \sum q_i \\ &= \sum q_i \mathbf{r}_i \end{aligned} \quad (8)$$

The result is unchanged from (7) as the constant  $\mathbf{r}_0$  could be taken outside the summation.

If we have a continuous distribution of charge (7) is further generalized to

$$\mathbf{p} = \int_{\text{all } q} \mathbf{r} dq \quad (9)$$

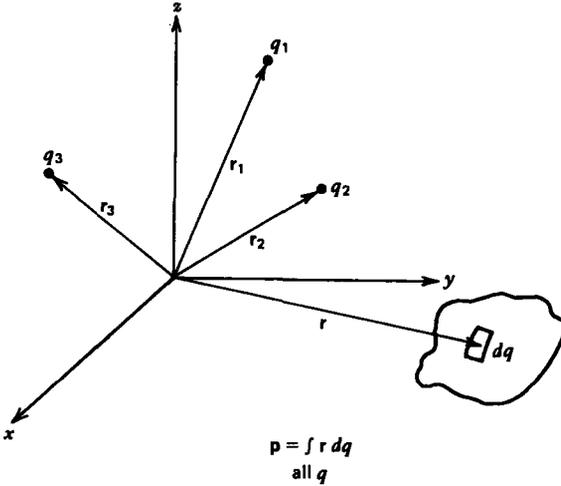


Figure 3-3 The dipole moment can be defined for any distribution of charge. If the net charge in the system is zero, the dipole moment is independent of the location of the origin.

Then the potential and electric field far away from any dipole distribution is given by the coordinate independent expressions in (3) and (5) where the dipole moment  $\mathbf{p}$  is given by (7) and (9).

### 3-1-2 Polarization Charge

We enclose a large number of dipoles within a dielectric medium with the differential-sized rectangular volume  $\Delta x \Delta y \Delta z$  shown in Figure 3-4a. All totally enclosed dipoles, being charge neutral, contribute no net charge within the volume. Only those dipoles within a distance  $\mathbf{d} \cdot \mathbf{n}$  of each surface are cut by the volume and thus contribute a net charge where  $\mathbf{n}$  is the unit normal to the surface at each face, as in Figure 3-4b. If the number of dipoles per unit volume is  $N$ , it is convenient to define the number density of dipoles as the polarization vector  $\mathbf{P}$ :

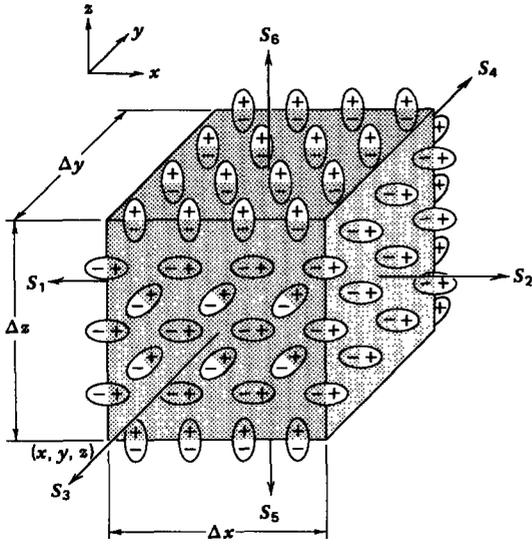
$$\mathbf{P} = N\mathbf{p} = Nq\mathbf{d} \quad (10)$$

The net charge enclosed near surface 1 is

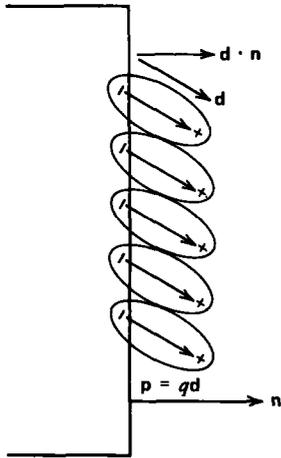
$$dq_1 = (Nqd_x)_{|x} \Delta y \Delta z = P_x(x) \Delta y \Delta z \quad (11)$$

while near the opposite surface 2

$$dq_2 = -(Nqd_x)_{|x+\Delta x} \Delta y \Delta z = -P_x(x + \Delta x) \Delta y \Delta z \quad (12)$$



(a)



(b)

Figure 3-4 (a) The net charge enclosed within a differential-sized volume of dipoles has contributions only from the dipoles that are cut by the surfaces. All totally enclosed dipoles contribute no net charge. (b) Only those dipoles within a distance  $d \cdot n$  of the surface are cut by the volume.

where we assume that  $\Delta y$  and  $\Delta z$  are small enough that the polarization  $\mathbf{P}$  is essentially constant over the surface. The polarization can differ at surface 1 at coordinate  $x$  from that at surface 2 at coordinate  $x + \Delta x$  if either the number density

$N$ , the charge  $q$ , or the displacement  $\mathbf{d}$  is a function of  $\mathbf{x}$ . The difference in sign between (11) and (12) is because near  $S_1$  the positive charge is within the volume, while near  $S_2$  negative charge remains in the volume. Note also that only the component of  $\mathbf{d}$  normal to the surface contributes to the volume of net charge.

Similarly, near the surfaces  $S_3$  and  $S_4$  the net charge enclosed is

$$\begin{aligned} dq_3 &= (Nqd_y)_{|y} \Delta x \Delta z = P_y(y) \Delta x \Delta z \\ dq_4 &= -(Nqd_y)_{|y+\Delta y} \Delta x \Delta z = -P_y(y+\Delta y) \Delta x \Delta z \end{aligned} \quad (13)$$

while near the surfaces  $S_5$  and  $S_6$  with normal in the  $z$  direction the net charge enclosed is

$$\begin{aligned} dq_5 &= (Nqd_z)_{|z} \Delta x \Delta y = P_z(z) \Delta x \Delta y \\ dq_6 &= -(Nqd_z)_{|z+\Delta z} \Delta x \Delta y = -P_z(z+\Delta z) \Delta x \Delta y \end{aligned} \quad (14)$$

The total charge enclosed within the volume is the sum of (11)–(14):

$$\begin{aligned} dq_T &= dq_1 + dq_2 + dq_3 + dq_4 + dq_5 + dq_6 \\ &= \left( \frac{P_x(x) - P_x(x + \Delta x)}{\Delta x} + \frac{P_y(y) - P_y(y + \Delta y)}{\Delta y} \right. \\ &\quad \left. + \frac{P_z(z) - P_z(z + \Delta z)}{\Delta z} \right) \Delta x \Delta y \Delta z \end{aligned} \quad (15)$$

As the volume shrinks to zero size, the polarization terms in (15) define partial derivatives so that the polarization volume charge density is

$$\rho_p = \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0 \\ \Delta z \rightarrow 0}} \frac{dq_T}{\Delta x \Delta y \Delta z} = - \left( \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z} \right) = -\nabla \cdot \mathbf{P} \quad (16)$$

This volume charge is also a source of the electric field and needs to be included in Gauss's law

$$\nabla \cdot (\epsilon_0 \mathbf{E}) = \rho_f + \rho_p = \rho_f - \nabla \cdot \mathbf{P} \quad (17)$$

where we subscript the free charge  $\rho_f$  with the letter  $f$  to distinguish it from the polarization charge  $\rho_p$ . The total polarization charge within a region is obtained by integrating (16) over the volume,

$$q_p = \int_V \rho_p dV = - \int_V \nabla \cdot \mathbf{P} dV = - \oint_S \mathbf{P} \cdot d\mathbf{S} \quad (18)$$

where we used the divergence theorem to relate the polarization charge to a surface integral of the polarization vector.

### 3-1-3 The Displacement Field

Since we have no direct way of controlling the polarization charge, it is convenient to cast Gauss's law only in terms of free charge by defining a new vector  $\mathbf{D}$  as

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad (19)$$

This vector  $\mathbf{D}$  is called the displacement field because it differs from  $\epsilon_0 \mathbf{E}$  due to the slight charge displacements in electric dipoles. Using (19), (17) can be rewritten as

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \nabla \cdot \mathbf{D} = \rho_f \quad (20)$$

where  $\rho_f$  only includes the free charge and not the bound polarization charge. By integrating both sides of (20) over a volume and using the divergence theorem, the new integral form of Gauss's law is

$$\int_V \nabla \cdot \mathbf{D} \, dV = \oint_S \mathbf{D} \cdot \mathbf{dS} = \int_V \rho_f \, dV \quad (21)$$

In free space, the polarization  $\mathbf{P}$  is zero so that  $\mathbf{D} = \epsilon_0 \mathbf{E}$  and (20)–(21) reduce to the free space laws used in Chapter 2.

### 3-1-4 Linear Dielectrics

It is now necessary to find the constitutive law relating the polarization  $\mathbf{P}$  to the applied electric field  $\mathbf{E}$ . An accurate discussion would require the use of quantum mechanics, which is beyond the scope of this text. However, a simplified classical model can be used to help us qualitatively understand the most interesting case of a linear dielectric.

#### (a) Polarizability

We model the atom as a fixed positive nucleus with a surrounding uniform spherical negative electron cloud, as shown in Figure 3-5a. In the absence of an applied electric field, the dipole moment is zero because the center of charge for the electron cloud is coincident with the nucleus. More formally, we can show this using (9), picking our origin at the position of the nucleus:

$$\mathbf{p} = Q \int_0^0 \mathbf{i}_r \rho_0 r^3 \sin \theta \, dr \, d\theta \, d\phi \quad (22)$$

Since the radial unit vector  $\mathbf{i}_r$  changes direction in space, it is necessary to use Table 1-2 to write  $\mathbf{i}_r$  in terms of the constant Cartesian unit vectors:

$$\mathbf{i}_r = \sin \theta \cos \phi \mathbf{i}_x + \sin \theta \sin \phi \mathbf{i}_y + \cos \theta \mathbf{i}_z \quad (23)$$

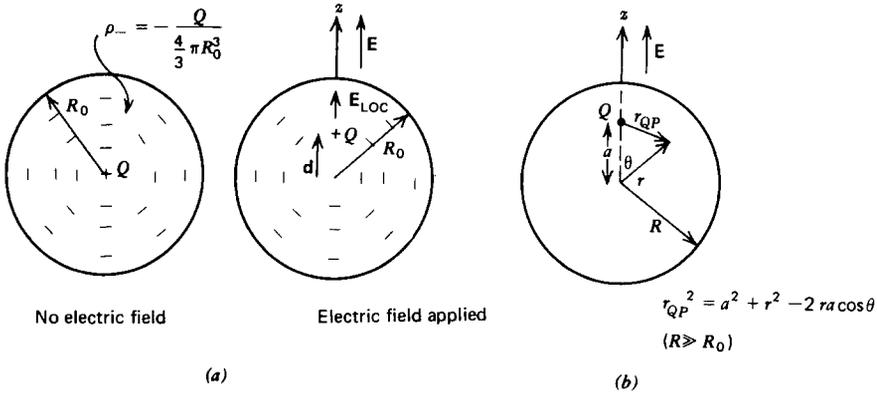


Figure 3-5 (a) A simple atomic classical model has a negative spherical electron cloud of small radius  $R_0$  centered about a positive nucleus when no external electric field is present. An applied electric field tends to move the positive charge in the direction of the field and the negative charge in the opposite direction creating an electric dipole. (b) The average electric field within a large sphere of radius  $R$  ( $R \gg R_0$ ) enclosing many point dipoles is found by superposing the average fields due to each point charge.

When (23) is used in (22) the  $x$  and  $y$  components integrate to zero when integrated over  $\phi$ , while the  $z$  component is zero when integrated over  $\theta$  so that  $\mathbf{p} = 0$ .

An applied electric field tends to push the positive charge in the direction of the field and the negative charge in the opposite direction causing a slight shift  $\mathbf{d}$  between the center of the spherical cloud and the positive nucleus, as in Figure 3-5a. Opposing this movement is the attractive coulombic force. Considering the center of the spherical cloud as our origin, the self-electric field within the cloud is found from Section 2.4.3b as

$$E_r = -\frac{Qr}{4\pi\epsilon_0 R_0^3} \tag{24}$$

In equilibrium the net force  $\mathbf{F}$  on the positive charge is zero,

$$\mathbf{F} = Q\left(\mathbf{E}_{Loc} - \frac{Q\mathbf{d}}{4\pi\epsilon_0 R_0^3}\right) = 0 \tag{25}$$

where we evaluate (24) at  $r = d$  and  $\mathbf{E}_{Loc}$  is the local polarizing electric field acting on the dipole. From (25) the equilibrium dipole spacing is

$$\mathbf{d} = \frac{4\pi\epsilon_0 R_0^3}{Q} \mathbf{E}_{Loc} \tag{26}$$

so that the dipole moment is written as

$$\mathbf{p} = Q\mathbf{d} = \alpha \mathbf{E}_{Loc}, \quad \alpha = 4\pi\epsilon_0 R_0^3 \tag{27}$$

where  $\alpha$  is called the polarizability.

**(b) The Local Electric Field**

If this dipole were isolated, the local electric field would equal the applied macroscopic field. However, a large number density  $N$  of neighboring dipoles also contributes to the polarizing electric field. The electric field changes drastically from point to point within a small volume containing many dipoles, being equal to the superposition of fields due to each dipole given by (5). The macroscopic field is then the average field over this small volume.

We calculate this average field by first finding the average field due to a single point charge  $Q$  a distance  $a$  along the  $z$  axis from the center of a spherical volume with radius  $R$  much larger than the radius of the electron cloud ( $R \gg R_0$ ) as in Figure 3-5*b*. The average field due to this charge over the spherical volume is

$$\langle \mathbf{E} \rangle = \frac{1}{\frac{4}{3}\pi R^3} \int_{r=0}^R \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{Q(r\mathbf{i}_r - a\mathbf{i}_z)r^2 \sin \theta \, dr \, d\theta \, d\phi}{4\pi\epsilon_0[a^2 + r^2 - 2ra \cos \theta]^{3/2}} \quad (28)$$

where we used the relationships

$$r_{QP}^2 = a^2 + r^2 - 2ra \cos \theta, \quad \mathbf{r}_{QP} = r\mathbf{i}_r - a\mathbf{i}_z \quad (29)$$

Using (23) in (28) again results in the  $x$  and  $y$  components being zero when integrated over  $\phi$ . Only the  $z$  component is now nonzero:

$$\langle E_z \rangle = \frac{Q}{\frac{4}{3}\pi R^3} \frac{2\pi}{(4\pi\epsilon_0)} \int_{\theta=0}^{\pi} \int_{r=0}^R \frac{r^3 (\cos \theta - a/r) \sin \theta \, dr \, d\theta}{[a^2 + r^2 - 2ra \cos \theta]^{3/2}} \quad (30)$$

We introduce the change of variable from  $\theta$  to  $u$

$$u = r^2 + a^2 - 2ar \cos \theta, \quad du = 2ar \sin \theta \, d\theta \quad (31)$$

so that (30) can be integrated over  $u$  and  $r$ . Performing the  $u$  integration first we have

$$\begin{aligned} \langle E_z \rangle &= \frac{3Q}{8\pi R^3 \epsilon_0} \int_{r=0}^R \int_{(r-a)^2}^{(r+a)^2} \frac{r}{4a^2} \frac{(r^2 - a^2 - u)}{u^{3/2}} \, dr \, du \\ &= \frac{3Q}{8\pi R^3 \epsilon_0} \int_{r=0}^R \frac{r}{4a^2} \left( -\frac{2(r^2 - a^2 + u)}{u^{1/2}} \right) \Big|_{u=(r-a)^2}^{(r+a)^2} \, dr \\ &= -\frac{3Q}{8\pi R^3 \epsilon_0 a^2} \int_{r=0}^R r^2 \left( 1 - \frac{r-a}{|r-a|} \right) \, dr \quad (32) \end{aligned}$$

We were careful to be sure to take the positive square root in the lower limit of  $u$ . Then for  $r > a$ , the integral is zero so

that the integral limits over  $r$  range from 0 to  $a$ :

$$\langle E_z \rangle = -\frac{3Q}{8\pi R^3 \epsilon_0 a^2} \int_{r=0}^a 2r^2 dr = \frac{-Qa}{4\pi\epsilon_0 R^3} \quad (33)$$

To form a dipole we add a negative charge  $-Q$ , a small distance  $d$  below the original charge. The average electric field due to the dipole is then the superposition of (33) for both charges:

$$\langle E_z \rangle = -\frac{Q}{4\pi\epsilon_0 R^3} [a - (a-d)] = -\frac{Qd}{4\pi\epsilon_0 R^3} = -\frac{p}{4\pi\epsilon_0 R^3} \quad (34)$$

If we have a number density  $N$  of such dipoles within the sphere, the total number of dipoles enclosed is  $\frac{4}{3}\pi R^3 N$  so that superposition of (34) gives us the average electric field due to all the dipoles in terms of the polarization vector  $\mathbf{P} = N\mathbf{p}$ :

$$\langle \mathbf{E} \rangle = -\frac{\frac{4}{3}\pi R^3 N\mathbf{p}}{4\pi\epsilon_0 R^3} = -\frac{\mathbf{P}}{3\epsilon_0} \quad (35)$$

The total macroscopic field is then the sum of the local field seen by each dipole and the average resulting field due to all the dipoles

$$\mathbf{E} = \langle \mathbf{E} \rangle + \mathbf{E}_{\text{Loc}} = -\frac{\mathbf{P}}{3\epsilon_0} + \mathbf{E}_{\text{Loc}} \quad (36)$$

so that the polarization  $\mathbf{P}$  is related to the macroscopic electric field from (27) as

$$\mathbf{P} = N\mathbf{p} = N\alpha\mathbf{E}_{\text{Loc}} = N\alpha\left(\mathbf{E} + \frac{\mathbf{P}}{3\epsilon_0}\right) \quad (37)$$

which can be solved for  $\mathbf{P}$  as

$$\mathbf{P} = \frac{N\alpha}{1 - N\alpha/3\epsilon_0} \mathbf{E} = \chi_e \epsilon_0 \mathbf{E}, \quad \chi_e = \frac{N\alpha/\epsilon_0}{1 - N\alpha/3\epsilon_0} \quad (38)$$

where we introduce the electric susceptibility  $\chi_e$  as the proportionality constant between  $\mathbf{P}$  and  $\epsilon_0 \mathbf{E}$ . Then, use of (38) in (19) relates the displacement field  $\mathbf{D}$  linearly to the electric field:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 (1 + \chi_e) \mathbf{E} = \epsilon_0 \epsilon_r \mathbf{E} = \epsilon \mathbf{E} \quad (39)$$

where  $\epsilon_r = 1 + \chi_e$  is called the relative dielectric constant and  $\epsilon = \epsilon_r \epsilon_0$  is the permittivity of the dielectric, also simply called the dielectric constant. In free space the susceptibility is zero ( $\chi_e = 0$ ) so that  $\epsilon_r = 1$  and the permittivity is that of free space,  $\epsilon = \epsilon_0$ . The last relation in (39) is usually the most convenient to use as all the results of Chapter 2 are also correct within

linear dielectrics if we replace  $\epsilon_0$  by  $\epsilon$ . Typical values of relative permittivity are listed in Table 3-1 for various common substances. High dielectric constant materials are usually composed of highly polar molecules.

**Table 3-1 The relative permittivity for various common substances at room temperature**

	$\epsilon_r = \epsilon/\epsilon_0$
Carbon Tetrachloride <sup>a</sup>	2.2
Ethanol <sup>a</sup>	24
Methanol <sup>a</sup>	33
<i>n</i> -Hexane <sup>a</sup>	1.9
Nitrobenzene <sup>a</sup>	35
Pure Water <sup>a</sup>	80
Barium Titanate <sup>b</sup> (with 20% Strontium Titanate)	>2100
Borosilicate Glass <sup>b</sup>	4.0
Ruby Mica (Muscovite) <sup>b</sup>	5.4
Polyethylene <sup>b</sup>	2.2
Polyvinyl Chloride <sup>b</sup>	6.1
Teflon <sup>b</sup> (Polytetrafluoroethylene)	2.1
Plexiglas <sup>b</sup>	3.4
Paraffin Wax <sup>b</sup>	2.2

<sup>a</sup> From Lange's Handbook of Chemistry, 10th ed., McGraw-Hill, New York, 1961, pp. 1234-37.

<sup>b</sup> From A. R. von Hippel (Ed.) Dielectric Materials and Applications, M.I.T., Cambridge, Mass., 1966, pp. 301-370

The polarizability and local electric field were only introduced so that we could relate microscopic and macroscopic fields. For most future problems we will describe materials by their permittivity  $\epsilon$  because this constant is most easily measured. The polarizability is then easily found as

$$\epsilon - \epsilon_0 = \frac{N\alpha}{1 - N\alpha/3\epsilon_0} \Rightarrow \frac{N\alpha}{3\epsilon_0} = \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \quad (40)$$

It then becomes simplest to work with the field vectors  $\mathbf{D}$  and  $\mathbf{E}$ . The polarization can always be obtained if needed from the definition

$$\mathbf{P} = \mathbf{D} - \epsilon_0\mathbf{E} = (\epsilon - \epsilon_0)\mathbf{E} \quad (41)$$

### EXAMPLE 3-1 POINT CHARGE WITHIN A DIELECTRIC SPHERE

Find all the fields and charges due to a point charge  $q$  within a linear dielectric sphere of radius  $R$  and permittivity  $\epsilon$  surrounded by free space, as in Figure 3-6.

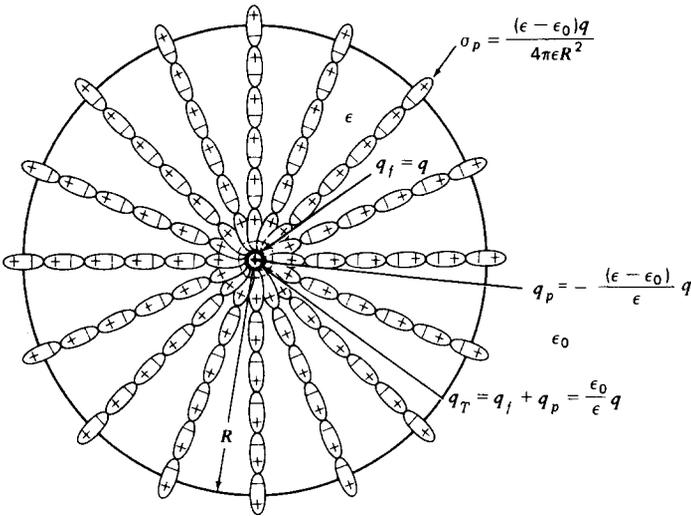


Figure 3-6 The electric field due to a point charge within a dielectric sphere is less than the free space field because of the partial neutralization of the point charge by the accumulation of dipole ends of opposite charge. The total polarization charge on the sphere remains zero as an equal magnitude but opposite sign polarization charge appears at the spherical interface.

**SOLUTION**

Applying Gauss's law of (21) to a sphere of any radius  $r$  whether inside or outside the sphere, the enclosed free charge is always  $q$ :

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = D_r 4\pi r^2 = q \Rightarrow D_r = \frac{q}{4\pi r^2} \quad \text{all } r$$

The electric field is then discontinuous at  $r = R$ ,

$$E_r = \begin{cases} \frac{D_r}{\epsilon} = \frac{q}{4\pi\epsilon r^2}, & r < R \\ \frac{D_r}{\epsilon_0} = \frac{q}{4\pi\epsilon_0 r^2}, & r > R \end{cases}$$

due to the abrupt change of permittivities. The polarization field is

$$P_r = D_r - \epsilon_0 E_r = \begin{cases} \frac{(\epsilon - \epsilon_0)q}{4\pi\epsilon r^2}, & r < R \\ 0, & r > R \end{cases}$$

The volume polarization charge  $\rho_p$  is zero everywhere,

$$\rho_p = -\nabla \cdot \mathbf{P} = -\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 P_r) = 0, \quad 0 < r < R$$

except at  $r=0$  where a point polarization charge is present, and at  $r=R$  where we have a surface polarization charge found by using (18) for concentric Gaussian spheres of radius  $r$  inside and outside the dielectric sphere:

$$q_p = -\oint_S \mathbf{P} \cdot d\mathbf{S} = -P_r 4\pi r^2 = \begin{cases} -(\epsilon - \epsilon_0)q/\epsilon, & r < R \\ 0, & r > R \end{cases}$$

We know that for  $r < R$  this polarization charge must be a point charge at the origin as there is no volume charge contribution yielding a total point charge at the origin:

$$q_T = q_p + q = \frac{\epsilon_0}{\epsilon} q$$

This reduction of net charge is why the electric field within the sphere is less than the free space value. The opposite polarity ends of the dipoles are attracted to the point charge, partially neutralizing it. The total polarization charge enclosed by the sphere with  $r > R$  is zero as there is an opposite polarity surface polarization charge at  $r=R$  with density,

$$\sigma_p = \frac{(\epsilon - \epsilon_0)q}{4\pi\epsilon R^2}$$

The total surface charge  $\sigma_p 4\pi R^2 = (\epsilon - \epsilon_0)q/\epsilon$  is equal in magnitude but opposite in sign to the polarization point charge at the origin. The total polarization charge always sums to zero.

### 3-1-5 Spontaneous Polarization

#### (a) Ferro-electrics

Examining (38) we see that when  $N\alpha/3\epsilon_0 = 1$  the polarization can be nonzero even if the electric field is zero. We can just meet this condition using the value of polarizability in (27) for electronic polarization if the whole volume is filled with contacting dipole spheres of the type in Figure 3-5a so that we have one dipole for every volume of  $\frac{4}{3}\pi R_0^3$ . Then any slight fluctuation in the local electric field increases the polarization, which in turn increases the local field resulting in spontaneous polarization so that all the dipoles over a region are aligned. In a real material dipoles are not so

densely packed. Furthermore, more realistic statistical models including thermally induced motions have shown that most molecules cannot meet the conditions for spontaneous polarization.

However, some materials have sufficient additional contributions to the polarizabilities due to ionic and orientational polarization that the condition for spontaneous polarization is met. Such materials are called ferro-electrics, even though they are not iron compounds, because of their similarity in behavior to iron compound ferro-magnetic materials, which we discuss in Section 5.5.3*c*. Ferro-electrics are composed of permanently polarized regions, called domains, as illustrated in Figure 3-7*a*. In the absence of an electric field, these domains are randomly distributed so that the net macroscopic polarization field is zero. When an electric field is applied, the dipoles tend to align with the field so that domains with a polarization component along the field grow at the expense of nonaligned domains. Ferro-electrics typically have very high permittivities such as barium titanate listed in Table 3-1.

The domains do not respond directly with the electric field as domain orientation and growth is not a reversible process but involves losses. The polarization  $\mathbf{P}$  is then nonlinearly related to the electric field  $\mathbf{E}$  by the hysteresis curve shown in Figure 3-8. The polarization of an initially unpolarized sample increases with electric field in a nonlinear way until the saturation value  $\mathbf{P}_{\text{sat}}$  is reached when all the domains are completely aligned with the field. A further increase in  $\mathbf{E}$  does not increase  $\mathbf{P}$  as all the dipoles are completely aligned.

As the field decreases, the polarization does not retrace its path but follows a new path as the dipoles tend to stick to their previous positions. Even when the electric field is zero, a

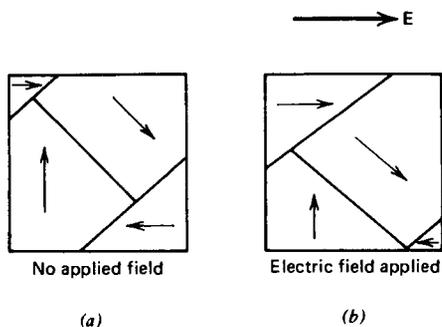


Figure 3-7 (a) In the absence of an applied electric field, a ferro-electric material generally has randomly distributed permanently polarized domains. Over a macroscopic volume, the net polarization due to all the domains is zero. (b) When an electric field is applied, domains with a polarization component in the direction of the field grow at the expense of nonaligned domains so that a net polarization can result.

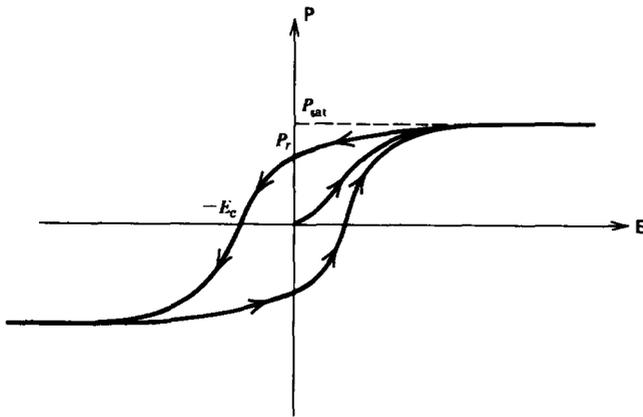


Figure 3-8 A typical ferro-electric hysteresis curve shows a saturation value  $P_{\text{sat}}$  when all the domains align with the field, a remanent polarization  $P_r$ , when the electric field is removed, and a negative coercive electric field  $-E_c$ , necessary to bring the polarization back to zero.

remanent polarization  $P_r$  remains. To bring the polarization to zero requires a negative coercive field  $-E_c$ . Further magnitude increases in negative electric field continues the symmetric hysteresis loop until a negative saturation is reached where all the dipoles have flipped over. If the field is now brought to zero and continued to positive field values, the whole hysteresis curve is traversed.

#### (b) Electrets

There are a class of materials called electrets that also exhibit a permanent polarization even in the absence of an applied electric field. Electrets are typically made using certain waxes or plastics that are heated until they become soft. They are placed within an electric field, tending to align the dipoles in the same direction as the electric field, and then allowed to harden. The dipoles are then frozen in place so that even when the electric field is removed a permanent polarization remains.

