

# chapter 8

*guided electromagnetic  
waves*

The uniform plane wave solutions developed in Chapter 7 cannot in actuality exist throughout all space, as an infinite amount of energy would be required from the sources. However, TEM waves can also propagate in the region of finite volume between electrodes. Such electrode structures, known as transmission lines, are used for electromagnetic energy flow from power (60 Hz) to microwave frequencies, as delay lines due to the finite speed  $c$  of electromagnetic waves, and in pulse forming networks due to reflections at the end of the line. Because of the electrode boundaries, more general wave solutions are also permitted where the electric and magnetic fields are no longer perpendicular. These new solutions also allow electromagnetic power flow in closed single conductor structures known as waveguides.

## 8-1 THE TRANSMISSION LINE EQUATIONS

### 8-1-1 The Parallel Plate Transmission Line

The general properties of transmission lines are illustrated in Figure 8-1 by the parallel plate electrodes a small distance  $d$  apart enclosing linear media with permittivity  $\epsilon$  and permeability  $\mu$ . Because this spacing  $d$  is much less than the width  $w$  or length  $l$ , we neglect fringing field effects and assume that the fields only depend on the  $z$  coordinate.

The perfectly conducting electrodes impose the boundary conditions:

- (i) The tangential component of  $\mathbf{E}$  is zero.
- (ii) The normal component of  $\mathbf{B}$  (and thus  $\mathbf{H}$  in the linear media) is zero.

With these constraints and the neglect of fringing near the electrode edges, the fields cannot depend on  $x$  or  $y$  and thus are of the following form:

$$\begin{aligned}\mathbf{E} &= E_x(z, t)\mathbf{i}_x \\ \mathbf{H} &= H_y(z, t)\mathbf{i}_y\end{aligned}\tag{1}$$

which when substituted into Maxwell's equations yield

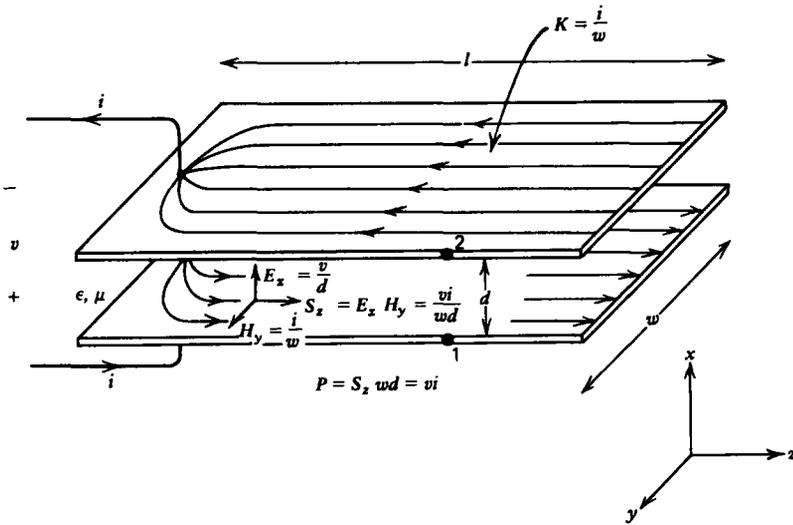


Figure 8-1 The simplest transmission line consists of two parallel perfectly conducting plates a small distance  $d$  apart.

$$\begin{aligned} \nabla \times \mathbf{E} &= -\mu \frac{\partial \mathbf{H}}{\partial t} \Rightarrow \frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} \\ \nabla \times \mathbf{H} &= \epsilon \frac{\partial \mathbf{E}}{\partial t} \Rightarrow \frac{\partial H_y}{\partial z} = -\epsilon \frac{\partial E_x}{\partial t} \end{aligned} \tag{2}$$

We recognize these equations as the same ones developed for plane waves in Section 7-3-1. The wave solutions found there are also valid here. However, now it is more convenient to introduce the circuit variables of voltage and current along the transmission line, which will depend on  $z$  and  $t$ .

Kirchoff's voltage and current laws will not hold along the transmission line as the electric field in (2) has nonzero curl and the current along the electrodes will have a divergence due to the time varying surface charge distribution,  $\sigma_f = \pm \epsilon E_x(z, t)$ . Because  $\mathbf{E}$  has a curl, the voltage difference measured between any two points is not unique, as illustrated in Figure 8-2, where we see time varying magnetic flux passing through the contour  $L_1$ . However, no magnetic flux passes through the path  $L_2$ , where the potential difference is measured between the two electrodes at the same value of  $z$ , as the magnetic flux is parallel to the surface. Thus, the voltage can be uniquely defined between the two electrodes at the same value of  $z$ :

$$v(z, t) = \int_{z=\text{const}}^2 \mathbf{E} \cdot d\mathbf{l} = E_x(z, t)d \tag{3}$$

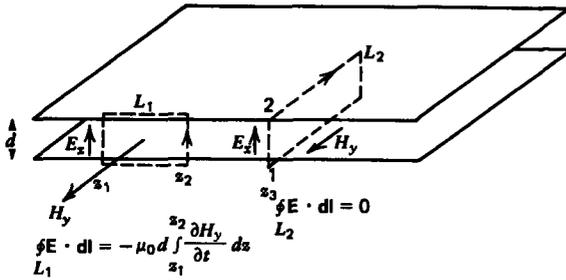


Figure 8-2 The potential difference measured between any two arbitrary points at different positions  $z_1$  and  $z_2$  on the transmission line is not unique—the line integral  $L_1$  of the electric field is nonzero since the contour has magnetic flux passing through it. If the contour  $L_2$  lies within a plane of constant  $z$  such as at  $z_3$ , no magnetic flux passes through it so that the voltage difference between the two electrodes at the same value of  $z$  is unique.

Similarly, the tangential component of  $\mathbf{H}$  is discontinuous at each plate by a surface current  $\pm \mathbf{K}$ . Thus, the total current  $i(z, t)$  flowing in the  $z$  direction on the lower plate is

$$i(z, t) = K_z w = H_y w \tag{4}$$

Substituting (3) and (4) back into (2) results in the transmission line equations:

$$\begin{aligned} \frac{\partial v}{\partial z} &= -L \frac{\partial i}{\partial t} \\ \frac{\partial i}{\partial z} &= -C \frac{\partial v}{\partial t} \end{aligned} \tag{5}$$

where  $L$  and  $C$  are the inductance and capacitance per unit length of the parallel plate structure:

$$L = \frac{\mu d}{w} \text{ henry/m}, \quad C = \frac{\epsilon w}{d} \text{ farad/m} \tag{6}$$

If both quantities are multiplied by the length of the line  $l$ , we obtain the inductance of a single turn plane loop if the line were short circuited, and the capacitance of a parallel plate capacitor if the line were open circuited.

It is no accident that the  $LC$  product

$$LC = \epsilon \mu = 1/c^2 \tag{7}$$

is related to the speed of light in the medium.

### 8-1-2 General Transmission Line Structures

The transmission line equations of (5) are valid for any two-conductor structure of arbitrary shape in the transverse

$xy$  plane but whose cross-sectional area does not change along its axis in the  $z$  direction.  $L$  and  $C$  are the inductance and capacitance per unit length as would be calculated in the quasi-static limits. Various simple types of transmission lines are shown in Figure 8-3. Note that, in general, the field equations of (2) must be extended to allow for  $x$  and  $y$  components but still no  $z$  components:

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_T(x, y, z, t) = E_x \mathbf{i}_x + E_y \mathbf{i}_y, & E_z &= 0 \\ \mathbf{H} &= \mathbf{H}_T(x, y, z, t) = H_x \mathbf{i}_x + H_y \mathbf{i}_y, & H_z &= 0 \end{aligned} \quad (8)$$

We use the subscript  $T$  in (8) to remind ourselves that the fields lie purely in the transverse  $xy$  plane. We can then also distinguish between spatial derivatives along the  $z$  axis ( $\partial/\partial z$ ) from those in the transverse plane ( $\partial/\partial x, \partial/\partial y$ ):

$$\nabla = \underbrace{\nabla_T}_{\mathbf{i}_x \frac{\partial}{\partial x} + \mathbf{i}_y \frac{\partial}{\partial y}} + \mathbf{i}_z \frac{\partial}{\partial z} \quad (9)$$

We may then write Maxwell's equations as

$$\begin{aligned} \nabla_T \times \mathbf{E}_T + \frac{\partial}{\partial z} (\mathbf{i}_z \times \mathbf{E}_T) &= -\mu \frac{\partial \mathbf{H}_T}{\partial t} \\ \nabla_T \times \mathbf{H}_T + \frac{\partial}{\partial z} (\mathbf{i}_z \times \mathbf{H}_T) &= \epsilon \frac{\partial \mathbf{E}_T}{\partial t} \\ \nabla_T \cdot \mathbf{E}_T &= 0 \\ \nabla_T \cdot \mathbf{H}_T &= 0 \end{aligned} \quad (10)$$

The following vector properties for the terms in (10) apply:

- (i)  $\nabla_T \times \mathbf{H}_T$  and  $\nabla_T \times \mathbf{E}_T$  lie purely in the  $z$  direction.
- (ii)  $\mathbf{i}_z \times \mathbf{E}_T$  and  $\mathbf{i}_z \times \mathbf{H}_T$  lie purely in the  $xy$  plane.

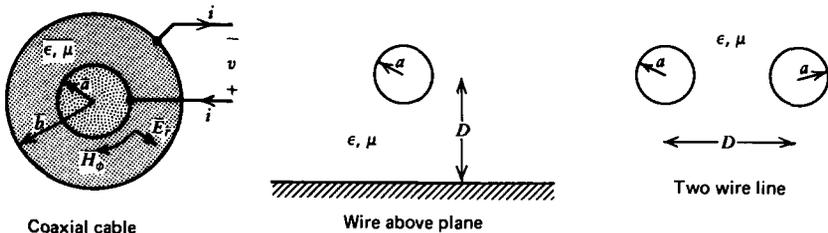


Figure 8-3 Various types of simple transmission lines.

Thus, the equations in (10) may be separated by equating vector components:

$$\begin{aligned}\nabla_T \times \mathbf{E}_T &= 0, & \nabla_T \times \mathbf{H}_T &= 0 \\ \nabla_T \cdot \mathbf{E}_T &= 0, & \nabla_T \cdot \mathbf{H}_T &= 0\end{aligned}\quad (11)$$

$$\begin{aligned}\frac{\partial}{\partial z}(\mathbf{i}_z \times \mathbf{E}_T) &= -\mu \frac{\partial \mathbf{H}_T}{\partial t} \Rightarrow \frac{\partial \mathbf{E}_T}{\partial z} = \mu \frac{\partial}{\partial t}(\mathbf{i}_z \times \mathbf{H}_T) \\ \frac{\partial}{\partial z}(\mathbf{i}_z \times \mathbf{H}_T) &= \epsilon \frac{\partial \mathbf{E}_T}{\partial t}\end{aligned}\quad (12)$$

where the Faraday's law equalities are obtained by crossing with  $\mathbf{i}_z$  and expanding the double cross product

$$\mathbf{i}_z \times (\mathbf{i}_z \times \mathbf{E}_T) = \mathbf{i}_z (\mathbf{i}_z \cdot \mathbf{E}_T) - \mathbf{E}_T (\mathbf{i}_z \cdot \mathbf{i}_z) = -\mathbf{E}_T \quad (13)$$

and remembering that  $\mathbf{i}_z \cdot \mathbf{E}_T = 0$ .

The set of equations in (11) tell us that the field dependences on the transverse coordinates are the same as if the system were static and source free. Thus, all the tools developed for solving static field solutions, including the two-dimensional Laplace's equations and the method of images, can be used to solve for  $\mathbf{E}_T$  and  $\mathbf{H}_T$  in the transverse  $xy$  plane.

We need to relate the fields to the voltage and current defined as a function of  $z$  and  $t$  for the transmission line of arbitrary shape shown in Figure 8-4 as

$$\begin{aligned}v(z, t) &= \int_{z=\text{const}}^2 \mathbf{E}_T \cdot d\mathbf{l} \\ i(z, t) &= \oint_{\text{contour } L \text{ at constant } z \text{ enclosing the inner conductor}} \mathbf{H}_T \cdot d\mathbf{s}\end{aligned}\quad (14)$$

The related quantities of charge per unit length  $q$  and flux per unit length  $\lambda$  along the transmission line are

$$\begin{aligned}q(z, t) &= \epsilon \oint_{z=\text{const}} \mathbf{E}_T \cdot \mathbf{n} \, ds \\ \lambda(z, t) &= \mu \int_{z=\text{const}}^2 \mathbf{H}_T \cdot (\mathbf{i}_z \times d\mathbf{l})\end{aligned}\quad (15)$$

The capacitance and inductance per unit length are then defined as the ratios:

$$\begin{aligned}C &= \frac{q(z, t)}{v(z, t)} = \frac{\epsilon \oint_L \mathbf{E}_T \cdot d\mathbf{s}}{\int_1^2 \mathbf{E}_T \cdot d\mathbf{l}} \Big|_{z=\text{const}} \\ L &= \frac{\lambda(z, t)}{i(z, t)} = \frac{\mu \int_1^2 \mathbf{H}_T \cdot (\mathbf{i}_z \times d\mathbf{l})}{\oint_L \mathbf{H}_T \cdot d\mathbf{s}} \Big|_{z=\text{const}}\end{aligned}\quad (16)$$

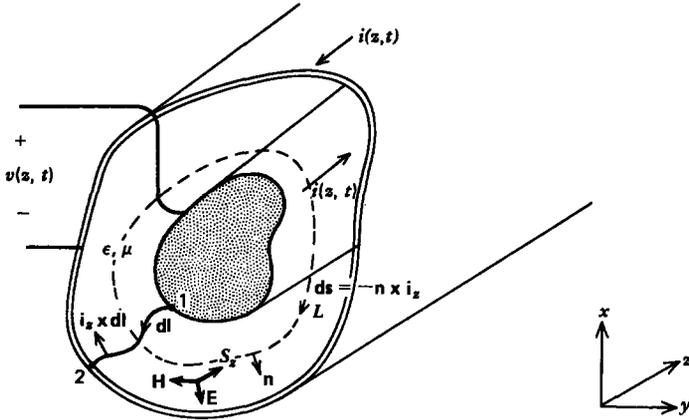


Figure 8-4 A general transmission line has two perfect conductors whose cross-sectional area does not change in the direction along its  $z$  axis, but whose shape in the transverse  $xy$  plane is arbitrary. The electric and magnetic fields are perpendicular, lie in the transverse  $xy$  plane, and have the same dependence on  $x$  and  $y$  as if the fields were static.

which are constants as the geometry of the transmission line does not vary with  $z$ . Even though the fields change with  $z$ , the ratios in (16) do not depend on the field amplitudes.

To obtain the general transmission line equations, we dot the upper equation in (12) with  $d\mathbf{l}$ , which can be brought inside the derivatives since  $d\mathbf{l}$  only varies with  $x$  and  $y$  and not  $z$  or  $t$ . We then integrate the resulting equation over a line at constant  $z$  joining the two electrodes:

$$\begin{aligned} \frac{\partial}{\partial z} \left( \int_1^2 \mathbf{E}_T \cdot d\mathbf{l} \right) &= \frac{\partial}{\partial t} \left( \mu \int_1^2 (\mathbf{i}_z \times \mathbf{H}_T) \cdot d\mathbf{l} \right) \\ &= -\frac{\partial}{\partial t} \left( \mu \int_1^2 \mathbf{H}_T \cdot (\mathbf{i}_z \times d\mathbf{l}) \right) \end{aligned} \quad (17)$$

where the last equality is obtained using the scalar triple product allowing the interchange of the dot and the cross:

$$(\mathbf{i}_z \times \mathbf{H}_T) \cdot d\mathbf{l} = -(\mathbf{H}_T \times \mathbf{i}_z) \cdot d\mathbf{l} = -\mathbf{H}_T \cdot (\mathbf{i}_z \times d\mathbf{l}) \quad (18)$$

We recognize the left-hand side of (17) as the  $z$  derivative of the voltage defined in (14), while the right-hand side is the negative time derivative of the flux per unit length defined in (15):

$$\frac{\partial v}{\partial z} = -\frac{\partial \lambda}{\partial t} = -L \frac{\partial i}{\partial t} \quad (19)$$

We could also have derived this last relation by dotting the upper equation in (12) with the normal  $\mathbf{n}$  to the inner

conductor and then integrating over the contour  $L$  surrounding the inner conductor:

$$\frac{\partial}{\partial z} \left( \oint_L \mathbf{n} \cdot \mathbf{E}_T ds \right) = \frac{\partial}{\partial t} \left( \mu \oint_L \mathbf{n} \cdot (\mathbf{i}_z \times \mathbf{H}_T) ds \right) = - \frac{\partial}{\partial t} \left( \mu \oint_L \mathbf{H}_T \cdot d\mathbf{s} \right) \quad (20)$$

where the last equality was again obtained by interchanging the dot and the cross in the scalar triple product identity:

$$\mathbf{n} \cdot (\mathbf{i}_z \times \mathbf{H}_T) = (\mathbf{n} \times \mathbf{i}_z) \cdot \mathbf{H}_T = -\mathbf{H}_T \cdot d\mathbf{s} \quad (21)$$

The left-hand side of (20) is proportional to the charge per unit length defined in (15), while the right-hand side is proportional to the current defined in (14):

$$\frac{1}{\epsilon} \frac{\partial q}{\partial z} = -\mu \frac{\partial i}{\partial t} \Rightarrow C \frac{\partial v}{\partial z} = -\epsilon \mu \frac{\partial i}{\partial t} \quad (22)$$

Since (19) and (22) must be identical, we obtain the general result previously obtained in Section 6-5-6 that the inductance and capacitance per unit length of any arbitrarily shaped transmission line are related as

$$LC = \epsilon \mu \quad (23)$$

We obtain the second transmission line equation by dotting the lower equation in (12) with  $d\mathbf{l}$  and integrating between electrodes:

$$\frac{\partial}{\partial t} \left( \epsilon \int_1^2 \mathbf{E}_T \cdot d\mathbf{l} \right) = \frac{\partial}{\partial z} \left( \int_1^2 (\mathbf{i}_z \times \mathbf{H}_T) \cdot d\mathbf{l} \right) = - \frac{\partial}{\partial z} \left( \int_1^2 \mathbf{H}_T \cdot (\mathbf{i}_z \times d\mathbf{l}) \right) \quad (24)$$

to yield from (14)–(16) and (23)

$$\epsilon \frac{\partial v}{\partial t} = - \frac{1}{\mu} \frac{\partial \lambda}{\partial z} = - \frac{L}{\mu} \frac{\partial i}{\partial z} \Rightarrow \frac{\partial i}{\partial z} = -C \frac{\partial v}{\partial t} \quad (25)$$

### EXAMPLE 8-1 THE COAXIAL TRANSMISSION LINE

Consider the coaxial transmission line shown in Figure 8-3 composed of two perfectly conducting concentric cylinders of radii  $a$  and  $b$  enclosing a linear medium with permittivity  $\epsilon$  and permeability  $\mu$ . We solve for the transverse dependence of the fields as if the problem were static, independent of time. If the voltage difference between cylinders is  $v$  with the inner cylinder carrying a total current  $i$  the static fields are

$$\mathbf{E}_r = \frac{v}{r \ln(b/a)}, \quad \mathbf{H}_\phi = \frac{i}{2\pi r}$$

The surface charge per unit length  $q$  and magnetic flux per unit length  $\lambda$  are

$$q = \epsilon E_r(r=a)2\pi a = \frac{2\pi\epsilon v}{\ln(b/a)}$$

$$\lambda = \int_a^b \mu H_\phi dr = \frac{\mu i}{2\pi} \ln \frac{b}{a}$$

so that the capacitance and inductance per unit length of this structure are

$$C = \frac{q}{v} = \frac{2\pi\epsilon}{\ln(b/a)}, \quad L = \frac{\lambda}{i} = \frac{\mu}{2\pi} \ln \frac{b}{a}$$

where we note that as required

$$LC = \epsilon\mu$$

Substituting  $E_r$  and  $H_\phi$  into (12) yields the following transmission line equations:

$$\frac{\partial E_r}{\partial z} = -\mu \frac{\partial H_\phi}{\partial t} \Rightarrow \frac{\partial v}{\partial z} = -L \frac{\partial i}{\partial t}$$

$$\frac{\partial H_\phi}{\partial z} = -\epsilon \frac{\partial E_r}{\partial t} \Rightarrow \frac{\partial i}{\partial z} = -C \frac{\partial v}{\partial t}$$

### 8-1-3 Distributed Circuit Representation

Thus far we have emphasized the field theory point of view from which we have derived relations for the voltage and current. However, we can also easily derive the transmission line equations using a distributed equivalent circuit derived from the following criteria:

- (i) The flow of current through a lossless medium between two conductors is entirely by displacement current, in exactly the same way as a capacitor.
- (ii) The flow of current along lossless electrodes generates a magnetic field as in an inductor.

Thus, we may discretize the transmission line into many small incremental sections of length  $\Delta z$  with series inductance  $L \Delta z$  and shunt capacitance  $C \Delta z$ , where  $L$  and  $C$  are the inductance and capacitance per unit lengths. We can also take into account the small series resistance of the electrodes  $R \Delta z$ , where  $R$  is the resistance per unit length (ohms per meter) and the shunt conductance loss in the dielectric  $G \Delta z$ , where  $G$  is the conductance per unit length (siemens per meter). If the transmission line and dielectric are lossless,  $R = 0$ ,  $G = 0$ .

The resulting equivalent circuit for a lossy transmission line shown in Figure 8-5 shows that the current at  $z + \Delta z$  and  $z$  differ by the amount flowing through the shunt capacitance and conductance:

$$i(z, t) - i(z + \Delta z, t) = C \Delta z \frac{\partial v(z, t)}{\partial t} + G \Delta z v(z, t) \quad (26)$$

Similarly, the voltage difference at  $z + \Delta z$  from  $z$  is due to the drop across the series inductor and resistor:

$$v(z, t) - v(z + \Delta z, t) = L \Delta z \frac{\partial i(z + \Delta z, t)}{\partial t} + i(z + \Delta z, t) R \Delta z \quad (27)$$

By dividing (26) and (27) through by  $\Delta z$  and taking the limit as  $\Delta z \rightarrow 0$ , we obtain the lossy transmission line equations:

$$\lim_{\Delta z \rightarrow 0} \frac{i(z + \Delta z, t) - i(z, t)}{\Delta z} = \frac{\partial i}{\partial z} = -C \frac{\partial v}{\partial t} - Gv \quad (28)$$

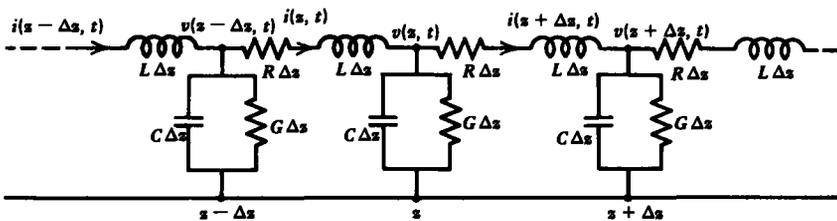
$$\lim_{\Delta z \rightarrow 0} \frac{v(z + \Delta z, t) - v(z, t)}{\Delta z} = \frac{\partial v}{\partial z} = -L \frac{\partial i}{\partial t} - iR$$

which reduce to (19) and (25) when  $R$  and  $G$  are zero.

8-1-4 Power Flow

Multiplying the upper equation in (28) by  $v$  and the lower by  $i$  and then adding yields the circuit equivalent form of Poynting's theorem:

$$\frac{\partial}{\partial z} (vi) = -\frac{\partial}{\partial t} (\frac{1}{2} Cv^2 + \frac{1}{2} Li^2) - Gv^2 - i^2 R \quad (29)$$



$$v(z, t) - v(z + \Delta z, t) = L \Delta z \frac{\partial}{\partial t} i(z + \Delta z, t) + i(z + \Delta z, t) R \Delta z$$

$$i(z, t) - i(z + \Delta z, t) = C \Delta z \frac{\partial}{\partial t} v(z, t) + G \Delta z v(z, t)$$

Figure 8-5 Distributed circuit model of a transmission line including small series and shunt resistive losses.

The power flow  $vi$  is converted into energy storage ( $\frac{1}{2}Cv^2 + \frac{1}{2}Li^2$ ) or is dissipated in the resistance and conductance per unit lengths.

From the fields point of view the total electromagnetic power flowing down the transmission line at any position  $z$  is

$$P(z, t) = \int_S (\mathbf{E}_T \times \mathbf{H}_T) \cdot \mathbf{i}_z dS = \int_S \mathbf{E}_T \cdot (\mathbf{H}_T \times \mathbf{i}_z) dS \quad (30)$$

where  $S$  is the region between electrodes in Figure 8-4. Because the transverse electric field is curl free, we can define the scalar potential

$$\nabla \times \mathbf{E}_T = 0 \Rightarrow \mathbf{E}_T = -\nabla_T V \quad (31)$$

so that (30) can be rewritten as

$$P(z, t) = \int_S (\mathbf{i}_z \times \mathbf{H}_T) \cdot \nabla_T V dS \quad (32)$$

It is useful to examine the vector expansion

$$\nabla_T \cdot [V(\mathbf{i}_z \times \mathbf{H}_T)] = (\mathbf{i}_z \times \mathbf{H}_T) \cdot \nabla_T V + V \nabla_T \cdot (\mathbf{i}_z \times \overset{0}{\mathbf{H}_T}) \quad (33)$$

where the last term is zero because  $\mathbf{i}_z$  is a constant vector and  $\mathbf{H}_T$  is also curl free:

$$\nabla_T \cdot (\mathbf{i}_z \times \mathbf{H}_T) = \mathbf{H}_T \cdot (\nabla_T \times \mathbf{i}_z) - \mathbf{i}_z \cdot (\nabla_T \times \mathbf{H}_T) = 0 \quad (34)$$

Then (32) can be converted to a line integral using the two-dimensional form of the divergence theorem:

$$\begin{aligned} P(z, t) &= \int_S \nabla_T \cdot [V(\mathbf{i}_z \times \mathbf{H}_T)] dS \\ &= - \int_{\substack{\text{contours on} \\ \text{the surfaces of} \\ \text{both electrodes}}} V(\mathbf{i}_z \times \mathbf{H}_T) \cdot \mathbf{n} ds \end{aligned} \quad (35)$$

where the line integral is evaluated at constant  $z$  along the surface of both electrodes. The minus sign arises in (35) because  $\mathbf{n}$  is defined inwards in Figure 8-4 rather than outwards as is usual in the divergence theorem. Since we are free to pick our zero potential reference anywhere, we take the outer conductor to be at zero voltage. Then the line integral in (35) is only nonzero over the inner conductor,

where  $V = v$ :

$$\begin{aligned}
 P(z, t) &= -v \oint_{\text{inner conductor}} (\mathbf{i}_z \times \mathbf{H}_T) \cdot \mathbf{n} \, ds \\
 &= v \oint_{\text{inner conductor}} (\mathbf{H}_T \times \mathbf{i}_z) \cdot \mathbf{n} \, ds \\
 &= v \oint_{\text{inner conductor}} \mathbf{H}_T \cdot (\mathbf{i}_z \times \mathbf{n}) \, ds \\
 &= v \oint_{\text{inner conductor}} \mathbf{H}_T \cdot \mathbf{d}\mathbf{s} \\
 &= vi
 \end{aligned} \tag{36}$$

where we realized that  $(\mathbf{i}_z \times \mathbf{n}) \, ds = \mathbf{d}\mathbf{s}$ , defined in Figure 8-4 if  $L$  lies along the surface of the inner conductor. The electromagnetic power flowing down a transmission line just equals the circuit power.

### 8-1-5 The Wave Equation

Restricting ourselves now to lossless transmission lines so that  $R = G = 0$  in (28), the two coupled equations in voltage and current can be reduced to two single wave equations in  $v$  and  $i$ :

$$\begin{aligned}
 \frac{\partial^2 v}{\partial t^2} &= c^2 \frac{\partial^2 v}{\partial z^2} \\
 \frac{\partial^2 i}{\partial t^2} &= c^2 \frac{\partial^2 i}{\partial z^2}
 \end{aligned} \tag{37}$$

where the speed of the waves is

$$c = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\epsilon\mu}} \text{ m/sec} \tag{38}$$

As we found in Section 7-3-2 the solutions to (37) are propagating waves in the  $\pm z$  directions at the speed  $c$ :

$$\begin{aligned}
 v(z, t) &= V_+(t - z/c) + V_-(t + z/c) \\
 i(z, t) &= I_+(t - z/c) + I_-(t + z/c)
 \end{aligned} \tag{39}$$

where the functions  $V_+$ ,  $V_-$ ,  $I_+$ , and  $I_-$  are determined by boundary conditions imposed by sources and transmission

line terminations. By substituting these solutions back into (28) with  $R = G = 0$ , we find the voltage and current functions related as

$$\begin{aligned} V_+ &= I_+ Z_0 \\ V_- &= -I_- Z_0 \end{aligned} \quad (40)$$

where

$$Z_0 = \sqrt{L/C} \text{ ohm} \quad (41)$$

is known as the characteristic impedance of the transmission line, analogous to the wave impedance  $\eta$  in Chapter 7. Its inverse  $Y_0 = 1/Z_0$  is also used and is termed the characteristic admittance. In practice, it is difficult to measure  $L$  and  $C$  of a transmission line directly. It is easier to measure the wave speed  $c$  and characteristic impedance  $Z_0$  and then calculate  $L$  and  $C$  from (38) and (41).

The most useful form of the transmission line solutions of (39) that we will use is

$$\begin{aligned} v(z, t) &= V_+(t - z/c) + V_-(t + z/c) \\ i(z, t) &= Y_0[V_+(t - z/c) - V_-(t + z/c)] \end{aligned} \quad (42)$$

Note the complete duality between these voltage-current solutions and the plane wave solutions in Section 7-3-2 for the electric and magnetic fields.

## 8-2 TRANSMISSION LINE TRANSIENT WAVES

The easiest way to solve for transient waves on transmission lines is through use of physical reasoning as opposed to mathematical rigor. Since the waves travel at a speed  $c$ , once generated they cannot reach any position  $z$  until a time  $z/c$  later. Waves traveling in the positive  $z$  direction are described by the function  $V_+(t - z/c)$  and waves traveling in the  $-z$  direction by  $V_-(t + z/c)$ . However, at any time  $t$  and position  $z$ , the voltage is equal to the sum of both solutions while the current is proportional to their difference.

### 8-2-1 Transients on Infinitely Long Transmission Lines

The transmission line shown in Figure 8-6a extends to infinity in the positive  $z$  direction. A time varying voltage source  $V(t)$  that is turned on at  $t = 0$  is applied at  $z = 0$  to the line which is initially unexcited. A positively traveling wave  $V_+(t - z/c)$  propagates away from the source. There is no negatively traveling wave,  $V_-(t + z/c) = 0$ . These physical

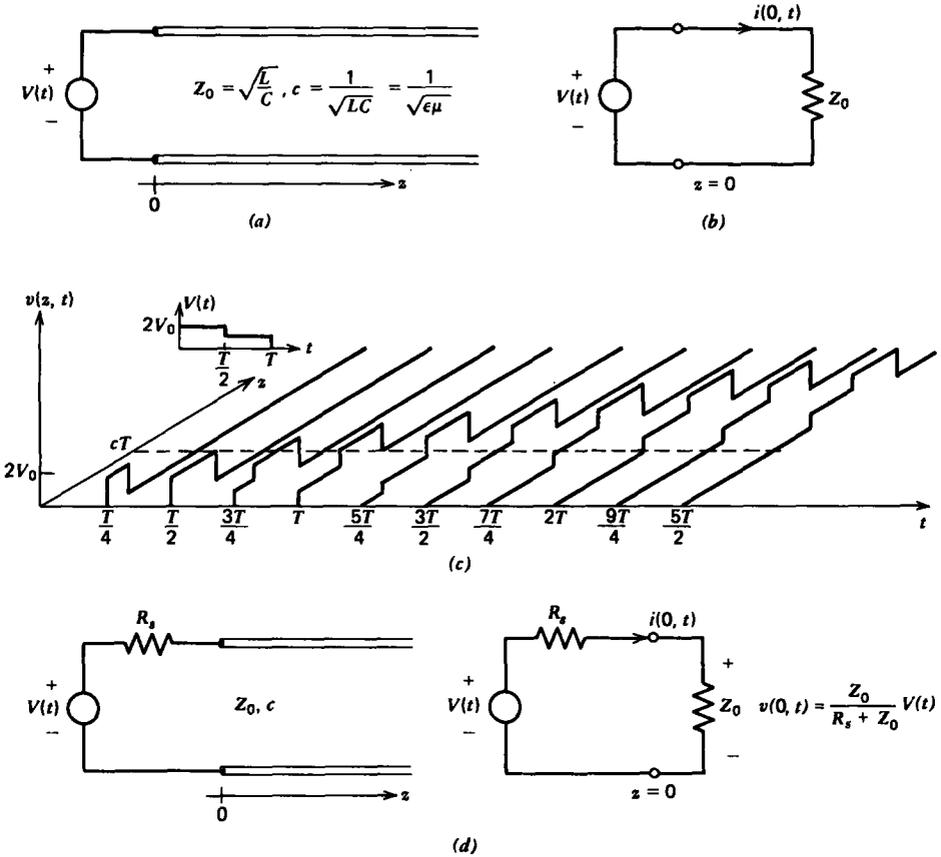


Figure 8-6 (a) A semi-infinite transmission line excited by a voltage source at  $z = 0$ . (b) To the source, the transmission line looks like a resistor  $Z_0$  equal to the characteristic impedance. (c) The spatial distribution of the voltage  $v(z, t)$  at various times for a staircase pulse of  $V(t)$ . (d) If the voltage source is applied to the transmission line through a series resistance  $R_s$ , the voltage across the line at  $z = 0$  is given by the voltage divider relation.

arguments are verified mathematically by realizing that at  $t = 0$  the voltage and current are zero for  $z > 0$ ,

$$v(z, t = 0) = V_+(-z/c) + V_-(-z/c) = 0 \tag{1}$$

$$i(z, t = 0) = Y_0[V_+(-z/c) - V_-(-z/c)] = 0$$

which only allows the trivial solutions

$$V_+(-z/c) = 0, \quad V_-(-z/c) = 0 \tag{2}$$

Since  $z$  can only be positive, whenever the argument of  $V_+$  is negative and of  $V_-$  positive, the functions are zero. Since  $t$  can only be positive, the argument of  $V_-(t + z/c)$  is always positive

so that the function is always zero. The argument of  $V_+(t - z/c)$  can be positive, allowing a nonzero solution if  $t > z/c$  agreeing with our conclusions reached by physical arguments.

With  $V_-(t + z/c) = 0$ , the voltage and current are related as

$$\begin{aligned}v(z, t) &= V_+(t - z/c) \\i(z, t) &= Y_0 V_+(t - z/c)\end{aligned}\quad (3)$$

The line voltage and current have the same shape as the source, delayed in time for any  $z$  by  $z/c$  with the current scaled in amplitude by  $Y_0$ . Thus as far as the source is concerned, the transmission line looks like a resistor of value  $Z_0$  yielding the equivalent circuit at  $z = 0$  shown in Figure 8-6*b*. At  $z = 0$ , the voltage equals that of the source

$$v(0, t) = V(t) = V_+(t) \quad (4)$$

If  $V(t)$  is the staircase pulse of total duration  $T$  shown in Figure 8-6*c*, the pulse extends in space over the spatial interval:

$$\begin{aligned}0 \leq z \leq ct, & \quad 0 \leq t \leq T \\c(t - T) \leq z \leq ct, & \quad t > T\end{aligned}\quad (5)$$

The analysis is the same even if the voltage source is in series with a source resistance  $R_s$ , as in Figure 8-6*d*. At  $z = 0$  the transmission line still looks like a resistor of value  $Z_0$  so that the transmission line voltage divides in the ratio given by the equivalent circuit shown:

$$\begin{aligned}v(z = 0, t) &= \frac{Z_0}{R_s + Z_0} V(t) = V_+(t) \\i(z = 0, t) &= Y_0 V_+(t) = \frac{V(t)}{R_s + Z_0}\end{aligned}\quad (6)$$

The total solution is then identical to that of (3) and (4) with the voltage and current amplitudes reduced by the voltage divider ratio  $Z_0/(R_s + Z_0)$ .

## 8-2-2 Reflections from Resistive Terminations

### (a) Reflection Coefficient

All transmission lines must have an end. In Figure 8-7 we see a positively traveling wave incident upon a load resistor  $R_L$  at  $z = l$ . The reflected wave will travel back towards the source at  $z = 0$  as a  $V_-$  wave. At the  $z = l$  end the following circuit

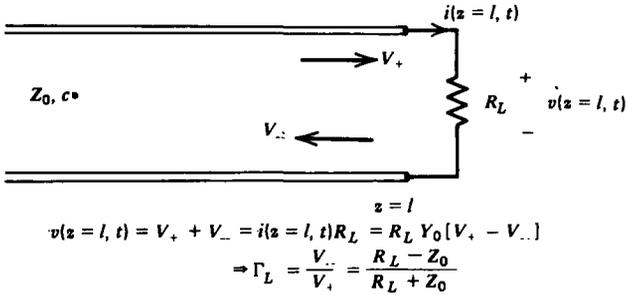


Figure 8-7 A  $V_+$  wave incident upon the end of a transmission line with a load resistor  $R_L$  is reflected as a  $V_-$  wave.

relations hold:

$$\begin{aligned} v(l, t) &= V_+(t - l/c) + V_-(t + l/c) \\ &= i(l, t)R_L \\ &= Y_0 R_L [V_+(t - l/c) - V_-(t + l/c)] \end{aligned} \quad (7)$$

We then find the amplitude of the negatively traveling wave in terms of the incident positively traveling wave as

$$\Gamma_L = \frac{V_-(t + l/c)}{V_+(t - l/c)} = \frac{R_L - Z_0}{R_L + Z_0} \quad (8)$$

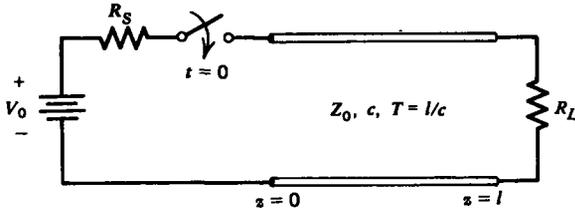
where  $\Gamma_L$  is known as the reflection coefficient that is of the same form as the reflection coefficient  $R$  in Section 7-6-1 for normally incident uniform plane waves on a dielectric.

The reflection coefficient gives us the relative amplitude of the returning  $V_-$  wave compared to the incident  $V_+$  wave. There are several important limits of (8):

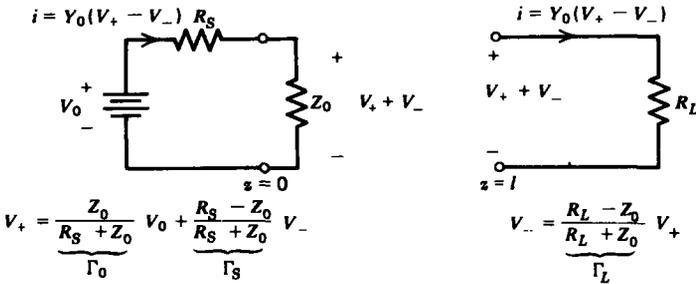
- (i) If  $R_L = Z_0$ , the reflection coefficient is zero ( $\Gamma_L = 0$ ) so that there is no reflected wave and the line is said to be matched.
- (ii) If the line is short circuited ( $R_L = 0$ ), then  $\Gamma_L = -1$ . The reflected wave is equal in amplitude but opposite in sign to the incident wave. In general, if  $R_L < Z_0$ , the reflected voltage wave has its polarity reversed.
- (iii) If the line is open circuited ( $R_L = \infty$ ), then  $\Gamma_L = +1$ . The reflected wave is identical to the incident wave. In general, if  $R_L > Z_0$ , the reflected voltage wave is of the same polarity as the incident wave.

#### (b) Step Voltage

A dc battery of voltage  $V_0$  with series resistance  $R$ , is switched onto the transmission line at  $t=0$ , as shown in Figure 8-8a. At  $z=0$ , the source has no knowledge of the



(a)



(b)

Figure 8-8 (a) A dc voltage  $V_0$  is switched onto a resistively loaded transmission line through a source resistance  $R_s$ . (b) The equivalent circuits at  $z = 0$  and  $z = l$  allow us to calculate the reflected voltage wave amplitudes in terms of the incident waves.

line's length or load termination, so as for an infinitely long line the transmission line looks like a resistor of value  $Z_0$  to the source. There is no  $V_-$  wave initially. The  $V_+$  wave is determined by the voltage divider ratio of the series source resistance and transmission line characteristic impedance as given by (6).

This  $V_+$  wave travels down the line at speed  $c$  where it is reflected at  $z = l$  for  $t > T$ , where  $T = l/c$  is the transit time for a wave propagating between the two ends. The new  $V_-$  wave generated is related to the incident  $V_+$  wave by the reflection coefficient  $\Gamma_L$ . As the  $V_+$  wave continues to propagate in the positive  $z$  direction, the  $V_-$  wave propagates back towards the source. The total voltage at any point on the line is equal to the sum of  $V_+$  and  $V_-$  while the current is proportional to their difference.