Other interesting polarization phenomena are:

**Electrostriction**—slight change in size of a dielectric due to the electrical force on the dipoles.

**Piezo-electricity**—when the electrostrictive effect is reversible so that a mechanical strain also creates a field.

**Pyro-electricity**—induced polarization due to heating or cooling.

3-2 CONDUCTION

3-2-1 Conservation of Charge

In contrast to dielectrics, most metals have their outermost band of electrons only weakly bound to the nucleus and are free to move in an applied electric field. In electrolytic solutions, ions of both sign are free to move. The flow of charge, called a current, is defined as the total charge flowing through a surface per unit time. In Figure 3-9a a single species of free charge with density  $\rho_f$  and velocity  $\mathbf{v}_i$  flows through a small differential sized surface  $d\mathbf{S}$ . The total charge flowing through this surface in a time  $\Delta t$  depends only on the velocity component perpendicular to the surface:

$$\Delta Q_i = \rho_f \Delta t \mathbf{v}_i \cdot d\mathbf{S} \tag{1}$$

The tangential component of the velocity parallel to the surface  $d\mathbf{S}$  only results in charge flow along the surface but not through it. The total differential current through  $d\mathbf{S}$  is then defined as

$$dI_i = \frac{\Delta Q_i}{\Delta t} = \rho_f \mathbf{v}_i \cdot d\mathbf{S} = \mathbf{J}_f \cdot d\mathbf{S} \text{ ampere} \tag{2}$$

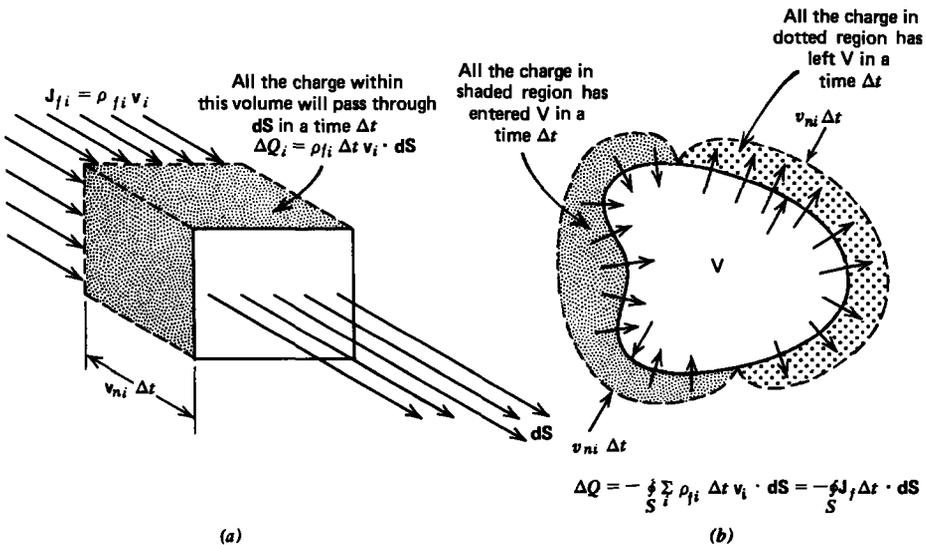


Figure 3-9 The current through a surface is defined as the number of charges per second passing through the surface. (a) The current is proportional to the component of charge velocity perpendicular to the surface. (b) The net change of total charge within a volume is equal to the difference of the charge entering to that leaving in a small time  $\Delta t$ .

where the free current density of these charges  $\mathbf{J}_f$  is a vector and is defined as

$$\mathbf{J}_f = \rho_f \mathbf{v}_i \text{ amp/m}^2 \quad (3)$$

If there is more than one type of charge carrier, the net charge density is equal to the algebraic sum of all the charge densities, while the net current density equals the vector sum of the current densities due to each carrier:

$$\rho_f = \sum \rho_{fi}, \quad \mathbf{J}_f = \sum \rho_{fi} \mathbf{v}_i \quad (4)$$

Thus, even if we have charge neutrality so that  $\rho_f = 0$ , a net current can flow if the charges move with different velocities. For example, two oppositely charged carriers with densities  $\rho_1 = -\rho_2 \equiv \rho_0$  moving with respective velocities  $\mathbf{v}_1$  and  $\mathbf{v}_2$  have

$$\rho_f = \rho_1 + \rho_2 = 0, \quad \mathbf{J}_f = \rho_1 \mathbf{v}_1 + \rho_2 \mathbf{v}_2 = \rho_0 (\mathbf{v}_1 - \mathbf{v}_2) \quad (5)$$

With  $\mathbf{v}_1 \neq \mathbf{v}_2$  a net current flows with zero net charge. This is typical in metals where the electrons are free to flow while the oppositely charged nuclei remain stationary.

The total current  $I$ , a scalar, flowing through a macroscopic surface  $S$ , is then just the sum of the total differential currents of all the charge carriers through each incremental size surface element:

$$I = \int_S \mathbf{J}_f \cdot d\mathbf{S} \quad (6)$$

Now consider the charge flow through the closed volume  $V$  with surface  $S$  shown in Figure 3-9b. A time  $\Delta t$  later, that charge within the volume near the surface with the velocity component outward will leave the volume, while that charge just outside the volume with a velocity component inward will just enter the volume. The difference in total charge is transported by the current:

$$\begin{aligned} \Delta Q &= \int_V [\rho_f(t + \Delta t) - \rho_f(t)] dV \\ &= -\oint_S \sum \rho_{fi} \mathbf{v}_i \Delta t \cdot d\mathbf{S} = -\oint_S \mathbf{J}_f \Delta t \cdot d\mathbf{S} \end{aligned} \quad (7)$$

The minus sign on the right is necessary because when  $\mathbf{v}_i$  is in the direction of  $d\mathbf{S}$ , charge has left the volume so that the enclosed charge decreases. Dividing (7) through by  $\Delta t$  and taking the limit as  $\Delta t \rightarrow 0$ , we use (3) to derive the integral conservation of charge equation:

$$\oint_S \mathbf{J}_f \cdot d\mathbf{S} + \int_V \frac{\partial \rho_f}{\partial t} dV = 0 \quad (8)$$

Using the divergence theorem, the surface integral can be converted to a volume integral:

$$\int_V \left[ \nabla \cdot \mathbf{J}_f + \frac{\partial \rho_f}{\partial t} \right] dV = 0 \Rightarrow \nabla \cdot \mathbf{J}_f + \frac{\partial \rho_f}{\partial t} = 0 \quad (9)$$

where the differential point form is obtained since the integral must be true for any volume so that the bracketed term must be zero at each point. From Gauss's law ( $\nabla \cdot \mathbf{D} = \rho_f$ ) (8) and (9) can also be written as

$$\oint_S \left( \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S} = 0, \quad \nabla \cdot \left( \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \right) = 0 \quad (10)$$

where  $\mathbf{J}_f$  is termed the conduction current density and  $\partial \mathbf{D} / \partial t$  is called the displacement current density.

This is the field form of Kirchoff's circuit current law that the algebraic sum of currents at a node sum to zero. Equation (10) equivalently tells us that the net flux of total current, conduction plus displacement, is zero so that all the current that enters a surface must leave it. The displacement current does not involve any charge transport so that time-varying current can be transmitted through space without charge carriers. Under static conditions, the displacement current is zero.

### 3-2-2 Charged Gas Conduction Models

#### (a) Governing Equations.

In many materials, including good conductors like metals, ionized gases, and electrolytic solutions as well as poorer conductors like lossy insulators and semiconductors, the charge carriers can be classically modeled as an ideal gas within the medium, called a plasma. We assume that we have two carriers of equal magnitude but opposite sign  $\pm q$  with respective masses  $m_{\pm}$  and number densities  $n_{\pm}$ . These charges may be holes and electrons in a semiconductor, oppositely charged ions in an electrolytic solution, or electrons and nuclei in a metal. When an electric field is applied, the positive charges move in the direction of the field while the negative charges move in the opposite direction. These charges collide with the host medium at respective frequencies  $\nu_+$  and  $\nu_-$ , which then act as a viscous or frictional dissipation opposing the motion. In addition to electrical and frictional forces, the particles exert a force on themselves through a pressure term due to thermal agitation that would be present even if the particles were uncharged. For an ideal gas the partial pressure  $p$  is

$$p = nkT \text{ Pascals } [\text{kg}\cdot\text{s}^{-2}\cdot\text{m}^{-1}] \quad (11)$$

where  $n$  is the number density of charges,  $T$  is the absolute temperature, and  $k = 1.38 \times 10^{-23}$  joule/°K is called Boltzmann's constant.

The net pressure force on the small rectangular volume shown in Figure 3-10 is

$$\mathbf{f}_p = \left( \frac{p(x - \Delta x) - p(x)}{\Delta x} \mathbf{i}_x + \frac{p(y) - p(y + \Delta y)}{\Delta y} \mathbf{i}_y + \frac{p(z) - p(z + \Delta z)}{\Delta z} \mathbf{i}_z \right) \Delta x \Delta y \Delta z \quad (12)$$

where we see that the pressure only exerts a net force on the volume if it is different on each opposite surface. As the volume shrinks to infinitesimal size, the pressure terms in (12) define partial derivatives so that the volume force density becomes

$$\lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0 \\ \Delta z \rightarrow 0}} \frac{\mathbf{f}_p}{\Delta x \Delta y \Delta z} = - \left( \frac{\partial p}{\partial x} \mathbf{i}_x + \frac{\partial p}{\partial y} \mathbf{i}_y + \frac{\partial p}{\partial z} \mathbf{i}_z \right) = -\nabla p \quad (13)$$

Then using (11)–(13), Newton's force law for each charge carrier within the small volume is

$$m_{\pm} \frac{\partial \mathbf{v}_{\pm}}{\partial t} = \pm q \mathbf{E} - m_{\pm} \nu_{\pm} \mathbf{v}_{\pm} - \frac{1}{n_{\pm}} \nabla(n_{\pm} k T) \quad (14)$$

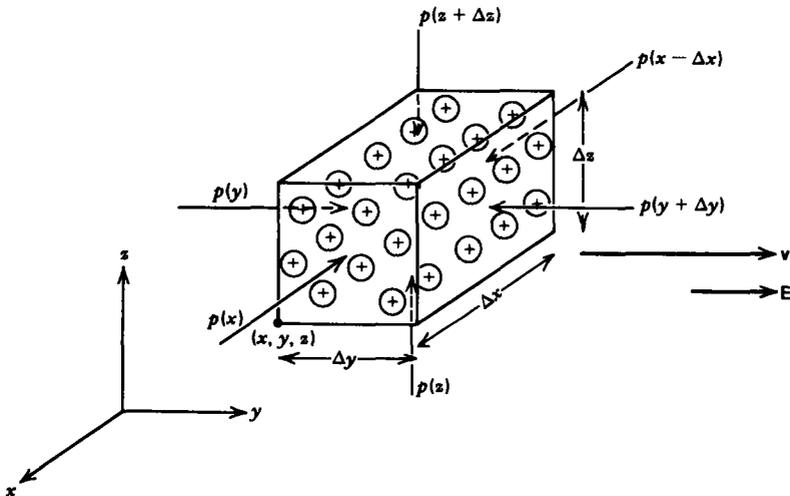


Figure 3-10 Newton's force law, applied to a small rectangular volume  $\Delta x \Delta y \Delta z$  moving with velocity  $\mathbf{v}$ , enclosing positive charges with number density  $n$ . The pressure is the force per unit area acting normally inward on each surface and only contributes to the net force if it is different on opposite faces.

where the electric field  $\mathbf{E}$  is due to the imposed field plus the field generated by the charges, as given by Gauss's law.

### (b) Drift-Diffusion Conduction

Because in many materials the collision frequencies are typically on the order of  $\nu \approx 10^{13}$  Hz, the inertia terms in (14) are often negligible. In this limit we can easily solve (14) for the velocity of each carrier as

$$\lim_{\partial v_{\pm}/\partial t \ll \nu_{\pm} v_{\pm}} v_{\pm} = \frac{1}{m_{\pm} \nu_{\pm}} \left( \pm q \mathbf{E} - \frac{1}{n_{\pm}} \nabla(n_{\pm} k T) \right) \quad (15)$$

The charge and current density for each carrier are simply given as

$$\rho_{\pm} = \pm q n_{\pm}, \quad \mathbf{J}_{\pm} = \rho_{\pm} v_{\pm} = \pm q n_{\pm} v_{\pm} \quad (16)$$

Multiplying (15) by the charge densities then gives us the constitutive law for each current as

$$\mathbf{J}_{\pm} = \pm q n_{\pm} v_{\pm} = \pm \rho_{\pm} \mu_{\pm} \mathbf{E} - D_{\pm} \nabla \rho_{\pm} \quad (17)$$

where  $\mu_{\pm}$  are called the particle mobilities and  $D_{\pm}$  are their diffusion coefficients

$$\mu_{\pm} = \frac{q}{m_{\pm} \nu_{\pm}} [\text{A} \cdot \text{kg}^{-1} \cdot \text{s}^{-2}], \quad D_{\pm} = \frac{k T}{m_{\pm} \nu_{\pm}} [\text{m}^2 \cdot \text{s}^{-1}] \quad (18)$$

assuming that the system is at constant temperature. We see that the ratio  $D_{\pm}/\mu_{\pm}$  for each carrier is the same having units of voltage, thus called the thermal voltage:

$$\frac{D_{\pm}}{\mu_{\pm}} = \frac{k T}{q} \text{ volts } [\text{kg} \cdot \text{m}^2 \cdot \text{A}^{-1} \cdot \text{s}^{-3}] \quad (19)$$

This equality is known as Einstein's relation.

In equilibrium when the net current of each carrier is zero, (17) can be written in terms of the potential as ( $\mathbf{E} = -\nabla V$ )

$$\mathbf{J}_{+} = \mathbf{J}_{-} = 0 = -\rho_{\pm} \mu_{\pm} \nabla V \mp D_{\pm} \nabla \rho_{\pm} \quad (20)$$

which can be rewritten as

$$\nabla \left[ \pm \frac{\mu_{\pm}}{D_{\pm}} V + \ln \rho_{\pm} \right] = 0 \quad (21)$$

The bracketed term can then only be a constant, so the charge density is related to the potential by the Boltzmann distribution:

$$\rho_{\pm} = \pm \rho_0 e^{\mp q V / k T} \quad (22)$$

where we use the Einstein relation of (19) and  $\pm \rho_0$  is the equilibrium charge density of each carrier when  $V = 0$  and are of equal magnitude because the system is initially neutral.

To find the spatial dependence of  $\rho$  and  $V$  we use (22) in Poisson's equation derived in Section 2.5.6:

$$\nabla^2 V = -\frac{(\rho_+ + \rho_-)}{\epsilon} = -\frac{\rho_0}{\epsilon} (e^{-qV/kT} - e^{qV/kT}) = \frac{2\rho_0}{\epsilon} \sinh \frac{qV}{kT} \tag{23}$$

This equation is known as the Poisson-Boltzmann equation because the charge densities obey Boltzmann distributions.

Consider an electrode placed at  $x = 0$  raised to the potential  $V_0$  with respect to a zero potential at  $x = \pm\infty$ , as in Figure 3-11a. Because the electrode is long, the potential only varies

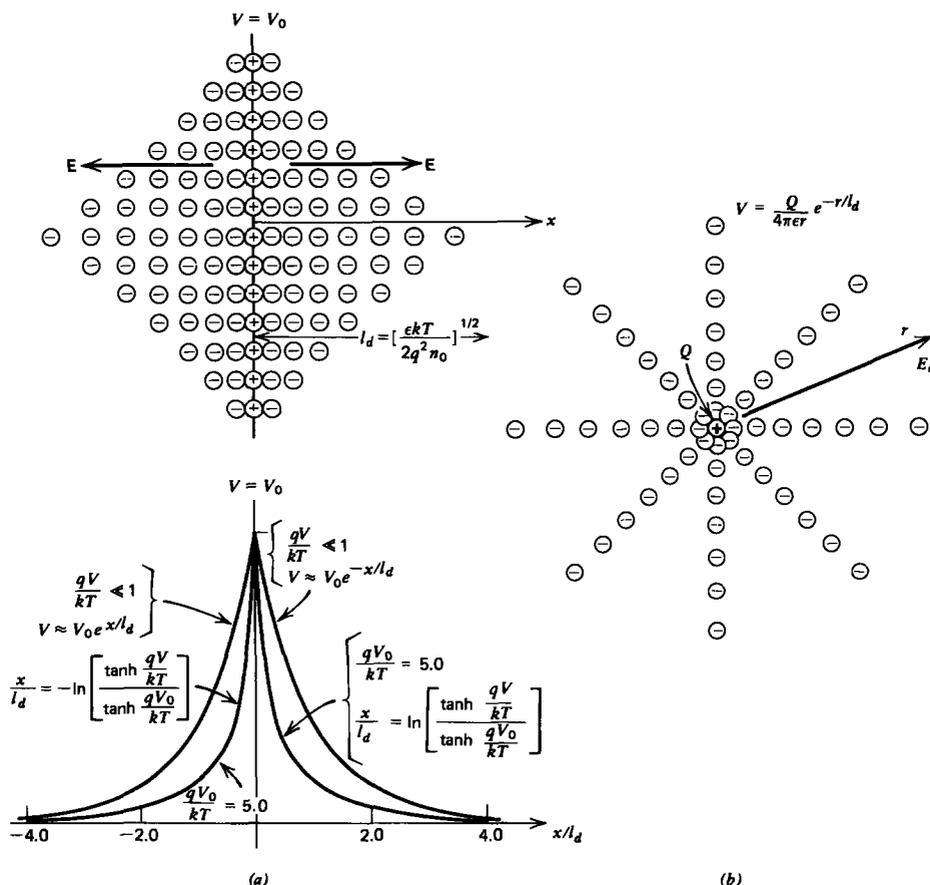


Figure 3-11 Opposite polarity mobile charges accumulate around any net charge inserted into a conductor described by the drift-diffusion equations, and tend to shield out its field for distances larger than the Debye length. (a) Electrode at potential  $V_0$  with respect to a zero potential at  $x = \pm\infty$ . The spatial decay is faster for larger values of  $V_0$ . (b) Point charge.

with the  $x$  coordinate so that (23) becomes

$$\frac{d^2 \tilde{V}}{dx^2} - \frac{1}{l_d^2} \sinh \tilde{V} = 0, \quad \tilde{V} = \frac{qV}{kT}, \quad l_d^2 = \frac{\epsilon kT}{2\rho_0 q} \quad (24)$$

where we normalize the voltage to the thermal voltage  $kT/q$  and  $l_d$  is called the Debye length.

If (24) is multiplied by  $d\tilde{V}/dx$ , it can be written as an exact differential:

$$\frac{d}{dx} \left[ \frac{1}{2} \left( \frac{d\tilde{V}}{dx} \right)^2 - \frac{\cosh \tilde{V}}{l_d^2} \right] = 0 \quad (25)$$

The bracketed term must then be a constant that is evaluated far from the electrode where the potential and electric field  $\tilde{E}_x = -d\tilde{V}/dx$  are zero:

$$\frac{d\tilde{V}}{dx} = -\tilde{E}_x = \left[ \frac{2}{l_d^2} (\cosh \tilde{V} - 1) \right]^{1/2} = \mp \frac{2}{l_d} \sinh \frac{\tilde{V}}{2} \begin{cases} x > 0 \\ x < 0 \end{cases} \quad (26)$$

The different signs taken with the square root are necessary because the electric field points in opposite directions on each side of the electrode. The potential is then implicitly found by direct integration as

$$\frac{\tanh(\tilde{V}/4)}{\tanh(\tilde{V}_0/4)} = e^{\mp x/l_d} \begin{cases} x > 0 \\ x < 0 \end{cases} \quad (27)$$

The Debye length thus describes the characteristic length over which the applied potential exerts influence. In many materials the number density of carriers is easily of the order of  $n_0 \approx 10^{20}/\text{m}^3$ , so that at room temperature ( $T \approx 293^\circ\text{K}$ ),  $l_d$  is typically  $10^{-7}$  m.

Often the potentials are very small so that  $qV/kT \ll 1$ . Then, the hyperbolic terms in (27), as well as in the governing equation of (23), can be approximated by their arguments:

$$\nabla^2 V - \frac{V}{l_d^2} = 0 \quad (28)$$

This approximation is only valid when the potentials are much less than the thermal voltage  $kT/q$ , which, at room temperature is about 25 mv. In this limit the solution of (27) shows that the voltage distribution decreases exponentially. At higher values of  $V_0$ , the decay is faster, as shown in Figure 3-11a.

If a point charge  $Q$  is inserted into the plasma medium, as in Figure 3-11b, the potential only depends on the radial distance  $r$ . In the small potential limit, (28) in spherical coordinates is

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) - \frac{V}{l_d^2} = 0 \quad (29)$$

Realizing that this equation can be rewritten as

$$\frac{\partial^2}{\partial r^2} (rV) - \frac{(rV)}{l_d^2} = 0 \quad (30)$$

we have a linear constant coefficient differential equation in the variable  $(rV)$  for which solutions are

$$rV = A_1 e^{-r/l_d} + A_2 e^{+r/l_d} \quad (31)$$

Because the potential must decay and not grow far from the charge,  $A_2 = 0$  and the solution is

$$V = \frac{Q}{4\pi\epsilon r} e^{-r/l_d} \quad (32)$$

where we evaluated  $A_1$  by realizing that as  $r \rightarrow 0$  the potential must approach that of an isolated point charge. Note that for small  $r$  the potential becomes very large and the small potential approximation is violated.

### (c) Ohm's Law

We have seen that the mobile charges in a system described by the drift-diffusion equations accumulate near opposite polarity charge and tend to shield out its effect for distances larger than the Debye length. Because this distance is usually so much smaller than the characteristic system dimensions, most regions of space outside the Debye sheath are charge neutral with equal amounts of positive and negative charge density  $\pm\rho_0$ . In this region, the diffusion term in (17) is negligible because there are no charge density gradients. Then the total current density is proportional to the electric field:

$$\mathbf{J} = \mathbf{J}_+ + \mathbf{J}_- = \rho_0(\mathbf{v}_+ - \mathbf{v}_-) = qn_0(\mu_+ + \mu_-)\mathbf{E} = \sigma\mathbf{E} \quad (33)$$

where  $\sigma$  [siemens/m ( $\text{m}^{-3}\text{-kg}^{-1}\text{-s}^3\text{-A}^2$ )] is called the Ohmic conductivity and (33) is the point form of Ohm's law. Sometimes it is more convenient to work with the reciprocal conductivity  $\rho_r = (1/\sigma)$  (ohm-m) called the resistivity. We will predominantly use Ohm's law to describe most media in this text, but it is important to remember that it is often not valid within the small Debye distances near charges. When Ohm's law is valid, the net charge is zero, thus giving no contribution to Gauss's law. Table 3-2 lists the Ohmic conductivities for various materials. We see that different materials vary over wide ranges in their ability to conduct charges.

The Ohmic conductivity of "perfect conductors" is large and is idealized to be infinite. Since all physical currents in (33) must remain finite, the electric field within the conductor

is zero so that it imposes an equipotential surface:

$$\lim_{\sigma \rightarrow \infty} \mathbf{J} = \sigma \mathbf{E} \Rightarrow \begin{cases} \mathbf{E} = 0 \\ V = \text{const} \\ \mathbf{J} = \text{finite} \end{cases} \quad (34)$$

**Table 3-2 The Ohmic conductivity for various common substances at room temperature**

	$\sigma$ [siemen/m]
Silver <sup>a</sup>	$6.3 \times 10^7$
Copper <sup>a</sup>	$5.9 \times 10^7$
Gold <sup>a</sup>	$4.2 \times 10^7$
Lead <sup>a</sup>	$0.5 \times 10^7$
Tin <sup>a</sup>	$0.9 \times 10^7$
Zinc <sup>a</sup>	$1.7 \times 10^7$
Carbon <sup>a</sup>	$7.3 \times 10^{-4}$
Mercury <sup>b</sup>	$1.06 \times 10^6$
Pure Water <sup>b</sup>	$4 \times 10^{-6}$
Nitrobenzene <sup>b</sup>	$5 \times 10^{-7}$
Methanol <sup>b</sup>	$4 \times 10^{-5}$
Ethanol <sup>b</sup>	$1.3 \times 10^{-7}$
Hexane <sup>b</sup>	$< 1 \times 10^{-18}$

<sup>a</sup> From Handbook of Chemistry and Physics, 49th ed., The Chemical Rubber Co., 1968, p. E80.

<sup>b</sup> From Lange's Handbook of Chemistry, 10th ed., McGraw-Hill, New York, 1961, pp. 1220-21.

Throughout this text electrodes are generally assumed to be perfectly conducting and thus are at a constant potential. The external electric field must then be incident perpendicularly to the surface.

#### (d) Superconductors

One notable exception to Ohm's law is for superconducting materials at cryogenic temperatures. Then, with collisions negligible ( $\nu_{\pm} = 0$ ) and the absolute temperature low ( $T \approx 0$ ), the electrical force on the charges is only balanced by their inertia so that (14) becomes simply

$$\frac{\partial \mathbf{v}_{\pm}}{\partial t} = \pm \frac{q}{m_{\pm}} \mathbf{E} \quad (35)$$

We multiply (35) by the charge densities that we assume to be constant so that the constitutive law relating the current

density to the electric field is

$$\frac{\partial(\pm qn_{\pm} v_{\pm})}{\partial t} = \frac{\partial J_{\pm}}{\partial t} = \frac{q^2 n_{\pm}}{m_{\pm}} \mathbf{E} = \omega_{p_{\pm}}^2 \epsilon \mathbf{E}, \quad \omega_{p_{\pm}}^2 = \frac{q^2 n_{\pm}}{m_{\pm} \epsilon} \quad (36)$$

where  $\omega_{p_{\pm}}$  is called the plasma frequency for each carrier.

For electrons ( $q = -1.6 \times 10^{-19}$  coul,  $m_- = 9.1 \times 10^{-31}$  kg) of density  $n_- \approx 10^{20}/\text{m}^3$  within a material with the permittivity of free space,  $\epsilon = \epsilon_0 \approx 8.854 \times 10^{-12}$  farad/m, the plasma frequency is

$$\begin{aligned} \omega_{p_-} &= \sqrt{q^2 n_- / m_- \epsilon} \approx 5.6 \times 10^{11} \text{ radian/sec} \\ \Rightarrow f_{p_-} &= \omega_{p_-} / 2\pi \approx 9 \times 10^{10} \text{ Hz} \end{aligned} \quad (37)$$

If such a material is placed between parallel plate electrodes that are open circuited, the electric field and current density  $\mathbf{J} = \mathbf{J}_+ + \mathbf{J}_-$  must be perpendicular to the electrodes, which we take as the  $x$  direction. If the electrode spacing is small compared to the width, the interelectrode fields far from the ends must then be  $x$  directed and be only a function of  $x$ . Then the time derivative of the charge conservation equation in (10) is

$$\frac{\partial}{\partial x} \left[ \frac{\partial}{\partial t} (J_+ + J_-) + \epsilon \frac{\partial^2 E}{\partial t^2} \right] = 0 \quad (38)$$

The bracketed term is just the time derivative of the total current density, which is zero because the electrodes are open circuited so that using (36) in (38) yields

$$\frac{\partial^2 E}{\partial t^2} + \omega_p^2 E = 0, \quad \omega_p^2 = \omega_{p_+}^2 + \omega_{p_-}^2 \quad (39)$$

which has solutions

$$E = A_1 \sin \omega_p t + A_2 \cos \omega_p t \quad (40)$$

Any initial perturbation causes an oscillatory electric field at the composite plasma frequency  $\omega_p$ . The charges then execute simple harmonic motion about their equilibrium position.

### 3-3 FIELD BOUNDARY CONDITIONS

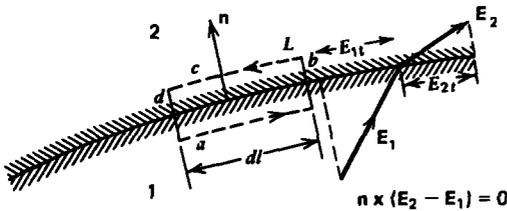
In many problems there is a surface of discontinuity separating dissimilar materials, such as between a conductor and a dielectric, or between different dielectrics. We must determine how the fields change as we cross the interface where the material properties change abruptly.

3-3-1 Tangential Component of  $\mathbf{E}$

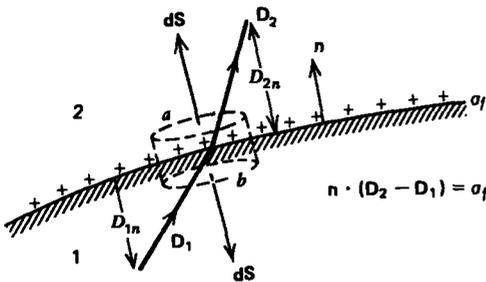
We apply the line integral of the electric field around a contour of differential size enclosing the interface between dissimilar materials, as shown in Figure 3-12a. The long sections  $a$  and  $c$  of length  $dl$  are tangential to the surface and the short joining sections  $b$  and  $d$  are of zero length as the interface is assumed to have zero thickness. Applying the line integral of the electric field around this contour, from Section 2.5.6 we obtain

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = (E_{1t} - E_{2t}) dl = 0 \tag{1}$$

where  $E_{1t}$  and  $E_{2t}$  are the components of the electric field tangential to the interface. We get no contribution from the normal components of field along sections  $b$  and  $d$  because the contour lengths are zero. The minus sign arises along  $c$  because the electric field is in the opposite direction of the contour traversal. We thus have that the tangential



(a)



(b)

Figure 3-12 (a) Stokes' law applied to a line integral about an interface of discontinuity shows that the tangential component of electric field is continuous across the boundary. (b) Gauss's law applied to a pill-box volume straddling the interface shows that the normal component of displacement vector is discontinuous in the free surface charge density  $\sigma_f$ .

components of the electric field are continuous across the interface

$$E_{1t} = E_{2t} \Rightarrow \mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0 \quad (2)$$

where  $\mathbf{n}$  is the interfacial normal shown in Figure 3-12a.

Within a perfect conductor the electric field is zero. Therefore, from (2) we know that the tangential component of  $\mathbf{E}$  outside the conductor is also zero. Thus the electric field must always terminate perpendicularly to a perfect conductor.

### 3-3-2 Normal Component of $\mathbf{D}$

We generalize the results of Section 2.4.6 to include dielectric media by again choosing a small Gaussian surface whose upper and lower surfaces of area  $dS$  are parallel to a surface charged interface and are joined by an infinitely thin cylindrical surface with zero area, as shown in Figure 3-12b. Then only faces  $a$  and  $b$  contribute to Gauss's law:

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = (D_{2n} - D_{1n}) dS = \sigma_f dS \quad (3)$$

where the interface supports a free surface charge density  $\sigma_f$  and  $D_{2n}$  and  $D_{1n}$  are the components of the displacement vector on either side of the interface in the direction of the normal  $\mathbf{n}$  shown, pointing from region 1 to region 2. Reducing (3) and using more compact notation we have

$$D_{2n} - D_{1n} = \sigma_f, \quad \mathbf{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma_f \quad (4)$$

where the minus sign in front of  $\mathbf{D}_1$  arises because the normal on the lower surface  $b$  is  $-\mathbf{n}$ . The normal components of the displacement vector are discontinuous if the interface has a surface charge density. If there is no surface charge ( $\sigma_f = 0$ ), the normal components of  $\mathbf{D}$  are continuous. If each medium has no polarization, (4) reduces to the free space results of Section 2.4.6.

At the interface between two different lossless dielectrics, there is usually no surface charge ( $\sigma_f = 0$ ), unless it was deliberately placed, because with no conductivity there is no current to transport charge. Then, even though the normal component of the  $\mathbf{D}$  field is continuous, the normal component of the electric field is discontinuous because the dielectric constant in each region is different.

At the interface between different conducting materials, free surface charge may exist as the current may transport charge to the surface discontinuity. Generally for such cases, the surface charge density is nonzero. In particular, if one region is a perfect conductor with zero internal electric field,

the surface charge density on the surface is just equal to the normal component of  $\mathbf{D}$  field at the conductor's surface,

$$\sigma_f = \mathbf{n} \cdot \mathbf{D} \quad (5)$$

where  $\mathbf{n}$  is the outgoing normal from the perfect conductor.

### 3-3-3 Point Charge Above a Dielectric Boundary

If a point charge  $q$  within a region of permittivity  $\epsilon_1$  is a distance  $d$  above a planar boundary separating region I from region II with permittivity  $\epsilon_2$ , as in Figure 3-13, the tangential component of  $\mathbf{E}$  and in the absence of free surface charge the normal component of  $\mathbf{D}$ , must be continuous across the interface. Let us try to use the method of images by placing an image charge  $q'$  at  $y = -d$  so that the solution in region I is due to this image charge plus the original point charge  $q$ . The solution for the field in region II will be due to an image charge  $q''$  at  $y = d$ , the position of the original point charge. Note that the appropriate image charge is always outside the region where the solution is desired. At this point we do not know if it is possible to satisfy the boundary conditions with these image charges, but we will try to find values of  $q'$  and  $q''$  to do so.

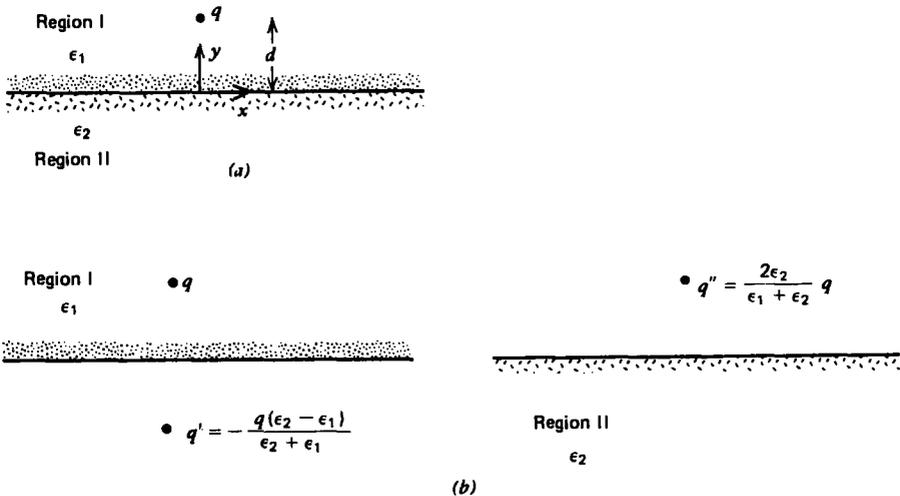


Figure 3-13 (a) A point charge  $q$  above a flat dielectric boundary requires different sets of image charges to solve for the fields in each region. (b) The field in region I is due to the original charge and the image charge  $q'$  while the field in region II is due only to image charge  $q''$ .

The potential in each region is

$$V_I = \frac{1}{4\pi\epsilon_1} \left( \frac{q}{[x^2 + (y-d)^2 + z^2]^{1/2}} + \frac{q'}{[x^2 + (y+d)^2 + z^2]^{1/2}} \right), \quad y \geq 0 \quad (6)$$

$$V_{II} = \frac{1}{4\pi\epsilon_2} \frac{q''}{[x^2 + (y-d)^2 + z^2]^{1/2}}, \quad y \leq 0$$

with resultant electric field

$$\begin{aligned} E_I &= -\nabla V_I \\ &= \frac{1}{4\pi\epsilon_1} \left( \frac{q[x\mathbf{i}_x + (y-d)\mathbf{i}_y + z\mathbf{i}_z]}{[x^2 + (y-d)^2 + z^2]^{3/2}} + \frac{q'[x\mathbf{i}_x + (y+d)\mathbf{i}_y + z\mathbf{i}_z]}{[x^2 + (y+d)^2 + z^2]^{3/2}} \right) \quad (7) \end{aligned}$$

$$E_{II} = -\nabla V_{II} = \frac{q''}{4\pi\epsilon_2} \frac{(x\mathbf{i}_x + (y-d)\mathbf{i}_y + z\mathbf{i}_z)}{[x^2 + (y-d)^2 + z^2]^{3/2}}$$

To satisfy the continuity of tangential electric field at  $y=0$  we have

$$\begin{aligned} E_{xI} &= E_{xII} \\ E_{zI} &= E_{zII} \end{aligned} \Rightarrow \frac{q+q'}{\epsilon_1} = \frac{q''}{\epsilon_2} \quad (8)$$

With no surface charge, the normal component of  $\mathbf{D}$  must be continuous at  $y=0$ ,

$$\epsilon_1 E_{yI} = \epsilon_2 E_{yII} \Rightarrow -q + q' = -q'' \quad (9)$$

Solving (8) and (9) for the unknown charges we find

$$\begin{aligned} q' &= -\frac{(\epsilon_2 - \epsilon_1)}{\epsilon_1 + \epsilon_2} q \\ q'' &= \frac{2\epsilon_2}{(\epsilon_1 + \epsilon_2)} q \end{aligned} \quad (10)$$

The force on the point charge  $q$  is due only to the field from image charge  $q'$ :

$$\mathbf{f} = \frac{qq'}{4\pi\epsilon_1(2d)^2} \mathbf{i}_y = -\frac{q^2(\epsilon_2 - \epsilon_1)}{16\pi\epsilon_1(\epsilon_1 + \epsilon_2)d^2} \mathbf{i}_y \quad (11)$$

### 3-3-4 Normal Component of $\mathbf{P}$ and $\epsilon_0\mathbf{E}$

By integrating the flux of polarization over the same Gaussian pillbox surface, shown in Figure 3-12b, we relate the discontinuity in normal component of polarization to the surface polarization charge density  $\sigma_p$  using the relations

from Section 3.1.2:

$$\oint_S \mathbf{P} \cdot d\mathbf{S} = - \int_S \sigma_p dS \Rightarrow P_{2n} - P_{1n} = -\sigma_p \Rightarrow \mathbf{n} \cdot (\mathbf{P}_2 - \mathbf{P}_1) = -\sigma_p \quad (12)$$

The minus sign in front of  $\sigma_p$  results because of the minus sign relating the volume polarization charge density to the divergence of  $\mathbf{P}$ .

To summarize, polarization charge is the source of  $\mathbf{P}$ , free charge is the source of  $\mathbf{D}$ , and the total charge is the source of  $\epsilon_0 \mathbf{E}$ . Using (4) and (12), the electric field interfacial discontinuity is

$$\mathbf{n} \cdot (\mathbf{E}_2 - \mathbf{E}_1) = \frac{\mathbf{n} \cdot [(\mathbf{D}_2 - \mathbf{D}_1) - (\mathbf{P}_2 - \mathbf{P}_1)]}{\epsilon_0} = \frac{\sigma_f + \sigma_p}{\epsilon_0} \quad (13)$$

For linear dielectrics it is often convenient to lump polarization effects into the permittivity  $\epsilon$  and never use the vector  $\mathbf{P}$ , only  $\mathbf{D}$  and  $\mathbf{E}$ .

For permanently polarized materials, it is usually convenient to replace the polarization  $\mathbf{P}$  by the equivalent polarization volume charge density and surface charge density of (12) and solve for  $\mathbf{E}$  using the coulombic superposition integral of Section 2.3.2. In many dielectric problems, there is no volume polarization charge, but at surfaces of discontinuity a surface polarization charge is present as given by (12).

### EXAMPLE 3-2 CYLINDER PERMANENTLY POLARIZED ALONG ITS AXIS

A cylinder of radius  $a$  and height  $L$  is centered about the  $z$  axis and has a uniform polarization along its axis,  $\mathbf{P} = P_0 \mathbf{i}_z$ , as shown in Figure 3-14. Find the electric field  $\mathbf{E}$  and displacement vector  $\mathbf{D}$  everywhere on its axis.

#### SOLUTION

With a constant polarization  $\mathbf{P}$ , the volume polarization charge density is zero:

$$\rho_p = -\nabla \cdot \mathbf{P} = 0$$

Since  $\mathbf{P} = 0$  outside the cylinder, the normal component of  $\mathbf{P}$  is discontinuous at the upper and lower surfaces yielding uniform surface polarization charges:

$$\sigma_p(z = L/2) = P_0, \quad \sigma_p(z = -L/2) = -P_0$$

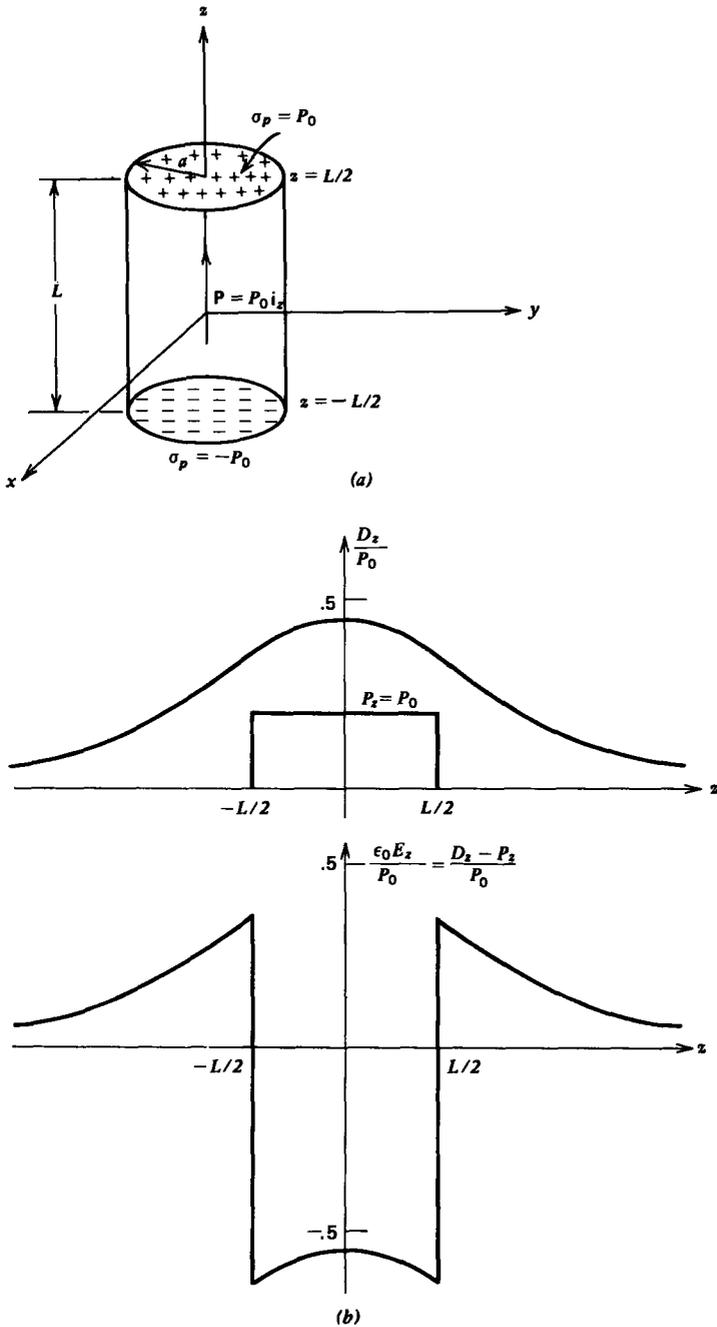


Figure 3-14 (a) The electric field due to a uniformly polarized cylinder of length  $L$  is the same as for two disks of surface charge of opposite polarity  $\pm P_0$  at  $z = L/2$ . (b) The perpendicular displacement field  $D_z$  is continuous across the interfaces at  $z = \pm L/2$  while the electric field  $E_z$  is discontinuous.

The solution for a single disk of surface charge was obtained in Section 2.3.5*b*. We superpose the results for the two disks taking care to shift the axial distance appropriately by  $\pm L/2$  yielding the concise solution for the displacement field:

$$D_z = \frac{P_0}{2} \left( \frac{(z + L/2)}{[a^2 + (z + L/2)^2]^{1/2}} - \frac{(z - L/2)}{[a^2 + (z - L/2)^2]^{1/2}} \right)$$

The electric field is then

$$E_z = \begin{cases} D_z/\epsilon_0, & |z| > L/2 \\ (D_z - P_0)/\epsilon_0 & |z| < L/2 \end{cases}$$

These results can be examined in various limits. If the radius  $a$  becomes very large, the electric field should approach that of two parallel sheets of surface charge  $\pm P_0$ , as in Section 2.3.4*b*:

$$\lim_{a \rightarrow \infty} E_z = \begin{cases} 0, & |z| > L/2 \\ -P_0/\epsilon_0, & |z| < L/2 \end{cases}$$

with a zero displacement field everywhere.

In the opposite limit, for large  $z$  ( $z \gg a$ ,  $z \gg L$ ) far from the cylinder, the axial electric field dies off as the dipole field with  $\theta = 0$

$$\lim_{z \rightarrow \infty} E_z = \frac{p}{2\pi\epsilon_0 z^3}, \quad p = P_0\pi a^2 L$$

with effective dipole moment  $p$  given by the product of the total polarization charge at  $z = L/2$ , ( $P_0\pi a^2$ ), and the length  $L$ .

### 3-3-5 Normal Component of $\mathbf{J}$

Applying the conservation of total current equation in Section 3.2.1 to the same Gaussian pillbox surface in Figure 3-12*b* results in contributions again only from the upper and lower surfaces labeled "a" and "b":

$$\mathbf{n} \cdot \left( \mathbf{J}_2 - \mathbf{J}_1 + \frac{\partial}{\partial t} (\mathbf{D}_2 - \mathbf{D}_1) \right) = 0 \quad (14)$$

where we assume that no surface currents flow along the interface. From (4), relating the surface charge density to the discontinuity in normal  $\mathbf{D}$ , this boundary condition can also be written as

$$\mathbf{n} \cdot (\mathbf{J}_2 - \mathbf{J}_1) + \frac{\partial \sigma_f}{\partial t} = 0 \quad (15)$$

which tells us that if the current entering a surface is different from the current leaving, charge has accumulated at the

interface. In the dc steady state the normal component of  $\mathbf{J}$  is continuous across a boundary.

### 3-4 RESISTANCE

#### 3-4-1 Resistance Between Two Electrodes

Two conductors maintained at a potential difference  $V$  within a conducting medium will each pass a total current  $I$ , as shown in Figure 3-15. By applying the surface integral form of charge conservation in Section 3.2.1 to a surface  $S'$  which surrounds both electrodes but is far enough away so that  $\mathbf{J}$  and  $\mathbf{D}$  are negligibly small, we see that the only nonzero current contributions are from the terminal wires that pass through the surface. These must sum to zero so that the

$$\mathbf{J}, \mathbf{E} \propto \frac{1}{r^3} \text{ far from the electrodes}$$

$$\lim_{r \rightarrow \infty} \oint_{S'} \mathbf{J} \cdot d\mathbf{S} = 0$$

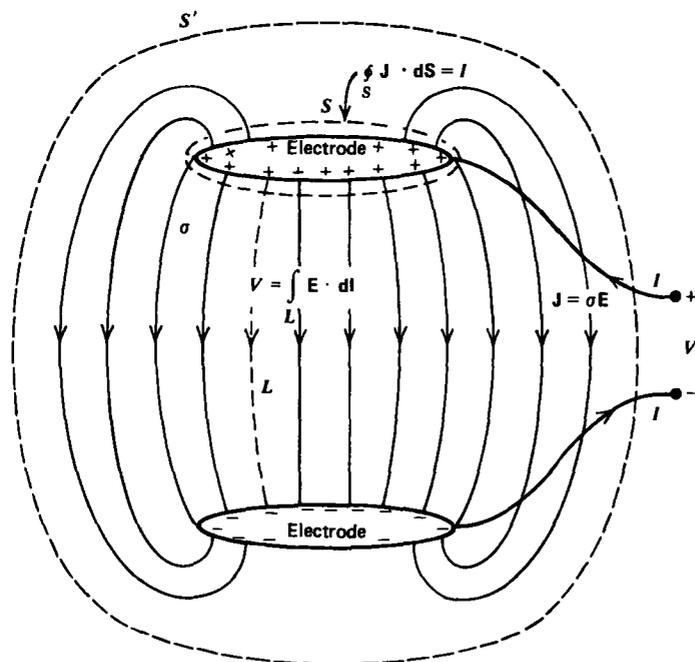


Figure 3-15 A voltage applied across two electrodes within an ohmic medium causes a current to flow into one electrode and out the other. The electrodes have equal magnitude but opposite polarity charges so that far away the fields die off as a dipole  $\propto (1/r^3)$ . Then, even though the surface  $S'$  is increasing as  $r^2$ , the flux of current goes to zero as  $1/r$ .

currents have equal magnitudes but flow in opposite directions. Similarly, applying charge conservation to a surface  $S$  just enclosing the upper electrode shows that the current  $I$  entering the electrode via the wire must just equal the total current (conduction plus displacement) leaving the electrode. This total current travels to the opposite electrode and leaves via the connecting wire.

The dc steady-state ratio of voltage to current between the two electrodes in Figure 3-15 is defined as the resistance:

$$R = \frac{V}{I} \text{ ohm } [\text{kg}\cdot\text{m}^2\cdot\text{s}^{-3}\cdot\text{A}^{-2}] \quad (1)$$

For an arbitrary geometry, (1) can be expressed in terms of the fields as

$$R = \frac{\int_L \mathbf{E} \cdot d\mathbf{l}}{\oint_S \mathbf{J} \cdot d\mathbf{S}} = \frac{\int_L \mathbf{E} \cdot d\mathbf{l}}{\oint_S \sigma \mathbf{E} \cdot d\mathbf{S}} \quad (2)$$

where  $S$  is a surface completely surrounding an electrode and  $L$  is any path joining the two electrodes. Note that the field line integral is taken along the line from the high to low potential electrode so that the voltage difference  $V$  is equal to the positive line integral. From (2), we see that the resistance only depends on the geometry and conductivity  $\sigma$  and not on the magnitude of the electric field itself. If we were to increase the voltage by any factor, the field would also increase by this same factor everywhere so that this factor would cancel out in the ratio of (2). The conductivity  $\sigma$  may itself be a function of position.

### 3-4-2 Parallel Plate Resistor

Two perfectly conducting parallel plate electrodes of arbitrarily shaped area  $A$  and spacing  $l$  enclose a cylinder of material with Ohmic conductivity  $\sigma$ , as in Figure 3-16a. The current must flow tangential to the outer surface as the outside medium being free space has zero conductivity so that no current can pass through the interface. Because the tangential component of electric field is continuous, a field does exist in the free space region that decreases with increasing distance from the resistor. This three-dimensional field is difficult to calculate because it depends on three coordinates.

The electric field within the resistor is much simpler to calculate because it is perpendicular to the electrodes in the  $x$  direction. Gauss's law with no volume charge then tells us that

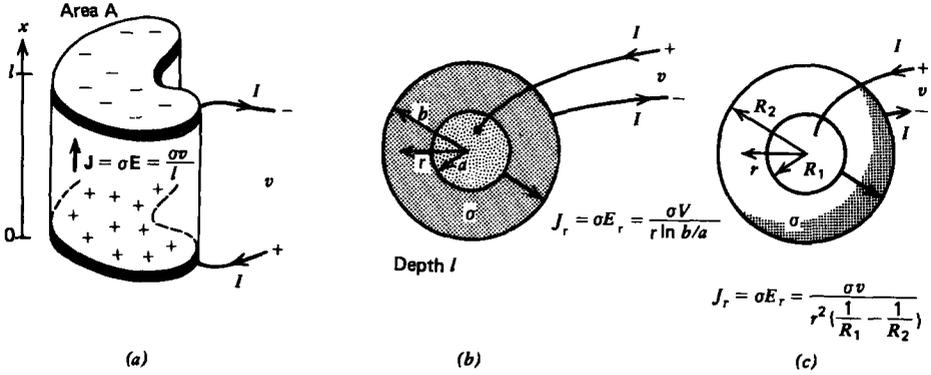


Figure 3-16 Simple resistor electrode geometries. (a) Parallel plates. (b) Coaxial cylinders. (c) Concentric spheres.

this field is constant:

$$\nabla \cdot (\epsilon \mathbf{E}) = 0 \Rightarrow \frac{dE_x}{dx} = 0 \Rightarrow E_x = E_0 \quad (3)$$

However, the line integral of  $\mathbf{E}$  between the electrodes must be the applied voltage  $v$ :

$$\int_0^l E_x dx = v \Rightarrow E_0 = v/l \quad (4)$$

The current density is then

$$\mathbf{J} = \sigma E_0 \mathbf{i}_x = (\sigma v/l) \mathbf{i}_x \quad (5)$$

so that the total current through the electrodes is

$$I = \oint_S \mathbf{J} \cdot d\mathbf{S} = (\sigma v/l) A \quad (6)$$

where the surface integral is reduced to a pure product because the constant current density is incident perpendicularly on the electrodes. The resistance is then

$$R = \frac{v}{I} = \frac{l}{\sigma A} = \frac{\text{spacing}}{(\text{conductivity})(\text{electrode area})} \quad (7)$$

Typical resistance values can vary over many orders of magnitude. If the electrodes have an area  $A = 1 \text{ cm}^2 (10^{-4} \text{ m}^2)$  with spacing  $l = 1 \text{ mm} (10^{-3} \text{ m})$  a material like copper has a resistance  $R \approx 0.17 \times 10^{-6} \text{ ohm}$  while carbon would have a resistance  $R \approx 1.4 \times 10^4 \text{ ohm}$ . Because of this large range of resistance values sub-units often used are micro-ohms ( $1 \mu\Omega = 10^{-6} \Omega$ ), milli-ohms ( $1 \text{ m}\Omega = 10^{-3} \Omega$ ), kilohm ( $1 \text{ k}\Omega = 10^3 \Omega$ ), and megohms ( $1 \text{ M}\Omega = 10^6 \Omega$ ), where the symbol  $\Omega$  is used to represent the unit of ohms.

Although the field outside the resistor is difficult to find, we do know that for distances far from the resistor the field approaches that of a point dipole due to the oppositely charged electrodes with charge density

$$\sigma_f(x=0) = -\sigma_f(x=l) = \epsilon E_0 = \epsilon v/l \quad (8)$$

and thus dipole moment

$$\mathbf{p} = -\sigma_f(x=0) A l \mathbf{i}_x = -\epsilon A v l \mathbf{i}_x \quad (9)$$

The minus sign arises because the dipole moment points from negative to positive charge. Note that (8) is only approximate because all of the external field lines in the free space region must terminate on the side and back of the electrodes giving further contributions to the surface charge density. Generally, if the electrode spacing  $l$  is much less than any of the electrode dimensions, this extra contribution is very small.

### 3-4-3 Coaxial Resistor

Two perfectly conducting coaxial cylinders of length  $l$ , inner radius  $a$ , and outer radius  $b$  are maintained at a potential difference  $v$  and enclose a material with Ohmic conductivity  $\sigma$ , as in Figure 3-16*b*. The electric field must then be perpendicular to the electrodes so that with no free charge Gauss's law requires

$$\nabla \cdot (\epsilon \mathbf{E}) = 0 \Rightarrow \frac{1}{r} \frac{\partial}{\partial r} (r E_r) = 0 \Rightarrow E_r = \frac{c}{r} \quad (10)$$

where  $c$  is an integration constant found from the voltage condition

$$\int_a^b E_r dr = c \ln r \Big|_a^b = v \Rightarrow c = \frac{v}{\ln(b/a)} \quad (11)$$

The current density is then

$$\mathbf{J}_r = \sigma \mathbf{E}_r = \frac{\sigma v}{r \ln(b/a)} \quad (12)$$

with the total current at any radius  $r$  being a constant

$$I = \int_{z=0}^l \int_{\phi=0}^{2\pi} J_r r d\phi dz = \frac{\sigma v 2\pi l}{\ln(b/a)} \quad (13)$$

so that the resistance is

$$R = \frac{v}{I} = \frac{\ln(b/a)}{2\pi\sigma l} \quad (14)$$

### 3-4-4 Spherical Resistor

We proceed in the same way for two perfectly conducting concentric spheres at a potential difference  $v$  with inner radius  $R_1$  and outer radius  $R_2$ , as in Figure 3-16c. With no free charge, symmetry requires the electric field to be purely radial so that Gauss's law yields

$$\nabla \cdot (\epsilon \mathbf{E}) = 0 \Rightarrow \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 E_r) = 0 \Rightarrow E_r = \frac{c}{r^2} \quad (15)$$

where  $c$  is a constant found from the voltage condition as

$$\int_{R_1}^{R_2} E_r dr = -\frac{c}{r} \Big|_{R_1}^{R_2} = v \Rightarrow c = \frac{v}{(1/R_1 - 1/R_2)} \quad (16)$$

The electric field and current density are inversely proportional to the square of the radius

$$J_r = \sigma E_r = \frac{\sigma v}{r^2(1/R_1 - 1/R_2)} \quad (17)$$

so that the current density is constant at any radius  $r$

$$I = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} J_r r^2 \sin \theta d\theta d\phi = \frac{4\pi\sigma v}{(1/R_1 - 1/R_2)} \quad (18)$$

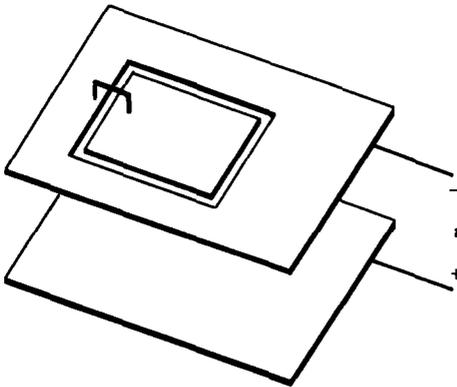
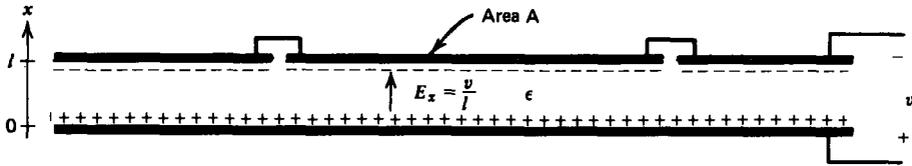
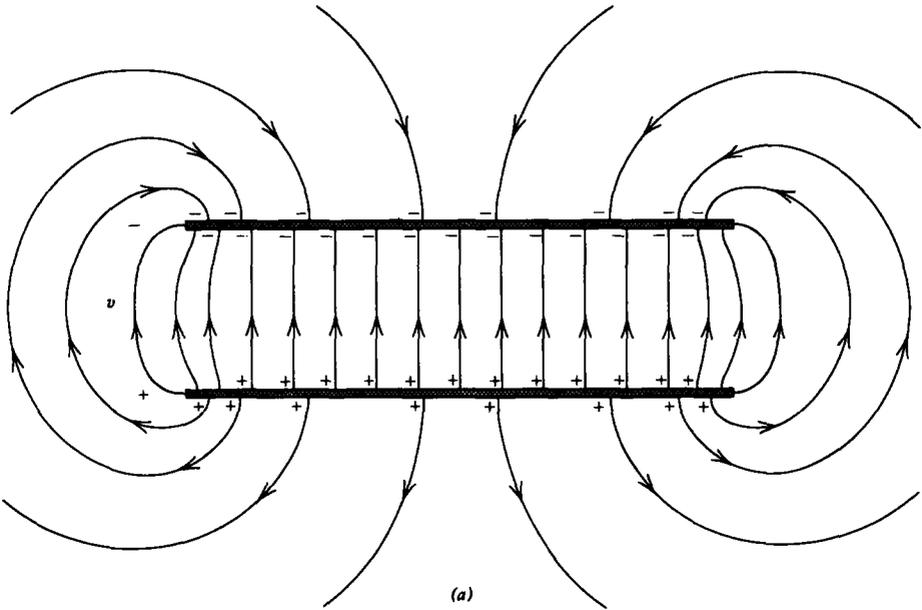
with resistance

$$R = \frac{v}{I} = \frac{(1/R_1 - 1/R_2)}{4\pi\sigma} \quad (19)$$

## 3-5 CAPACITANCE

### 3-5-1 Parallel Plate Electrodes

Parallel plate electrodes of finite size constrained to potential difference  $v$  enclose a dielectric medium with permittivity  $\epsilon$ . The surface charge density does not distribute itself uniformly, as illustrated by the fringing field lines for infinitely thin parallel plate electrodes in Figure 3-17a. Near the edges the electric field is highly nonuniform decreasing in magnitude on the back side of the electrodes. Between the electrodes, far from the edges the electric field is uniform, being the same as if the electrodes were infinitely long. Fringing field effects can be made negligible if the electrode spacing  $l$  is much less than the depth  $d$  or width  $w$ . For more accurate work, end effects can be made even more negligible by using a guard ring encircling the upper electrode, as in Figure 3-17b. The guard ring is maintained at the same potential as the electrode, thus except for the very tiny gap, the field between



(b)

Figure 3-17 (a) Two infinitely thin parallel plate electrodes of finite area at potential difference  $v$  have highly nonuniform fields outside the interelectrode region. (b) A guard ring around one electrode removes end effects so that the field between the electrodes is uniform. The end effects now arise at the edge of the guard ring, which is far from the region of interest.

the electrodes is as if the end effects were very far away and not just near the electrode edges.