When the  $V_-$  wave reaches the end of the transmission line at  $z = 0$  at time  $2T$ , in general a new  $V_+$  wave is generated, which can be found by solving the equivalent circuit shown in Figure 8-8b:

$$v(0, t) + i(0, t)R_s = V_0 \Rightarrow V_+(0, t) + V_-(0, t) + Y_0R_s[V_+(0, t) - V_-(0, t)] = V_0 \quad (9)$$

to yield

$$V_+(0, t) = \Gamma_s V_-(0, t) + \frac{Z_0 V_0}{Z_0 + R_s}, \quad \Gamma_s = \frac{R_s - Z_0}{R_s + Z_0} \quad (10)$$

where  $\Gamma_s$  is just the reflection coefficient at the source end. This new  $V_+$  wave propagates towards the load again generating a new  $V_-$  wave as the reflections continue.

If the source resistance is matched to the line,  $R_s = Z_0$  so that  $\Gamma_s = 0$ , then  $V_+$  is constant for all time and the steady state is reached for  $t > 2T$ . If the load was matched, the steady state is reached for  $t > T$  no matter the value of  $R_s$ . There are no further reflections from the end of a matched line. In Figure 8-9 we plot representative voltage and current spatial distributions for various times assuming the source is matched to the line for the load being matched, open, or short circuited.

(i) Matched Line

When  $R_L = Z_0$  the load reflection coefficient is zero so that  $V_+ = V_0/2$  for all time. The wavefront propagates down the line with the voltage and current being identical in shape. The system is in the dc steady state for  $t \geq T$ .

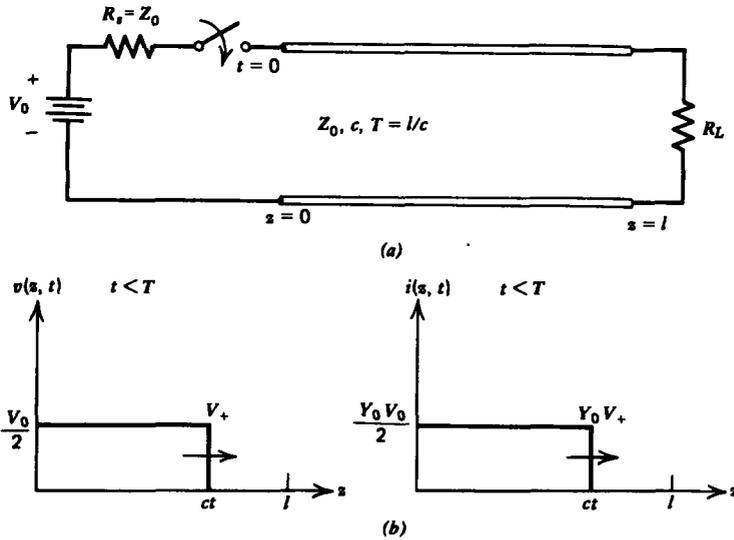


Figure 8-9 (a) A dc voltage is switched onto a transmission line with load resistance  $R_L$  through a source resistance  $R_s$  matched to the line. (b) Regardless of the load resistance, half the source voltage propagates down the line towards the load. If the load is also matched to the line ( $R_L = Z_0$ ), there are no reflections and the steady state of  $v(z, t \geq T) = V_0/2$ ,  $i(z, t \geq T) = Y_0 V_0/2$  is reached for  $t \geq T$ . (c) If the line is short circuited ( $R_L = 0$ ), then  $\Gamma_L = -1$  so that the  $V_+$  and  $V_-$  waves cancel for the voltage but add for the current wherever they overlap in space. Since the source end is matched, no further reflections arise at  $z = 0$  so that the steady state is reached for  $t \geq 2T$ . (d) If the line is open circuited ( $R_L = \infty$ ) so that  $\Gamma_L = +1$ , the  $V_+$  and  $V_-$  waves add for the voltage but cancel for the current.

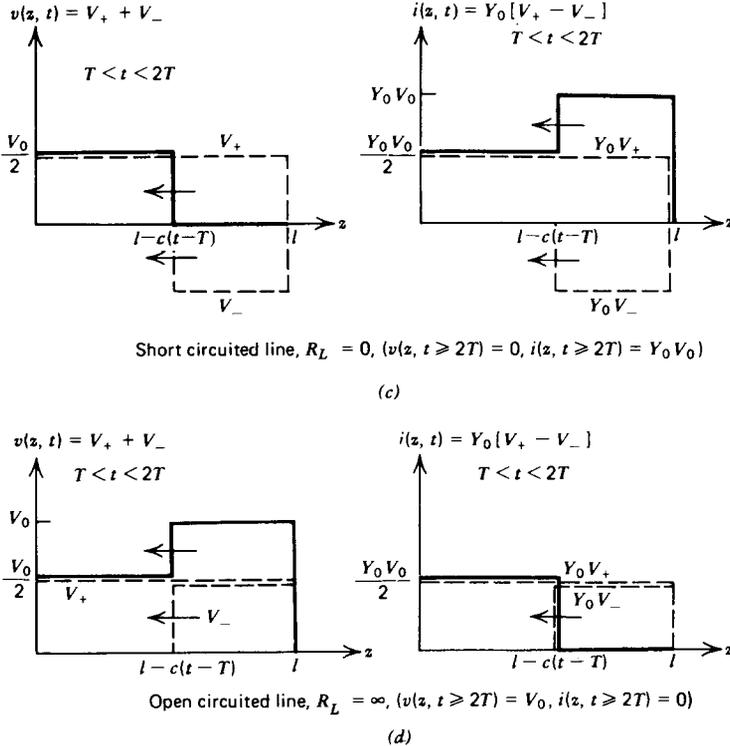


Figure 8-9

**(ii) Open Circuited Line**

When  $R_L = \infty$  the reflection coefficient is unity so that  $V_+ = V_-$ . When the incident and reflected waves overlap in space the voltages add to a staircase pulse shape while the current is zero. For  $t \geq 2T$ , the voltage is  $V_0$  everywhere on the line while the current is zero.

**(iii) Short Circuited Line**

When  $R_L = 0$  the load reflection coefficient is  $-1$  so that  $V_+ = -V_-$ . When the incident and reflected waves overlap in space, the total voltage is zero while the current is now a staircase pulse shape. For  $t \geq 2T$  the voltage is zero everywhere on the line while the current is  $V_0/Z_0$ .

**8-2-3 Approach to the dc Steady State**

If the load end is matched, the steady state is reached after one transit time  $T = l/c$  for the wave to propagate from the source to the load. If the source end is matched, after one

round trip  $2T = 2l/c$  no further reflections occur. If neither end is matched, reflections continue on forever. However, for nonzero and noninfinite source and load resistances, the reflection coefficient is always less than unity in magnitude so that each successive reflection is reduced in amplitude. After a few round-trips, the changes in  $V_+$  and  $V_-$  become smaller and eventually negligible. If the source resistance is zero and the load resistance is either zero or infinite, the transient pulses continue to propagate back and forth forever in the lossless line, as the magnitude of the reflection coefficients are unity.

Consider again the dc voltage source in Figure 8-8a switched through a source resistance  $R_s$  at  $t=0$  onto a transmission line loaded at its  $z=l$  end with a load resistor  $R_L$ . We showed in (10) that the  $V_+$  wave generated at the  $z=0$  end is related to the source and an incoming  $V_-$  wave as

$$V_+ = \Gamma_0 V_0 + \Gamma_s V_-, \quad \Gamma_0 = \frac{Z_0}{R_s + Z_0}, \quad \Gamma_s = \frac{R_s - Z_0}{R_s + Z_0} \quad (11)$$

Similarly, at  $z=l$ , an incident  $V_+$  wave is converted into a  $V_-$  wave through the load reflection coefficient:

$$V_- = \Gamma_L V_+, \quad \Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} \quad (12)$$

We can now tabulate the voltage at  $z=l$  using the following reasoning:

- (i) For the time interval  $t < T$  the voltage at  $z=l$  is zero as no wave has yet reached the end.
- (ii) At  $z=0$  for  $0 \leq t \leq 2T$ ,  $V_- = 0$  resulting in a  $V_+$  wave emanating from  $z=0$  with amplitude  $V_+ = \Gamma_0 V_0$ .
- (iii) When this  $V_+$  wave reaches  $z=l$ , a  $V_-$  wave is generated with amplitude  $V_- = \Gamma_L V_+$ . The incident  $V_+$  wave at  $z=l$  remains unchanged until another interval of  $2T$ , whereupon the just generated  $V_-$  wave after being reflected from  $z=0$  as a new  $V_+$  wave given by (11) again returns to  $z=l$ .
- (iv) Thus, the voltage at  $z=l$  only changes at times  $(2n-1)T$ ,  $n=1, 2, \dots$ , while the voltage at  $z=0$  changes at times  $2(n-1)T$ . The resulting voltage waveforms at the ends are stairstep patterns with steps at these times.

The  $n$ th traveling  $V_+$  wave is then related to the source and the  $(n-1)$ th  $V_-$  wave at  $z=0$  as

$$V_{+n} = \Gamma_0 V_0 + \Gamma_s V_{-(n-1)} \quad (13)$$

while the  $(n - 1)$ th  $V_-$  wave is related to the incident  $(n - 1)$ th  $V_+$  wave at  $z = l$  as

$$V_{-(n-1)} = \Gamma_L V_{+(n-1)} \tag{14}$$

Using (14) in (13) yields a single linear constant coefficient difference equation in  $V_{+n}$ :

$$V_{+n} - \Gamma_s \Gamma_L V_{+(n-1)} = \Gamma_0 V_0 \tag{15}$$

For a particular solution we see that  $V_{+n}$  being a constant satisfies (15):

$$V_{+n} = C \Rightarrow C(1 - \Gamma_s \Gamma_L) = \Gamma_0 V_0 \Rightarrow C = \frac{\Gamma_0}{1 - \Gamma_s \Gamma_L} V_0 \tag{16}$$

To this solution we can add any homogeneous solution assuming the right-hand side of (15) is zero:

$$V_{+n} - \Gamma_s \Gamma_L V_{+(n-1)} = 0 \tag{17}$$

We try a solution of the form

$$V_{+n} = A \lambda^n \tag{18}$$

which when substituted into (17) requires

$$A \lambda^{n-1} (\lambda - \Gamma_s \Gamma_L) = 0 \Rightarrow \lambda = \Gamma_s \Gamma_L \tag{19}$$

The total solution is then a sum of the particular and homogeneous solutions:

$$V_{+n} = \frac{\Gamma_0}{1 - \Gamma_s \Gamma_L} V_0 + A (\Gamma_s \Gamma_L)^n \tag{20}$$

The constant  $A$  is found by realizing that the first transient wave is

$$\dot{V}_{+1} = \Gamma_0 V_0 = \frac{\Gamma_0}{1 - \Gamma_s \Gamma_L} V_0 + A (\Gamma_s \Gamma_L) \tag{21}$$

which requires  $A$  to be

$$A = -\frac{\Gamma_0 V_0}{1 - \Gamma_s \Gamma_L} \tag{22}$$

so that (20) becomes

$$V_{+n} = \frac{\Gamma_0 V_0}{1 - \Gamma_s \Gamma_L} [1 - (\Gamma_s \Gamma_L)^n] \tag{23}$$

Raising the index of (14) by one then gives the  $n$ th  $V_-$  wave as

$$V_{-n} = \Gamma_L V_{+n} \tag{24}$$

so that the total voltage at  $z = l$  after  $n$  reflections at times  $(2n - 1)T$ ,  $n = 1, 2, \dots$ , is

$$V_n = V_{+n} + V_{-n} = \frac{V_0 \Gamma_0 (1 + \Gamma_L)}{1 - \Gamma_s \Gamma_L} [1 - (\Gamma_s \Gamma_L)^n] \quad (25)$$

or in terms of the source and load resistances

$$V_n = \frac{R_L}{R_L + R_s} V_0 [1 - (\Gamma_s \Gamma_L)^n] \quad (26)$$

The steady-state results as  $n \rightarrow \infty$ . If either  $R_s$  or  $R_L$  are nonzero or noninfinite, the product of  $\Gamma_s \Gamma_L$  must be less than unity. Under these conditions

$$\lim_{\substack{n \rightarrow \infty \\ (|\Gamma_s \Gamma_L| < 1)}} (\Gamma_s \Gamma_L)^n = 0 \quad (27)$$

so that in the steady state

$$\lim_{n \rightarrow \infty} V_n = \frac{R_L}{R_s + R_L} V_0 \quad (28)$$

which is just the voltage divider ratio as if the transmission line was just a pair of zero-resistance connecting wires. Note also that if either end is matched so that either  $\Gamma_s$  or  $\Gamma_L$  is zero, the voltage at the load end is immediately in the steady state after the time  $T$ .

In Figure 8-10 the load is plotted versus time with  $R_s = 0$  and  $R_L = 3Z_0$  so that  $\Gamma_s \Gamma_L = -\frac{1}{2}$  and with  $R_L = \frac{1}{3}Z_0$  so that

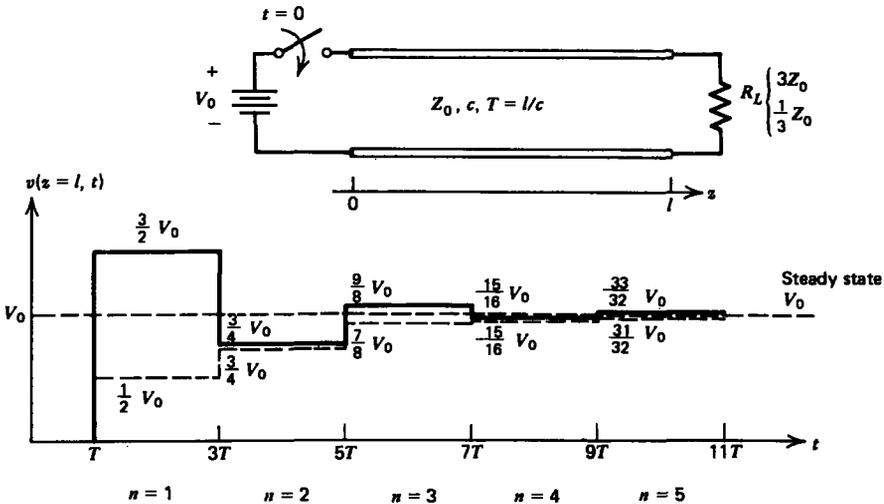


Figure 8-10 The load voltage as a function of time when  $R_s = 0$  and  $R_L = 3Z_0$  so that  $\Gamma_s \Gamma_L = -\frac{1}{2}$  (solid) and with  $R_L = \frac{1}{3}Z_0$  so that  $\Gamma_s \Gamma_L = \frac{1}{2}$  (dashed). The dc steady state is the same as if the transmission line were considered a pair of perfectly conducting wires in a circuit.

$\Gamma_s \Gamma_L = +\frac{1}{2}$ . Then (26) becomes

$$V_n = \begin{cases} V_0 [1 - (-\frac{1}{2})^n], & R_L = 3Z_0 \\ V_0 [1 - (\frac{1}{2})^n], & R_L = \frac{1}{3}Z_0 \end{cases} \quad (29)$$

The step changes in load voltage oscillate about the steady-state value  $V_\infty = V_0$ . The steps rapidly become smaller having less than one-percent variation for  $n > 7$ .

If the source resistance is zero and the load resistance is either zero or infinite (short or open circuits), a lossless transmission line never reaches a dc steady state as the limit of (27) does not hold with  $\Gamma_s \Gamma_L = \pm 1$ . Continuous reflections with no decrease in amplitude results in pulse waveforms for all time. However, in a real transmission line, small losses in the conductors and dielectric allow a steady state to be eventually reached.

Consider the case when  $R_s = 0$  and  $R_L = \infty$  so that  $\Gamma_s \Gamma_L = -1$ . Then from (26) we have

$$V_n = \begin{cases} 0, & n \text{ even} \\ 2V_0, & n \text{ odd} \end{cases} \quad (30)$$

which is sketched in Figure 8-11a.

For any source and load resistances the current through the load resistor at  $z = l$  is

$$\begin{aligned} I_n &= \frac{V_n}{R_L} = \frac{V_0 \Gamma_0 (1 + \Gamma_L)}{R_L (1 - \Gamma_s \Gamma_L)} [1 - (\Gamma_s \Gamma_L)^n] \\ &= \frac{2V_0 \Gamma_0}{R_L + Z_0} \frac{[1 - (\Gamma_s \Gamma_L)^n]}{(1 - \Gamma_s \Gamma_L)} \end{aligned} \quad (31)$$

If both  $R_s$  and  $R_L$  are zero so that  $\Gamma_s \Gamma_L = 1$ , the short circuit current in (31) is in the indeterminate form  $0/0$ , which can be evaluated using l'Hôpital's rule:

$$\begin{aligned} \lim_{\Gamma_s \Gamma_L \rightarrow 1} I_n &= \frac{2V_0 \Gamma_0}{R_L + Z_0} \frac{[-n(\Gamma_s \Gamma_L)^{n-1}]}{(-1)} \\ &= \frac{2V_0 n}{Z_0} \end{aligned} \quad (32)$$

As shown by the solid line in Figure 8-11b, the current continually increases in a stepwise fashion. As  $n$  increases to infinity, the current also becomes infinite, which is expected for a battery connected across a short circuit.

## 8-2-4 Inductors and Capacitors as Quasi-static Approximations to Transmission Lines

If the transmission line was one meter long with a free space dielectric medium, the round trip transit time  $2T = 2l/c$

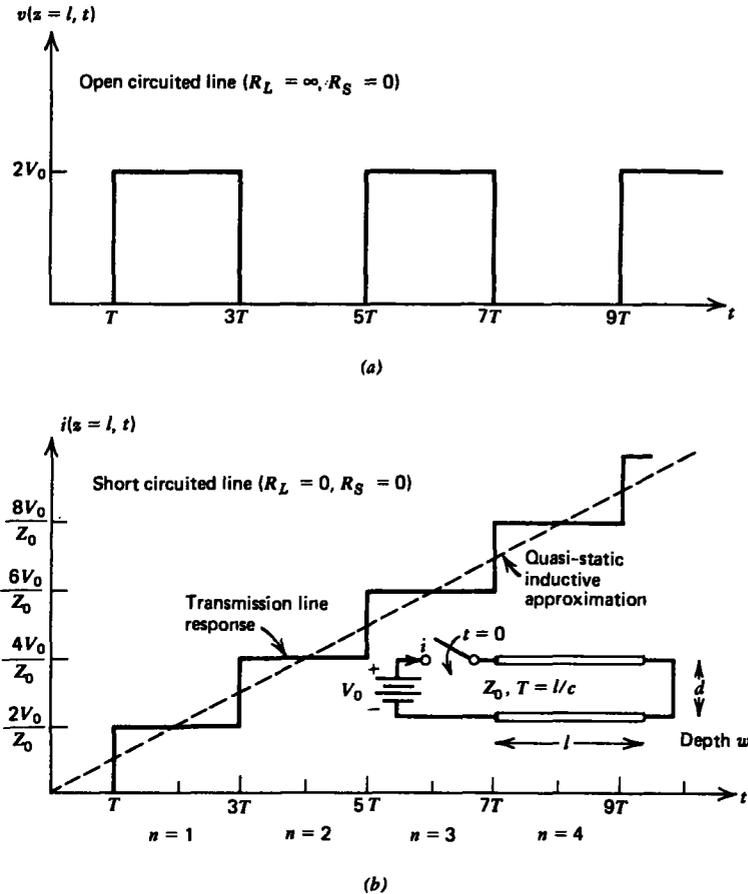


Figure 8-11 The (a) open circuit voltage and (b) short circuit current at the  $z = l$  end of the transmission line for  $R_L = 0$ . No dc steady state is reached because the system is lossless. If the short circuited transmission line is modeled as an inductor in the quasi-static limit, a step voltage input results in a linearly increasing current (shown dashed). The exact transmission line response is the solid staircase waveform.

is approximately 6 nsec. For many circuit applications this time is so fast that it may be considered instantaneous. In this limit the quasi-static circuit element approximation is valid.

For example, consider again the short circuited transmission line ( $R_L = 0$ ) of length  $l$  with zero source resistance. In the magnetic quasi-static limit we would call the structure an inductor with inductance  $Ll$  (remember,  $L$  is the inductance per unit length) so that the terminal voltage and current are related as

$$v = (Ll) \frac{di}{dt} \tag{33}$$

If a constant voltage  $V_0$  is applied at  $t=0$ , the current is obtained by integration of (33) as

$$i = \frac{V_0}{Ll} t \quad (34)$$

where we use the initial condition of zero current at  $t=0$ . The linear time dependence of the current, plotted as the dashed line in Figure 8-11*b*, approximates the rising staircase waveform obtained from the exact transmission line analysis of (32).