We often use the phrase “neglect fringing” to mean that the nonuniform field effects near corners and edges are negligible.

With the neglect of fringing field effects near the electrode ends, the electric field is perpendicular to the electrodes and related to the voltage as

$$\int_0^l E_x dx = v \Rightarrow E_x = v/l \quad (1)$$

The displacement vector is then proportional to the electric field terminating on each electrode with an equal magnitude but opposite polarity surface charge density given by

$$D_x = \epsilon E_x = \sigma_f(x=0) = -\sigma_f(x=l) = \epsilon v/l \quad (2)$$

The charge is positive where the voltage polarity is positive, and vice versa, with the electric field directed from the positive to negative electrode. The magnitude of total free charge on each electrode is

$$q_f = \sigma_f(x=0)A = \frac{\epsilon A}{l}v \quad (3)$$

The capacitance  $C$  is defined as the magnitude of the ratio of total free charge on either electrode to the voltage difference between electrodes:

$$\begin{aligned} C &= \frac{q_f}{v} = \frac{\epsilon A}{l} \\ &= \frac{\text{(permittivity) (electrode area)}}{\text{spacing}} \text{ farad } [A^2 \cdot s^4 \cdot \text{kg}^{-1} \cdot \text{m}^{-2}] \end{aligned} \quad (4)$$

Even though the system remains neutral, mobile electrons on the lower electrode are transported through the voltage source to the upper electrode in order to terminate the displacement field at the electrode surfaces, thus keeping the fields zero inside the conductors. Note that no charge is transported through free space. The charge transport between electrodes is due to work by the voltage source and results in energy stored in the electric field.

In SI units, typical capacitance values are very small. If the electrodes have an area of  $A = 1 \text{ cm}^2 (10^{-4} \text{ m}^2)$  with spacing of  $l = 1 \text{ mm} (10^{-3} \text{ m})$ , the free space capacitance is  $C \approx 0.9 \times 10^{-12}$  farad. For this reason usual capacitance values are expressed in microfarads ( $1 \mu\text{f} = 10^{-6}$  farad), nanofarads ( $1 \text{ nf} = 10^{-9}$  farad), and picofarads ( $1 \text{ pf} = 10^{-12}$  farad).

With a linear dielectric of permittivity  $\epsilon$  as in Figure 3-18a, the field of (1) remains unchanged for a given voltage but the charge on the electrodes and thus the capacitance increases with the permittivity, as given by (3). However, if the total free charge on each electrode were constrained, the voltage difference would decrease by the same factor.

These results arise because of the presence of polarization charges on the electrodes that partially cancel the free charge. The polarization vector within the dielectric-filled parallel plate capacitor is a constant

$$P_x = D_x - \epsilon_0 E_x = (\epsilon - \epsilon_0) E_x = (\epsilon - \epsilon_0) v / l \quad (5)$$

so that the volume polarization charge density is zero. However, with zero polarization in the electrodes, there is a discontinuity in the normal component of polarization at the electrode surfaces. The boundary condition of Section 3.3.4 results in an equal magnitude but opposite polarity surface polarization charge density on each electrode, as illustrated in

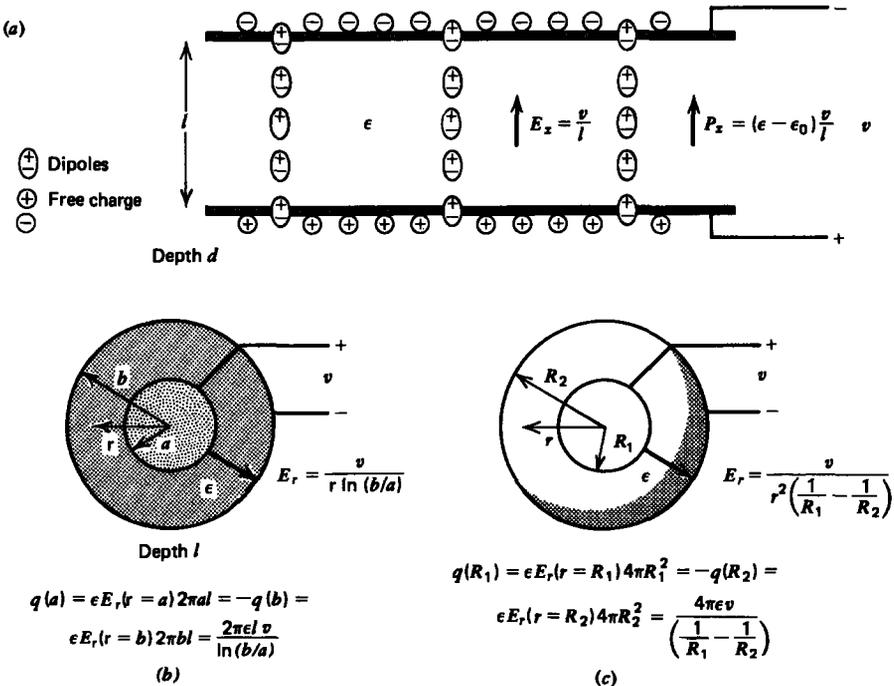


Figure 3-18 The presence of a dielectric between the electrodes increases the capacitance because for a given voltage additional free charge is needed on each electrode to overcome the partial neutralization of the attracted opposite polarity dipole ends. (a) Parallel plate electrodes. (b) Coaxial cylinders. (c) Concentric spheres.

Figure 3-18a:

$$\sigma_p(x=0) = -\sigma_p(x=l) = -P_x = -(\epsilon - \epsilon_0)v/l \quad (6)$$

Note that negative polarization charge appears on the positive polarity electrode and vice versa. This is because opposite charges attract so that the oppositely charged ends of the dipoles line up along the electrode surface partially neutralizing the free charge.

### 3-5-2 Capacitance for any Geometry

We have based our discussion around a parallel plate capacitor. Similar results hold for any shape electrodes in a dielectric medium with the capacitance defined as the magnitude of the ratio of total free charge on an electrode to potential difference. The capacitance is always positive by definition and for linear dielectrics is only a function of the geometry and dielectric permittivity and not on the voltage levels,

$$C = \frac{q_f}{v} = \frac{\int_S \mathbf{D} \cdot d\mathbf{S}}{\int_L \mathbf{E} \cdot d\mathbf{l}} = \frac{\int_S \epsilon \mathbf{E} \cdot d\mathbf{S}}{\int_L \mathbf{E} \cdot d\mathbf{l}} \quad (7)$$

as multiplying the voltage by a constant factor also increases the electric field by the same factor so that the ratio remains unchanged.

The integrals in (7) are similar to those in Section 3.4.1 for an Ohmic conductor. For the same geometry filled with a homogenous Ohmic conductor or a linear dielectric, the resistance-capacitance product is a constant independent of the geometry:

$$RC = \frac{\int_L \mathbf{E} \cdot d\mathbf{l}}{\sigma \int_S \mathbf{E} \cdot d\mathbf{S}} \frac{\epsilon \int_S \mathbf{E} \cdot d\mathbf{S}}{\int_L \mathbf{E} \cdot d\mathbf{l}} = \frac{\epsilon}{\sigma} \quad (8)$$

Thus, for a given geometry, if either the resistance or capacitance is known, the other quantity is known immediately from (8). We can thus immediately write down the capacitance of the geometries shown in Figure 3-18 assuming the medium between electrodes is a linear dielectric with permittivity  $\epsilon$  using the results of Sections 3.4.2–3.4.4:

$$\text{Parallel Plate } R = \frac{l}{\sigma A} \Rightarrow C = \frac{\epsilon A}{l}$$

$$\text{Coaxial } R = \frac{\ln(b/a)}{2\pi\sigma l} \Rightarrow C = \frac{2\pi\epsilon l}{\ln(b/a)} \quad (9)$$

$$\text{Spherical } R = \frac{1/R_1 - 1/R_2}{4\pi\sigma} \Rightarrow C = \frac{4\pi\epsilon}{(1/R_1 - 1/R_2)}$$

3-5-3 Current Flow Through a Capacitor

From the definition of capacitance in (7), the current to an electrode is

$$i = \frac{dq_f}{dt} = \frac{d}{dt}(Cv) = C\frac{dv}{dt} + v\frac{dC}{dt} \tag{10}$$

where the last term only arises if the geometry or dielectric permittivity changes with time. For most circuit applications, the capacitance is independent of time and (10) reduces to the usual voltage-current circuit relation.

In the capacitor of arbitrary geometry, shown in Figure 3-19, a conduction current  $i$  flows through the wires into the upper electrode and out of the lower electrode changing the amount of charge on each electrode, as given by (10). There is no conduction current flowing in the dielectric between the electrodes. As discussed in Section 3.2.1 the total current, displacement plus conduction, is continuous. Between the electrodes in a lossless capacitor, this current is entirely displacement current. The displacement field is itself related to the time-varying surface charge distribution on each electrode as given by the boundary condition of Section 3.3.2.

3-5-4 Capacitance of Two Contacting Spheres

If the outer radius  $R_2$  of the spherical capacitor in (9) is put at infinity, we have the capacitance of an isolated sphere of radius  $R$  as

$$C = 4\pi\epsilon R \tag{11}$$

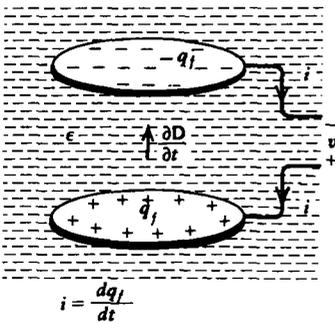


Figure 3-19 The conduction current  $i$  that travels through the connecting wire to an electrode in a lossless capacitor is transmitted through the dielectric medium to the opposite electrode via displacement current. No charge carriers travel through the lossless dielectric.

If the surrounding medium is free space ( $\epsilon = \epsilon_0$ ) for  $R = 1$  m, we have that  $C \approx \frac{1}{9} \times 10^{-9}$  farad  $\approx 111$  pf.

We wish to find the self-capacitance of two such contacting spheres raised to a potential  $V_0$ , as shown in Figure 3-20. The capacitance is found by first finding the total charge on the two spheres. We can use the method of images by first placing an image charge  $q_1 = Q = 4\pi\epsilon RV_0$  at the center of each sphere to bring each surface to potential  $V_0$ . However, each of these charges will induce an image charge  $q_2$  in the other sphere at distance  $b_2$  from the center,

$$q_2 = -\frac{Q}{2}, \quad b_2 = \frac{R^2}{D} = \frac{R}{2} \quad (12)$$

where we realize that the distance from inducing charge to the opposite sphere center is  $D = 2R$ . This image charge does not raise the potential of either sphere. Similarly, each of these image charges induces another image charge  $q_3$  in the other sphere at distance  $b_3$ ,

$$q_3 = -\frac{q_2 R}{D - b_2} = \frac{Q}{3}, \quad b_3 = \frac{R^2}{D - b_2} = \frac{2}{3}R \quad (13)$$

which will induce a further image charge  $q_4$ , ad infinitum. An infinite number of image charges will be necessary, but with the use of difference equations we will be able to add all the image charges to find the total charge and thus the capacitance.

The  $n$ th image charge  $q_n$  and its distance from the center  $b_n$  are related to the  $(n-1)$ th images as

$$q_n = -\frac{q_{n-1}R}{D - b_{n-1}}, \quad b_n = \frac{R^2}{D - b_{n-1}} \quad (14)$$

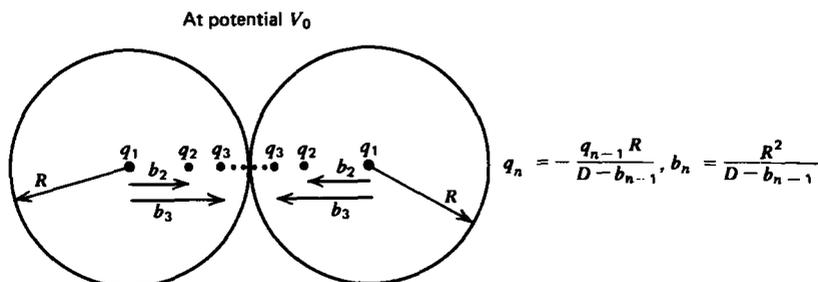


Figure 3-20 Two identical contacting spheres raised to a potential  $V_0$  with respect to infinity are each described by an infinite number of image charges  $q_n$  each a distance  $b_n$  from the sphere center.

where  $D = 2R$ . We solve the first relation for  $b_{n-1}$  as

$$D - b_{n-1} = -\frac{q_{n-1}}{q_n} R \quad (15)$$

$$b_n = \frac{q_n}{q_{n+1}} R + D$$

where the second relation is found by incrementing  $n$  in the first relation by 1. Substituting (15) into the second relation of (14) gives us a single equation in the  $q_n$ 's:

$$\frac{q_n R}{q_{n+1}} + D = -\frac{R q_n}{q_{n-1}} \Rightarrow \frac{1}{q_{n+1}} + \frac{2}{q_n} + \frac{1}{q_{n-1}} = 0 \quad (16)$$

If we define the reciprocal charges as

$$p_n = 1/q_n \quad (17)$$

then (16) becomes a homogeneous linear constant coefficient difference equation

$$p_{n+1} + 2p_n + p_{n-1} = 0 \quad (18)$$

Just as linear constant coefficient differential equations have exponential solutions, (18) has power law solutions of the form

$$p_n = A\lambda^n \quad (19)$$

where the characteristic roots  $\lambda$ , analogous to characteristic frequencies, are found by substitution back into (18),

$$\lambda^{n+1} + 2\lambda^n + \lambda^{n-1} = 0 \Rightarrow \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0 \quad (20)$$

to yield a double root with  $\lambda = -1$ . Because of the double root, the superposition of both solutions is of the form

$$p_n = A_1(-1)^n + A_2 n(-1)^n \quad (21)$$

similar to the behavior found in differential equations with double characteristic frequencies. The correctness of (21) can be verified by direct substitution back into (18). The constants  $A_1$  and  $A_2$  are determined from  $q_1$  and  $q_2$  as

$$\left. \begin{aligned} p_1 = 1/Q &= -A_1 - A_2 \\ p_2 = \frac{1}{q_2} &= -\frac{2}{Q} = +A_1 + 2A_2 \end{aligned} \right\} \Rightarrow \begin{cases} A_1 = 0 \\ A_2 = -\frac{1}{Q} \end{cases} \quad (22)$$

so that the  $n$ th image charge is

$$q_n = \frac{1}{p_n} = \frac{1}{-(-1)^n n / Q} = \frac{-(-1)^n Q}{n} \quad (23)$$

The capacitance is then given as the ratio of the total charge on the two spheres to the voltage,

$$C = \frac{2 \sum_{n=0}^{\infty} q_n}{V_0} = -\frac{2Q}{V_0} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = +\frac{2Q}{V_0} [1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots] \\ = 8\pi\epsilon R \ln 2 \quad (24)$$

where we recognize the infinite series to be the Taylor series expansion of  $\ln(1+x)$  with  $x=1$ . The capacitance of two contacting spheres is thus  $2 \ln 2 \approx 1.39$  times the capacitance of a single sphere given by (11).

The distance from the center to each image charge is obtained from (23) substituted into (15) as

$$b_n = \left( \frac{(-1)^n (n+1)}{n(-1)^{n+1}} + 2 \right) R = \frac{(n-1)}{n} R \quad (25)$$

We find the force of attraction between the spheres by taking the sum of the forces on each image charge on one of the spheres due to all the image charges on the other sphere. The force on the  $n$ th image charge on one sphere due to the  $m$ th image charge in the other sphere is

$$f_{nm} = \frac{-q_n q_m}{4\pi\epsilon [2R - b_n - b_m]^2} = \frac{-Q_0^2 (-1)^{n+m}}{4\pi\epsilon R^2} \frac{nm}{(m+n)^2} \quad (26)$$

where we used (23) and (25). The total force on the left sphere is then found by summing over all values of  $m$  and  $n$ ,

$$f = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{nm} = \frac{-Q_0^2}{4\pi\epsilon R^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+m} nm}{(n+m)^2} \\ = \frac{-Q_0^2}{4\pi\epsilon R^3} \frac{1}{6} [\ln 2 - \frac{1}{4}] \quad (27)$$

where the double series can be explicitly expressed.\* The force is negative because the like charge spheres repel each other. If  $Q_0 = 1$  coul with  $R = 1$  m, in free space this force is  $f \approx 6.6 \times 10^8$  nt, which can lift a mass in the earth's gravity field of  $6.8 \times 10^7$  kg ( $\approx 3 \times 10^7$  lb).

### 3-6 LOSSY MEDIA

Many materials are described by both a constant permittivity  $\epsilon$  and constant Ohmic conductivity  $\sigma$ . When such a material is placed between electrodes do we have a capacitor

\* See Albert D. Wheelon, Tables of Summable Series and Integrals Involving Bessel Functions, Holden Day, (1968) pp. 55, 56.

or a resistor? We write the governing equations of charge conservation and Gauss's law with linear constitutive laws:

$$\nabla \cdot \mathbf{J}_f + \frac{\partial \rho_f}{\partial t} = 0, \quad \mathbf{J}_f = \sigma \mathbf{E} + \rho_f \mathbf{U} \quad (1)$$

$$\nabla \cdot \mathbf{D} = \rho_f, \quad \mathbf{D} = \epsilon \mathbf{E} \quad (2)$$

We have generalized Ohm's law in (1) to include convection currents if the material moves with velocity  $\mathbf{U}$ . In addition to the conduction charges, any free charges riding with the material also contribute to the current. Using (2) in (1) yields a single partial differential equation in  $\rho_f$ :

$$\underbrace{\sigma(\nabla \cdot \mathbf{E})}_{\rho_f/\epsilon} + \nabla \cdot (\rho_f \mathbf{U}) + \frac{\partial \rho_f}{\partial t} = 0 \Rightarrow \frac{\partial \rho_f}{\partial t} + \nabla \cdot (\rho_f \mathbf{U}) + \frac{\sigma}{\epsilon} \rho_f = 0 \quad (3)$$

### 3-6-1 Transient Charge Relaxation

Let us first assume that the medium is stationary so that  $\mathbf{U} = 0$ . Then the solution to (3) for any initial possibly spatially varying charge distribution  $\rho_0(x, y, z, t = 0)$  is

$$\rho_f = \rho_0(x, y, z, t = 0) e^{-t/\tau}, \quad \tau = \epsilon/\sigma \quad (4)$$

where  $\tau$  is the relaxation time. This solution is the continuum version of the resistance-capacitance (RC) decay time in circuits.

The solution of (4) tells us that at all positions within a conductor, any initial charge density dies off exponentially with time. It does not spread out in space. This is our justification of not considering any net volume charge in conducting media. If a system has no volume charge at  $t = 0$  ( $\rho_0 = 0$ ), it remains uncharged for all further time. Charge is transported through the region by the Ohmic current, but the net charge remains zero. Even if there happens to be an initial volume charge distribution, for times much longer than the relaxation time the volume charge density becomes negligibly small. In metals,  $\tau$  is on the order of  $10^{-19}$  sec, which is the justification of assuming the fields are zero within an electrode. Even though their large conductivity is not infinite, for times longer than the relaxation time  $\tau$ , the field solutions are the same as if a conductor were perfectly conducting.

The question remains as to where the relaxed charge goes. The answer is that it is carried by the conduction current to surfaces of discontinuity where the conductivity abruptly changes.

### 3-6-2 Uniformly Charged Sphere

A sphere of radius  $R_2$  with constant permittivity  $\epsilon$  and Ohmic conductivity  $\sigma$  is uniformly charged up to the radius  $R_1$  with charge density  $\rho_0$  at time  $t = 0$ , as in Figure 3-21. From  $R_1$  to  $R_2$  the sphere is initially uncharged so that it remains uncharged for all time. The sphere is surrounded by free space with permittivity  $\epsilon_0$  and zero conductivity.

From (4) we can immediately write down the volume charge distribution for all time,

$$\rho_f = \begin{cases} \rho_0 e^{-t/\tau}, & r < R_1 \\ 0, & r > R_1 \end{cases} \quad (5)$$

where  $\tau = \epsilon/\sigma$ . The total charge on the sphere remains constant,  $Q = \frac{4}{3}\pi R_1^3 \rho_0$ , but the volume charge is transported by the Ohmic current to the interface at  $r = R_2$  where it becomes a surface charge. Enclosing the system by a Gaussian surface with  $r > R_2$  shows that the external electric field is time independent,

$$E_r = \frac{Q}{4\pi\epsilon_0 r^2}, \quad r > R_2 \quad (6)$$

Similarly, applying Gaussian surfaces for  $r < R_1$  and  $R_1 < r < R_2$  yields

$$E_r = \begin{cases} \frac{\rho_0 r e^{-t/\tau}}{3\epsilon} = \frac{Q r e^{-t/\tau}}{4\pi\epsilon R_1^3}, & 0 < r < R_1 \\ \frac{Q e^{-t/\tau}}{4\pi\epsilon r^2}, & R_1 < r < R_2 \end{cases} \quad (7)$$

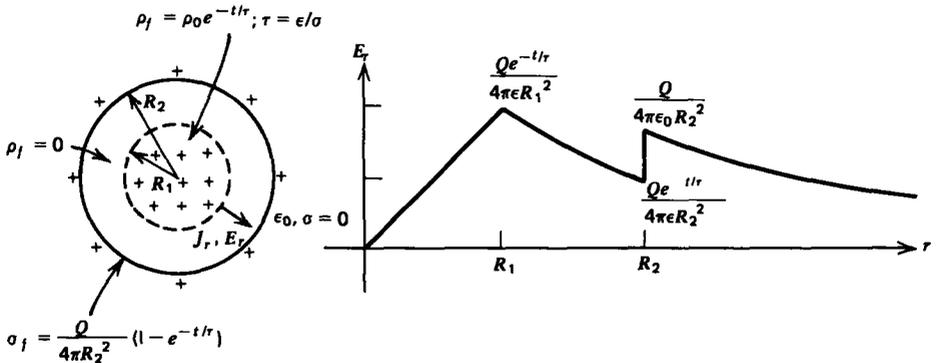


Figure 3-21 An initial volume charge distribution within an Ohmic conductor decays exponentially towards zero with relaxation time  $\tau = \epsilon/\sigma$  and appears as a surface charge at an interface of discontinuity. Initially uncharged regions are always uncharged with the charge transported through by the current.

The surface charge density at  $r = R_2$  builds up exponentially with time:

$$\begin{aligned}\sigma_f(r = R_2) &= \epsilon_0 E_r(r = R_{2+}) - \epsilon E_r(r = R_{2-}) \\ &= \frac{Q}{4\pi R_2^2} (1 - e^{-t/\tau})\end{aligned}\quad (8)$$

The charge is carried from the charged region ( $r < R_1$ ) to the surface at  $r = R_2$  via the conduction current with the charge density inbetween ( $R_1 < r < R_2$ ) remaining zero:

$$J_c = \sigma E_r = \begin{cases} \frac{\sigma Q r}{4\pi \epsilon R_1^3} e^{-t/\tau}, & 0 < r < R_1 \\ \frac{\sigma Q e^{-t/\tau}}{4\pi \epsilon r^2}, & R_1 < r < R_2 \\ 0, & r > R_2 \end{cases}\quad (9)$$

Note that the total current, conduction plus displacement, is zero everywhere:

$$-J_c = J_d = \epsilon \frac{\partial E_r}{\partial t} = \begin{cases} -\frac{Q r \sigma e^{-t/\tau}}{4\pi \epsilon R_1^3}, & 0 < r < R_1 \\ -\frac{\sigma Q e^{-t/\tau}}{4\pi \epsilon r^2}, & R_1 < r < R_2 \\ 0, & r > R_2 \end{cases}\quad (10)$$

### 3-6-3 Series Lossy Capacitor

#### (a) Charging transient

To exemplify the difference between resistive and capacitive behavior we examine the case of two different materials in series stressed by a step voltage first turned on at  $t = 0$ , as shown in Figure 3-22a. Since it takes time to charge up the interface, the interfacial surface charge cannot instantaneously change at  $t = 0$  so that it remains zero at  $t = 0_+$ . With no surface charge density, the displacement field is continuous across the interface so that the solution at  $t = 0_+$  is the same as for two lossless series capacitors independent of the conductivities:

$$D_x = \epsilon_1 E_1 = \epsilon_2 E_2\quad (11)$$

The voltage constraint requires that

$$\int_0^{a+b} E_x dx = E_1 a + E_2 b = V\quad (12)$$

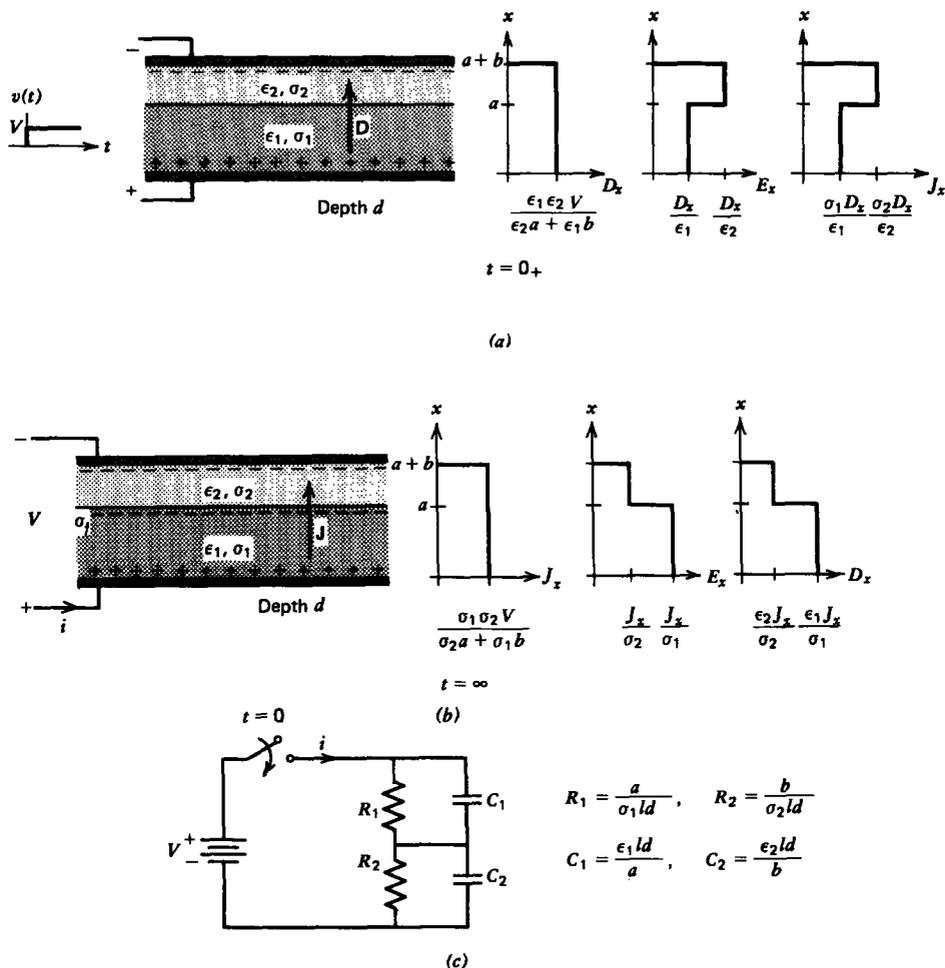


Figure 3-22 Two different lossy dielectric materials in series between parallel plate electrodes have permittivities and Ohmic conductivities that change abruptly across the interface. (a) At  $t = 0_+$ , right after a step voltage is applied, the interface is uncharged so that the displacement field is continuous with the solution the same as for two lossless dielectrics in series. (b) Since the current is discontinuous across the boundary between the materials, the interface will charge up. In the dc steady state the current is continuous. (c) Each region is equivalent to a resistor and capacitor in parallel.

so that the displacement field is

$$D_x(t = 0_+) = \frac{\epsilon_1 \epsilon_2 V}{\epsilon_2 a + \epsilon_1 b} \tag{13}$$

The total current from the battery is due to both conduction and displacement currents. At  $t = 0$ , the displacement current

is infinite (an impulse) as the displacement field instantaneously changes from zero to (13) to produce the surface charge on each electrode:

$$\sigma_f(x=0) = -\sigma_f(x=a+b) = D_x \quad (14)$$

After the voltage has been on a long time, the fields approach their steady-state values, as in Figure 3-22*b*. Because there are no more time variations, the current density must be continuous across the interface just the same as for two series resistors independent of the permittivities,

$$J_x(t \rightarrow \infty) = \sigma_1 E_1 = \sigma_2 E_2 = \frac{\sigma_1 \sigma_2 V}{\sigma_2 a + \sigma_1 b} \quad (15)$$

where we again used (12). The interfacial surface charge is now

$$\sigma_f(x=a) = \epsilon_2 E_2 - \epsilon_1 E_1 = \frac{(\epsilon_2 \sigma_1 - \epsilon_1 \sigma_2) V}{\sigma_2 a + \sigma_1 b} \quad (16)$$

What we have shown is that for early times the system is purely capacitive, while for long times the system is purely resistive. The inbetween transient interval is found by using (12) with charge conservation applied at the interface:

$$\begin{aligned} \mathbf{n} \cdot \left( \mathbf{J}_2 - \mathbf{J}_1 + \frac{d}{dt} (\mathbf{D}_2 - \mathbf{D}_1) \right) &= 0 \\ \Rightarrow \sigma_2 E_2 - \sigma_1 E_1 + \frac{d}{dt} [\epsilon_2 E_2 - \epsilon_1 E_1] &= 0 \end{aligned} \quad (17)$$

With (12) to relate  $E_2$  to  $E_1$  we obtain a single ordinary differential equation in  $E_1$ ,

$$\frac{dE_1}{dt} + \frac{E_1}{\tau} = \frac{\sigma_2 V}{\epsilon_2 a + \epsilon_1 b} \quad (18)$$

where the relaxation time is a weighted average of relaxation times of each material:

$$\tau = \frac{\epsilon_1 b + \epsilon_2 a}{\sigma_1 b + \sigma_2 a} \quad (19)$$

Using the initial condition of (13) the solutions for the fields are

$$\begin{aligned} E_1 &= \frac{\sigma_2 V}{\sigma_2 a + \sigma_1 b} (1 - e^{-t/\tau}) + \frac{\epsilon_2 V}{\epsilon_2 a + \epsilon_1 b} e^{-t/\tau} \\ E_2 &= \frac{\sigma_1 V}{\sigma_2 a + \sigma_1 b} (1 - e^{-t/\tau}) + \frac{\epsilon_1 V}{\epsilon_2 a + \epsilon_1 b} e^{-t/\tau} \end{aligned} \quad (20)$$

Note that as  $t \rightarrow \infty$  the solutions approach those of (15). The interfacial surface charge is

$$\sigma_f(x = a) = \varepsilon_2 E_2 - \varepsilon_1 E_1 = \frac{(\varepsilon_2 \sigma_1 - \varepsilon_1 \sigma_2)}{\sigma_2 a + \sigma_1 b} (1 - e^{-t/\tau}) V \quad (21)$$

which is zero at  $t = 0$  and agrees with (16) for  $t \rightarrow \infty$ .

The total current delivered by the voltage source is

$$\begin{aligned} i &= \left( \sigma_1 E_1 + \varepsilon_1 \frac{dE_1}{dt} \right) ld = \left( \sigma_2 E_2 + \varepsilon_2 \frac{dE_2}{dt} \right) ld \\ &= \left[ \frac{\sigma_1 \sigma_2}{\sigma_2 a + \sigma_1 b} + \left( \sigma_1 - \frac{\varepsilon_1}{\tau} \right) \left( \frac{\varepsilon_2}{\varepsilon_2 a + \varepsilon_1 b} - \frac{\sigma_2}{\sigma_2 a + \sigma_1 b} \right) e^{-t/\tau} \right. \\ &\quad \left. + \frac{\varepsilon_1 \varepsilon_2}{\varepsilon_2 a + \varepsilon_1 b} \delta(t) \right] ldV \end{aligned} \quad (22)$$

where the last term is the impulse current that instantaneously puts charge on each electrode in zero time at  $t = 0$ :

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases} \Rightarrow \int_{0^-}^{0^+} \delta(t) dt = 1$$

To reiterate, we see that for early times the capacitances dominate and that in the steady state the resistances dominate with the transition time depending on the relaxation times and geometry of each region. The equivalent circuit for the system is shown in Figure 3-22c as a series combination of a parallel resistor-capacitor for each region.

### (b) Open Circuit

Once the system is in the dc steady state, we instantaneously open the circuit so that the terminal current is zero. Then, using (22) with  $i = 0$ , we see that the fields decay independently in each region with the relaxation time of each region:

$$\begin{aligned} E_1 &= \frac{\sigma_2 V}{\sigma_2 a + \sigma_1 b} e^{-t/\tau_1}, \quad \tau_1 = \frac{\varepsilon_1}{\sigma_1} \\ E_2 &= \frac{\sigma_1 V}{\sigma_2 a + \sigma_1 b} e^{-t/\tau_2}, \quad \tau_2 = \frac{\varepsilon_2}{\sigma_2} \end{aligned} \quad (23)$$

The open circuit voltage and interfacial charge then decay as

$$\begin{aligned} V_\infty &= E_1 a + E_2 b = \frac{V}{\sigma_2 a + \sigma_1 b} [\sigma_2 a e^{-t/\tau_1} + \sigma_1 b e^{-t/\tau_2}] \\ \sigma_f &= \varepsilon_2 E_2 - \varepsilon_1 E_1 = \frac{V}{\sigma_2 a + \sigma_1 b} [\varepsilon_2 \sigma_1 e^{-t/\tau_2} - \varepsilon_1 \sigma_2 e^{-t/\tau_1}] \end{aligned} \quad (24)$$

**(c) Short Circuit**

If the dc steady-state system is instead short circuited, we set  $V = 0$  in (12) and (18),

$$\begin{aligned} E_1 a + E_2 b &= 0 \\ \frac{dE_1}{dt} + \frac{E_1}{\tau} &= 0 \end{aligned} \quad (25)$$

where  $\tau$  is still given by (19). Since at  $t = 0$  the interfacial surface charge cannot instantaneously change, the initial fields must obey the relation

$$\lim_{t \rightarrow 0} (\varepsilon_2 E_2 - \varepsilon_1 E_1) = -\left(\frac{\varepsilon_2 a}{b} + \varepsilon_1\right) E_1 = \frac{(\varepsilon_2 \sigma_1 - \varepsilon_1 \sigma_2) V}{\sigma_2 a + \sigma_1 b} \quad (26)$$

to yield the solutions

$$E_1 = -\frac{E_2 b}{a} = -\frac{(\varepsilon_2 \sigma_1 - \varepsilon_1 \sigma_2) b V}{(\varepsilon_2 a + \varepsilon_1 b)(\sigma_2 a + \sigma_1 b)} e^{-t/\tau} \quad (27)$$

The short circuit current and surface charge are then

$$\begin{aligned} i &= \left[ \left( \frac{\sigma_1 \varepsilon_2 - \varepsilon_1 \sigma_2}{\varepsilon_1 b + \varepsilon_2 a} \right)^2 \frac{abV}{(\sigma_2 a + \sigma_1 b)} e^{-t/\tau} - \frac{V \varepsilon_1 \varepsilon_2}{\varepsilon_2 a + \varepsilon_1 b} \delta(t) \right] ld \\ \sigma_f &= \varepsilon_2 E_2 - \varepsilon_1 E_1 = \frac{(\varepsilon_2 \sigma_1 - \varepsilon_1 \sigma_2)}{\sigma_2 a + \sigma_1 b} V e^{-t/\tau} \end{aligned} \quad (28)$$

The impulse term in the current is due to the instantaneous change in displacement field from the steady-state values found from (15) to the initial values of (26).

**(d) Sinusoidal Steady State**

Now rather than a step voltage, we assume that the applied voltage is sinusoidal,

$$v(t) = V_0 \cos \omega t \quad (29)$$

and has been on a long time.

The fields in each region are still only functions of time and not position. It is convenient to use complex notation so that all quantities are written in the form

$$\begin{aligned} v(t) &= \text{Re} (V_0 e^{j\omega t}) \\ E_1(t) &= \text{Re} (\hat{E}_1 e^{j\omega t}), \quad E_2(t) = \text{Re} (\hat{E}_2 e^{j\omega t}) \end{aligned} \quad (30)$$

Using carets above a term to designate a complex amplitude, the applied voltage condition of (12) requires

$$\hat{E}_1 a + \hat{E}_2 b = V_0 \quad (31)$$

while the interfacial charge conservation equation of (17) becomes

$$\begin{aligned} \sigma_2 \hat{E}_2 - \sigma_1 \hat{E}_1 + j\omega (\varepsilon_2 \hat{E}_2 - \varepsilon_1 \hat{E}_1) &= [\sigma_2 + j\omega \varepsilon_2] \hat{E}_2 \\ -[\sigma_1 + j\omega \varepsilon_1] \hat{E}_1 &= 0 \end{aligned} \quad (32)$$

The solutions are

$$\frac{\hat{E}_1}{(j\omega \varepsilon_2 + \sigma_2)} = \frac{\hat{E}_2}{(j\omega \varepsilon_1 + \sigma_1)} = \frac{V_0}{[b(\sigma_1 + j\omega \varepsilon_1) + a(\sigma_2 + j\omega \varepsilon_2)]} \quad (33)$$

which gives the interfacial surface charge amplitude as

$$\hat{\sigma}_f = \varepsilon_2 \hat{E}_2 - \varepsilon_1 \hat{E}_1 = \frac{(\varepsilon_2 \sigma_1 - \varepsilon_1 \sigma_2) V_0}{[b(\sigma_1 + j\omega \varepsilon_1) + a(\sigma_2 + j\omega \varepsilon_2)]} \quad (34)$$

As the frequency becomes much larger than the reciprocal relaxation times,

$$\omega \gg \frac{\sigma_1}{\varepsilon_1}, \quad \omega \gg \frac{\sigma_2}{\varepsilon_2} \quad (35)$$

the surface charge density goes to zero. This is because the surface charge cannot keep pace with the high-frequency alternations, and thus the capacitive component dominates. Thus, in experimental work charge accumulations can be prevented if the excitation frequencies are much faster than the reciprocal charge relaxation times.

The total current through the electrodes is

$$\begin{aligned} \hat{I} &= (\sigma_1 + j\omega \varepsilon_1) \hat{E}_1 ld = (\sigma_2 + j\omega \varepsilon_2) \hat{E}_2 ld \\ &= \frac{ld(\sigma_1 + j\omega \varepsilon_1)(\sigma_2 + j\omega \varepsilon_2) V_0}{[b(\sigma_1 + j\omega \varepsilon_1) + a(\sigma_2 + j\omega \varepsilon_2)]} \\ &= \frac{V_0}{\frac{R_2}{R_2 C_2 j\omega + 1} + \frac{R_1}{R_1 C_1 j\omega + 1}} \end{aligned} \quad (36)$$

with the last result easily obtained from the equivalent circuit in Figure 3-22c.

### 3-6-4 Distributed Systems

#### (a) Governing Equations

In all our discussions we have assumed that the electrodes are perfectly conducting so that they have no resistance and the electric field terminates perpendicularly. Consider now the parallel plate geometry shown in Figure 3-23a, where the electrodes have a large but finite conductivity  $\sigma_c$ . The electrodes are no longer equi-potential surfaces since as the current passes along the conductor an Ohmic  $iR$  drop results.

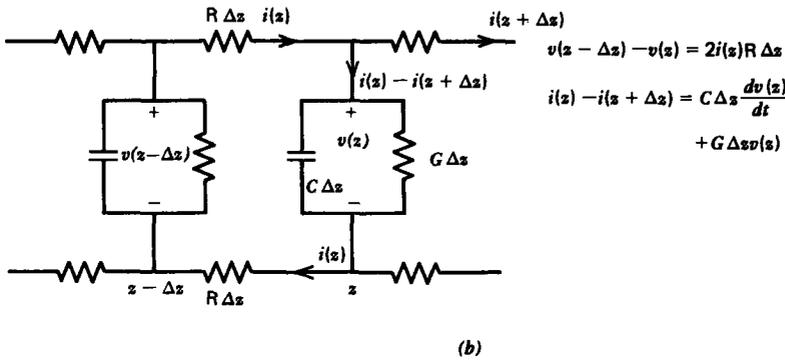
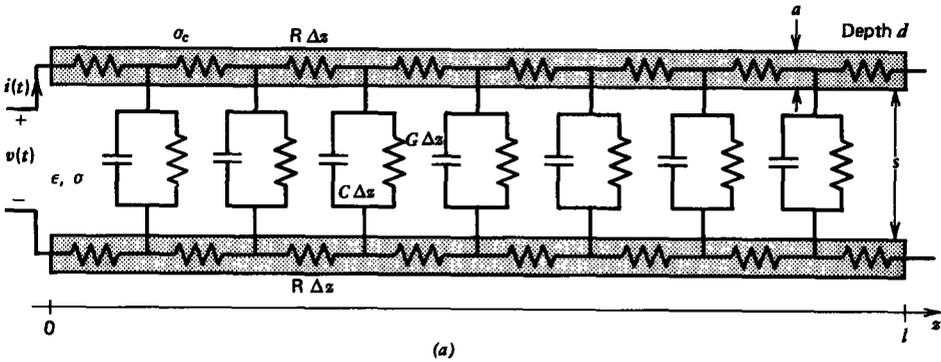


Figure 3-23 Lossy parallel plate electrodes with finite Ohmic conductivity  $\sigma_c$  enclose a lossy dielectric with permittivity  $\epsilon$  and conductivity  $\sigma$ . (a) This system can be modeled by a distributed resistor-capacitor network. (b) Kirchoff's voltage and current laws applied to a section of length  $\Delta z$  allow us to describe the system by partial differential equations.

The current is also shunted through the lossy dielectric so that less current flows at the far end of the conductor than near the source. We can find approximate solutions by breaking the continuous system into many small segments of length  $\Delta z$ . The electrode resistance of this small section is

$$R \Delta z = \frac{\Delta z}{\sigma_c a d} \tag{37}$$

where  $R = 1/(\sigma_c a d)$  is just the resistance per unit length. We have shown in the previous section that the dielectric can be modeled as a parallel resistor-capacitor combination,

$$C \Delta z = \frac{\epsilon d \Delta z}{s}, \quad \frac{1}{G \Delta z} = \frac{s}{\sigma d \Delta z} \tag{38}$$

$C$  is the capacitance per unit length and  $G$  is the conductance per unit length where the conductance is the reciprocal of the

resistance. It is more convenient to work with the conductance because it is in parallel with the capacitance.

We apply Kirchoff's voltage and current laws for the section of equivalent circuit shown in Figure 3-23b:

$$v(z - \Delta z) - v(z) = 2i(z)R \Delta z \quad (39)$$

$$i(z) - i(z + \Delta z) = C \Delta z \frac{dv(z)}{dt} + G \Delta z v(z)$$

The factor of 2 in the upper equation arises from the equal series resistances of the upper and lower conductors. Dividing through by  $\Delta z$  and taking the limit as  $\Delta z$  becomes infinitesimally small yields the partial differential equations

$$-\frac{\partial v}{\partial z} = 2iR \quad (40)$$

$$-\frac{\partial i}{\partial z} = C \frac{\partial v}{\partial t} + Gv$$

Taking  $\partial/\partial z$  of the upper equation allows us to substitute in the lower equation to eliminate  $i$ ,

$$\frac{\partial^2 v}{\partial z^2} = 2RC \frac{\partial v}{\partial t} + 2RGv \quad (41)$$

which is called a transient diffusion equation. Equations (40) and (41) are also valid for any geometry whose cross sectional area remains constant over its length. The  $2R$  represents the series resistance per unit length of both electrodes, while  $C$  and  $G$  are the capacitance and conductance per unit length of the dielectric medium.

#### (b) Steady State

If a dc voltage  $V_0$  is applied, the steady-state voltage is

$$\frac{d^2 v}{dz^2} - 2RGv = 0 \Rightarrow v = A_1 \sinh \sqrt{2RG}z + A_2 \cosh \sqrt{2RG}z \quad (42)$$

where the constants are found by the boundary conditions at  $z = 0$  and  $z = l$ ,

$$v(z = 0) = V_0, \quad i(z = l) = 0 \quad (43)$$

We take the  $z = l$  end to be open circuited. Solutions are

$$v(z) = V_0 \frac{\cosh \sqrt{2RG}(z - l)}{\cosh \sqrt{2RG}l} \quad (44)$$

$$i(z) = -\frac{1}{2R} \frac{dv}{dz} = V_0 \sqrt{\frac{G}{2R}} \frac{\sinh \sqrt{2RG}(z - l)}{\cosh \sqrt{2RG}l}$$

**(c) Transient Solution**

If this dc voltage is applied as a step at  $t = 0$ , it takes time for the voltage and current to reach these steady-state distributions. Because (41) is linear, we can use superposition and guess a solution for the voltage that is the sum of the steady-state solution in (44) and a transient solution that dies off with time:

$$v(z, t) = \frac{V_0 \cosh \sqrt{2RG}(z-l)}{\cosh \sqrt{2RG}l} + \hat{v}(z) e^{-\alpha t} \quad (45)$$

At this point we do not know the function  $\hat{v}(z)$  or  $\alpha$ . Substituting the assumed solution of (45) back into (41) yields the ordinary differential equation

$$\frac{d^2 \hat{v}}{dz^2} + p^2 \hat{v} = 0, \quad p^2 = 2RC\alpha - 2RG \quad (46)$$

which has the trigonometric solutions

$$\hat{v}(z) = a_1 \sin pz + a_2 \cos pz \quad (47)$$

Since the time-independent part of (45) already satisfies the boundary conditions at  $z = 0$ , the transient part must be zero there so that  $a_2 = 0$ . The transient contribution to the current  $i$ , found from (40),

$$i(z, t) = V_0 \sqrt{\frac{G}{2R}} \frac{\sinh \sqrt{2RG}(z-l)}{\cosh \sqrt{2RG}l} + \hat{i}(z) e^{-\alpha t} \quad (48)$$

$$\hat{i}(z) = -\frac{1}{2R} \frac{d\hat{v}(z)}{dz} = -\frac{pa_1}{2R} \cos pz$$

must still be zero at  $z = l$ , which means that  $pl$  must be an odd integer multiple of  $\pi/2$ ,

$$pl = (2n+1) \frac{\pi}{2} \Rightarrow \alpha_n = \frac{1}{2RC} \left( (2n+1) \frac{\pi}{2l} \right)^2 + \frac{G}{C}, \quad n = 0, 1, 2, \dots \quad (49)$$

Since the boundary conditions allow an infinite number of values of  $\alpha$ , the most general solution is the superposition of all allowed solutions:

$$v(z, t) = V_0 \frac{\cosh \sqrt{2RG}(z-l)}{\cosh \sqrt{2RG}l} + \sum_{n=0}^{\infty} A_n \sin (2n+1) \frac{\pi z}{2l} e^{-\alpha_n t} \quad (50)$$

This solution satisfies the boundary conditions but not the initial conditions at  $t = 0$  when the voltage is first turned on. Before the voltage source is applied, the voltage distribution throughout the system is zero. It must remain zero right after

being turned on otherwise the time derivative in (40) would be infinite, which requires nonphysical infinite currents. Thus we impose the initial condition

$$v(z, t = 0) = 0 = V_0 \frac{\cosh \sqrt{2RG}(z-l)}{\cosh \sqrt{2RG}l} + \sum_{n=0}^{\infty} A_n \sin(2n+1) \frac{\pi z}{2l} \quad (51)$$

We can solve for the amplitudes  $A_n$  by multiplying (51) through by  $\sin(2m+1) \pi z/2l$  and then integrating over  $z$  from 0 to  $l$ :

$$0 = \frac{V_0}{\cosh \sqrt{2RG}l} \int_0^l \cosh \sqrt{2RG}(z-l) \sin(2m+1) \frac{\pi z}{2l} dz + \int_0^l \sum_{n=0}^{\infty} A_n \sin(2n+1) \frac{\pi z}{2l} \sin(2m+1) \frac{\pi z}{2l} dz \quad (52)$$

The first term is easily integrated by writing the hyperbolic cosine in terms of exponentials,\* while the last term integrates to zero for all values of  $m$  not equal to  $n$  so that the amplitudes are

$$A_n = -\frac{1}{l^2} \frac{\pi V_0 (2n+1)}{2RG + [(2n+1) \pi/2l]^2} \quad (53)$$

The total solutions are then

$$v(z, t) = \frac{V_0 \cosh \sqrt{2RG}(z-l)}{\cosh \sqrt{2RG}l} - \frac{\pi V_0}{l^2} \sum_{n=0}^{\infty} \frac{(2n+1) \sin[(2n+1) (\pi z/2l)] e^{-\alpha_n t}}{2RG + [(2n+1) (\pi/2l)]^2}$$

$$i(z, t) = -\frac{1}{2R} \frac{\partial v}{\partial z} = -\frac{V_0 \sqrt{G/2R} \sinh \sqrt{2RG}(z-l)}{\cosh \sqrt{2RG}l} + \frac{\pi^2 V_0}{4l^3 R} \sum_{n=0}^{\infty} \frac{(2n+1)^2 \cos[(2n+1) (\pi z/2l)] e^{-\alpha_n t}}{2RG + [(2n+1) (\pi/2l)]^2} \quad (54)$$

\*  $\int \cosh a(z-l) \sin bz \, dz$

$$= \frac{1}{a^2 + b^2} [a \sin bz \sinh a(z-l) - b \cos bz \cosh a(z-l)]$$

$$\int_0^l \sin(2n+1)bz \sin(2m+1)bz \, dz = \begin{cases} 0 & m \neq n \\ l/2 & m = n \end{cases}$$

The fundamental time constant corresponds to the smallest value of  $\alpha$ , which is when  $n = 0$ :

$$\tau_0 = \frac{1}{\alpha_0} = \frac{C}{G + \frac{1}{2R} \left(\frac{\pi}{2l}\right)^2} \tag{55}$$

For times long compared to  $\tau_0$  the system is approximately in the steady state. Because of the fast exponential decrease for times greater than zero, the infinite series in (54) can often be approximated by the first term. These solutions are plotted in Figure 3-24 for the special case where  $G = 0$ . Then the voltage distribution builds up from zero to a constant value diffusing in from the left. The current near  $z = 0$  is initially very large. As time increases, with  $G = 0$ , the current everywhere decreases towards a zero steady state.

### 3-6-5 Effects of Convection

We have seen that in a stationary medium any initial charge density decays away to a surface of discontinuity. We now wish to focus attention on a dc steady-state system of a conducting medium moving at constant velocity  $U\hat{i}_x$ , as in Figure 3-25. A source at  $x = 0$  maintains a constant charge density  $\rho_0$ . Then (3) in the dc steady state with constant

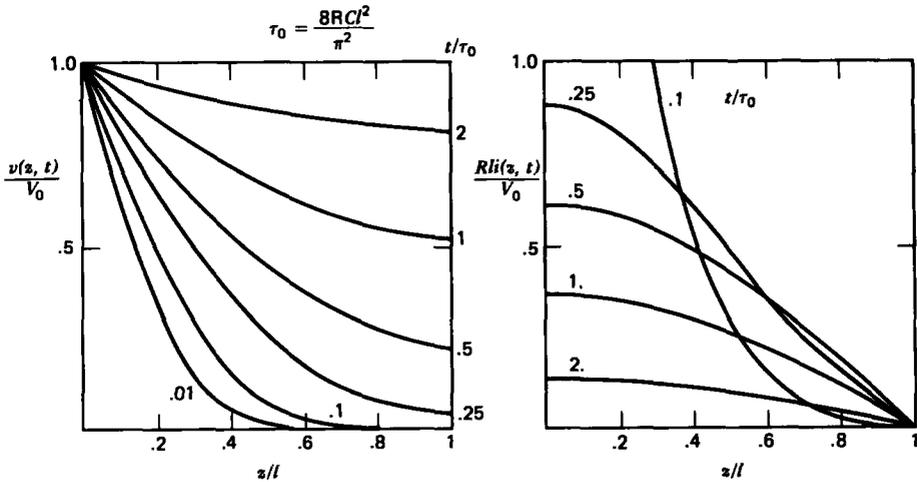


Figure 3-24 The transient voltage and current spatial distributions for various times for the lossy line in Figure 3-23a with  $G = 0$  for a step voltage excitation at  $z = 0$  with the  $z = l$  end open circuited. The diffusion effects arise because of the lossy electrodes where the longest time constant is  $\tau_0 = 8RCl^2/\pi^2$ .

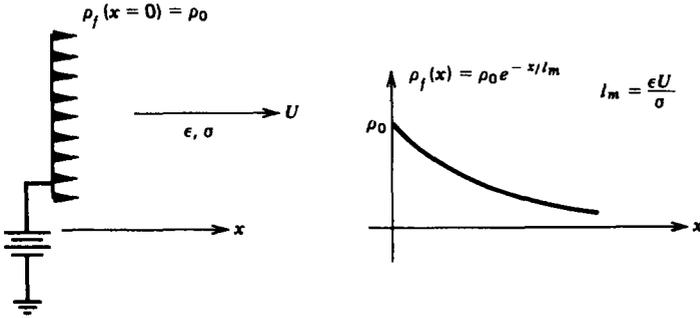


Figure 3-25 A moving conducting material with velocity  $U\mathbf{i}_x$  tends to take charge injected at  $x=0$  with it. The steady-state charge density decreases exponentially from the source.

velocity becomes

$$\frac{d\rho_f}{dx} + \frac{\sigma}{\epsilon U} \rho_f = 0 \quad (56)$$

which has exponentially decaying solutions

$$\rho_f = \rho_0 e^{-x/l_m}, \quad l_m = \frac{\epsilon U}{\sigma} \quad (57)$$

where  $l_m$  represents a characteristic spatial decay length. If the system has cross-sectional area  $A$ , the total charge  $q$  in the system is

$$q = \int_0^{\infty} \rho_f A \, dx = \rho_0 l_m A \quad (58)$$

### 3-6-6 The Earth and its Atmosphere as a Leaky Spherical Capacitor\*

In fair weather, at the earth's surface exists a dc electric field with approximate strength of 100 V/m directed radially toward the earth's center. The magnitude of the electric field decreases with height above the earth's surface because of the nonuniform electrical conductivity  $\sigma(r)$  of the atmosphere approximated as

$$\sigma(r) = \sigma_0 + a(r - R)^2 \text{ siemen/m} \quad (59)$$

where measurements have shown that

$$\begin{aligned} \sigma_0 &\approx 3 \times 10^{-14} \\ a &\approx .5 \times 10^{-20} \end{aligned} \quad (60)$$

\* M. A. Uman, "The Earth and Its Atmosphere as a Leaky Spherical Capacitor," *Am. J. Phys.* V. 42, Nov. 1974, pp. 1033-1035.

and  $R \approx 6 \times 10^6$  meter is the earth's radius. The conductivity increases with height because of cosmic radiation in the lower atmosphere. Because of solar radiation the atmosphere acts as a perfect conductor above 50 km.