Similarly, if the transmission line were open circuited with  $R_L = \infty$ , it would be a capacitor of value  $Cl$  in the electric quasi-static limit so that the voltage on the line charges up through the source resistance  $R_s$  with time constant  $\tau = R_s Cl$  as

$$v(t) = V_0(1 - e^{-t/\tau}) \quad (35)$$

The exact transmission line voltage at the  $z=l$  end is given by (26) with  $R_L = \infty$  so that  $\Gamma_L = 1$ :

$$V_n = V_0(1 - \Gamma_s^n) \quad (36)$$

where the source reflection coefficient can be written as

$$\begin{aligned} \Gamma_s &= \frac{R_s - Z_0}{R_s + Z_0} \\ &= \frac{R_s - \sqrt{L/C}}{R_s + \sqrt{L/C}} \end{aligned} \quad (37)$$

If we multiply the numerator and denominator of (37) through by  $Cl$ , we have

$$\begin{aligned} \Gamma_s &= \frac{R_s Cl - l\sqrt{LC}}{R_s Cl + l\sqrt{LC}} \\ &= \frac{\tau - T}{\tau + T} = \frac{1 - T/\tau}{1 + T/\tau} \end{aligned} \quad (38)$$

where

$$T = l\sqrt{LC} = \ell c \quad (39)$$

For the quasi-static limit to be valid, the wave transit time  $T$  must be much faster than any other time scale of interest so that  $T/\tau \ll 1$ . In Figure 8-12 we plot (35) and (36) for two values of  $T/\tau$  and see that the quasi-static and transmission line results approach each other as  $T/\tau$  becomes small.

When the roundtrip wave transit time is so small compared to the time scale of interest so as to appear to be instantaneous, the circuit treatment is an excellent approximation.

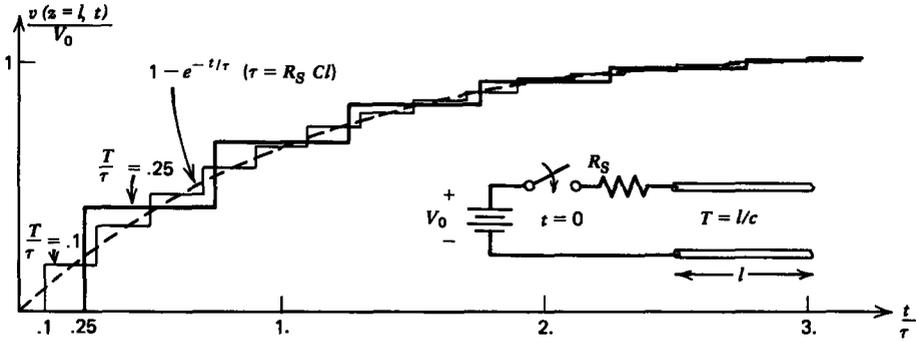


Figure 8-12 The open circuit voltage at  $z=l$  for a step voltage applied at  $t=0$  through a source resistance  $R_s$ , for various values of  $T/\tau$ , which is the ratio of propagation time  $T=l/c$  to quasi-static charging time  $\tau=R_s C l$ . The dashed curve shows the exponential rise obtained by a circuit analysis assuming the open circuited transmission line is a capacitor.

If this propagation time is significant, then the transmission line equations must be used.

8-2-5 Reflections from Arbitrary Terminations

For resistive terminations we have been able to relate reflected wave amplitudes in terms of an incident wave amplitude through the use of a reflection coefficient because the voltage and current in the resistor are algebraically related. For an arbitrary termination, which may include any component such as capacitors, inductors, diodes, transistors, or even another transmission line with perhaps a different characteristic impedance, it is necessary to solve a circuit problem at the end of the line. For the arbitrary element with voltage  $V_L$  and current  $I_L$  at  $z=l$ , shown in Figure 8-13a, the voltage and current at the end of line are related as

$$v(z=l, t) = V_L(t) = V_+(t-l/c) + V_-(t+l/c) \tag{40}$$

$$i(z=l, t) = I_L(t) = Y_0[V_+(t-l/c) - V_-(t+l/c)] \tag{41}$$

We assume that we know the incident  $V_+$  wave and wish to find the reflected  $V_-$  wave. We then eliminate the unknown  $V_-$  in (40) and (41) to obtain

$$2V_+(t-l/c) = V_L(t) + I_L(t)Z_0 \tag{42}$$

which suggests the equivalent circuit in Figure 8-13b.

For a particular lumped termination we solve the equivalent circuit for  $V_L(t)$  or  $I_L(t)$ . Since  $V_+(t-l/c)$  is already known as it is incident upon the termination, once  $V_L(t)$  or

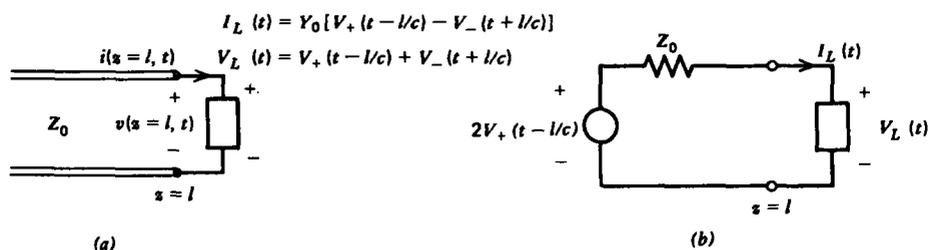


Figure 8-13 A transmission line with an (a) arbitrary load at the  $z=l$  end can be analyzed from the equivalent circuit in (b). Since  $V_+$  is known, calculation of the load current or voltage yields the reflected wave  $V_-$ .

$I_L(t)$  is calculated from the equivalent circuit,  $V_-(t+l/c)$  can be calculated as  $V_- = V_L - V_+$ .

For instance, consider the lossless transmission lines of length  $l$  shown in Figure 8-14a terminated at the end with either a lumped capacitor  $C_L$  or an inductor  $L_L$ . A step voltage at  $t=0$  is applied at  $z=0$  through a source resistor matched to the line.

The source at  $z=0$  is unaware of the termination at  $z=l$  until a time  $2T$ . Until this time it launches a  $V_+$  wave of amplitude  $V_0/2$ . At  $z=l$ , the equivalent circuit for the capacitive termination is shown in Figure 8-14b. Whereas resistive terminations just altered wave amplitudes upon reflection, inductive and capacitive terminations introduce differential equations.

From (42), the voltage across the capacitor  $v_c$  obeys the differential equation

$$Z_0 C_L \frac{dv_c}{dt} + v_c = 2V_+ = V_0, \quad t > T \quad (43)$$

with solution

$$v_c(t) = V_0 [1 - e^{-(t-T)/Z_0 C_L}], \quad t > T \quad (44)$$

Note that the voltage waveform plotted in Figure 8-14b begins at time  $T=l/c$ .

Thus, the returning  $V_-$  wave is given as

$$V_- = v_c - V_+ = V_0/2 + V_0 e^{-(t-T)/Z_0 C_L} \quad (45)$$

This reflected wave travels back to  $z=0$ , where no further reflections occur since the source end is matched. The current at  $z=l$  is then

$$i_c = C_L \frac{dv_c}{dt} = \frac{V_0}{Z_0} e^{-(t-T)/Z_0 C_L}, \quad t > T \quad (46)$$

and is also plotted in Figure 8-14b.

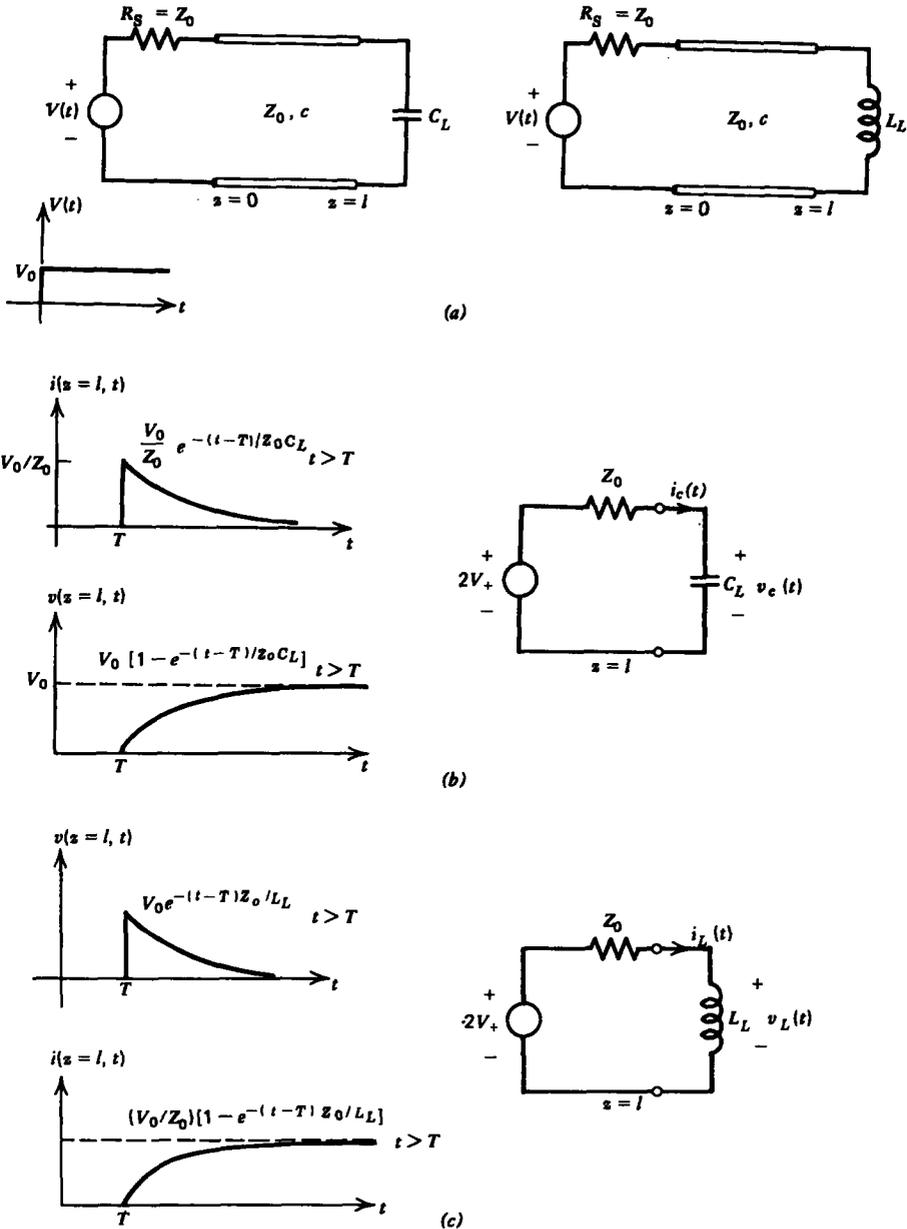


Figure 8-14 (a) A step voltage is applied to transmission lines loaded at  $z = l$  with a capacitor  $C_L$  or inductor  $L_L$ . The load voltage and current are calculated from the (b) resistive-capacitive or (c) resistive-inductive equivalent circuits at  $z = l$  to yield exponential waveforms with respective time constants  $\tau = Z_0 C_L$  and  $\tau = L_L / Z_0$  as the solutions approach the dc steady state. The waveforms begin after the initial  $V_+$  wave arrives at  $z = l$  after a time  $T = l/c$ . There are no further reflections as the source end is matched.

If the end at  $z = 0$  were not matched, a new  $V_+$  would be generated. When it reached  $z = l$ , we would again solve the  $RC$  circuit with the capacitor now initially charged. The reflections would continue, eventually becoming negligible if  $R_1$  is nonzero.

Similarly, the governing differential equation for the inductive load obtained from the equivalent circuit in Figure 8-14c is

$$L_L \frac{di_L}{dt} + i_L Z_0 = 2V_+ = V_0, \quad t > T \tag{47}$$

with solution

$$i_L = \frac{V_0}{Z_0} (1 - e^{-(t-T)Z_0/L_L}), \quad t > T \tag{48}$$

The voltage across the inductor is

$$v_L = L_L \frac{di_L}{dt} = V_0 e^{-(t-T)Z_0/L_L}, \quad t > T \tag{49}$$

Again since the end at  $z = 0$  is matched, the returning  $V_-$  wave from  $z = l$  is not reflected at  $z = 0$ . Thus the total voltage and current for all time at  $z = l$  is given by (48) and (49) and is sketched in Figure 8-14c.

### 8-3 SINUSOIDAL TIME VARIATIONS

#### 8-3-1 Solutions to the Transmission Line Equations

Often transmission lines are excited by sinusoidally varying sources so that the line voltage and current also vary sinusoidally with time:

$$\begin{aligned} v(z, t) &= \text{Re} [\hat{v}(z) e^{j\omega t}] \\ i(z, t) &= \text{Re} [\hat{i}(z) e^{j\omega t}] \end{aligned} \tag{1}$$

Then as we found for TEM waves in Section 7-4, the voltage and current are found from the wave equation solutions of Section 8-1-5 as linear combinations of exponential functions with arguments  $t - z/c$  and  $t + z/c$ :

$$\begin{aligned} v(z, t) &= \text{Re} [\hat{V}_+ e^{j\omega(t-z/c)} + \hat{V}_- e^{j\omega(t+z/c)}] \\ i(z, t) &= Y_0 \text{Re} [\hat{V}_+ e^{j\omega(t-z/c)} - \hat{V}_- e^{j\omega(t+z/c)}] \end{aligned} \tag{2}$$

Now the phasor amplitudes  $\hat{V}_+$  and  $\hat{V}_-$  are complex numbers and do not depend on  $z$  or  $t$ .

By factoring out the sinusoidal time dependence in (2), the spatial dependences of the voltage and current are

$$\begin{aligned}\hat{v}(z) &= \hat{V}_+ e^{-jkz} + \hat{V}_- e^{+jkz} \\ \hat{i}(z) &= Y_0(\hat{V}_+ e^{-jkz} - \hat{V}_- e^{+jkz})\end{aligned}\quad (3)$$

where the wavenumber is again defined as

$$k = \omega/c \quad (4)$$

### 8-3-2 Lossless Terminations

#### (a) Short Circuited Line

The transmission line shown in Figure 8-15a is excited by a sinusoidal voltage source at  $z = -l$  imposing the boundary condition

$$\begin{aligned}v(z = -l, t) &= V_0 \cos \omega t \\ &= \text{Re}(V_0 e^{j\omega t}) \Rightarrow \hat{v}(z = -l) = V_0 = \hat{V}_+ e^{jkl} + \hat{V}_- e^{-jkl}\end{aligned}\quad (5)$$

Note that to use (3) we must write all sinusoids in complex notation. Then since all time variations are of the form  $e^{j\omega t}$ , we may suppress writing it each time and work only with the spatial variations of (3).

Because the transmission line is short circuited, we have the additional boundary condition

$$v(z = 0, t) = 0 \Rightarrow \hat{v}(z = 0) = 0 = \hat{V}_+ + \hat{V}_- \quad (6)$$

which when simultaneously solved with (5) yields

$$\hat{V}_+ = -\hat{V}_- = \frac{V_0}{2j \sin kl} \quad (7)$$

The spatial dependences of the voltage and current are then

$$\begin{aligned}\hat{v}(z) &= \frac{V_0(e^{-jkz} - e^{jkz})}{2j \sin kl} = -\frac{V_0 \sin kz}{\sin kl} \\ \hat{i}(z) &= \frac{V_0 Y_0(e^{-jkz} + e^{jkz})}{2j \sin kl} = -j \frac{V_0 Y_0 \cos kz}{\sin kl}\end{aligned}\quad (8)$$

The instantaneous voltage and current as functions of space and time are then

$$\begin{aligned}v(z, t) &= \text{Re}[\hat{v}(z) e^{j\omega t}] = -V_0 \frac{\sin kz}{\sin kl} \cos \omega t \\ i(z, t) &= \text{Re}[\hat{i}(z) e^{j\omega t}] = \frac{V_0 Y_0 \cos kz \sin \omega t}{\sin kl}\end{aligned}\quad (9)$$

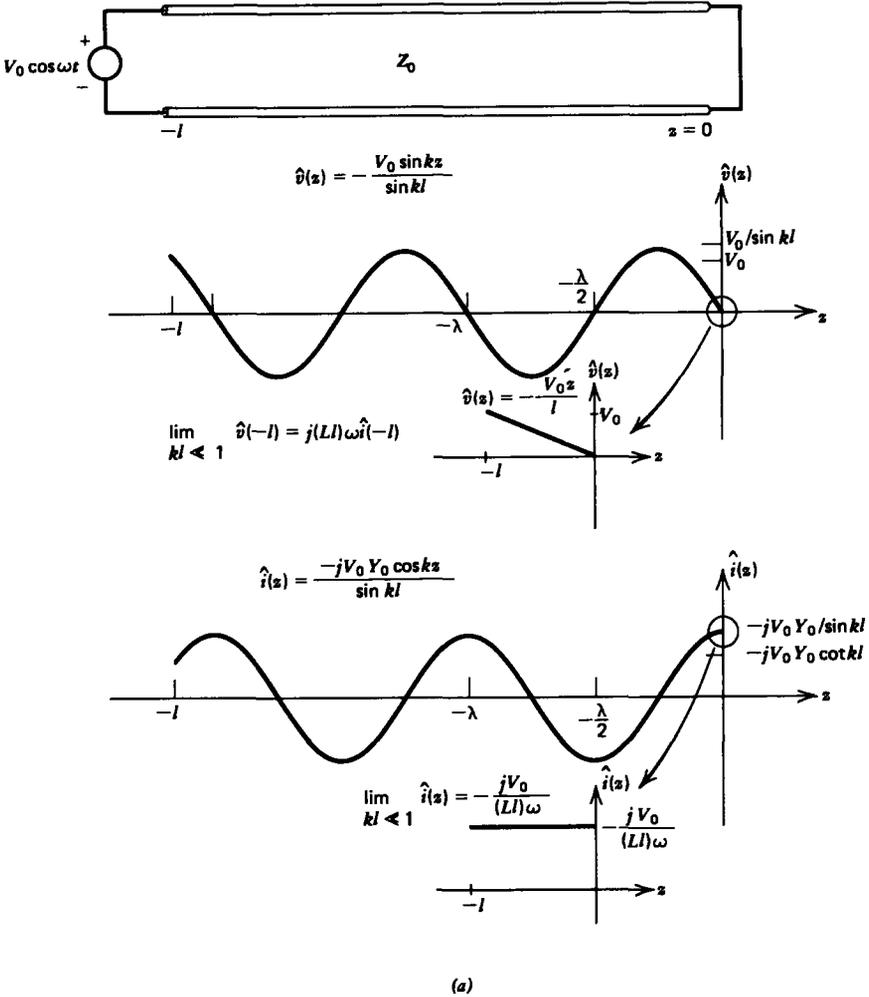


Figure 8-15 The voltage and current distributions on a (a) short circuited and (b) open circuited transmission line excited by sinusoidal voltage sources at  $z = -l$ . If the lines are much shorter than a wavelength, they act like reactive circuit elements. (c) As the frequency is raised, the impedance reflected back as a function of  $z$  can look capacitive or inductive making the transition through open or short circuits.