In the dc steady state, charge conservation of Section 3-2-1 with spherical symmetry requires

$$\nabla \cdot \mathbf{J} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 J_r) = 0 \Rightarrow J_r = \sigma(r) E_r = \frac{C}{r^2} \quad (61)$$

where the constant of integration  $C$  is found by specifying the surface electric field  $E_r(R) \approx -100$  V/m

$$J_r(r) = \frac{\sigma(R) E_r(R) R^2}{r^2} \quad (62)$$

At the earth's surface the current density is then

$$J_r(R) = \sigma(R) E_r(R) = \sigma_0 E_r(R) \approx -3 \times 10^{-12} \text{ amp/m}^2 \quad (63)$$

The total current directed radially inwards over the whole earth is then

$$I = |J_r(R) 4\pi R^2| \approx 1350 \text{ amp} \quad (64)$$

The electric field distribution throughout the atmosphere is found from (62) as

$$E_r(r) = \frac{J_r(r)}{\sigma(r)} = \frac{\sigma(R) E_r(R) R^2}{r^2 \sigma(r)} \quad (65)$$

The surface charge density on the earth's surface is

$$\sigma_f(r=R) = \epsilon_0 E_r(R) \approx -8.85 \times 10^{-10} \text{ Coul/m}^2 \quad (66)$$

This negative surface charge distribution (remember:  $E_r(r) < 0$ ) is balanced by positive volume charge distribution throughout the atmosphere

$$\begin{aligned} \rho_f(r) &= \epsilon_0 \nabla \cdot \mathbf{E} = \frac{\epsilon_0}{r^2} \frac{\partial}{\partial r} (r^2 E_r) = \frac{\epsilon_0 \sigma(R) E_r(R) R^2}{r^2} \frac{d}{dr} \left( \frac{1}{\sigma(r)} \right) \\ &= \frac{-\epsilon_0 \sigma(R) E_r(R) R^2}{r^2 (\sigma(r))^2} 2a(r-R) \end{aligned} \quad (67)$$

The potential difference between the upper atmosphere and the earth's surface is

$$\begin{aligned} V &= - \int_R^\infty E_r(r) dr \\ &= -\sigma(R) E_r(R) R^2 \int_R^\infty \frac{dr}{r^2 [\sigma_0 + a(r-R)]^2} \end{aligned}$$

$$\begin{aligned}
 &= -\frac{\sigma(R)E_r(R)R^2}{a} \left\{ -\frac{R}{\left(R^2 + \frac{\sigma_0}{a}\right)^2} \ln \left[ \frac{(r-R)^2 + \frac{\sigma_0}{a}}{r^2} \right] \right. \\
 &\quad \left. - \frac{1}{r\left(R^2 + \frac{\sigma_0}{a}\right)} + \frac{\left(R^2 - \frac{\sigma_0}{a}\right)}{\sqrt{\frac{\sigma_0}{a}\left(R^2 + \frac{\sigma_0}{a}\right)^2}} \tan^{-1} \frac{(r-R)}{\sqrt{\frac{\sigma_0}{a}}} \right\} \Bigg|_{r=R}^{\infty} \\
 &= -\frac{\sigma(R)E_r(R)}{a\left(R^2 + \frac{\sigma_0}{a}\right)^2} R^2 \left\{ R \ln \frac{\sigma_0}{aR^2} + \frac{\left(R^2 + \frac{\sigma_0}{a}\right)}{R} + \frac{\frac{\pi}{2}\left(R^2 - \frac{\sigma_0}{a}\right)}{\sqrt{\frac{\sigma_0}{a}}} \right\} \quad (68)
 \end{aligned}$$

Using the parameters of (60), we see that  $\sigma_0/a \ll R^2$  so that (68) approximately reduces to

$$V \approx -\frac{\sigma_0 E_r(R)}{aR^2} \left\{ R \left( \ln \frac{\sigma_0}{aR^2} + 1 \right) + \frac{\pi R^2}{2\sqrt{\frac{\sigma_0}{a}}} \right\} \quad (69)$$

≈ 384,000 volts

If the earth's charge were not replenished, the current flow would neutralize the charge at the earth's surface with a time constant of order

$$\tau = \frac{\epsilon_0}{\sigma_0} \approx 300 \text{ seconds} \quad (70)$$

It is thought that localized stormy regions simultaneously active all over the world serve as "batteries" to keep the earth charged via negatively charged lightning to ground and corona at ground level, producing charge that moves from ground to cloud. This thunderstorm current must be upwards and balances the downwards fair weather current of (64).

### 3.7 FIELD-DEPENDENT SPACE CHARGE DISTRIBUTIONS

A stationary Ohmic conductor with constant conductivity was shown in Section 3-6-1 to not support a steady-state volume charge distribution. This occurs because in our classical Ohmic model in Section 3-2-2c one species of charge (e.g., electrons in metals) move relative to a stationary background species of charge with opposite polarity so that charge neutrality is maintained. However, if only one species of

charge is injected into a medium, a net steady-state volume charge distribution can result.

Because of the electric force, this distribution of volume charge  $\rho_f$  contributes to and also in turn depends on the electric field. It now becomes necessary to simultaneously satisfy the coupled electrical and mechanical equations.

### 3-7-1 Space Charge Limited Vacuum Tube Diode

In vacuum tube diodes, electrons with charge  $-e$  and mass  $m$  are boiled off the heated cathode, which we take as our zero potential reference. This process is called thermionic emission. A positive potential  $V_0$  applied to the anode at  $x=l$  accelerates the electrons, as in Figure 3-26. Newton's law for a particular electron is

$$m \frac{dv}{dt} = -eE = e \frac{dV}{dx} \tag{1}$$

In the dc steady state the velocity of the electron depends only on its position  $x$  so that

$$m \frac{dv}{dt} = m \frac{dv}{dx} \frac{dx}{dt} = mv \frac{dv}{dx} \Rightarrow \frac{d}{dx} \left( \frac{1}{2} mv^2 \right) = \frac{d}{dx} (eV) \tag{2}$$

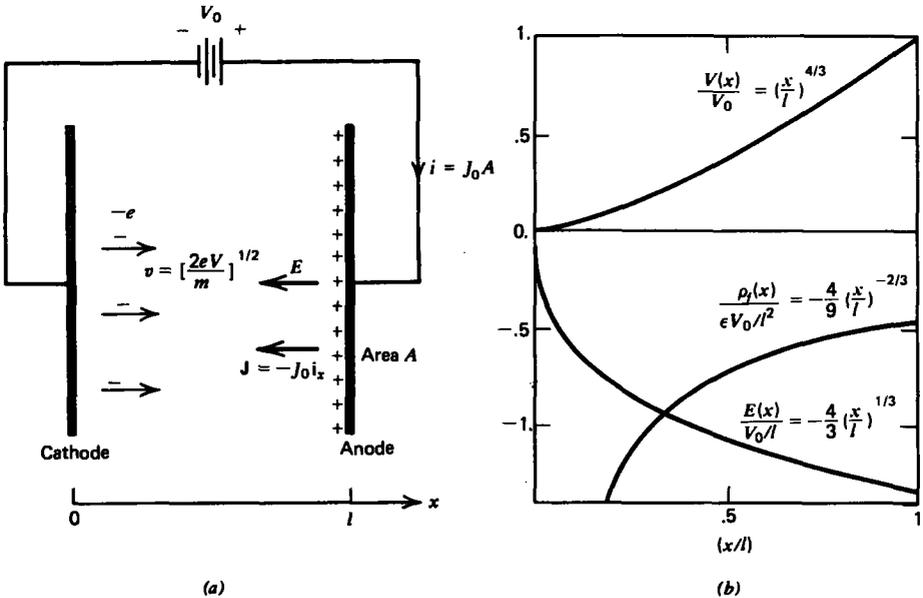


Figure 3-26 Space charge limited vacuum tube diode. (a) Thermionic injection of electrons from the heated cathode into vacuum with zero initial velocity. The positive anode potential attracts the electrons whose acceleration is proportional to the local electric field. (b) Steady-state potential, electric field, and volume charge distributions.

With this last equality, we have derived the energy conservation theorem

$$\frac{d}{dx} [\frac{1}{2}mv^2 - eV] = 0 \Rightarrow \frac{1}{2}mv^2 - eV = \text{const} \quad (3)$$

where we say that the kinetic energy  $\frac{1}{2}mv^2$  plus the potential energy  $-eV$  is the constant total energy. We limit ourselves here to the simplest case where the injected charge at the cathode starts out with zero velocity. Since the potential is also chosen to be zero at the cathode, the constant in (3) is zero. The velocity is then related to the electric potential as

$$v = \left( \frac{2e}{m} V \right)^{1/2} \quad (4)$$

In the time-independent steady state the current density is constant,

$$\nabla \cdot \mathbf{J} = 0 \Rightarrow \frac{dJ_x}{dx} = 0 \Rightarrow \mathbf{J} = -J_0 \mathbf{i}_x \quad (5)$$

and is related to the charge density and velocity as

$$J_0 = -\rho_f v \Rightarrow \rho_f = -J_0 \left( \frac{m}{2e} \right)^{1/2} V^{-1/2} \quad (6)$$

Note that the current flows from anode to cathode, and thus is in the negative  $x$  direction. This minus sign is incorporated in (5) and (6) so that  $J_0$  is positive. Poisson's equation then requires that

$$\nabla^2 V = \frac{-\rho_f}{\epsilon} \Rightarrow \frac{d^2 V}{dx^2} = \frac{J_0}{\epsilon} \left( \frac{m}{2e} \right)^{1/2} V^{-1/2} \quad (7)$$

Power law solutions to this nonlinear differential equation are guessed of the form

$$V = Bx^p \quad (8)$$

which when substituted into (7) yields

$$Bp(p-1)x^{p-2} = \frac{J_0}{\epsilon} \left( \frac{m}{2e} \right)^{1/2} B^{-1/2} x^{-p/2} \quad (9)$$

For this assumed solution to hold for all  $x$  we require that

$$p-2 = -\frac{p}{2} \Rightarrow p = \frac{4}{3} \quad (10)$$

which then gives us the amplitude  $B$  as

$$B = \left[ \frac{9}{4} \frac{J_0}{\epsilon} \left( \frac{m}{2e} \right)^{1/2} \right]^{2/3} \quad (11)$$

so that the potential is

$$V(x) = \left[ \frac{9}{4} \frac{J_0}{\epsilon} \left( \frac{m}{2e} \right)^{1/2} \right]^{2/3} x^{4/3} \quad (12)$$

The potential is zero at the cathode, as required, while the anode potential  $V_0$  requires the current density to be

$$\begin{aligned} V(x=l) = V_0 &= \left[ \frac{9}{4} \frac{J_0}{\epsilon} \left( \frac{m}{2e} \right)^{1/2} \right]^{2/3} l^{4/3} \\ \Rightarrow J_0 &= \frac{4}{9} \frac{\epsilon}{l^2} \left( \frac{2e}{m} \right)^{1/2} V_0^{3/2} \end{aligned} \quad (13)$$

which is called the Langmuir-Child law.

The potential, electric field, and charge distributions are then concisely written as

$$\begin{aligned} V(x) &= V_0 \left( \frac{x}{l} \right)^{4/3} \\ E(x) &= -\frac{dV(x)}{dx} = -\frac{4}{3} \frac{V_0}{l} \left( \frac{x}{l} \right)^{1/3} \\ \rho_f(x) &= \epsilon \frac{dE(x)}{dx} = -\frac{4}{9} \epsilon \frac{V_0}{l^2} \left( \frac{x}{l} \right)^{-2/3} \end{aligned} \quad (14)$$

and are plotted in Figure 3-26*b*. We see that the charge density at the cathode is infinite but that the total charge between the electrodes is finite,

$$q_T = \int_{x=0}^l \rho_f(x) A dx = -\frac{4}{3} \epsilon \frac{V_0}{l} A \quad (15)$$

being equal in magnitude but opposite in sign to the total surface charge on the anode:

$$q_A = \sigma_f(x=l)A = -\epsilon E(x=l)A = +\frac{4}{3} \epsilon \frac{V_0}{l} A \quad (16)$$

There is no surface charge on the cathode because the electric field is zero there.

This displacement  $x$  of each electron can be found by substituting the potential distribution of (14) into (4),

$$v = \frac{dx}{dt} = \left( \frac{2eV_0}{m} \right)^{1/2} \left( \frac{x}{l} \right)^{2/3} \Rightarrow \frac{dx}{x^{2/3}} = \left( \frac{2eV_0}{ml^{4/3}} \right)^{1/2} dt \quad (17)$$

which integrates to

$$x = \frac{1}{27} \left( \frac{2eV_0}{ml^{4/3}} \right)^{3/2} t^3 \quad (18)$$

The charge transit time  $\tau$  between electrodes is found by solving (18) with  $x = l$ :

$$\tau = 3l \left( \frac{m}{2eV_0} \right)^{1/2} \quad (19)$$

For an electron ( $m = 9.1 \times 10^{-31}$  kg,  $e = 1.6 \times 10^{-19}$  coul) with 100 volts applied across  $l = 1$  cm ( $10^{-2}$  m) this time is  $\tau \approx 5 \times 10^{-9}$  sec. The peak electron velocity when it reaches the anode is  $v(x=l) \approx 6 \times 10^6$  m/sec, which is approximately 50 times less than the vacuum speed of light.

Because of these fast response times vacuum tube diodes are used in alternating voltage applications for rectification as current only flows when the anode is positive and as nonlinear circuit elements because of the three-halves power law of (13) relating current and voltage.

### 3-7-2 Space Charge Limited Conduction in Dielectrics

Conduction properties of dielectrics are often examined by injecting charge. In Figure 3-27, an electron beam with current density  $\mathbf{J} = -J_0 \mathbf{i}_x$  is suddenly turned on at  $t = 0$ .<sup>\*</sup> In media, the acceleration of the charge is no longer proportional to the electric field. Rather, collisions with the medium introduce a frictional drag so that the velocity is proportional to the electric field through the electron mobility  $\mu$ :

$$\mathbf{v} = -\mu \mathbf{E} \quad (20)$$

As the electrons penetrate the dielectric, the space charge front is a distance  $s$  from the interface where (20) gives us

$$ds/dt = -\mu E(s) \quad (21)$$

Although the charge density is nonuniformly distributed behind the wavefront, the total charge  $Q$  within the dielectric behind the wave front at time  $t$  is related to the current density as

$$J_0 A = \rho_f \mu E_x A = -Q/t \Rightarrow Q = -J_0 A t \quad (22)$$

Gauss's law applied to the rectangular surface enclosing all the charge within the dielectric then relates the fields at the interface and the charge front to this charge as

$$\oint_S \epsilon \mathbf{E} \cdot d\mathbf{S} = [\epsilon E(s) - \epsilon_0 E(0)] A = Q = -J_0 A t \quad (23)$$

<sup>\*</sup> See P. K. Watson, J. M. Schneider, and H. R. Till, *Electrohydrodynamic Stability of Space Charge Limited Currents In Dielectric Liquids. II. Experimental Study*, Phys. Fluids **13** (1970), p. 1955.

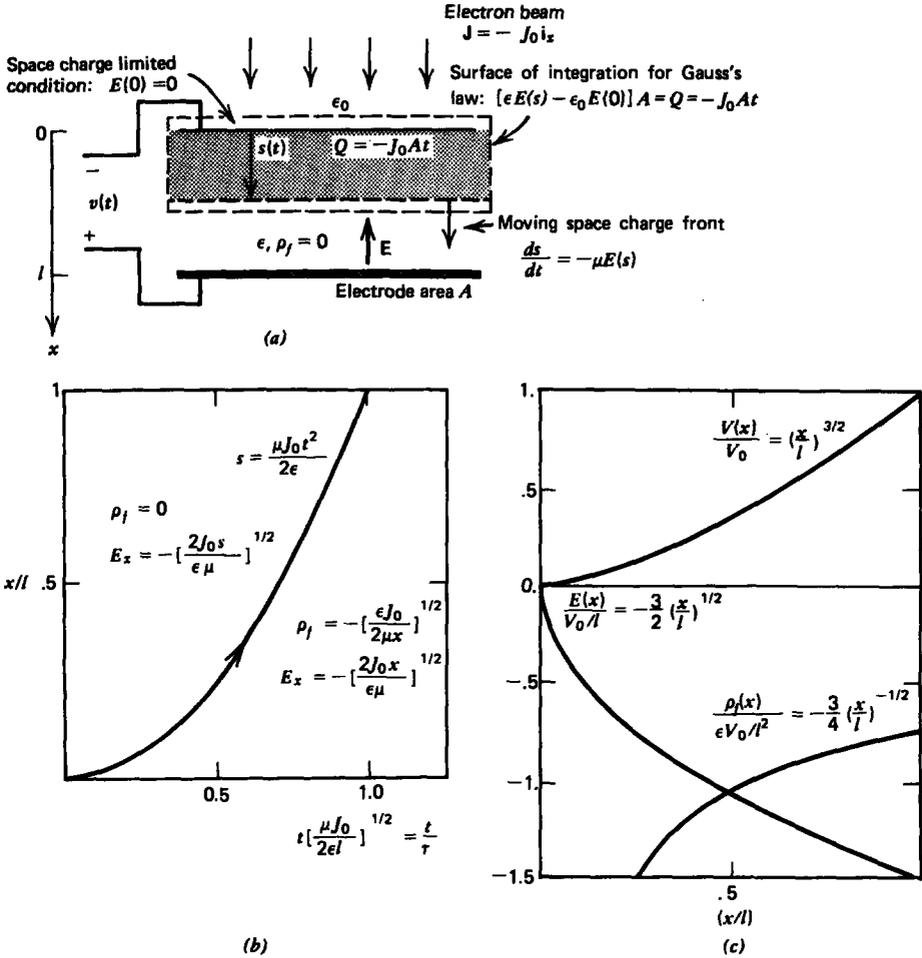


Figure 3-27 (a) An electron beam carrying a current  $-J_0 i_x$  is turned on at  $t = 0$ . The electrons travel through the dielectric with mobility  $\mu$ . (b) The space charge front, at a distance  $s$  in front of the space charge limited interface at  $x = 0$ , travels towards the opposite electrode. (c) After the transit time  $t_c = [2\epsilon l/\mu J_0]^{1/2}$  the steady-state potential, electric field, and space charge distributions.

The maximum current flows when  $E(0) = 0$ , which is called space charge limited conduction. Then using (23) in (21) gives us the time dependence of the space charge front:

$$\frac{ds}{dt} = \frac{\mu J_0 t}{\epsilon} \Rightarrow s(t) = \frac{\mu J_0 t^2}{2\epsilon} \quad (24)$$

Behind the front Gauss's law requires

$$\frac{dE_x}{dx} = \frac{\rho_f}{\epsilon} = \frac{J_0}{\epsilon\mu E_x} \Rightarrow E_x \frac{dE_x}{dx} = \frac{J_0}{\epsilon\mu} \quad (25)$$

while ahead of the moving space charge the charge density is zero so that the current is carried entirely by displacement current and the electric field is constant in space. The spatial distribution of electric field is then obtained by integrating (25) to

$$E_x = \begin{cases} -\sqrt{2J_0x/\epsilon\mu}, & 0 \leq x \leq s(t) \\ -\sqrt{2J_0s/\epsilon\mu}, & s(t) \leq x \leq l \end{cases} \quad (26)$$

while the charge distribution is

$$\rho_f = \epsilon \frac{dE_x}{dx} = \begin{cases} -\sqrt{\epsilon J_0/(2\mu x)}, & 0 \leq x \leq s(t) \\ 0, & s(t) \leq x \leq l \end{cases} \quad (27)$$

as indicated in Figure 3-27b.

The time dependence of the voltage across the dielectric is then

$$\begin{aligned} v(t) &= \int_0^l E_x dx = \int_0^{s(t)} \sqrt{\frac{2J_0x}{\epsilon\mu}} dx + \int_{s(t)}^l \sqrt{\frac{2J_0s}{\epsilon\mu}} dx \\ &= \frac{J_0 t}{\epsilon} - \frac{\mu J_0^2 t^3}{6\epsilon^2}, \quad s(t) \leq l \end{aligned} \quad (28)$$

These transient solutions are valid until the space charge front  $s$ , given by (24), reaches the opposite electrode with  $s = l$  at time

$$\tau = \sqrt{2\epsilon l/\mu J_0} \quad (29)$$

Thereafter, the system is in the dc steady state with the terminal voltage  $V_0$  related to the current density as

$$J_0 = \frac{9}{8} \frac{\epsilon\mu V_0^2}{l^3} \quad (30)$$

which is the analogous Langmuir-Child's law for collision dominated media. The steady-state electric field and space charge density are then concisely written as

$$E_x = -\frac{3}{2} \frac{V_0}{l} \left(\frac{x}{l}\right)^{1/2}, \quad \rho_f = \epsilon \frac{dE}{dx} = -\frac{3}{4} \frac{\epsilon V_0}{l^2} \left(\frac{x}{l}\right)^{-1/2} \quad (31)$$

and are plotted in Figure 3-27c.

In liquids a typical ion mobility is of the order of  $10^{-7} \text{ m}^2/(\text{volt-sec})$  with a permittivity of  $\epsilon = 2\epsilon_0 \approx 1.77 \times 10^{-11} \text{ farad/m}$ . For a spacing of  $l = 10^{-2} \text{ m}$  with a potential difference of  $V_0 = 10^4 \text{ V}$  the current density of (30) is  $J_0 \approx 2 \times 10^{-4} \text{ amp/m}^2$  with the transit time given by (29)  $\tau \approx 0.133 \text{ sec}$ . Charge transport times in collision dominated media are much larger than in vacuum.

### 3-8 ENERGY STORED IN A DIELECTRIC MEDIUM

The work needed to assemble a charge distribution is stored as potential energy in the electric field because if the charges are allowed to move this work can be regained as kinetic energy or mechanical work.

#### 3-8-1 Work Necessary to Assemble a Distribution of Point Charges

##### (a) Assembling the Charges

Let us compute the work necessary to bring three already existing free charges  $q_1$ ,  $q_2$ , and  $q_3$  from infinity to any position, as in Figure 3-28. It takes no work to bring in the first charge as there is no electric field present. The work necessary to bring in the second charge must overcome the field due to the first charge, while the work needed to bring in the third charge must overcome the fields due to both other charges. Since the electric potential developed in Section 2-5-3 is defined as the work per unit charge necessary to bring a point charge in from infinity, the total work necessary to bring in the three charges is

$$W = q_1(0) + q_2 \left( \frac{q_1}{4\pi\epsilon r_{12}} \right) + q_3 \left( \frac{q_1}{4\pi\epsilon r_{13}} + \frac{q_2}{4\pi\epsilon r_{23}} \right) \quad (1)$$

where the final distances between the charges are defined in Figure 3-28 and we use the permittivity  $\epsilon$  of the medium. We can rewrite (1) in the more convenient form

$$W = \frac{1}{2} \left\{ q_1 \left[ \frac{q_2}{4\pi\epsilon r_{12}} + \frac{q_3}{4\pi\epsilon r_{13}} \right] + q_2 \left[ \frac{q_1}{4\pi\epsilon r_{12}} + \frac{q_3}{4\pi\epsilon r_{23}} \right] + q_3 \left[ \frac{q_1}{4\pi\epsilon r_{13}} + \frac{q_2}{4\pi\epsilon r_{23}} \right] \right\} \quad (2)$$

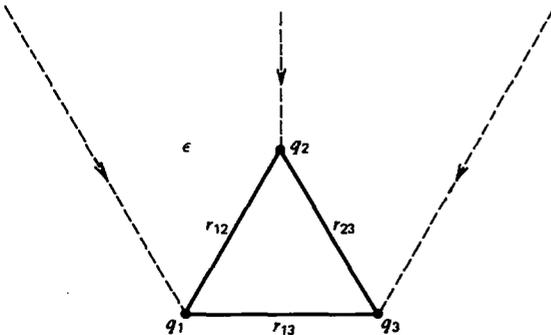


Figure 3-28 Three already existing point charges are brought in from an infinite distance to their final positions.

where we recognize that each bracketed term is just the potential at the final position of each charge and includes contributions from all the other charges, except the one located at the position where the potential is being evaluated:

$$W = \frac{1}{2}[q_1 V_1 + q_2 V_2 + q_3 V_3] \quad (3)$$

Extending this result for any number  $N$  of already existing free point charges yields

$$W = \frac{1}{2} \sum_{n=1}^N q_n V_n \quad (4)$$

The factor of  $\frac{1}{2}$  arises because the potential of a point charge at the time it is brought in from infinity is less than the final potential when all the charges are assembled.

### (b) Binding Energy of a Crystal

One major application of (4) is in computing the largest contribution to the binding energy of ionic crystals, such as salt (NaCl), which is known as the Madelung electrostatic energy. We take a simple one-dimensional model of a crystal consisting of an infinitely long string of alternating polarity point charges  $\pm q$  a distance  $a$  apart, as in Figure 3-29. The average work necessary to bring a positive charge as shown in Figure 3-29 from infinity to its position on the line is obtained from (4) as

$$W = \frac{1}{2} \frac{2q^2}{4\pi\epsilon a} \left[ -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \dots \right] \quad (5)$$

The extra factor of 2 in the numerator is necessary because the string extends to infinity on each side. The infinite series is recognized as the Taylor series expansion of the logarithm

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots \quad (6)$$

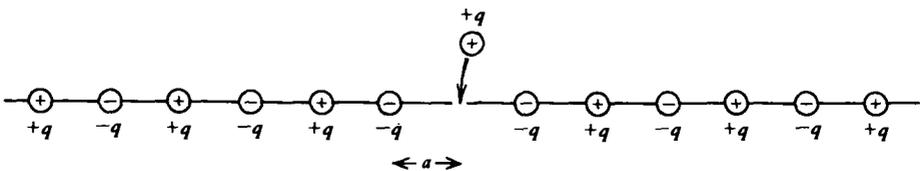


Figure 3-29 A one-dimensional crystal with alternating polarity charges  $\pm q$  a distance  $a$  apart.

where  $x = 1$  so that\*

$$W = \frac{-q^2}{4\pi\epsilon a} \ln 2 \quad (7)$$

This work is negative because the crystal pulls on the charge as it is brought in from infinity. This means that it would take positive work to remove the charge as it is bound to the crystal. A typical ion spacing is about  $3 \text{ \AA}$  ( $3 \times 10^{-10} \text{ m}$ ) so that if  $q$  is a single proton ( $q = 1.6 \times 10^{-19} \text{ coul}$ ), the binding energy is  $W \approx 5.3 \times 10^{-19} \text{ joule}$ . Since this number is so small it is usually more convenient to work with units of energy per unit electronic charge called electron volts (ev), which are obtained by dividing  $W$  by the charge on an electron so that, in this case,  $W \approx 3.3 \text{ ev}$ .

If the crystal was placed in a medium with higher permittivity, we see from (7) that the binding energy decreases. This is why many crystals are soluble in water, which has a relative dielectric constant of about 80.

### 3-8-2 Work Necessary to Form a Continuous Charge Distribution

Not included in (4) is the self-energy of each charge itself or, equivalently, the work necessary to assemble each point charge. Since the potential  $V$  from a point charge  $q$  is proportional to  $q$ , the self-energy is proportional  $q^2$ . However, evaluating the self-energy of a point charge is difficult because the potential is infinite at the point charge.

To understand the self-energy concept better it helps to model a point charge as a small uniformly charged spherical volume of radius  $R$  with total charge  $Q = \frac{4}{3}\pi R^3 \rho_0$ . We assemble the sphere of charge from spherical shells, as shown in Figure 3-30, each of thickness  $dr_n$  and incremental charge  $dq_n = 4\pi r_n^2 dr_n \rho_0$ . As we bring in the  $n$ th shell to be placed at radius  $r_n$  the total charge already present and the potential there are

$$q_n = \frac{4}{3} \pi r_n^3 \rho_0, \quad V_n = \frac{q_n}{4\pi\epsilon r_n} = \frac{r_n^2 \rho_0}{3\epsilon} \quad (8)$$

\* Strictly speaking, this series is only conditionally convergent for  $x = 1$  and its sum depends on the grouping of individual terms. If the series in (6) for  $x = 1$  is rewritten as

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{3} + \frac{1}{6} - \frac{1}{8} + \dots + \frac{1}{2k-1} - \frac{1}{4k-2} - \frac{1}{4k} \dots \quad k \geq 1$$

then its sum is  $\frac{1}{2} \ln 2$ . [See J. Pleines and S. Mahajan, *On Conditionally Divergent Series and a Point Charge Between Two Parallel Grounded Planes*, Am. J. Phys. 45 (1977) p. 868.]

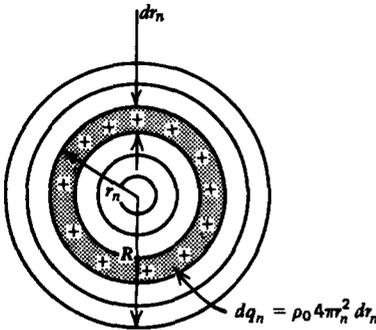


Figure 3-30 A point charge is modelled as a small uniformly charged sphere. It is assembled by bringing in spherical shells of differential sized surface charge elements from infinity.

so that the work required to bring in the  $n$ th shell is

$$dW_n = V_n dq_n = \frac{\rho_0^2 4\pi r_n^4}{3\epsilon} dr_n \quad (9)$$

The total work necessary to assemble the sphere is obtained by adding the work needed for each shell:

$$W = \int dW_n = \int_0^R \frac{4\pi\rho_0^2 r^4}{3\epsilon} dr = \frac{4\pi\rho_0^2 R^5}{15\epsilon} = \frac{3Q^2}{20\pi\epsilon R} \quad (10)$$

For a finite charge  $Q$  of zero radius the work becomes infinite. However, Einstein's theory of relativity tells us that this work necessary to assemble the charge is stored as energy that is related to the mass as

$$W = mc^2 = \frac{3Q^2}{20\pi\epsilon R} \Rightarrow R = \frac{3Q^2}{20\pi\epsilon mc^2} \quad (11)$$

which then determines the radius of the charge. For the case of an electron ( $Q = 1.6 \times 10^{-19}$  coul,  $m = 9.1 \times 10^{-31}$  kg) in free space ( $\epsilon = \epsilon_0 = 8.854 \times 10^{-12}$  farad/m), this radius is

$$R_{\text{electron}} = \frac{3(1.6 \times 10^{-19})^2}{20\pi(8.854 \times 10^{-12})(9.1 \times 10^{-31})(3 \times 10^8)^2} \approx 1.69 \times 10^{-15} \text{ m} \quad (12)$$

We can also obtain the result of (10) by using (4) where each charge becomes a differential element  $dq$ , so that the summation becomes an integration over the continuous free charge distribution:

$$W = \frac{1}{2} \int_{\text{all } q_r} V dq_r \quad (13)$$

For the case of the uniformly charged sphere,  $dq_f = \rho_0 dV$ , the final potential within the sphere is given by the results of Section 2-5-5b:

$$V = \frac{\rho_0}{2\epsilon} \left( R^2 - \frac{r^2}{3} \right) \quad (14)$$

Then (13) agrees with (10):

$$W = \frac{1}{2} \int \rho_0 V dV = \frac{4\pi\rho_0^2}{4\epsilon} \int_0^R \left( R^2 - \frac{r^2}{3} \right) r^2 dr = \frac{4\pi\rho_0^2 R^5}{15\epsilon} = \frac{3Q^2}{20\pi\epsilon R} \quad (15)$$

Thus, in general, we define (13) as the energy stored in the electric field, including the self-energy term. It differs from (4), which only includes interaction terms between different charges and not the infinite work necessary to assemble each point charge. Equation (13) is valid for line, surface, and volume charge distributions with the differential charge elements given in Section 2-3-1. Remember when using (4) and (13) that the zero reference for the potential is assumed to be at infinity. Adding a constant  $V_0$  to the potential will change the energy unless the total charge in the system is zero

$$\begin{aligned} W &= \frac{1}{2} \int (V + V_0) dq_f \\ &= \frac{1}{2} \int V dq_f + \frac{1}{2} V_0 \int dq_f^0 \\ &= \frac{1}{2} \int V dq_f \end{aligned} \quad (16)$$

### 3-8-3 Energy Density of the Electric Field

It is also convenient to express the energy  $W$  stored in a system in terms of the electric field. We assume that we have a volume charge distribution with density  $\rho_f$ . Then,  $dq_f = \rho_f dV$ , where  $\rho_f$  is related to the displacement field from Gauss's law:

$$W = \frac{1}{2} \int_V \rho_f V dV = \frac{1}{2} \int_V V(\nabla \cdot \mathbf{D}) dV \quad (17)$$

Let us examine the vector expansion

$$\nabla \cdot (V\mathbf{D}) = (\mathbf{D} \cdot \nabla)V + V(\nabla \cdot \mathbf{D}) \Rightarrow V(\nabla \cdot \mathbf{D}) = \nabla \cdot (V\mathbf{D}) + \mathbf{D} \cdot \mathbf{E} \quad (18)$$

where  $\mathbf{E} = -\nabla V$ . Then (17) becomes

$$W = \frac{1}{2} \int_V \mathbf{D} \cdot \mathbf{E} dV + \frac{1}{2} \int_V \nabla \cdot (V\mathbf{D}) dV \quad (19)$$

The last term on the right-hand side can be converted to a surface integral using the divergence theorem:

$$\int_V \nabla \cdot (VD) dV = \oint_S V \mathbf{D} \cdot d\mathbf{S} \quad (20)$$

If we let the volume  $V$  be of infinite extent so that the enclosing surface  $S$  is at infinity, the charge distribution that only extends over a finite volume looks like a point charge for which the potential decreases as  $1/r$  and the displacement vector dies off as  $1/r^2$ . Thus the term,  $VD$  at best dies off as  $1/r^3$ . Then, even though the surface area of  $S$  increases as  $r^2$ , the surface integral tends to zero as  $r$  becomes infinite as  $1/r$ . Thus, the second volume integral in (19) approaches zero:

$$\lim_{r \rightarrow \infty} \int_V \nabla \cdot (VD) dV = \oint_S V \mathbf{D} \cdot d\mathbf{S} = 0 \quad (21)$$

This conclusion is not true if the charge distribution is of infinite extent, since for the case of an infinitely long line or surface charge, the potential itself becomes infinite at infinity because the total charge on the line or surface is infinite. However, for finite size charge distributions, which is always the case in reality, (19) becomes

$$\begin{aligned} W &= \frac{1}{2} \int_{\text{all space}} \mathbf{D} \cdot \mathbf{E} dV \\ &= \int_{\text{all space}} \frac{1}{2} \epsilon \mathbf{E}^2 dV \end{aligned} \quad (22)$$

where the integration extends over all space. This result is true even if the permittivity  $\epsilon$  is a function of position. It is convenient to define the energy density as the positive-definite quantity:

$$w = \frac{1}{2} \epsilon \mathbf{E}^2 \text{ joule/m}^3 [\text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-2}] \quad (23)$$

where the total energy is

$$W = \int_{\text{all space}} w dV \quad (24)$$

Note that although (22) is numerically equal to (13), (22) implies that electric energy exists in those regions where a nonzero electric field exists even if no charge is present in that region, while (13) implies that electric energy exists only where the charge is nonzero. The answer as to where the energy is stored—in the charge distribution or in the electric field—is a matter of convenience since you cannot have one without the other. Numerically both equations yield the same answers but with contributions from different regions of space.

**3-8-4 Energy Stored in Charged Spheres**

**(a) Volume Charge**

We can also find the energy stored in a uniformly charged sphere using (22) since we know the electric field in each region from Section 2-4-3*b*. The energy density is then

$$w = \frac{\epsilon}{2} E_r^2 = \begin{cases} \frac{Q^2}{32\pi^2 \epsilon r^4}, & r > R \\ \frac{Q^2 r^2}{32\pi^2 \epsilon R^6}, & r < R \end{cases} \quad (25)$$

with total stored energy

$$\begin{aligned} W &= \int_{\text{all space}} w \, dV \\ &= \frac{Q^2}{8\pi\epsilon} \left( \int_0^R \frac{r^4}{R^6} \, dr + \int_R^\infty \frac{dr}{r^2} \right) = \frac{3}{20} \frac{Q^2}{\pi\epsilon R} \end{aligned} \quad (26)$$

which agrees with (10) and (15).

**(b) Surface Charge**

If the sphere is uniformly charged on its surface  $Q = 4\pi R^2 \sigma_0$ , the potential and electric field distributions are

$$V(r) = \begin{cases} \frac{Q}{4\pi\epsilon R} \\ \frac{Q}{4\pi\epsilon r} \end{cases}; \quad E_r = \begin{cases} 0, & r < R \\ \frac{Q}{4\pi\epsilon r^2}, & r > R \end{cases} \quad (27)$$

Using (22) the energy stored is

$$W = \frac{\epsilon}{2} \left( \frac{Q}{4\pi\epsilon} \right)^2 4\pi \int_R^\infty \frac{dr}{r^2} = \frac{Q^2}{8\pi\epsilon R} \quad (28)$$

This result is equally as easy obtained using (13):

$$\begin{aligned} W &= \frac{1}{2} \int_S \sigma_0 V(r=R) \, dS \\ &= \frac{1}{2} \sigma_0 V(r=R) 4\pi R^2 = \frac{Q^2}{8\pi\epsilon R} \end{aligned} \quad (29)$$

The energy stored in a uniformly charged sphere is 20% larger than the surface charged sphere for the same total charge  $Q$ . This is because of the additional energy stored throughout the sphere's volume. Outside the sphere ( $r > R$ ) the fields are the same as is the stored energy.

**(c) Binding Energy of an Atom**

In Section 3-1-4 we modeled an atom as a fixed positive point charge nucleus  $Q$  with a surrounding uniform spherical cloud of negative charge with total charge  $-Q$ , as in Figure 3-31. Potentials due to the positive point and negative volume charges are found from Section 2-5-5b as

$$V_+(r) = \frac{Q}{4\pi\epsilon_0 r}$$

$$V_-(r) = \begin{cases} -\frac{3Q}{8\pi\epsilon_0 R^3} \left( R^2 - \frac{r^2}{3} \right), & r < R \\ -\frac{Q}{4\pi\epsilon_0 r}, & r > R \end{cases} \quad (30)$$

The binding energy of the atom is easily found by superposition considering first the uniformly charged negative sphere with self-energy given in (10), (15), and (26) and then adding the energy of the positive point charge:

$$W = \frac{3Q^2}{20\pi\epsilon_0 R} + Q[V_-(r=0)] = -\frac{9Q^2}{40\pi\epsilon_0 R} \quad (31)$$

This is the work necessary to assemble the atom from charges at infinity. Once the positive nucleus is in place, it attracts the following negative charges so that the field does work on the charges and the work of assembly in (31) is negative. Equivalently, the magnitude of (31) is the work necessary for us to disassemble the atom by overcoming the attractive coulombic forces between the opposite polarity charges.

When alternatively using (4) and (13), we only include the potential of the negative volume charge at  $r=0$  acting on the positive charge, while we include the total potential due to both in evaluating the energy of the volume charge. We do

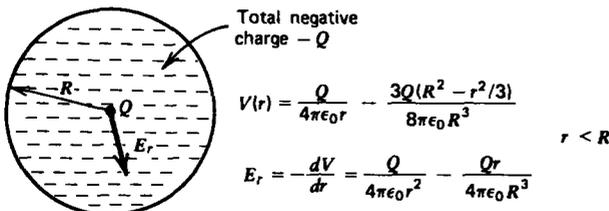


Figure 3-31 An atom can be modelled as a point charge  $Q$  representing the nucleus, surrounded by a cloud of uniformly distributed electrons with total charge  $-Q$  within a sphere of radius  $R$ .

not consider the infinite self-energy of the point charge that would be included if we used (22):

$$\begin{aligned}
 W &= \frac{1}{2} Q V_-(r=0) - \frac{1}{2} \int_{r=0}^R [V_+(r) + V_-(r)] \frac{3Qr^2}{R^3} dr \\
 &= -\frac{3Q^2}{16\pi\epsilon_0 R} - \frac{3Q^2}{8\pi\epsilon_0 R^3} \int_0^R \left( r - \frac{3r^2}{2R} + \frac{r^4}{2R^3} \right) dr \\
 &= -\frac{9Q^2}{40\pi\epsilon_0 R} \tag{32}
 \end{aligned}$$

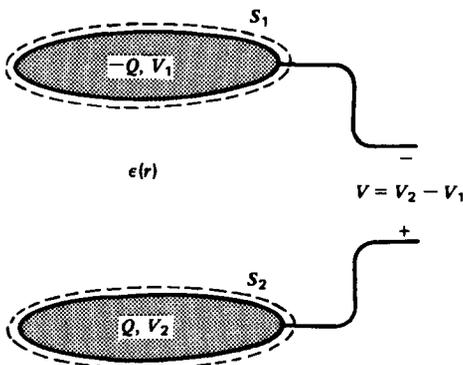
### 3-8-5 Energy Stored in a Capacitor

In a capacitor all the charge resides on the electrodes as a surface charge. Consider two electrodes at voltage  $V_1$  and  $V_2$  with respect to infinity, and thus at voltage difference  $V = V_2 - V_1$ , as shown in Figure 3-32. Each electrode carries opposite polarity charge with magnitude  $Q$ . Then (13) can be used to compute the total energy stored as

$$W = \frac{1}{2} \left[ \int_{S_1} V_1 \sigma_1 dS_1 + \int_{S_2} V_2 \sigma_2 dS_2 \right] \tag{33}$$

Since each surface is an equipotential, the voltages  $V_1$  and  $V_2$  may be taken outside the integrals. The integral then reduces to the total charge  $\pm Q$  on each electrode:

$$\begin{aligned}
 W &= \frac{1}{2} \left[ V_1 \int_{S_1} \underbrace{\sigma_1 dS_1}_{-Q} + V_2 \int_{S_2} \underbrace{\sigma_2 dS_2}_Q \right] \\
 &= \frac{1}{2} (V_2 - V_1) Q = \frac{1}{2} QV \tag{34}
 \end{aligned}$$



$$W = \frac{1}{2} QV = \frac{1}{2} CV^2 = \frac{1}{2} Q^2/C$$

Figure 3-32 A capacitor stores energy in the electric field.

Since in a capacitor the charge and voltage are linearly related through the capacitance

$$Q = CV \quad (35)$$

the energy stored in the capacitor can also be written as

$$W = \frac{1}{2} QV = \frac{1}{2} CV^2 = \frac{1}{2} \frac{Q^2}{C} \quad (36)$$

This energy is equivalent to (22) in terms of the electric field and gives us an alternate method to computing the capacitance if we know the electric field distribution.

### EXAMPLE 3-3 CAPACITANCE OF AN ISOLATED SPHERE

A sphere of radius  $R$  carries a uniformly distributed surface charge  $Q$ . What is its capacitance?

#### SOLUTION

The stored energy is given by (28) or (29) so that (36) gives us the capacitance:

$$C = Q^2/2W = 4\pi\epsilon R$$

## 3.9 FIELDS AND THEIR FORCES

### 3-9-1 Force Per Unit Area on a Sheet of Surface Charge

A confusion arises in applying Coulomb's law to find the perpendicular force on a sheet of surface charge as the normal electric field is different on each side of the sheet. Using the over-simplified argument that half the surface charge resides on each side of the sheet yields the correct force

$$\mathbf{f} = \frac{1}{2} \int_S \sigma_f (\mathbf{E}_1 + \mathbf{E}_2) dS \quad (1)$$

where, as shown in Figure 3-33a,  $\mathbf{E}_1$  and  $\mathbf{E}_2$  are the electric fields on each side of the sheet. Thus, the correct field to use is the average electric field  $\frac{1}{2}(\mathbf{E}_1 + \mathbf{E}_2)$  across the sheet.

For the tangential force, the tangential components of  $\mathbf{E}$  are continuous across the sheet ( $E_{1t} = E_{2t} = E_t$ ) so that

$$f_t = \frac{1}{2} \int_S \sigma_f (E_{1t} + E_{2t}) dS = \int_S \sigma_f E_t dS \quad (2)$$

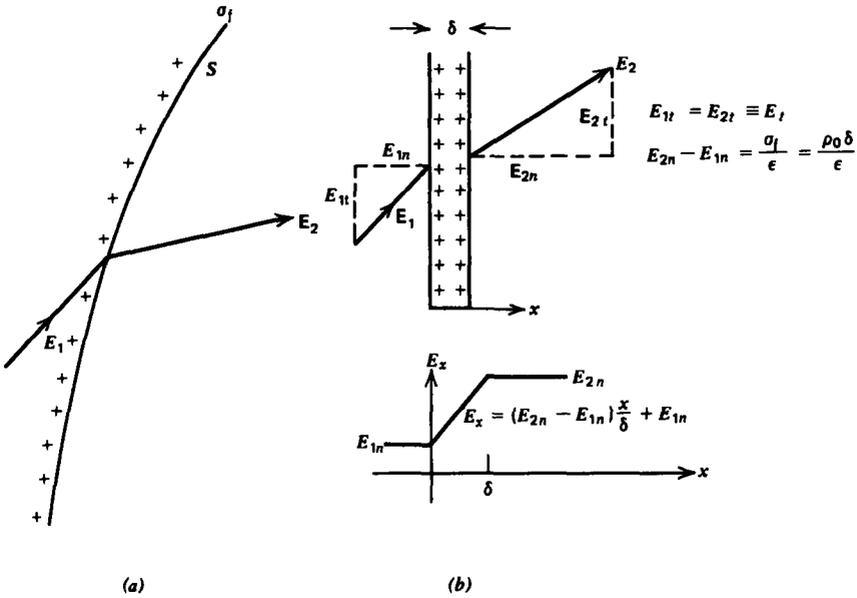


Figure 3-33 (a) The normal component of electric field is discontinuous across the sheet of surface charge. (b) The sheet of surface charge can be modeled as a thin layer of volume charge. The electric field then varies linearly across the volume.

The normal fields are discontinuous across the sheet so that the perpendicular force is

$$\begin{aligned} \sigma_f = \epsilon(E_{2n} - E_{1n}) \Rightarrow f_n &= \frac{1}{2} \int_S \epsilon(E_{2n} - E_{1n})(E_{1n} + E_{2n}) dS \\ &= \frac{1}{2} \int_S \epsilon(E_{2n}^2 - E_{1n}^2) dS \end{aligned} \quad (3)$$

To be mathematically rigorous we can examine the field transition through the sheet more closely by assuming the surface charge is really a uniform volume charge distribution  $\rho_0$  of very narrow thickness  $\delta$ , as shown in Figure 3-33b. Over the small surface element  $dS$ , the surface appears straight so that the electric field due to the volume charge can then only vary with the coordinate  $x$  perpendicular to the surface. Then the point form of Gauss's law within the volume yields

$$\frac{dE_x}{dx} = \frac{\rho_0}{\epsilon} \Rightarrow E_x = \frac{\rho_0 x}{\epsilon} + \text{const} \quad (4)$$

The constant in (4) is evaluated by the boundary conditions on the normal components of electric field on each side of the

sheet

$$E_x(x=0) = E_{1n}, \quad E_x(x=\delta) = E_{2n} \quad (5)$$

so that the electric field is

$$E_x = (E_{2n} - E_{1n}) \frac{x}{\delta} + E_{1n} \quad (6)$$

As the slab thickness  $\delta$  becomes very small, we approach a sheet charge relating the surface charge density to the discontinuity in electric fields as

$$\lim_{\substack{\rho_0 \rightarrow \infty \\ \delta \rightarrow 0}} \rho_0 \delta = \sigma_f = \epsilon (E_{2n} - E_{1n}) \quad (7)$$

Similarly the force per unit area on the slab of volume charge is

$$\begin{aligned} F_x &= \int_0^\delta \rho_0 E_x dx \\ &= \int_0^\delta \rho_0 \left[ (E_{2n} - E_{1n}) \frac{x}{\delta} + E_{1n} \right] dx \\ &= \left[ \rho_0 (E_{2n} - E_{1n}) \frac{x^2}{2\delta} + E_{1n} x \right] \Big|_0^\delta \\ &= \frac{\rho_0 \delta}{2} (E_{1n} + E_{2n}) \end{aligned} \quad (8)$$

In the limit of (7), the force per unit area on the sheet of surface charge agrees with (3):

$$\lim_{\rho_0 \delta = \sigma_f} F_x = \frac{\sigma_f}{2} (E_{1n} + E_{2n}) = \frac{\epsilon}{2} (E_{2n}^2 - E_{1n}^2) \quad (9)$$

### 3-9-2 Forces on a Polarized Medium

#### (a) Force Density

In a uniform electric field there is no force on a dipole because the force on each charge is equal in magnitude but opposite in direction, as in Figure 3-34a. However, if the dipole moment is not aligned with the field there is an aligning torque given by  $\mathbf{t} = \mathbf{p} \times \mathbf{E}$ . The torque per unit volume  $\mathbf{T}$  on a polarized medium with  $N$  dipoles per unit volume is then

$$\mathbf{T} = N\mathbf{t} = N\mathbf{p} \times \mathbf{E} = \mathbf{P} \times \mathbf{E} \quad (10)$$

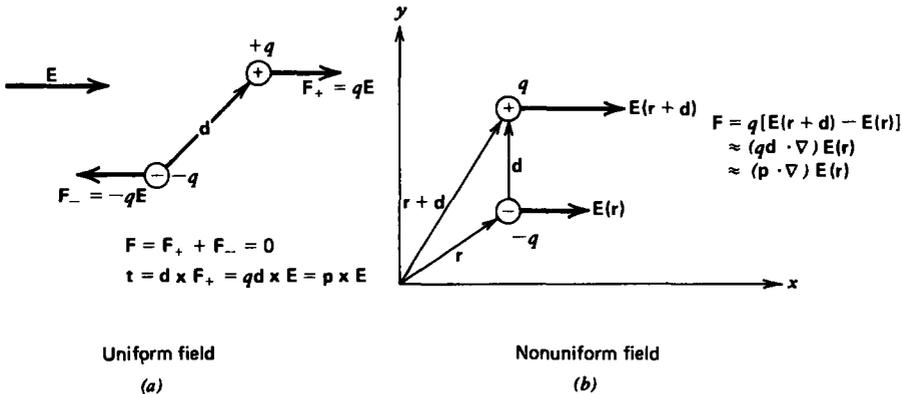


Figure 3-34 (a) A torque is felt by a dipole if its moment is not aligned with the electric field. In a uniform electric field there is no net force on a dipole because the force on each charge is equal in magnitude but opposite in direction. (b) There is a net force on a dipole only in a nonuniform field.

For a linear dielectric, this torque is zero because the polarization is induced by the field so that  $\mathbf{P}$  and  $\mathbf{E}$  are in the same direction.

A net force can be applied to a dipole if the electric field is different on each end, as in Figure 3-34b:

$$\mathbf{f} = -q[\mathbf{E}(\mathbf{r}) - \mathbf{E}(\mathbf{r} + \mathbf{d})] \tag{11}$$

For point dipoles, the dipole spacing  $\mathbf{d}$  is very small so that the electric field at  $\mathbf{r} + \mathbf{d}$  can be expanded in a Taylor series as

$$\begin{aligned} \mathbf{E}(\mathbf{r} + \mathbf{d}) &\approx \mathbf{E}(\mathbf{r}) + d_x \frac{\partial}{\partial x} \mathbf{E}(\mathbf{r}) + d_y \frac{\partial}{\partial y} \mathbf{E}(\mathbf{r}) + d_z \frac{\partial}{\partial z} \mathbf{E}(\mathbf{r}) \\ &= \mathbf{E}(\mathbf{r}) + (\mathbf{d} \cdot \nabla) \mathbf{E}(\mathbf{r}) \end{aligned} \tag{12}$$

Then the force on a point dipole is

$$\mathbf{f} = (q\mathbf{d} \cdot \nabla) \mathbf{E}(\mathbf{r}) = (\mathbf{p} \cdot \nabla) \mathbf{E}(\mathbf{r}) \tag{13}$$

If we have a distribution of such dipoles with number density  $N$ , the polarization force density is

$$\mathbf{F} = N\mathbf{f} = (N\mathbf{p} \cdot \nabla) \mathbf{E} = (\mathbf{P} \cdot \nabla) \mathbf{E} \tag{14}$$

Of course, if there is any free charge present we must also add the coulombic force density  $\rho_f \mathbf{E}$ .

**(b) Permanently Polarized Medium**

A permanently polarized material with polarization  $P_0 \mathbf{i}_y$  is free to slide between parallel plate electrodes, as is shown in Figure 3-35.

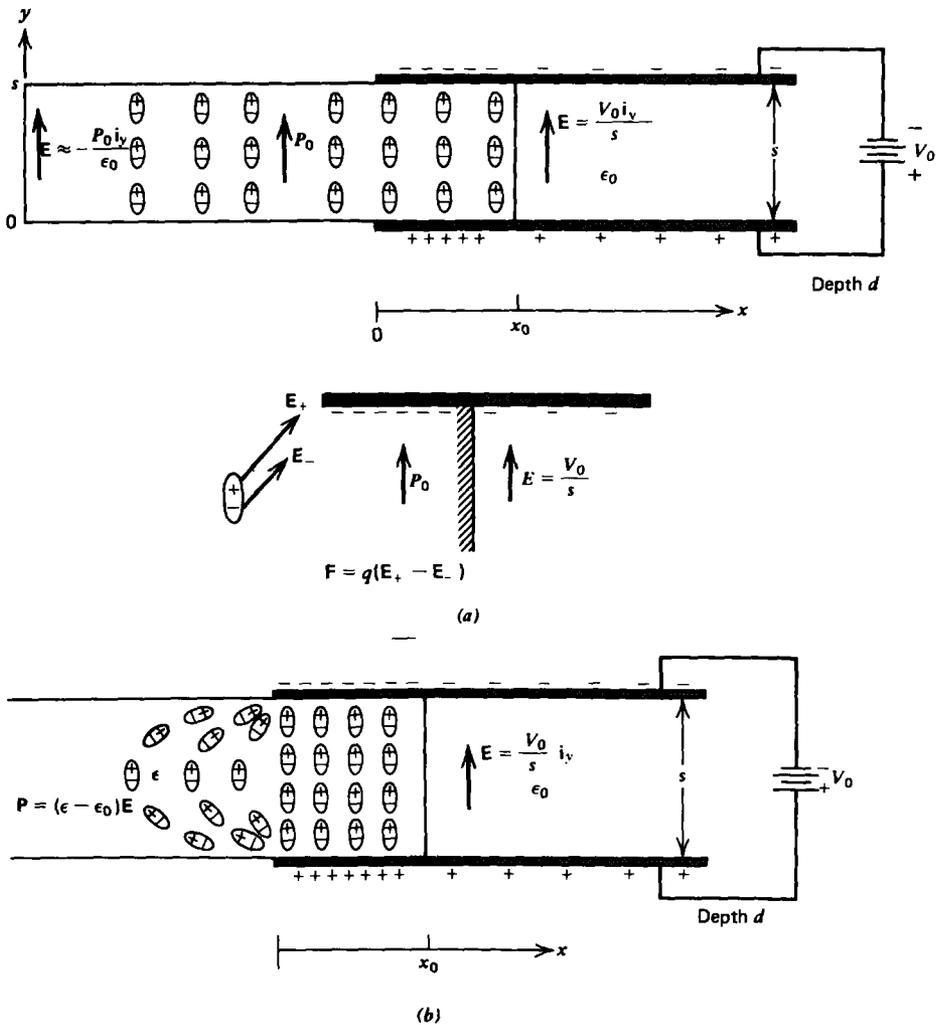


Figure 3-35 (a) A permanently polarized electret partially inserted into a capacitor has a force on it due to the Coulombic attraction between the dipole charges and the surface charge on the electrodes. The net force arises in the fringing field region as the end of the dipole further from the electrode edge feels a smaller electric field. Depending on the voltage magnitude and polarity, the electret can be pulled in or pushed out of the capacitor. (b) A linear dielectric is always attracted into a free space capacitor because of the net force on dipoles in the nonuniform field. The dipoles are now aligned with the electric field, no matter the voltage polarity.

We only know the electric field in the interelectrode region and from Example 3-2 far away from the electrodes:

$$E_y(x = x_0) = \frac{V_0}{s}, \quad E_y(x = -\infty) = -\frac{P_0}{\epsilon_0} \quad (15)$$

Unfortunately, neither of these regions contribute to the force because the electric field is uniform and (14) requires a field gradient for a force. The force arises in the fringing fields near the electrode edges where the field is nonuniform and, thus, exerts less of a force on the dipole end further from the electrode edges. At first glance it looks like we have a difficult problem because we do not know the fields where the force acts. However, because the electric field has zero curl,

$$\nabla \times \mathbf{E} = 0 \Rightarrow \frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x} \quad (16)$$

the  $x$  component of the force density can be written as

$$\begin{aligned} F_x &= P_y \frac{\partial E_x}{\partial y} \\ &= P_y \frac{\partial E_y}{\partial x} \\ &= \frac{\partial}{\partial x} (P_y E_y) - E_y \frac{\partial P_y}{\partial x} \end{aligned} \quad (17)$$

The last term in (17) is zero because  $P_y = P_0$  is a constant. The total  $x$  directed force is then

$$\begin{aligned} f_x &= \int F_x dx dy dz \\ &= \int_{x=-\infty}^{x_0} \int_{y=0}^s \int_{z=0}^d \frac{\partial}{\partial x} (P_y E_y) dx dy dz \end{aligned} \quad (18)$$

We do the  $x$  integration first so that the  $y$  and  $z$  integrations are simple multiplications as the fields at the limits of the  $x$  integration are independent of  $y$  and  $z$ :

$$f_x = P_0 E_y s d \Big|_{x=-\infty}^{x_0} = P_0 V_0 d + \frac{P_0^2 s d}{\epsilon_0} \quad (19)$$

There is a force pulling the electret between the electrodes even if the voltage were zero due to the field generated by the surface charge on the electrodes induced by the electret. This force is increased if the imposed electric field and polarization are in the same direction. If the voltage polarity is reversed, the force is negative and the electret is pushed out if the magnitude of the voltage exceeds  $P_0 s / \epsilon_0$ .

### (c) Linearly Polarized Medium

The problem is different if the slab is polarized by the electric field, as the polarization will then be in the direction of the electric field and thus have  $x$  and  $y$  components in the fringing fields near the electrode edges where the force

arises, as in Figure 3-35*b*. The dipoles tend to line up as shown with the positive ends attracted towards the negative electrode and the negative dipole ends towards the positive electrode. Because the farther ends of the dipoles are in a slightly weaker field, there is a net force to the right tending to draw the dielectric into the capacitor.

The force density of (14) is

$$F_x = P_x \frac{\partial E_x}{\partial x} + P_y \frac{\partial E_x}{\partial y} = (\epsilon - \epsilon_0) \left( E_x \frac{\partial E_x}{\partial x} + E_y \frac{\partial E_x}{\partial y} \right) \quad (20)$$

Because the electric field is curl free, as given in (16), the force density is further simplified to

$$F_x = \frac{(\epsilon - \epsilon_0)}{2} \frac{\partial}{\partial x} (E_x^2 + E_y^2) \quad (21)$$

The total force is obtained by integrating (21) over the volume of the dielectric:

$$\begin{aligned} f_x &= \int_{x=-\infty}^{x_0} \int_{y=0}^s \int_{z=0}^d \frac{(\epsilon - \epsilon_0)}{2} \frac{\partial}{\partial x} (E_x^2 + E_y^2) dx dy dz \\ &= \frac{(\epsilon - \epsilon_0)sd}{2} (E_x^2 + E_y^2) \Big|_{x=-\infty}^{x_0} = \frac{(\epsilon - \epsilon_0)}{2} \frac{V_0^2 d}{s} \end{aligned} \quad (22)$$

where we knew that the fields were zero at  $x = -\infty$  and uniform at  $x = x_0$ :

$$E_y(x_0) = V_0/s, \quad E_x(x_0) = 0 \quad (23)$$

The force is now independent of voltage polarity and always acts in the direction to pull the dielectric into the capacitor if  $\epsilon > \epsilon_0$ .

### 3-9-3 Forces on a Capacitor

Consider a capacitor that has one part that can move in the  $x$  direction so that the capacitance depends on the coordinate  $x$ :

$$q = C(x)v \quad (24)$$

The current is obtained by differentiating the charge with respect to time:

$$\begin{aligned} i &= \frac{dq}{dt} = \frac{d}{dt} [C(x)v] = C(x) \frac{dv}{dt} + v \frac{dC(x)}{dt} \\ &= C(x) \frac{dv}{dt} + v \frac{dC(x)}{dx} \frac{dx}{dt} \end{aligned} \quad (25)$$

Note that this relation has an extra term over the usual circuit formula, proportional to the speed of the moveable member, where we expanded the time derivative of the capacitance by the chain rule of differentiation. Of course, if the geometry is fixed and does not change with time ( $dx/dt = 0$ ), then (25) reduces to the usual circuit expression. The last term is due to the electro-mechanical coupling.