The spatial distributions of voltage and current as a function of  $z$  at a specific instant of time are plotted in Figure 8-15a and are seen to be  $90^\circ$  out of phase with one another in space with their distributions periodic with wavelength  $\lambda$  given by

$$\lambda = \frac{2\pi}{k} = \frac{2\pi c}{\omega} \tag{10}$$

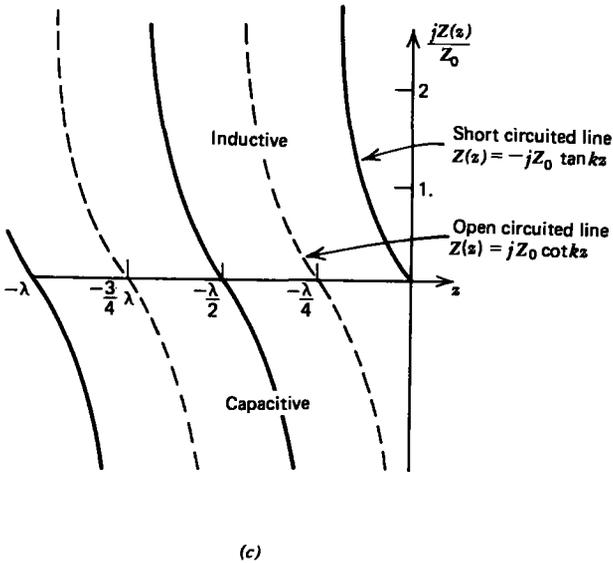
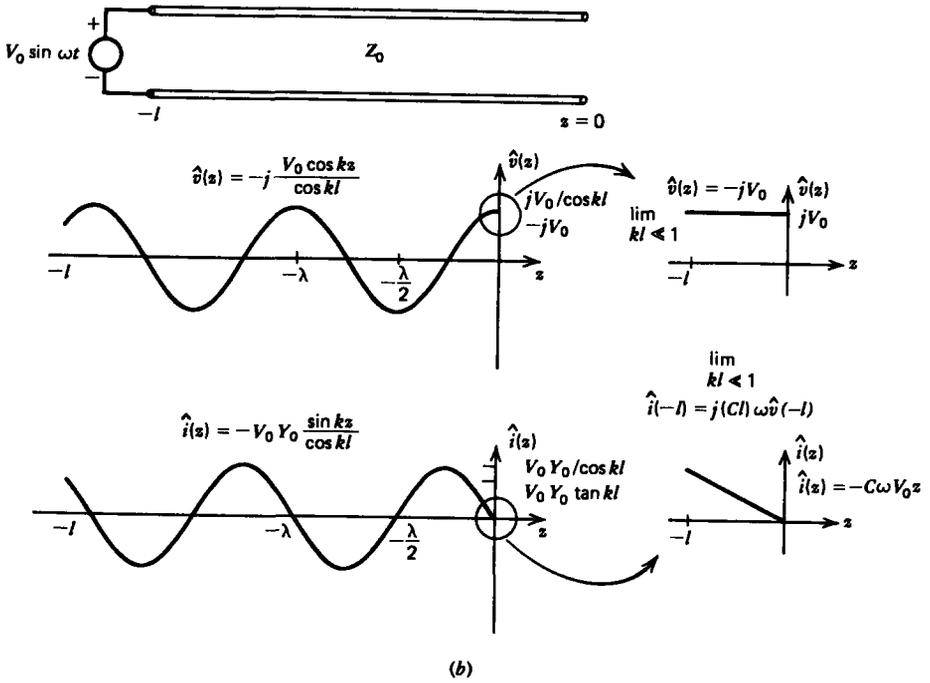


Figure 8-15

The complex impedance at any position  $z$  is defined as

$$Z(z) = \frac{\hat{v}(z)}{\hat{i}(z)} \quad (11)$$

which for this special case of a short circuited line is found from (8) as

$$Z(z) = -jZ_0 \tan kz \quad (12)$$

In particular, at  $z = -l$ , the transmission line appears to the generator as an impedance of value

$$Z(z = -l) = jZ_0 \tan kl \quad (13)$$

From the solid lines in Figure 8-15c we see that there are various regimes of interest:

- (i) When the line is an integer multiple of a half wavelength long so that  $kl = n\pi$ ,  $n = 1, 2, 3, \dots$ , the impedance at  $z = -l$  is zero and the transmission line looks like a short circuit.
- (ii) When the line is an odd integer multiple of a quarter wavelength long so that  $kl = (2n-1)\pi/2$ ,  $n = 1, 2, \dots$ , the impedance at  $z = -l$  is infinite and the transmission line looks like an open circuit.
- (iii) Between the short and open circuit limits  $(n-1)\pi < kl < (2n-1)\pi/2$ ,  $n = 1, 2, 3, \dots$ ,  $Z(z = -l)$  has a positive reactance and hence looks like an inductor.
- (iv) Between the open and short circuit limits  $(n-\frac{1}{2})\pi < kl < n\pi$ ,  $n = 1, 2, \dots$ ,  $Z(z = -l)$  has a negative reactance and so looks like a capacitor.

Thus, the short circuited transmission line takes on all reactive values, both positive (inductive) and negative (capacitive), including open and short circuits as a function of  $kl$ . Thus, if either the length of the line  $l$  or the frequency is changed, the impedance of the transmission line is changed.

Examining (8) we also notice that if  $\sin kl = 0$ , ( $kl = n\pi$ ,  $n = 1, 2, \dots$ ), the voltage and current become infinite (in practice the voltage and current become large limited only by losses). Under these conditions, the system is said to be resonant with the resonant frequencies given by

$$\omega_n = n\pi c/l, \quad n = 1, 2, 3, \dots \quad (14)$$

Any voltage source applied at these frequencies will result in very large voltages and currents on the line.

### (b) Open Circuited Line

If the short circuit is replaced by an open circuit, as in Figure 8-15b, and for variety we change the source at  $z = -l$  to

$V_0 \sin \omega t$  the boundary conditions are

$$\begin{aligned} i(z=0, t) &= 0 \\ v(z=-l, t) &= V_0 \sin \omega t = \operatorname{Re}(-jV_0 e^{j\omega t}) \end{aligned} \quad (15)$$

Using (3) the complex amplitudes obey the relations

$$\begin{aligned} \hat{i}(z=0) &= 0 = Y_0(\hat{V}_+ - \hat{V}_-) \\ \hat{v}(z=-l) &= -jV_0 = \hat{V}_+ e^{jkl} + \hat{V}_- e^{-jkl} \end{aligned} \quad (16)$$

which has solutions

$$\hat{V}_+ = \hat{V}_- = \frac{-jV_0}{2 \cos kl} \quad (17)$$

The spatial dependences of the voltage and current are then

$$\begin{aligned} \hat{v}(z) &= \frac{-jV_0}{2 \cos kl} (e^{-j kz} + e^{j kz}) = \frac{-jV_0}{\cos kl} \cos kz \\ \hat{i}(z) &= \frac{-jV_0 Y_0}{2 \cos kl} (e^{-j kz} - e^{j kz}) = -\frac{V_0 Y_0}{\cos kl} \sin kz \end{aligned} \quad (18)$$

with instantaneous solutions as a function of space and time:

$$\begin{aligned} v(z, t) &= \operatorname{Re}[\hat{v}(z) e^{j\omega t}] = \frac{V_0 \cos kz}{\cos kl} \sin \omega t \\ i(z, t) &= \operatorname{Re}[\hat{i}(z) e^{j\omega t}] = -\frac{V_0 Y_0}{\cos kl} \sin kz \cos \omega t \end{aligned} \quad (19)$$

The impedance at  $z = -l$  is

$$Z(z = -l) = \frac{\hat{v}(-l)}{\hat{i}(-l)} = -jZ_0 \cot kl \quad (20)$$

Again the impedance is purely reactive, as shown by the dashed lines in Figure 8-15c, alternating signs every quarter wavelength so that the open circuit load looks to the voltage source as an inductor, capacitor, short or open circuit depending on the frequency and length of the line.

Resonance will occur if

$$\cos kl = 0 \quad (21)$$

or

$$kl = (2n - 1)\pi/2, \quad n = 1, 2, 3, \dots \quad (22)$$

so that the resonant frequencies are

$$\omega_n = \frac{(2n - 1)\pi c}{2l} \quad (23)$$

### 8-3-3 Reactive Circuit Elements as Approximations to Short Transmission Lines

Let us re-examine the results obtained for short and open circuited lines in the limit when  $l$  is much shorter than the wavelength  $\lambda$  so that in this long wavelength limit the spatial trigonometric functions can be approximated as

$$\lim_{kl \ll 1} \begin{cases} \sin kz \approx kz \\ \cos kz \approx 1 \end{cases} \quad (24)$$

Using these approximations, the voltage, current, and impedance for the short circuited line excited by a voltage source  $V_0 \cos \omega t$  can be obtained from (9) and (13) as

$$\lim_{kl \ll 1} \begin{cases} v(z, t) = -\frac{V_0 z}{l} \cos \omega t, & v(-l, t) = V_0 \cos \omega t \\ i(z, t) = \frac{V_0 Y_0}{kl} \sin \omega t, & i(-l, t) = \frac{V_0 \sin \omega t}{(Ll)\omega} \\ Z(-l) = jZ_0 kl = j\frac{\omega Z_0 l}{c} = j\omega(Ll) \end{cases} \quad (25)$$

We see that the short circuited transmission line acts as an inductor of value  $(Ll)$  (remember that  $L$  is the inductance per unit length), where we used the relations

$$Z_0 = \frac{1}{Y_0} = \sqrt{\frac{L}{C}}, \quad c = \frac{1}{\sqrt{LC}} \quad (26)$$

Note that at  $z = -l$ ,

$$v(-l, t) = (Ll) \frac{di(-l, t)}{dt} \quad (27)$$

Similarly for the open circuited line we obtain:

$$\lim_{kl \ll 1} \begin{cases} v(z, t) = V_0 \sin \omega t \\ i(z, t) = -V_0 Y_0 kz \cos \omega t, & i(-l, t) = (Cl)\omega V_0 \cos \omega t \\ Z(-l) = \frac{-jZ_0}{kl} = \frac{-j}{(Cl)\omega} \end{cases} \quad (28)$$

For the open circuited transmission line, the terminal voltage and current are simply related as for a capacitor,

$$i(-l, t) = (Cl) \frac{dv(-l, t)}{dt} \quad (29)$$

with capacitance given by  $(Cl)$ .

In general, if the frequency of excitation is low enough so that the length of a transmission line is much shorter than the

wavelength, the circuit approximations of inductance and capacitance are appropriate. However, it must be remembered that if the frequencies of interest are so high that the length of a circuit element is comparable to the wavelength, it no longer acts like that element. In fact, as found in Section 8-3-2, a capacitor can even look like an inductor, a short circuit, or an open circuit at high enough frequency while vice versa an inductor can also look capacitive, a short or an open circuit.

In general, if the termination is neither a short nor an open circuit, the voltage and current distribution becomes more involved to calculate and is the subject of Section 8-4.

### 8-3-4 Effects of Line Losses

#### (a) Distributed Circuit Approach

If the dielectric and transmission line walls have Ohmic losses, the voltage and current waves decay as they propagate. Because the governing equations of Section 8-1-3 are linear with constant coefficients, in the sinusoidal steady state we assume solutions of the form

$$\begin{aligned} v(z, t) &= \text{Re} (\hat{V} e^{j(\omega t - kz)}) \\ i(z, t) &= \text{Re} (\hat{I} e^{j(\omega t - kz)}) \end{aligned} \quad (30)$$

where now  $\omega$  and  $k$  are not simply related as the nondispersive relation in (4). Rather we substitute (30) into Eq. (28) in Section 8-1-3:

$$\begin{aligned} \frac{\partial i}{\partial z} &= -C \frac{\partial v}{\partial t} - Gv \Rightarrow -jk\hat{I} = -(Cj\omega + G)\hat{V} \\ \frac{\partial v}{\partial z} &= -L \frac{\partial i}{\partial t} - iR \Rightarrow -jk\hat{V} = -(Lj\omega + R)\hat{I} \end{aligned} \quad (31)$$

which requires that

$$\frac{\hat{V}}{\hat{I}} = \frac{jk}{(Cj\omega + G)} = \frac{Lj\omega + R}{jk} \quad (32)$$

We solve (32) self-consistently for  $k$  as

$$k^2 = -(Lj\omega + R)(Cj\omega + G) = LC\omega^2 - j\omega(RC + LG) - RG \quad (33)$$

The wavenumber is thus complex so that we find the real and imaginary parts from (33) as

$$\begin{aligned} k^2 = k_r + jk_i &\Rightarrow k_r^2 - k_i^2 = LC\omega^2 - RG \\ 2k_r k_i &= -\omega(RC + LG) \end{aligned} \quad (34)$$

In the low loss limit where  $\omega RC \ll 1$  and  $\omega LG \ll 1$ , the spatial decay of  $k_i$  is small compared to the propagation wavenumber  $k_r$ . In this limit we have the following approximate solution:

$$\lim_{\substack{\omega RC \ll 1 \\ \omega LG \ll 1}} \begin{cases} k_r \approx \pm \omega \sqrt{LC} = \pm \omega/c \\ k_i = -\frac{\omega(RC + LG)}{2k_r} \approx \mp \frac{1}{2} \left[ R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}} \right] \\ \approx \mp \frac{1}{2} (RY_0 + GZ_0) \end{cases} \quad (35)$$

We use the upper sign for waves propagating in the  $+z$  direction and the lower sign for waves traveling in the  $-z$  direction.

### (b) Distortionless lines

Using the value of  $k$  of (33),

$$k = \pm [-(Lj\omega + R)(Cj\omega + G)]^{1/2} \quad (36)$$

in (32) gives us the frequency dependent wave impedance for waves traveling in the  $\pm z$  direction as

$$\frac{\hat{V}}{\hat{I}} = \pm \left( \frac{Lj\omega + R}{Cj\omega + G} \right)^{1/2} = \pm \sqrt{\frac{L}{C}} \left( \frac{j\omega + R/L}{j\omega + G/C} \right)^{1/2} \quad (37)$$

If the line parameters are adjusted so that

$$\frac{R}{L} = \frac{G}{C} \quad (38)$$

the impedance in (37) becomes frequency independent and equal to the lossless line impedance. Under the conditions of (38) the complex wavenumber reduces to

$$k_r = \pm \omega \sqrt{LC}, \quad k_i = \mp \sqrt{RG} \quad (39)$$

Although the waves are attenuated, all frequencies propagate at the same phase and group velocities as for a lossless line

$$v_p = \frac{\omega}{k_r} = \pm \frac{1}{\sqrt{LC}} \quad (40)$$

$$v_g = \frac{d\omega}{dk_r} = \pm \frac{1}{\sqrt{LC}}$$

Since all the Fourier components of a pulse excitation will travel at the same speed, the shape of the pulse remains unchanged as it propagates down the line. Such lines are called distortionless.

**(c) Fields Approach**

If  $R = 0$ , we can directly find the TEM wave solutions using the same solutions found for plane waves in Section 7-4-3. There we found that a dielectric with permittivity  $\epsilon$  and small Ohmic conductivity  $\sigma$  has a complex wavenumber:

$$\lim_{\sigma/\omega\epsilon \ll 1} k \approx \pm \left( \frac{\omega}{c} - \frac{j\sigma\eta}{2} \right) \quad (41)$$

Equating (41) to (35) with  $R = 0$  requires that  $GZ_0 = \sigma\eta$ .

The tangential component of  $\mathbf{H}$  at the perfectly conducting transmission line walls is discontinuous by a surface current. However, if the wall has a large but noninfinite Ohmic conductivity  $\sigma_w$ , the fields penetrate in with a characteristic distance equal to the skin depth  $\delta = \sqrt{2/\omega\mu\sigma_w}$ . The resulting  $z$ -directed current gives rise to a  $z$ -directed electric field so that the waves are no longer purely TEM.

Because we assume this loss to be small, we can use an approximate perturbation method to find the spatial decay rate of the fields. We assume that the fields between parallel plane electrodes are essentially the same as when the system is lossless except now being exponentially attenuated as  $e^{-\alpha z}$ , where  $\alpha = -k_i$ :

$$\begin{aligned} E_x(z, t) &= \text{Re} \left[ \hat{E} e^{j(\omega t - k_r z)} e^{-\alpha z} \right] \\ H_y(z, t) &= \text{Re} \left[ \frac{\hat{E}}{\eta} e^{j(\omega t - k_r z)} e^{-\alpha z} \right], \quad k_r = \frac{\omega}{c} \end{aligned} \quad (42)$$

From the real part of the complex Poynting's theorem derived in Section 7-2-4, we relate the divergence of the time-average electromagnetic power density to the time-average dissipated power:

$$\nabla \cdot \langle \mathbf{S} \rangle = -\langle P_d \rangle \quad (43)$$

Using the divergence theorem we integrate (43) over a volume of thickness  $\Delta z$  that encompasses the entire width and thickness of the line, as shown in Figure 8-16:

$$\begin{aligned} \int_V \nabla \cdot \langle \mathbf{S} \rangle dV &= \oint_S \langle \mathbf{S} \rangle \cdot d\mathbf{S} \\ &= \int_{z+\Delta z} \langle S_z(z+\Delta z) \rangle dS \\ &\quad - \int_z \langle S_z(z) \rangle dS = - \int_V \langle P_d \rangle dV \end{aligned} \quad (44)$$

The power  $\langle P_d \rangle$  is dissipated in the dielectric and in the walls. Defining the total electromagnetic power as

$$\langle P(z) \rangle = \int \langle S_z(z) \rangle dS \quad (45)$$

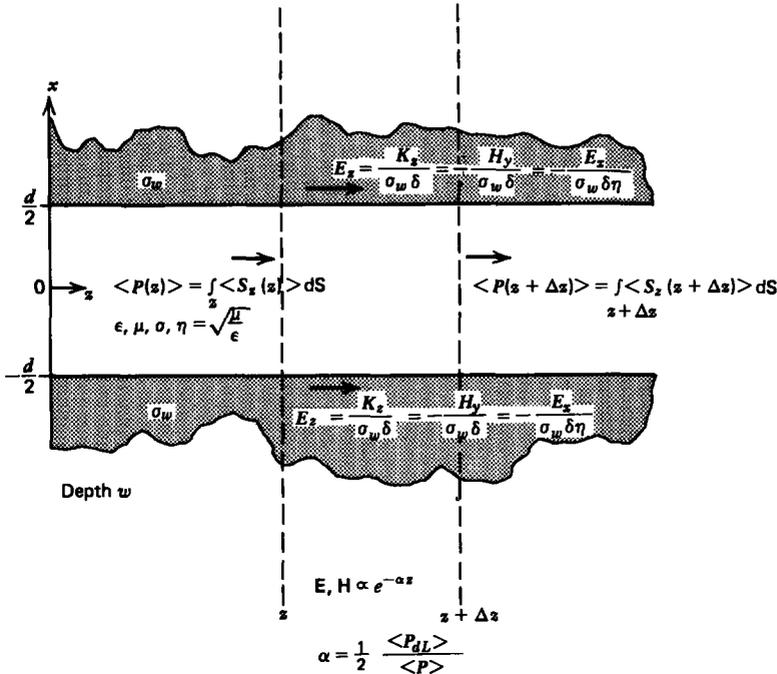


Figure 8-16 A transmission line with lossy walls and dielectric results in waves that decay as they propagate. The spatial decay rate  $\alpha$  of the fields is approximately proportional to the ratio of time average dissipated power per unit length  $\langle P_{dL} \rangle$  to the total time average electromagnetic power flow  $\langle P \rangle$  down the line.

(44) can be rewritten as

$$\langle P(z + \Delta z) \rangle - \langle P(z) \rangle = - \int \langle P_d \rangle dx dy dz \quad (46)$$

Dividing through by  $dz = \Delta z$ , we have in the infinitesimal limit

$$\lim_{\Delta z \rightarrow 0} \frac{\langle P(z + \Delta z) \rangle - \langle P(z) \rangle}{\Delta z} = \frac{d\langle P \rangle}{dz} = - \int_S \langle P_d \rangle dx dy = - \langle P_{dL} \rangle \quad (47)$$

where  $\langle P_{dL} \rangle$  is the power dissipated per unit length. Since the fields vary as  $e^{-\alpha z}$ , the power flow that is proportional to the square of the fields must vary as  $e^{-2\alpha z}$  so that

$$\frac{d\langle P \rangle}{dz} = -2\alpha \langle P \rangle = - \langle P_{dL} \rangle \quad (48)$$

which when solved for the spatial decay rate is proportional to the ratio of dissipated power per unit length to the total

electromagnetic power flowing down the transmission line:

$$\alpha = \frac{1}{2} \frac{\langle P_{dL} \rangle}{\langle P \rangle} \quad (49)$$

For our lossy transmission line, the power is dissipated both in the walls and in the dielectric. Fortunately, it is not necessary to solve the complicated field problem within the walls because we already approximately know the magnetic field at the walls from (42). Since the wall current is effectively confined to the skin depth  $\delta$ , the cross-sectional area through which the current flows is essentially  $w\delta$  so that we can define the surface conductivity as  $\sigma_w\delta$ , where the electric field at the wall is related to the lossless surface current as

$$\mathbf{K}_w = \sigma_w \delta \mathbf{E}_w \quad (50)$$

The surface current in the wall is approximately found from the magnetic field in (42) as

$$K_z = -H_y = -E_x / \eta \quad (51)$$

The time-average power dissipated in the wall is then

$$\langle P_{dL} \rangle_{\text{wall}} = \frac{w}{2} \text{Re} (\mathbf{E}_w \cdot \mathbf{K}_w^*) = \frac{1}{2} \frac{|\mathbf{K}_w|^2 w}{\sigma_w \delta} = \frac{1}{2} \frac{|\hat{E}|^2 w}{\sigma_w \delta \eta^2} \quad (52)$$

The total time-average dissipated power in the walls and dielectric per unit length for a transmission line system of depth  $w$  and plate spacing  $d$  is then

$$\begin{aligned} \langle P_{dL} \rangle &= 2 \langle P_{dL} \rangle_{\text{wall}} + \frac{1}{2} \sigma |\hat{E}|^2 w d \\ &= \frac{1}{2} |\hat{E}|^2 w \left( \frac{2}{\eta^2 \sigma_w \delta} + \sigma d \right) \end{aligned} \quad (53)$$

where we multiply (52) by two because of the losses in both electrodes. The time-average electromagnetic power is

$$\langle P \rangle = \frac{1}{2} \frac{|\hat{E}|^2}{\eta} w d \quad (54)$$

so that the spatial decay rate is found from (49) as

$$\alpha = -k_i = \frac{1}{2} \left( \frac{2}{\eta^2 \sigma_w \delta} + \sigma d \right) \frac{\eta}{d} = \frac{1}{2} \left( \sigma \eta + \frac{2}{\eta \sigma_w \delta d} \right) \quad (55)$$

Comparing (55) to (35) we see that

$$\begin{aligned} GZ_0 &= \sigma \eta, & RY_0 &= \frac{2}{\eta \sigma_w \delta d} \\ \Rightarrow Z_0 &= \frac{1}{Y_0} = \frac{d}{w} \eta, & G &= \frac{\sigma w}{d}, & R &= \frac{2}{\sigma_w w \delta} \end{aligned} \quad (56)$$

## 8-4 ARBITRARY IMPEDANCE TERMINATIONS

### 8-4-1 The Generalized Reflection Coefficient

A lossless transmission line excited at  $z = -l$  with a sinusoidal voltage source is now terminated at its other end at  $z = 0$  with an arbitrary impedance  $Z_L$ , which in general can be a complex number. Defining the load voltage and current at  $z = 0$  as

$$\begin{aligned} v(z = 0, t) &= v_L(t) = \text{Re}(V_L e^{j\omega t}) \\ i(z = 0, t) &= i_L(t) = \text{Re}(I_L e^{j\omega t}), \quad I_L = V_L/Z_L \end{aligned} \quad (1)$$

where  $V_L$  and  $I_L$  are complex amplitudes, the boundary conditions at  $z = 0$  are

$$V_+ + V_- = V_L \quad (2)$$

$$Y_0(V_+ - V_-) = I_L = V_L/Z_L$$

We define the reflection coefficient as the ratio

$$\Gamma_L = V_-/V_+ \quad (3)$$

and solve as

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \quad (4)$$

Here in the sinusoidal steady state with reactive loads,  $\Gamma_L$  can be a complex number as  $Z_L$  may be complex. For transient pulse waveforms,  $\Gamma_L$  was only defined for resistive loads. For capacitive and inductive terminations, the reflections were given by solutions to differential equations in time. Now that we are only considering sinusoidal time variations so that time derivatives are replaced by  $j\omega$ , we can generalize  $\Gamma_L$  for the sinusoidal steady state.

It is convenient to further define the generalized reflection coefficient as

$$\Gamma(z) = \frac{V_- e^{jkz}}{V_+ e^{-jkz}} = \frac{V_-}{V_+} e^{2jkz} = \Gamma_L e^{2jkz} \quad (5)$$

where  $\Gamma_L$  is just  $\Gamma(z = 0)$ . Then the voltage and current on the line can be expressed as

$$\begin{aligned} \hat{v}(z) &= V_+ e^{-jkz} [1 + \Gamma(z)] \\ \hat{i}(z) &= Y_0 V_+ e^{-jkz} [1 - \Gamma(z)] \end{aligned} \quad (6)$$

The advantage to this notation is that now the impedance along the line can be expressed as

$$Z_n(z) = \frac{Z(z)}{Z_0} = \frac{\hat{v}(z)}{\hat{i}(z)Z_0} = \frac{1 + \Gamma(z)}{1 - \Gamma(z)} \quad (7)$$

where  $Z_n$  is defined as the normalized impedance. We can now solve (7) for  $\Gamma(z)$  as

$$\Gamma(z) = \frac{Z_n(z) - 1}{Z_n(z) + 1} \quad (8)$$

Note the following properties of  $Z_n(z)$  and  $\Gamma(z)$  for passive loads:

- (i)  $Z_n(z)$  is generally complex. For passive loads its real part is allowed over the range from zero to infinity while its imaginary part can extend from negative to positive infinity.
- (ii) The magnitude of  $\Gamma(z)$ ,  $|\Gamma_L|$  must be less than or equal to 1 for passive loads.
- (iii) From (5), if  $z$  is increased or decreased by a half wavelength,  $\Gamma(z)$  and hence  $Z_n(z)$  remain unchanged. Thus, if the impedance is known at any position, the impedance of all-points integer multiples of a half wavelength away have the same impedance.
- (iv) From (5), if  $z$  is increased or decreased by a quarter wavelength,  $\Gamma(z)$  changes sign, while from (7)  $Z_n(z)$  goes to its reciprocal  $\Rightarrow 1/Z_n(z) = Y_n(z)$ .
- (v) If the line is matched,  $Z_L = Z_0$ , then  $\Gamma_L = 0$  and  $Z_n(z) = 1$ . The impedance is the same everywhere along the line.

### 8-4-2 Simple Examples

#### (a) Load Impedance Reflected Back to the Source

Properties (iii)–(v) allow simple computations for transmission line systems that have lengths which are integer multiples of quarter or half wavelengths. Often it is desired to maximize the power delivered to a load at the end of a transmission line by adding a lumped admittance  $Y$  across the line. For the system shown in Figure 8-17a, the impedance of the load is reflected back to the generator and then added in parallel to the lumped reactive admittance  $Y$ . The normalized load impedance of  $(R_L + jX_L)/Z_0$  inverts when reflected back to the source by a quarter wavelength to  $Z_0/(R_L + jX_L)$ . Since this is the normalized impedance the actual impedance is found by multiplying by  $Z_0$  to yield  $Z(z = -\lambda/4) = Z_0^2/(R_L + jX_L)$ . The admittance of this reflected load then adds in parallel to  $Y$  to yield a total admittance of  $Y + (R_L + jX_L)/Z_0^2$ . If  $Y$  is pure imaginary and of opposite sign to the reflected load susceptance with value  $-jX_L/Z_0^2$ , maximum power is delivered to the line if the source resistance  $R_S$  also equals the resulting line input impedance,  $R_S = Z_0^2/R_L$ . Since  $Y$  is purely

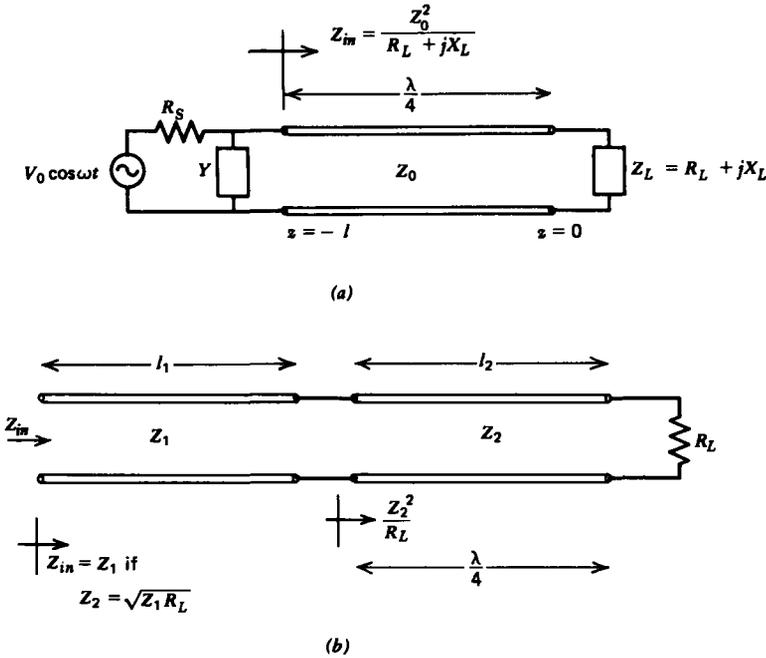


Figure 8-17 The normalized impedance reflected back through a quarter-wave-long line inverts. (a) The time-average power delivered to a complex load can be maximized if  $Y$  is adjusted to just cancel the reactive admittance of the load reflected back to the source with  $R_s$  equaling the resulting input resistance. (b) If the length  $l_2$  of the second transmission line shown is a quarter wave long or an odd integer multiple of  $\lambda/4$  and its characteristic impedance is equal to the geometric average of  $Z_1$  and  $R_L$ , the input impedance  $Z_{in}$  is matched to  $Z_1$ .

reactive and the transmission line is lossless, half the time-average power delivered by the source is dissipated in the load:

$$\langle P \rangle = \frac{1}{8} \frac{V_0^2}{R_S} = \frac{1}{8} \frac{R_L V_0^2}{Z_0^2} \tag{9}$$

Such a reactive element  $Y$  is usually made from a variable length short circuited transmission line called a stub. As shown in Section 8-3-2a, a short circuited lossless line always has a pure reactive impedance.

To verify that the power in (9) is actually dissipated in the load, we write the spatial distribution of voltage and current along the line as

$$\begin{aligned} \hat{v}(z) &= V_+ e^{-jkz} (1 + \Gamma_L e^{2jkz}) \\ \hat{i}(z) &= Y_0 V_+ e^{-jkz} (1 - \Gamma_L e^{2jkz}) \end{aligned} \tag{10}$$

where the reflection coefficient for this load is given by (4) as

$$\Gamma_L = \frac{R_L + jX_L - Z_0}{R_L + jX_L + Z_0} \quad (11)$$

At  $z = -l = -\lambda/4$  we have the boundary condition

$$\begin{aligned} \hat{v}(z = -l) &= V_0/2 = V_+ e^{jkl}(1 + \Gamma_L e^{-2jkl}) \\ &= jV_+(1 - \Gamma_L) \end{aligned} \quad (12)$$

which allows us to solve for  $V_+$  as

$$V_+ = \frac{-jV_0}{2(1 - \Gamma_L)} = \frac{-jV_0}{4Z_0}(R_L + jX_L + Z_0) \quad (13)$$

The time-average power dissipated in the load is then

$$\begin{aligned} \langle P_L \rangle &= \frac{1}{2} \text{Re} [\hat{v}(z = 0)\hat{i}^*(z = 0)] \\ &= \frac{1}{2} |\hat{i}(z = 0)|^2 R_L \\ &= \frac{1}{2} |V_+|^2 |1 - \Gamma_L|^2 Y_0^2 R_L \\ &= \frac{1}{8} V_0^2 Y_0^2 R_L \end{aligned} \quad (14)$$

which agrees with (9).

### (b) Quarter Wavelength Matching

It is desired to match the load resistor  $R_L$  to the transmission line with characteristic impedance  $Z_1$  for any value of its length  $l_1$ . As shown in Figure 8-17b, we connect the load to  $Z_1$  via another transmission line with characteristic impedance  $Z_2$ . We wish to find the values of  $Z_2$  and  $l_2$  necessary to match  $R_L$  to  $Z_1$ .

This problem is analogous to the dielectric coating problem of Example 7-1, where it was found that reflections could be eliminated if the coating thickness between two different dielectric media was an odd integer multiple of a quarter wavelength and whose wave impedance was equal to the geometric average of the impedance in each adjacent region. The normalized load on  $Z_2$  is then  $Z_{n2} = R_L/Z_2$ . If  $l_2$  is an odd integer multiple of a quarter wavelength long, the normalized impedance  $Z_{n2}$  reflected back to the first line inverts to  $Z_2/R_L$ . The actual impedance is obtained by multiplying this normalized impedance by  $Z_2$  to give  $Z_2^2/R_L$ . For  $Z_{in}$  to be matched to  $Z_1$  for any value of  $l_1$ , this impedance must be matched to  $Z_1$ :

$$Z_1 = Z_2^2/R_L \Rightarrow Z_2 = \sqrt{Z_1 R_L} \quad (15)$$

### 8-4-3 The Smith Chart

Because the range of allowed values of  $\Gamma_L$  must be contained within a unit circle in the complex plane, all values of  $Z_n(z)$  can be mapped by a transformation within this unit circle using (8). This transformation is what makes the substitutions of (3)–(8) so valuable. A graphical aid of this mathematical transformation was developed by P. H. Smith in 1939 and is known as the Smith chart. Using the Smith chart avoids the tedium in problem solving with complex numbers.

Let us define the real and imaginary parts of the normalized impedance at some value of  $z$  as

$$Z_n(z) = r + jx \quad (16)$$

The reflection coefficient similarly has real and imaginary parts given as

$$\Gamma(z) = \Gamma_r + j\Gamma_i \quad (17)$$

Using (7) we have

$$r + jx = \frac{1 + \Gamma_r + j\Gamma_i}{1 - \Gamma_r - j\Gamma_i} \quad (18)$$

Multiplying numerator and denominator by the complex conjugate of the denominator  $(1 - \Gamma_r + j\Gamma_i)$  and separating real and imaginary parts yields

$$\begin{aligned} r &= \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1 - \Gamma_r)^2 + \Gamma_i^2} \\ x &= \frac{2\Gamma_i}{(1 - \Gamma_r)^2 + \Gamma_i^2} \end{aligned} \quad (19)$$

Since we wish to plot (19) in the  $\Gamma_r - \Gamma_i$  plane we rewrite these equations as

$$\begin{aligned} \left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 &= \frac{1}{(1+r)^2} \\ (\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 &= \frac{1}{x^2} \end{aligned} \quad (20)$$

Both equations in (20) describe a family of orthogonal circles. The upper equation is that of a circle of radius  $1/(1+r)$  whose center is at the position  $\Gamma_i = 0$ ,  $\Gamma_r = r/(1+r)$ . The lower equation is a circle of radius  $|1/x|$  centered at the position  $\Gamma_r = 1$ ,  $\Gamma_i = 1/x$ . Figure 8-18a illustrates these circles for a particular value of  $r$  and  $x$ , while Figure 8-18b shows a few representative values of  $r$  and  $x$ . In Figure 8-19, we have a complete Smith chart. Only those parts of the circles that lie within the unit circle in the  $\Gamma$  plane are considered for passive

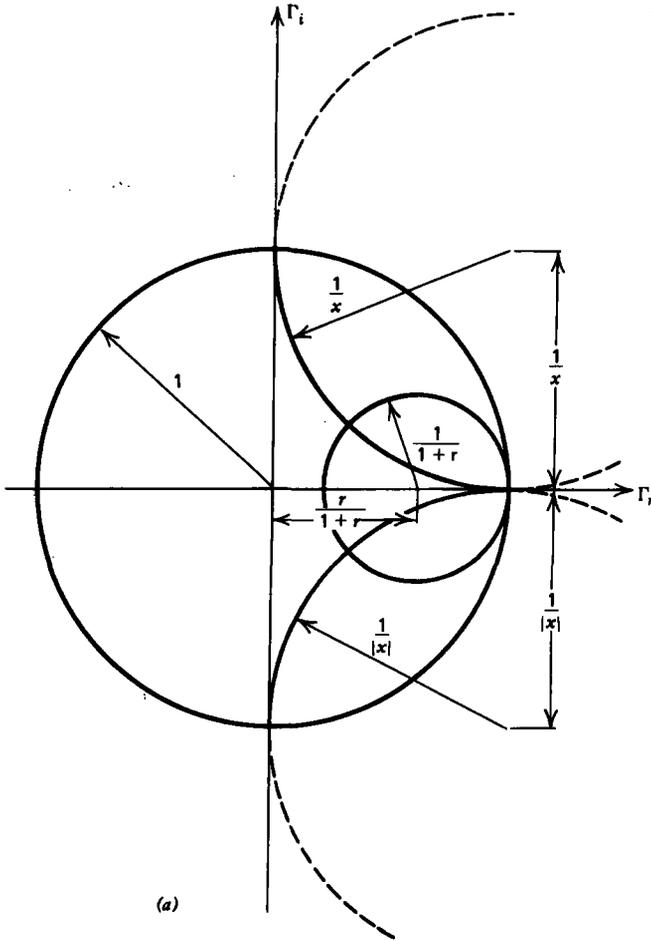


Figure 8-18 For passive loads the Smith chart is constructed within the unit circle in the complex  $\Gamma$  plane. (a) Circles of constant normalized resistance  $r$  and reactance  $x$  are constructed with the centers and radii shown. (b) Smith chart construction for various values of  $r$  and  $x$ .

resistive-reactive loads. The values of  $\Gamma(z)$  themselves are usually not important and so are not listed, though they can be easily found from (8). Note that all circles pass through the point  $\Gamma_r = 1, \Gamma_i = 0$ .

The outside of the circle is calibrated in wavelengths toward the generator, so if the impedance is known at any point on the transmission line (usually at the load end), the impedance at any other point on the line can be found using just a compass and a ruler. From the definition of  $\Gamma(z)$  in (5) with  $z$  negative, we move clockwise around the Smith chart when heading towards the source and counterclockwise when moving towards the load.

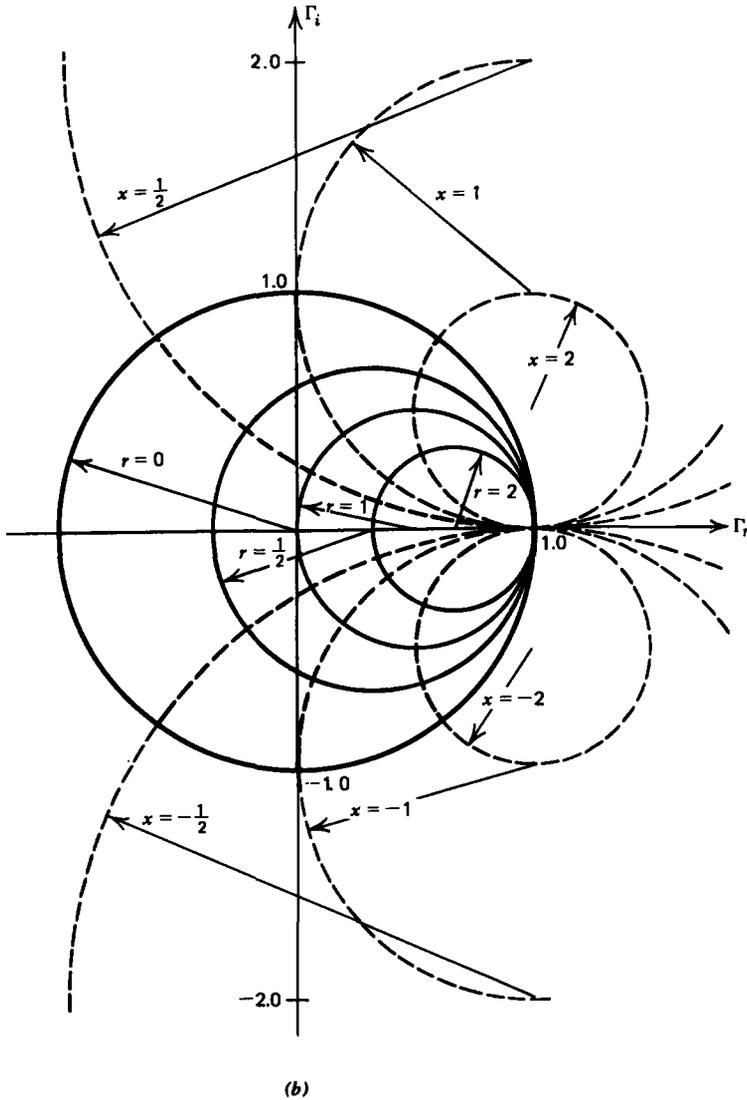


Figure 8-18

In particular, consider the transmission line system in Figure 8-20a. The normalized load impedance is  $Z_n = 1 + j$ . Using the Smith chart in Figure 8-20b, we find the load impedance at position A. The effective impedance reflected back to  $z = -l$  must lie on the circle of constant radius returning to A whenever  $l$  is an integer multiple of a half wavelength. The table in Figure 8-20 lists the impedance at  $z = -l$  for various line lengths. Note that at point C, where  $l = \lambda/4$ , that the normalized impedance is the reciprocal of

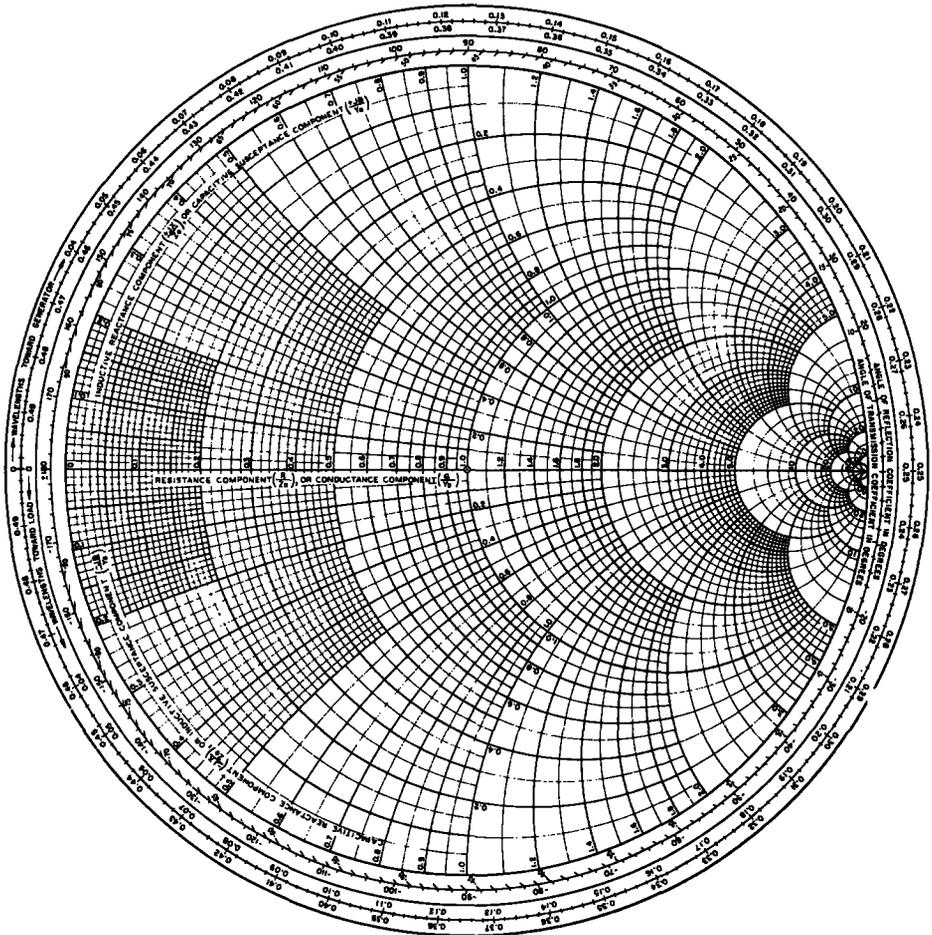


Figure 8-19 A complete Smith chart.

that at  $A$ . Similarly the normalized impedance at  $B$  is the reciprocal of that at  $D$ .