The power delivered to a time-dependent capacitance is

$$p = vi = v \frac{d}{dt} [C(x)v] \quad (26)$$

which can be expanded to the form

$$\begin{aligned} p &= \frac{d}{dt} \left[ \frac{1}{2} C(x)v^2 \right] + \frac{1}{2} v^2 \frac{dC(x)}{dt} \\ &= \frac{d}{dt} \left[ \frac{1}{2} C(x)v^2 \right] + \frac{1}{2} v^2 \frac{dC(x)}{dx} \frac{dx}{dt} \end{aligned} \quad (27)$$

where the last term is again obtained using the chain rule of differentiation. This expression can be put in the form

$$p = \frac{dW}{dt} + f_x \frac{dx}{dt} \quad (28)$$

where we identify the power  $p$  delivered to the capacitor as going into increasing the energy storage  $W$  and mechanical power  $f_x dx/dt$  in moving a part of the capacitor:

$$W = \frac{1}{2} C(x)v^2, \quad f_x = \frac{1}{2} v^2 \frac{dC(x)}{dx} \quad (29)$$

Using (24), the stored energy and force can also be expressed in terms of the charge as

$$W = \frac{1}{2} \frac{q^2}{C(x)}, \quad f_x = \frac{1}{2} \frac{q^2}{C^2(x)} \frac{dC(x)}{dx} = -\frac{1}{2} q^2 \frac{d[1/C(x)]}{dx} \quad (30)$$

To illustrate the ease in using (29) or (30) to find the force, consider again the partially inserted dielectric in Figure 3-35*b*. The capacitance when the dielectric extends a distance  $x$  into the electrodes is

$$C(x) = \frac{\epsilon x d}{s} + \epsilon_0 \frac{(l-x)d}{s} \quad (31)$$

so that the force on the dielectric given by (29) agrees with (22):

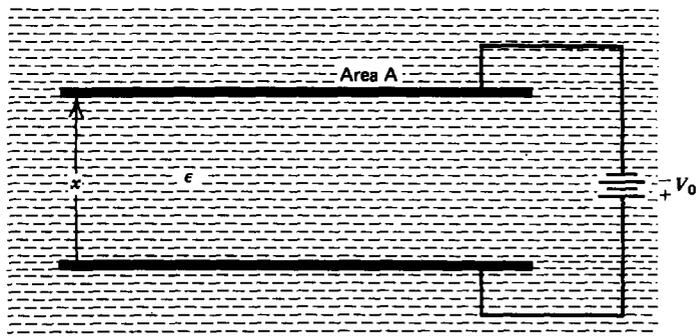
$$f_x = \frac{1}{2} V_0^2 \frac{dC(x)}{dx} = \frac{1}{2} (\epsilon - \epsilon_0) \frac{V_0^2 d}{s} \quad (32)$$

Note that we neglected the fringing field contributions to the capacitance in (31) even though they are the physical origin of the force. The results agree because this extra capacitance does not depend on the position  $x$  of the dielectric when  $x$  is far from the electrode edges.

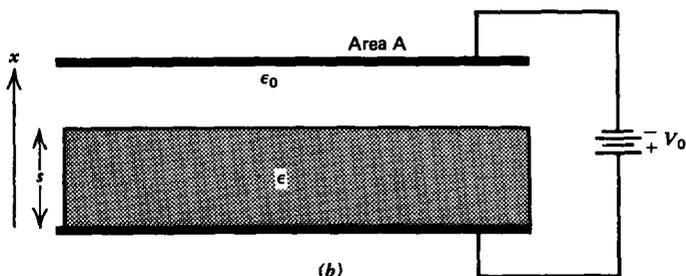
This method can only be used for linear dielectric systems described by (24). It is not valid for the electret problem treated in Section 3-9-2b because the electrode charge is not linearly related to the voltage, being in part induced by the electret.

**EXAMPLE 3-4 FORCE ON A PARALLEL PLATE CAPACITOR**

Two parallel, perfectly conducting electrodes of area  $A$  and a distance  $x$  apart are shown in Figure 3-36. For each of the following two configurations, find the force on the upper electrode in the  $x$  direction when the system is constrained to constant voltage  $V_0$  or constant charge  $Q_0$ .



(a)



(b)

Figure 3-36 A parallel plate capacitor (a) immersed within a dielectric fluid or with (b) a free space region in series with a solid dielectric.

**(a) Liquid Dielectric**

The electrodes are immersed within a liquid dielectric with permittivity  $\epsilon$ , as shown in Figure 3-36a.

**SOLUTION**

The capacitance of the system is

$$C(x) = \epsilon A/x$$

so that the force from (29) for constant voltage is

$$f_x = \frac{1}{2} V_0^2 \frac{dC(x)}{dx} = -\frac{1}{2} \frac{\epsilon A V_0^2}{x^2}$$

The force being negative means that it is in the direction opposite to increasing  $x$ , in this case downward. The capacitor plates attract each other because they are oppositely charged and opposite charges attract. The force is independent of voltage polarity and gets infinitely large as the plate spacing approaches zero. The result is also valid for free space with  $\epsilon = \epsilon_0$ . The presence of the dielectric increases the attractive force.

If the electrodes are constrained to a constant charge  $Q_0$  the force is then attractive but independent of  $x$ :

$$f_x = -\frac{1}{2} Q_0^2 \frac{d}{dx} \frac{1}{C(x)} = -\frac{1}{2} \frac{Q_0^2}{\epsilon A}$$

For both these cases, the numerical value of the force is the same because  $Q_0$  and  $V_0$  are related by the capacitance, but the functional dependence on  $x$  is different. The presence of a dielectric now decreases the force over that of free space.

**(b) Solid Dielectric**

A solid dielectric with permittivity  $\epsilon$  of thickness  $s$  is inserted between the electrodes with the remainder of space having permittivity  $\epsilon_0$ , as shown in Figure 3-36b.

**SOLUTION**

The total capacitance for this configuration is given by the series combination of capacitance due to the dielectric block and the free space region:

$$C(x) = \frac{\epsilon \epsilon_0 A}{\epsilon_0 s + \epsilon(x-s)}$$

The force on the upper electrode for constant voltage is

$$f_x = \frac{1}{2} V_0^2 \frac{d}{dx} C(x) = -\frac{\epsilon^2 \epsilon_0 A V_0^2}{2[\epsilon_0 s + \epsilon(x-s)]^2}$$

If the electrode just rests on the dielectric so that  $x = s$ , the force is

$$f_x = -\frac{\epsilon^2 A V_0^2}{2\epsilon_0 s^2}$$

This result differs from that of part (a) when  $x = s$  by the factor  $\epsilon_r = \epsilon/\epsilon_0$  because in this case moving the electrode even slightly off the dielectric leaves a free space region in between. In part (a) no free space gap develops as the liquid dielectric fills in the region, so that the dielectric is always in contact with the electrode. The total force on the electrode-dielectric interface is due to both free and polarization charge.

With the electrodes constrained to constant charge, the force on the upper electrode is independent of position and also independent of the permittivity of the dielectric block:

$$f_x = -\frac{1}{2} Q_0^2 \frac{d}{dx} \frac{1}{C(x)} = -\frac{1}{2} \frac{Q_0^2}{\epsilon_0 A}$$

### 3-10 ELECTROSTATIC GENERATORS

#### 3-10-1 Van de Graaff Generator

In the 1930s, reliable means of generating high voltages were necessary to accelerate charged particles in atomic studies. In 1931, Van de Graaff developed an electrostatic generator where charge is sprayed onto an insulating moving belt that transports this charge onto a conducting dome, as illustrated in Figure 3-37a. If the dome was considered an isolated sphere of radius  $R$ , the capacitance is given as  $C = 4\pi\epsilon_0 R$ . The transported charge acts as a current source feeding this capacitance, as in Figure 3-37b, so that the dome voltage builds up linearly with time:

$$i = C \frac{dv}{dt} \Rightarrow v = \frac{i}{C} t \quad (1)$$

This voltage increases until the breakdown strength of the surrounding atmosphere is reached, whereupon a spark discharge occurs. In air, the electric field breakdown strength  $E_b$  is  $3 \times 10^6$  V/m. The field near the dome varies as  $E_r = VR/r^2$ , which is maximum at  $r = R$ , which implies a maximum voltage of  $V_{\max} = E_b R$ . For  $V_{\max} = 10^6$  V, the radius of the sphere

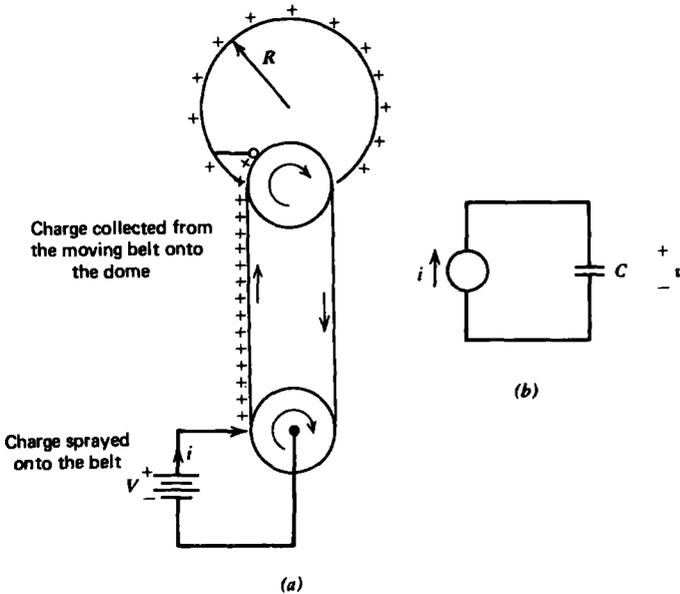


Figure 3-37 (a) A Van de Graaff generator consists of a moving insulating belt that transports injected charge onto a conducting dome which can thus rise to very high voltages, easily in excess of a million volts. (b) A simple equivalent circuit consists of the conveying charge modeled as a current source charging the capacitance of the dome.

must be  $R \approx \frac{1}{3} \text{ m}$  so that the capacitance is  $C \approx 37 \text{ pf}$ . With a charging current of one microampere, it takes  $t \approx 37 \text{ sec}$  to reach this maximum voltage.

### 3-10-2 Self-Excited Electrostatic Induction Machines

In the Van de Graaff generator, an external voltage source is necessary to deposit charge on the belt. In the late 1700s, self-excited electrostatic induction machines were developed that did not require any external electrical source. To understand how these devices work, we modify the Van de Graaff generator configuration, as in Figure 3-38a, by putting conducting segments on the insulating belt. Rather than spraying charge, we place an electrode at voltage  $V$  with respect to the lower conducting pulley so that opposite polarity charge is induced on the moving segments. As the segments move off the pulley, they carry their charge with them. So far, this device is similar to the Van de Graaff generator using induced charge rather than sprayed charge and is described by the same equivalent circuit where the current source now depends on the capacitance  $C_i$  between the inducing electrode and the segmented electrodes, as in Figure 3-38b.

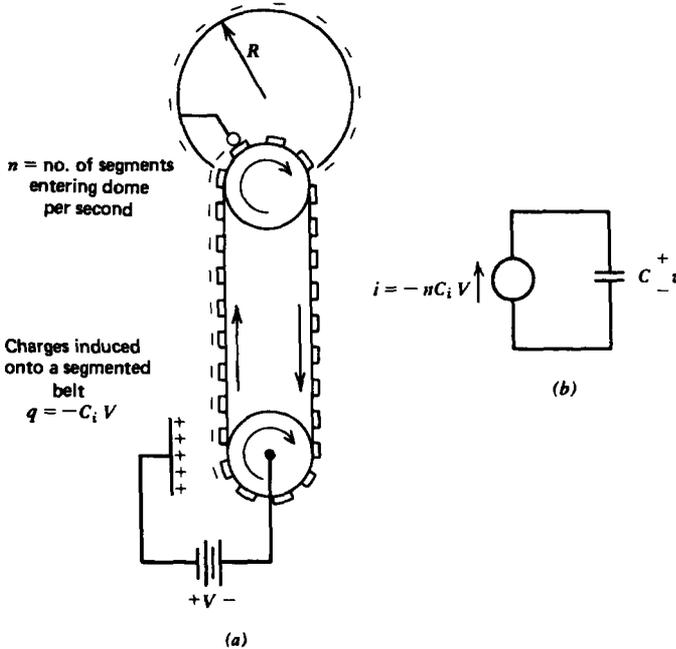


Figure 3-38 A modified Van de Graaff generator as an electrostatic induction machine. (a) Here charges are induced onto a segmented belt carrying insulated conductors as the belt passes near an electrode at voltage  $V$ . (b) Now the current source feeding the capacitor equivalent circuit depends on the capacitance  $C_i$  between the electrode and the belt.

Now the early researchers cleverly placed another induction machine nearby as in Figure 3-39a. Rather than applying a voltage source, because one had not been invented yet, they electrically connected the dome of each machine to the inducer electrode of the other. The induced charge on one machine was proportional to the voltage on the other dome. Although no voltage is applied, any charge imbalance on an inducer electrode due to random noise or stray charge will induce an opposite charge on the moving segmented belt that carries this charge to the dome of which some appears on the other inducer electrode where the process is repeated with opposite polarity charge. The net effect is that the charge on the original inducer has been increased.

More quantitatively, we use the pair of equivalent circuits in Figure 3-39b to obtain the coupled equations

$$-nC_i v_1 = C \frac{dv_2}{dt}, \quad -nC_i v_2 = C \frac{dv_1}{dt} \quad (2)$$

where  $n$  is the number of segments per second passing through the dome. All voltages are referenced to the lower pulleys that are electrically connected together. Because these

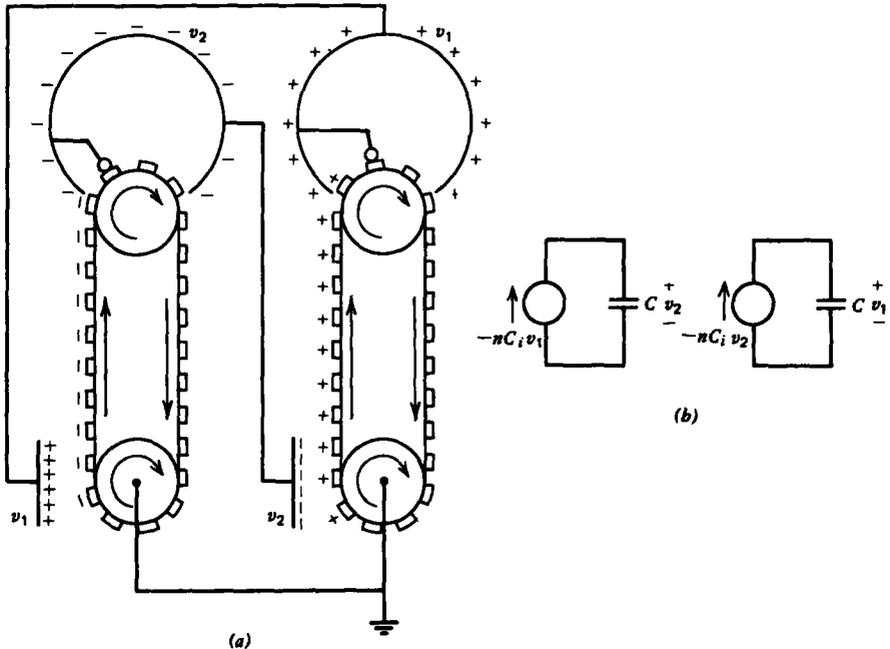


Figure 3-39 (a) A pair of coupled self-excited electrostatic induction machines generate their own inducing voltage. (b) The system is described by two simple coupled circuits.

are linear constant coefficient differential equations, the solutions must be exponentials:

$$v_1 = \hat{V}_1 e^{st}, \quad v_2 = \hat{V}_2 e^{st} \quad (3)$$

Substituting these assumed solutions into (2) yields the following characteristic roots:

$$s^2 = \left(\frac{nC_i}{C}\right)^2 \Rightarrow s = \pm \frac{nC_i}{C} \quad (4)$$

so that the general solution is

$$\begin{aligned} v_1 &= A_1 e^{(nC_i/C)t} + A_2 e^{-(nC_i/C)t} \\ v_2 &= -A_1 e^{(nC_i/C)t} + A_2 e^{-(nC_i/C)t} \end{aligned} \quad (5)$$

where  $A_1$  and  $A_2$  are determined from initial conditions.

The negative root of (4) represents the uninteresting decaying solutions while the positive root has solutions that grow unbounded with time. This is why the machine is self-excited. Any initial voltage perturbation, no matter how small, increases without bound until electrical breakdown is reached. Using representative values of  $n = 10$ ,  $C_i = 2$  pf, and  $C = 10$  pf, we have that  $s = \pm 2$  so that the time constant for voltage build-up is about one-half second.

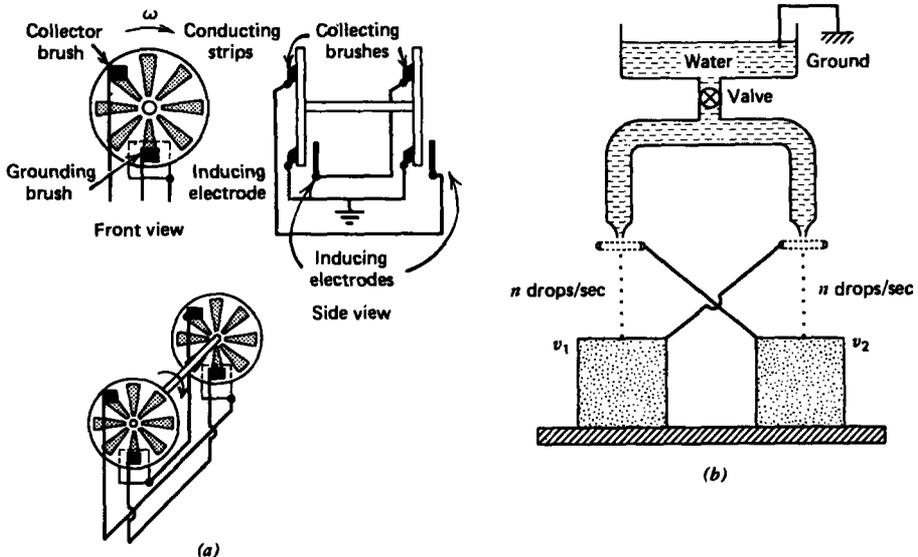


Figure 3-40 Other versions of self-excited electrostatic induction machines use (a) rotating conducting strips (Wimshurst machine) or (b) falling water droplets (Lord Kelvin's water dynamo). These devices are also described by the coupled equivalent circuits in Figure 3-39b.

The early electrical scientists did not use a segmented belt but rather conducting disks embedded in an insulating wheel that could be turned by hand, as shown for the Wimshurst machine in Figure 3-40a. They used the exponentially growing voltage to charge up a capacitor called a Leyden jar (credited to scientists from Leyden, Holland), which was a glass bottle silvered on the inside and outside to form two electrodes with the glass as the dielectric.

An analogous water drop dynamo was invented by Lord Kelvin (then Sir W. Thomson) in 1861, which replaced the rotating disks by falling water drops, as in Figure 3-40b. All these devices are described by the coupled equivalent circuits in Figure 3-39b.

### 3-10-3 Self-Excited Three-Phase Alternating Voltages

In 1967, Euerle\* modified Kelvin's original dynamo by adding a third stream of water droplets so that three-phase

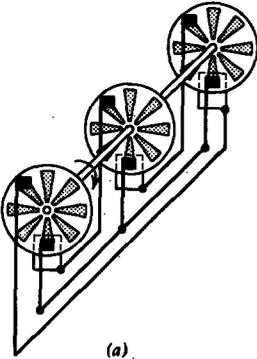
\* W. C. Euerle, "A Novel Method of Generating Polyphase Power," M.S. Thesis, Massachusetts Institute of Technology, 1967. See also J. R. Melcher, *Electric Fields and Moving Media*, IEEE Trans. Education *E-17* (1974), pp. 100-110, and the film by the same title produced for the National Committee on Electrical Engineering Films by the Educational Development Center, 39 Chapel St., Newton, Mass. 02160.

alternating voltages were generated. The analogous three-phase Wimshurst machine is drawn in Figure 3-41a with equivalent circuits in Figure 3-41b. Proceeding as we did in (2) and (3),

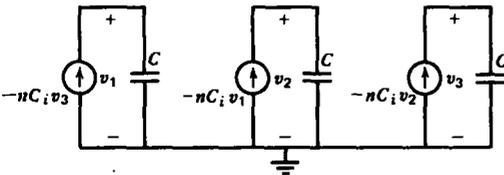
$$\begin{aligned} -nC_i v_1 &= C \frac{dv_2}{dt}, & v_1 &= \hat{V}_1 e^{st} \\ -nC_i v_2 &= C \frac{dv_3}{dt}, & v_2 &= \hat{V}_2 e^{st} \\ -nC_i v_3 &= C \frac{dv_1}{dt}, & v_3 &= \hat{V}_3 e^{st} \end{aligned} \quad (6)$$

equation (6) can be rewritten as

$$\begin{bmatrix} nC_i & C_s & 0 \\ 0 & nC_i & C_s \\ C_s & 0 & nC_i \end{bmatrix} \begin{bmatrix} \hat{V}_1 \\ \hat{V}_2 \\ \hat{V}_3 \end{bmatrix} = 0 \quad (7)$$



(a)



(b)

Figure 3-41 (a) Self-excited three-phase ac Wimshurst machine. (b) The coupled equivalent circuit is valid for any of the analogous machines discussed.

which requires that the determinant of the coefficients of  $\hat{V}_1$ ,  $\hat{V}_2$ , and  $\hat{V}_3$  be zero:

$$\begin{aligned} (nC_i)^3 + (Cs)^3 = 0 &\Rightarrow s = \left(\frac{nC_i}{C}\right)^{1/3} (-1)^{1/3} \\ &= \left(\frac{nC_i}{C}\right)^{1/3} e^{j(\pi/3)(2r-1)}, \quad r = 1, 2, 3 \quad (8) \\ \Rightarrow s_1 &= -\frac{nC_i}{C} \\ s_{2,3} &= \frac{nC_i}{2C} [1 \pm \sqrt{3} j] \end{aligned}$$

where we realized that  $(-1)^{1/3}$  has three roots in the complex plane. The first root is an exponentially decaying solution, but the other two are complex conjugates where the positive real part means exponential growth with time while the imaginary part gives the frequency of oscillation. We have a self-excited three-phase generator as each voltage for the unstable modes is  $120^\circ$  apart in phase from the others:

$$\frac{\hat{V}_2}{\hat{V}_1} = \frac{\hat{V}_3}{\hat{V}_2} = \frac{\hat{V}_1}{\hat{V}_3} = -\frac{nC_i}{Cs_{2,3}} = -\frac{1}{2}(1 \pm \sqrt{3}j) = e^{\pm j(2/3)\pi} \quad (9)$$

Using our earlier typical values following (5), we see that the oscillation frequencies are very low,  $f = (1/2\pi) \text{Im}(s) \approx 0.28 \text{ Hz}$ .

### 3-10-4 Self-Excited Multi-frequency Generators

If we have  $N$  such generators, as in Figure 3-42, with the last one connected to the first one, the  $k$ th equivalent circuit yields

$$-nC_i \hat{V}_k = Cs \hat{V}_{k+1} \quad (10)$$

This is a linear constant coefficient difference equation. Analogously to the exponential time solutions in (3) valid for linear constant coefficient differential equations, solutions to (10) are of the form

$$\hat{V}_k = A\lambda^k \quad (11)$$

where the characteristic root  $\lambda$  is found by substitution back into (10) to yield

$$-nC_i A\lambda^k = Cs A\lambda^{k+1} \Rightarrow \lambda = -nC_i/Cs \quad (12)$$

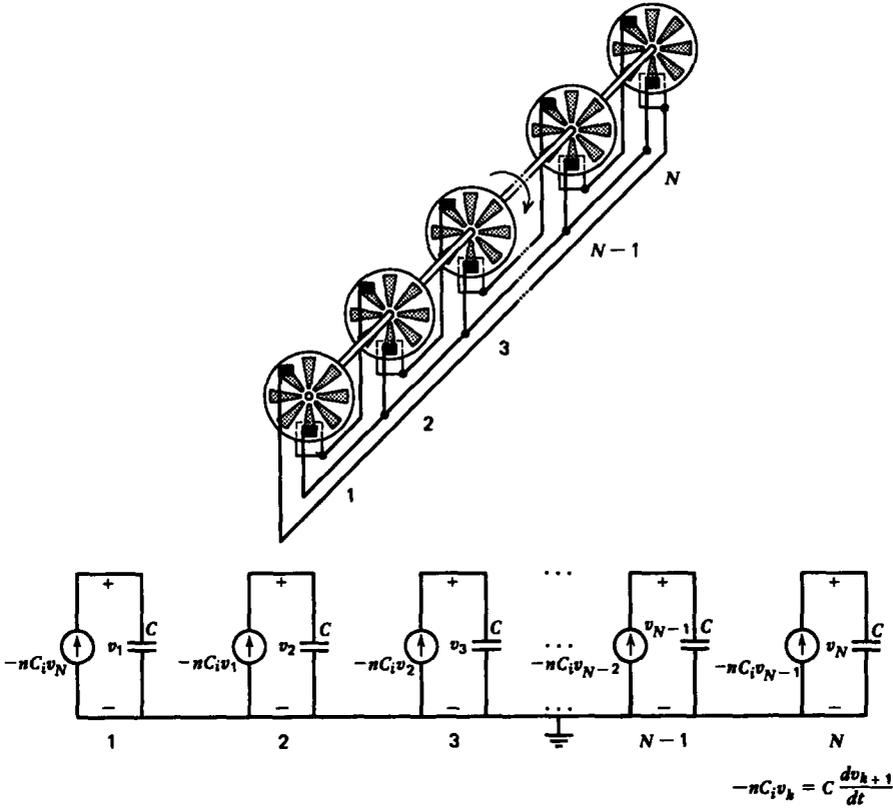


Figure 3-42 Multi-frequency, polyphase self-excited Wimshurst machine with equivalent circuit.

Since the last generator is coupled to the first one, we must have that

$$\begin{aligned}
 V_{N+1} &= V_1 \Rightarrow \lambda^{N+1} = \lambda^1 \\
 &\Rightarrow \lambda^N = 1 \\
 &\Rightarrow \lambda = 1^{1/N} = e^{j2\pi r/N}, \quad r = 1, 2, 3, \dots, N \quad (13)
 \end{aligned}$$

where we realize that unity has  $N$  complex roots.

The system natural frequencies are then obtained from (12) and (13) as

$$s = -\frac{nC_i}{C\lambda} = -\frac{nC_i}{C} e^{-j2\pi r/N}, \quad r = 1, 2, \dots, N \quad (14)$$

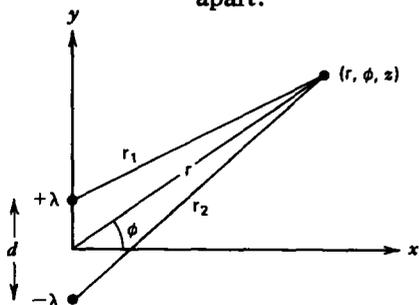
We see that for  $N=2$  and  $N=3$  we recover the results of (4) and (8). All the roots with a positive real part of  $s$  are unstable and the voltages spontaneously build up in time with oscillation frequencies  $\omega_0$  given by the imaginary part of  $s$ .

$$\omega_0 = |\text{Im}(s)| = \frac{nC_i}{C} |\sin 2\pi r/N| \quad (15)$$

**PROBLEMS**

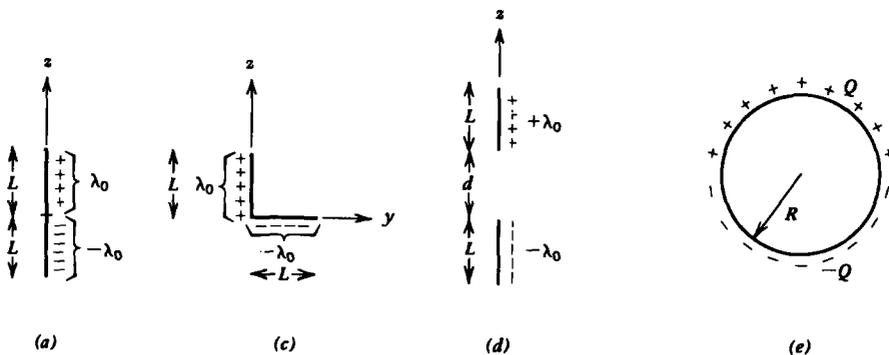
**Section 3-1**

1. A two-dimensional dipole is formed by two infinitely long parallel line charges of opposite polarity  $\pm \lambda$  a small distance  $d$ , apart.



- (a) What is the potential at any coordinate  $(r, \phi, z)$ ?
- (b) What are the potential and electric field far from the dipole ( $r \gg d$ )? What is the dipole moment per unit length?
- (c) What is the equation of the field lines?

2. Find the dipole moment for each of the following charge distributions:



- (a) Two uniform colinear opposite polarity line charges  $\pm \lambda_0$  each a small distance  $L$  along the  $z$  axis.
- (b) Same as (a) with the line charge distribution as

$$\lambda(z) = \begin{cases} \lambda_0(1 - z/L), & 0 < z < L \\ -\lambda_0(1 + z/L), & -L < z < 0 \end{cases}$$

- (c) Two uniform opposite polarity line charges  $\pm \lambda_0$  each of length  $L$  but at right angles.
- (d) Two parallel uniform opposite polarity line charges  $\pm \lambda_0$  each of length  $L$  a distance  $d$  apart.

MIT OpenCourseWare  
<http://ocw.mit.edu>

Resource: [OpenStax: Introduction to Psychology](#)  
Textbook: [OpenStax: Introduction to Psychology](#)

The following may not correspond to a particular course on MIT OpenCourseWare, but has been provided by the author as an individual learning resource.

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.