The current from the voltage source is found using the equivalent circuit shown in Figure 8-20c as

$$i = |\hat{I}| \sin(\omega t - \phi) \quad (21)$$

where the current magnitude and phase angle are

$$|\hat{I}| = \frac{V_0}{|50 + Z(z = -l)|}, \quad \phi = \tan^{-1} \frac{\text{Im}[Z(z = -l)]}{50 + \text{Re}[Z(z = -l)]} \quad (22)$$

Representative numerical values are listed in Figure 8-20.

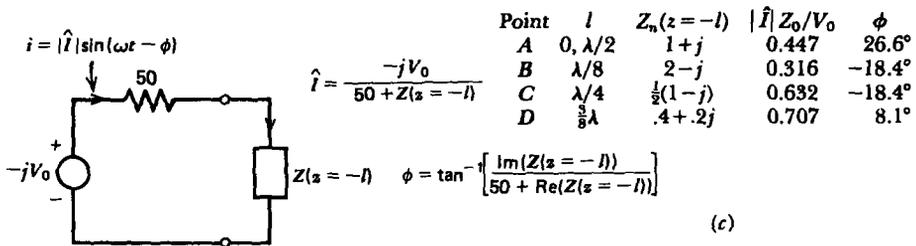
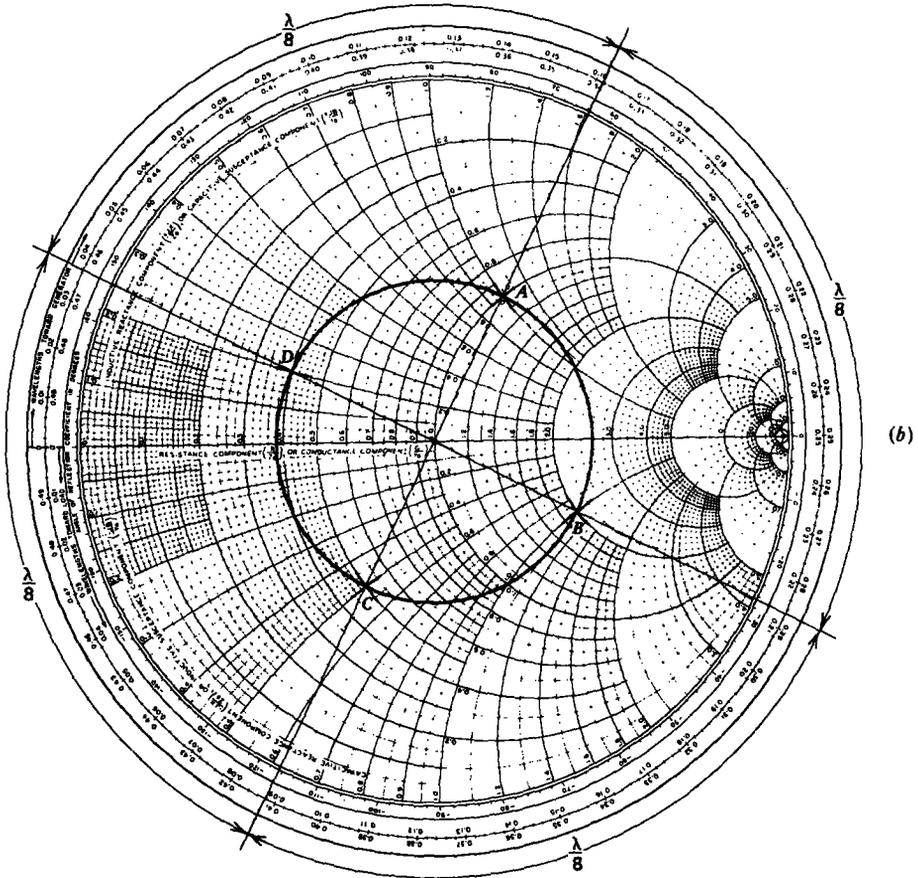
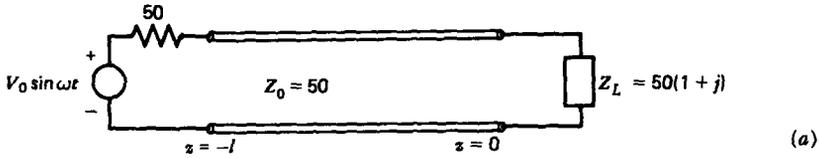


Figure 8-20 (a) The load impedance at  $z = 0$  reflected back to the source is found using the (b) Smith chart for various line lengths. Once this impedance is known the source current is found by solving the simple series circuit in (c).

8-4-4 Standing Wave Parameters

The impedance and reflection coefficient are not easily directly measured at microwave frequencies. In practice, one slides an ac voltmeter across a slotted transmission line and measures the magnitude of the peak or rms voltage and not its phase angle.

From (6) the magnitude of the voltage and current at any position  $z$  is

$$\begin{aligned} |\hat{v}(z)| &= |V_+| |1 + \Gamma(z)| \\ |\hat{i}(z)| &= Y_0 |V_+| |1 - \Gamma(z)| \end{aligned} \tag{23}$$

From (23), the variations of the voltage and current magnitudes can be drawn by a simple construction in the  $\Gamma$  plane, as shown in Figure 8-21. Note again that  $|V_+|$  is just a real number independent of  $z$  and that  $|\Gamma(z)| \leq 1$  for a passive termination. We plot  $|1 + \Gamma(z)|$  and  $|1 - \Gamma(z)|$  since these terms are proportional to the voltage and current magnitudes, respectively. The following properties from this con-

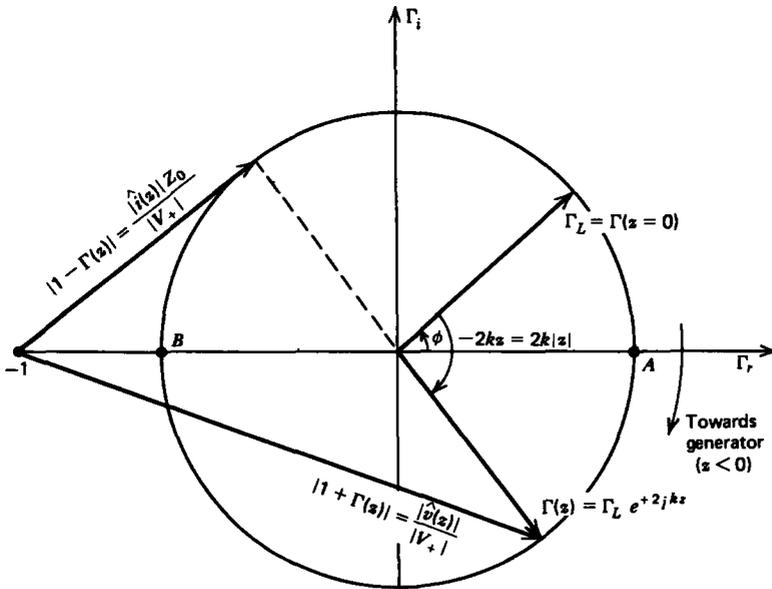


Figure 8-21 The voltage and current magnitudes along a transmission line are respectively proportional to the lengths of the vectors  $|1 + \Gamma(z)|$  and  $|1 - \Gamma(z)|$  in the complex  $\Gamma$  plane.

struction are apparent:

- (i) The magnitude of the current is smallest and the voltage magnitude largest when  $\Gamma(z) = 1$  at point *A* and vice versa when  $\Gamma(z) = -1$  at point *B*.
- (ii) The voltage and current are in phase at the points of maximum or minimum magnitude of either at points *A* or *B*.
- (iii) A rotation of  $\Gamma(z)$  by an angle  $\pi$  corresponds to a change of  $\lambda/4$  in  $z$ , thus any voltage (or current) maximum is separated by  $\lambda/4$  from its nearest minima on either side.

By plotting the lengths of the phasors  $|1 \pm \Gamma(z)|$ , as in Figure 8-22, we obtain a plot of what is called the standing wave pattern on the line. Observe that the curves are not sinusoidal. The minima are sharper than the maxima so the minima are usually located in position more precisely by measurement than the maxima.

From Figures 8-21 and 8-22, the ratio of the maximum voltage magnitude to the minimum voltage magnitude is defined as the voltage standing wave ratio, or VSWR for short:

$$\frac{|\hat{v}(z)|_{\max}}{|\hat{v}(z)|_{\min}} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} = \text{VSWR} \quad (24)$$

The VSWR is measured by simply recording the largest and smallest readings of a sliding voltmeter. Once the VSWR is measured, the reflection coefficient magnitude can be calculated from (24) as

$$|\Gamma_L| = \frac{\text{VSWR} - 1}{\text{VSWR} + 1} \quad (25)$$

The angle  $\phi$  of the reflection coefficient

$$\Gamma_L = |\Gamma_L| e^{j\phi} \quad (26)$$

can also be determined from these standing wave measurements. According to Figure 8-21,  $\Gamma(z)$  must swing clockwise through an angle  $\phi + \pi$  as we move from the load at  $z = 0$  toward the generator to the first voltage minimum at *B*. The shortest distance  $d_{\min}$  that we must move to reach the first voltage minimum is given by

$$2kd_{\min} = \phi + \pi \quad (27)$$

or

$$\frac{\phi}{\pi} = 4 \frac{d_{\min}}{\lambda} - 1 \quad (28)$$

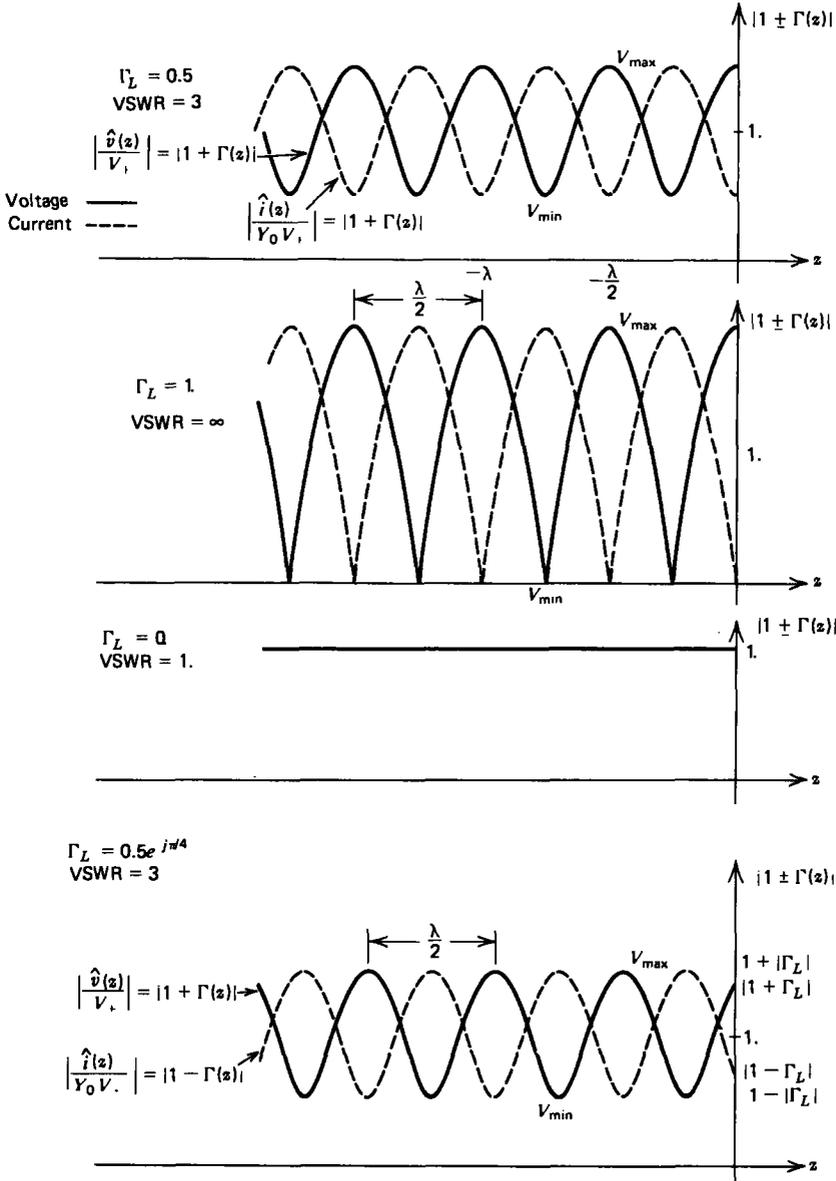


Figure 8-22 Voltage and current standing wave patterns plotted for various values of the VSWR.

A measurement of  $d_{min}$ , as well as a determination of the wavelength (the distance between successive minima or maxima is  $\lambda/2$ ) yields the complex reflection coefficient of the load using (25) and (28). Once we know the complex reflection coefficient we can calculate the load impedance

from (7). These standing wave measurements are sufficient to determine the terminating load impedance  $Z_L$ . These measurement properties of the load reflection coefficient and its relation to the load impedance are of great importance at high frequencies where the absolute measurement of voltage or current may be difficult. Some special cases of interest are:

- (i) Matched line—If  $\Gamma_L = 0$ , then  $VSWR = 1$ . The voltage magnitude is constant everywhere on the line.
- (ii) Short or open circuited line—If  $|\Gamma_L| = 1$ , then  $VSWR = \infty$ . The minimum voltage on the line is zero.
- (iii) The peak normalized voltage  $|\hat{v}(z)/V_+|$  is  $1 + |\Gamma_L|$  while the minimum normalized voltage is  $1 - |\Gamma_L|$ .
- (iv) The normalized voltage at  $z = 0$  is  $|1 + \Gamma_L|$  while the normalized current  $|\hat{i}(z)/Y_0 V_+|$  at  $z = 0$  is  $|1 - \Gamma_L|$ .
- (v) If the load impedance is real ( $Z_L = R_L$ ), then (4) shows us that  $\Gamma_L$  is real. Then evaluating (7) at  $z = 0$ , where  $\Gamma(z = 0) = \Gamma_L$ , we see that when  $Z_L > Z_0$  that  $VSWR = Z_L/Z_0$  while if  $Z_L < Z_0$ ,  $VSWR = Z_0/Z_L$ .

For a general termination, if we know the VSWR and  $d_{\min}$ , we can calculate the load impedance from (7) as

$$\begin{aligned} Z_L &= Z_0 \frac{1 + |\Gamma_L| e^{j\phi}}{1 - |\Gamma_L| e^{j\phi}} \\ &= Z_0 \frac{[VSWR + 1 + (VSWR - 1) e^{j\phi}]}{[VSWR + 1 - (VSWR - 1) e^{j\phi}]} \end{aligned} \quad (29)$$

Multiplying through by  $e^{-j\phi/2}$  and then simplifying yields

$$\begin{aligned} Z_L &= \frac{Z_0 [VSWR - j \tan(\phi/2)]}{[1 - j VSWR \tan(\phi/2)]} \\ &= \frac{Z_0 [1 - j VSWR \tan kd_{\min}]}{[VSWR - j \tan kd_{\min}]} \end{aligned} \quad (30)$$

### EXAMPLE 8-2 VOLTAGE STANDING WAVE RATIO

The VSWR on a 50-Ohm (characteristic impedance) transmission line is 2. The distance between successive voltage minima is 40 cm while the distance from the load to the first minima is 10 cm. What is the reflection coefficient and load impedance?

**SOLUTION**

We are given

$$\text{VSWR} = 2$$

$$kd_{\min} = \frac{2\pi(10)}{2(40)} = \frac{\pi}{4}$$

The reflection coefficient is given from (25)–(28) as

$$\Gamma_L = \frac{1}{3} e^{-j\pi/2} = \frac{-j}{3}$$

while the load impedance is found from (30) as

$$\begin{aligned} Z_L &= \frac{50(1-2j)}{2-j} \\ &= 40 - 30j \text{ ohm} \end{aligned}$$

**8-5 STUB TUNING**

In practice, most sources are connected to a transmission line through a series resistance matched to the line. This eliminates transient reflections when the excitation is turned on or off. To maximize the power flow to a load, it is also necessary for the load impedance reflected back to the source to be equal to the source impedance and thus equal to the characteristic impedance of the line,  $Z_0$ . This matching of the load to the line for an arbitrary termination can only be performed by adding additional elements along the line.

Usually these elements are short circuited transmission lines, called stubs, whose lengths can be varied. The reactance of the stub can be changed over the range from  $-j\infty$  to  $j\infty$  simply by varying its length, as found in Section 8-3-2, for the short circuited line. Because stubs are usually connected in parallel to a transmission line, it is more convenient to work with admittances rather than impedances as admittances in parallel simply add.

**8-5-1 Use of the Smith Chart for Admittance Calculations**

Fortunately the Smith chart can also be directly used for admittance calculations where the normalized admittance is defined as

$$Y_n(z) = \frac{Y(z)}{Y_0} = \frac{1}{Z_n(z)} \quad (1)$$

If the normalized load admittance  $Y_{nL}$  is known, straightforward impedance calculations first require the computation

$$Z_{nL} = 1/Y_{nL} \quad (2)$$

so that we could enter the Smith chart at  $Z_{nL}$ . Then we rotate by the required angle corresponding to  $2kz$  and read the new  $Z_n(z)$ . Then we again compute its reciprocal to find

$$Y_n(z) = 1/Z_n(z) \quad (3)$$

The two operations of taking the reciprocal are tedious. We can use the Smith chart itself to invert the impedance by using the fact that the normalized impedance is inverted by a  $\lambda/4$  section of line, so that a rotation of  $\Gamma(z)$  by  $180^\circ$  changes a normalized impedance into its reciprocal. Hence, if the admittance is given, we enter the Smith chart with a given value of normalized admittance  $Y_n$  and rotate by  $180^\circ$  to find  $Z_n$ . We then rotate by the appropriate number of wavelengths to find  $Z_n(z)$ . Finally, we again rotate by  $180^\circ$  to find  $Y_n(z) = 1/Z_n(z)$ . We have actually rotated the reflection coefficient by an angle of  $2\pi + 2kz$ . Rotation by  $2\pi$  on the Smith chart, however, brings us back to wherever we started, so that only the  $2kz$  rotation is significant. As long as we do an even number of  $\pi$  rotations by entering the Smith chart with an admittance and leaving again with an admittance, we can use the Smith chart with normalized admittances exactly as if they were normalized impedances.

### EXAMPLE 8-3 USE OF THE SMITH CHART FOR ADMITTANCE CALCULATIONS

The load impedance on a 50-Ohm line is

$$Z_L = 50(1 + j)$$

What is the admittance of the load?

#### SOLUTION

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By direct computation we have

$$Y_L = \frac{1}{Z_L} = \frac{1}{50(1 + j)} = \frac{(1 - j)}{100}$$

To use the Smith chart we find the normalized impedance at A in Figure 8-23:

$$Z_{nL} = 1 + j$$

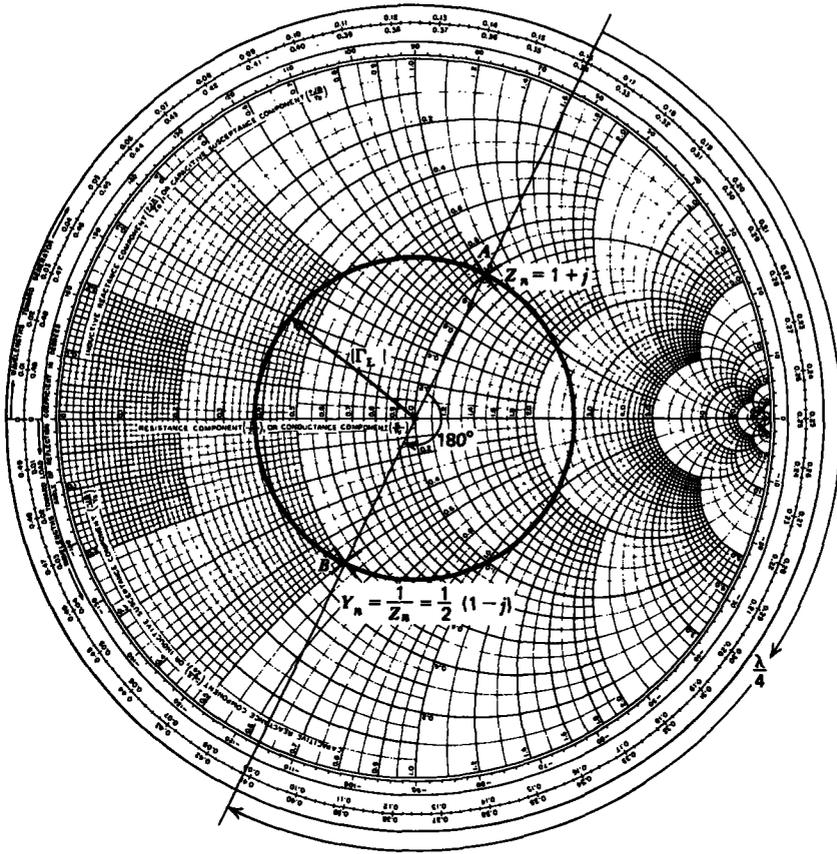


Figure 8-23 The Smith chart offers a convenient way to find the reciprocal of a complex number using the property that the normalized impedance reflected back by a quarter wavelength inverts. Thus, the normalized admittance is found by locating the normalized impedance and rotating this point by  $180^\circ$  about the constant  $|\Gamma_L|$  circle.

The normalized admittance that is the reciprocal of the normalized impedance is found by locating the impedance a distance  $\lambda/4$  away from the load end at  $B$ :

$$Y_{nL} = 0.5(1 - j) \Rightarrow Y_L = Y_n Y_0 = (1 - j)/100$$

Note that the point  $B$  is just  $180^\circ$  away from  $A$  on the constant  $|\Gamma_L|$  circle. For more complicated loads the Smith chart is a convenient way to find the reciprocal of a complex number.

### 8-5-2 Single-Stub Matching

A termination of value  $Z_L = 50(1+j)$  on a 50-Ohm transmission line is to be matched by means of a short circuited stub at a distance  $l_1$  from the load, as shown in Figure 8-24a. We need to find the line length  $l_1$  and the length of the stub  $l_2$  such that the impedance at the junction is matched to the line ( $Z_{in} = 50 \text{ Ohm}$ ). Then we know that all further points to the left of the junction have the same impedance of 50 Ohms.

Because of the parallel connection, it is simpler to use the Smith chart as an admittance transformation. The normalized load admittance can be computed using the Smith chart by rotating by  $180^\circ$  from the normalized load impedance at  $A$ , as was shown in Figure 8-23 and Example 8-3,

$$Z_{nL} = 1 + j \quad (4)$$

to yield

$$Y_{nL} = 0.5(1 - j) \quad (5)$$

at the point  $B$ .

Now we know from Section 8-3-2 that the short circuited stub can only add an imaginary component to the admittance. Since we want the total normalized admittance to be unity to the left of the stub in Figure 8-24

$$Y_{in} = Y_1 + Y_2 = 1 \quad (6)$$

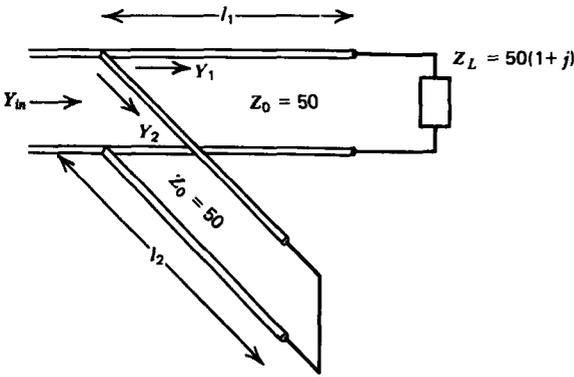
when  $Y_{nL}$  is reflected back to be  $Y_1$  it must wind up on the circle whose real part is 1 (as  $Y_2$  can only be imaginary), which occurs either at  $C$  or back at  $A$  allowing  $l_1$  to be either  $0.25\lambda$  at  $A$  or  $(0.25 + 0.177)\lambda = 0.427\lambda$  at  $C$  (or these values plus any integer multiple of  $\lambda/2$ ). Then  $Y_1$  is either of the following two conjugate values:

$$Y_1 = \begin{cases} 1 + j, & l_1 = 0.25\lambda \text{ (A)} \\ 1 - j, & l_1 = 0.427\lambda \text{ (C)} \end{cases} \quad (7)$$

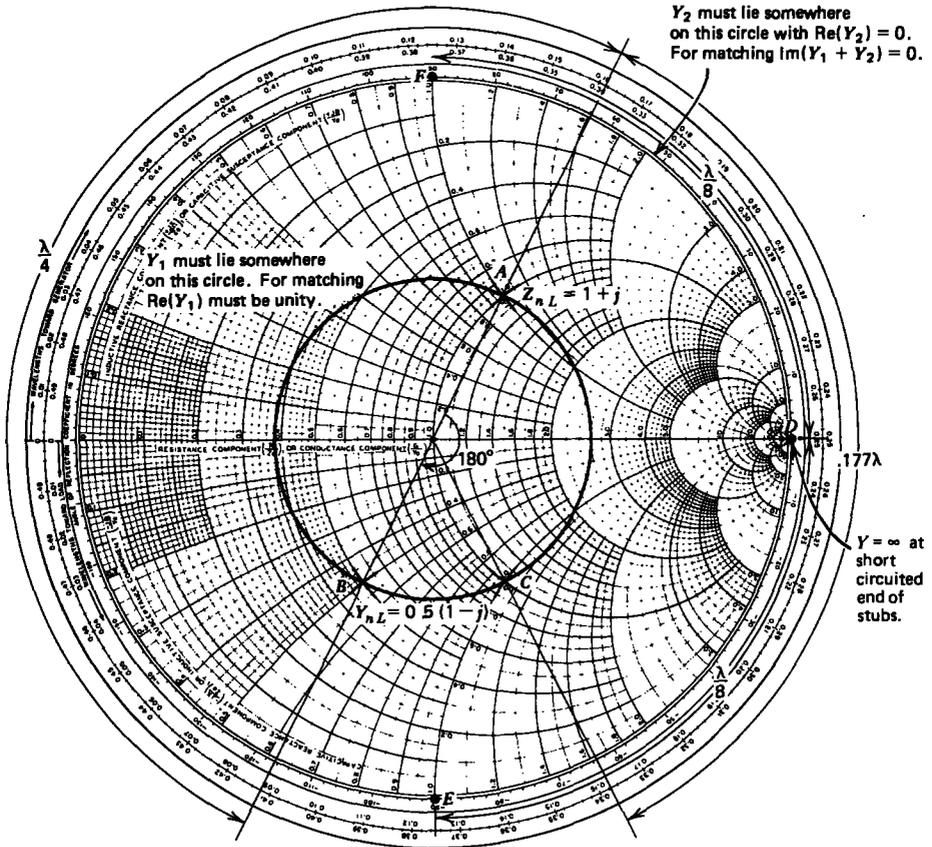
For  $Y_{in}$  to be unity we must pick  $Y_2$  to have an imaginary part to just cancel the imaginary part of  $Y_1$ :

$$Y_2 = \begin{cases} -j, & l_1 = 0.25\lambda \\ +j, & l_1 = 0.427\lambda \end{cases} \quad (8)$$

which means, since the shorted end has an infinite admittance at  $D$  that the stub must be of length such as to rotate the admittance to the points  $E$  or  $F$  requiring a stub length  $l_2$  of  $(\lambda/8)(E)$  or  $(3\lambda/8)(F)$  (or these values plus any integer multiple



(a)



(b)

Figure 8-24 (a) A single stub tuner consisting of a variable length short circuited line  $l_2$  can match any load to the line by putting the stub at the appropriate distance  $l_1$  from the load. (b) Smith chart construction. (c) Voltage standing wave pattern.

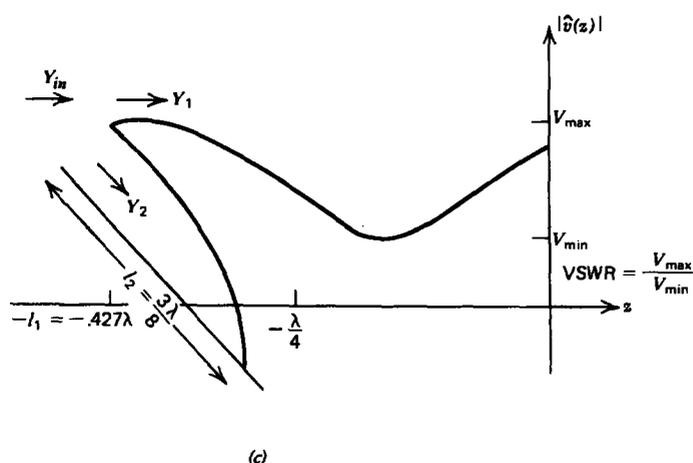


Figure 8-24

of  $\lambda/2$ ). Thus, the solutions can be summarized as

$$\begin{aligned} l_1 &= 0.25\lambda + n\lambda/2, & l_2 &= \lambda/8 + m\lambda/2 \\ \text{or} & & & \\ l_1 &= 0.427\lambda + n\lambda/2, & l_2 &= 3\lambda/8 + m\lambda/2 \end{aligned} \quad (9)$$

where  $n$  and  $m$  are any nonnegative integers (including zero).

When the load is matched by the stub to the line, the VSWR to the left of the stub is unity, while to the right of the stub over the length  $l_1$  the reflection coefficient is

$$\Gamma_L = \frac{Z_{nL} - 1}{Z_{nL} + 1} = \frac{j}{2 + j} \quad (10)$$

which has magnitude

$$|\Gamma_L| = 1/\sqrt{5} \approx 0.447 \quad (11)$$

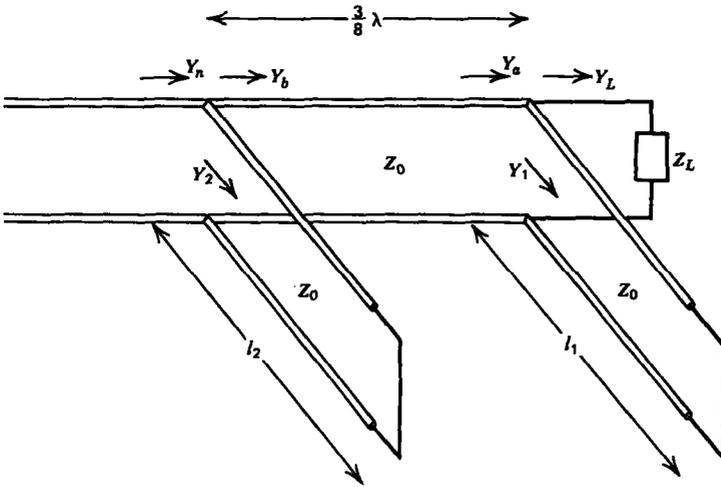
so that the voltage standing wave ratio is

$$\text{VSWR} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} \approx 2.62 \quad (12)$$

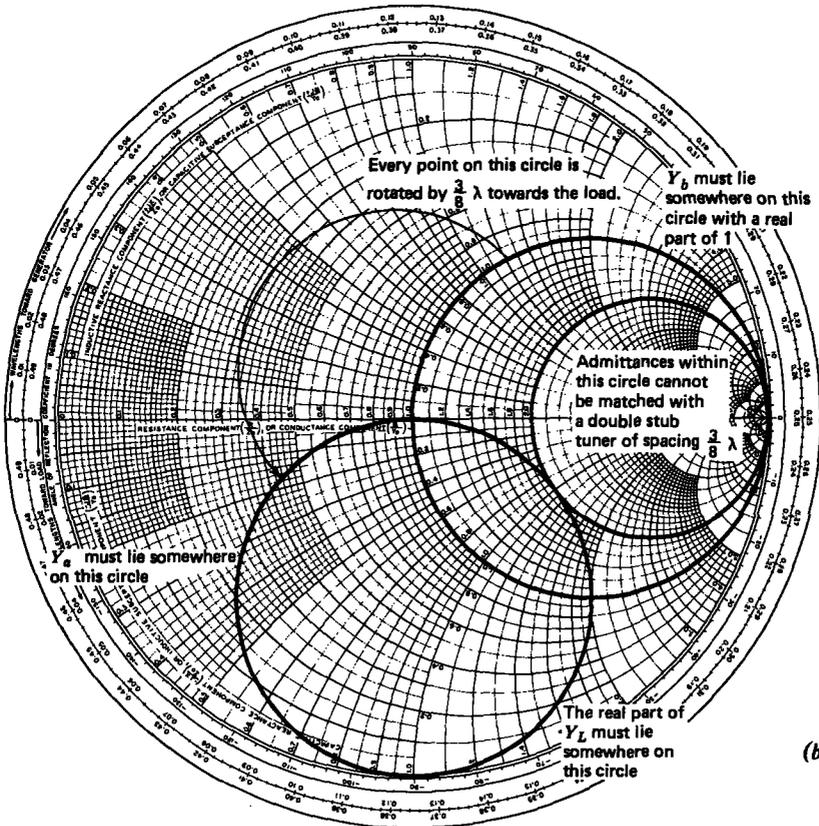
The disadvantage to single-stub tuning is that it is not easy to vary the length  $l_1$ . Generally new elements can only be connected at the ends of the line and not inbetween.

### 8-5-3 Double-Stub Matching

This difficulty of not having a variable length line can be overcome by using two short circuited stubs a fixed length apart, as shown in Figure 8-25a. This fixed length is usually  $\frac{3}{8}\lambda$ . A match is made by adjusting the length of the stubs  $l_1$  and



(a)



(b)

Figure 8-25 (a) A double stub tuner of fixed spacing cannot match all loads but is useful because additional elements can only be placed at transmission line terminations and not at any general position along a line as required for a single-stub tuner. (b) Smith chart construction. If the stubs are  $\frac{3}{8}\lambda$  apart, normalized load admittances whose real part exceeds 2 cannot be matched.

$l_2$ . One problem with the double-stub tuner is that not all loads can be matched for a given stub spacing.

The normalized admittances at each junction are related as

$$\begin{aligned} Y_a &= Y_1 + Y_L \\ Y_n &= Y_2 + Y_b \end{aligned} \quad (13)$$

where  $Y_1$  and  $Y_2$  are the purely reactive admittances of the stubs reflected back to the junctions while  $Y_b$  is the admittance of  $Y_a$  reflected back towards the load by  $\frac{3}{8}\lambda$ . For a match we require that  $Y_n$  be unity. Since  $Y_2$  is purely imaginary, the real part of  $Y_b$  must lie on the circle with a real part of unity. Then  $Y_a$  must lie somewhere on this circle when each point on the circle is reflected back by  $\frac{3}{8}\lambda$ . This generates another circle that is  $\frac{3}{2}\pi$  back in the counterclockwise direction as we are moving toward the load, as illustrated in Figure 8-25*b*. To find the conditions for a match, we work from left to right towards the load using the following reasoning:

- (i) Since  $Y_2$  is purely imaginary, the real part of  $Y_b$  must lie on the circle with a real part of unity, as in Figure 8-25*b*.
- (ii) Every possible point on  $Y_b$  must be reflected towards the load by  $\frac{3}{8}\lambda$  to find the locus of possible match for  $Y_a$ . This generates another circle that is  $\frac{3}{2}\pi$  back in the counterclockwise direction as we move towards the load, as in Figure 8-25*b*.

Again since  $Y_1$  is purely imaginary, the real part of  $Y_a$  must also equal the real part of the load admittance. This yields two possible solutions if the load admittance is outside the forbidden circle enclosing all load admittances with a real part greater than 2. Only loads with normalized admittances whose real part is less than 2 can be matched by the double-stub tuner of  $\frac{3}{8}\lambda$  spacing. Of course, if a load is within the forbidden circle, it can be matched by a double-stub tuner if the stub spacing is different than  $\frac{3}{8}\lambda$ .

#### EXAMPLE 8-4 DOUBLE-STUB MATCHING

The load impedance  $Z_L = 50(1 + j)$  on a 50-Ohm line is to be matched by a double-stub tuner of  $\frac{3}{8}\lambda$  spacing. What stub lengths  $l_1$  and  $l_2$  are necessary?

#### SOLUTION

The normalized load impedance  $Z_{nL} = 1 + j$  corresponds to a normalized load admittance:

$$Y_{nL} = 0.5(1 - j)$$



Then the two solutions for  $Y_a$  lie on the intersection of the circle shown in Figure 8-26a with the  $r = 0.5$  circle:

$$Y_{a1} = 0.5 - 0.14j$$

$$Y_{a2} = 0.5 - 1.85j$$

We then find  $Y_1$  by solving for the imaginary part of the upper equation in (13):

$$Y_1 = j \operatorname{Im}(Y_a - Y_L) = \begin{cases} 0.36j \Rightarrow l_1 = 0.305\lambda & (F) \\ -1.35j \Rightarrow l_1 = 0.1\lambda & (E) \end{cases}$$

By rotating the  $Y_a$  solutions by  $\frac{3}{8}\lambda$  back to the generator ( $270^\circ$  clockwise, which is equivalent to  $90^\circ$  counterclockwise), their intersection with the  $r = 1$  circle gives the solutions for  $Y_b$  as

$$Y_{b1} = 1.0 - 0.72j$$

$$Y_{b2} = 1.0 + 2.7j$$

This requires  $Y_2$  to be

$$Y_2 = -j \operatorname{Im}(Y_b) = \begin{cases} 0.72j \Rightarrow l_2 = 0.349\lambda & (G) \\ -2.7j \Rightarrow l_2 = 0.056\lambda & (H) \end{cases}$$

The voltage standing wave pattern along the line and stubs is shown in Figure 8.26b. Note the continuity of voltage at the junctions. The actual stub lengths can be those listed plus any integer multiple of  $\lambda/2$ .

## 8-6 THE RECTANGULAR WAVEGUIDE

We showed in Section 8-1-2 that the electric and magnetic fields for TEM waves have the same form of solutions in the plane transverse to the transmission line axis as for statics. The inner conductor within a closed transmission line structure such as a coaxial cable is necessary for TEM waves since it carries a surface current and a surface charge distribution, which are the source for the magnetic and electric fields. A hollow conducting structure, called a waveguide, cannot propagate TEM waves since the static fields inside a conducting structure enclosing no current or charge is zero.

However, new solutions with electric or magnetic fields along the waveguide axis as well as in the transverse plane are allowed. Such solutions can also propagate along transmission lines. Here the axial displacement current can act as a source

of the transverse magnetic field giving rise to transverse magnetic (TM) modes as the magnetic field lies entirely within the transverse plane. Similarly, an axial time varying magnetic field generates transverse electric (TE) modes. The most general allowed solutions on a transmission line are TEM, TM, and TE modes. Removing the inner conductor on a closed transmission line leaves a waveguide that can only propagate TM and TE modes.

### 8-6-1 Governing Equations

To develop these general solutions we return to Maxwell's equations in a linear source-free material:

$$\begin{aligned}\nabla \times \mathbf{E} &= -\mu \frac{\partial \mathbf{H}}{\partial t} \\ \nabla \times \mathbf{H} &= \epsilon \frac{\partial \mathbf{E}}{\partial t} \\ \epsilon \nabla \cdot \mathbf{E} &= 0 \\ \mu \nabla \cdot \mathbf{H} &= 0\end{aligned}\tag{1}$$

Taking the curl of Faraday's law, we expand the double cross product and then substitute Ampere's law to obtain a simple vector equation in  $\mathbf{E}$  alone:

$$\begin{aligned}\nabla \times (\nabla \times \mathbf{E}) &= \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \\ &= -\mu \frac{\partial}{\partial t} (\nabla \times \mathbf{H}) \\ &= -\epsilon \mu \frac{\partial^2 \mathbf{E}}{\partial t^2}\end{aligned}\tag{2}$$

Since  $\nabla \cdot \mathbf{E} = 0$  from Gauss's law when the charge density is zero, (2) reduces to the vector wave equation in  $\mathbf{E}$ :

$$\nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}, \quad c^2 = \frac{1}{\epsilon \mu}\tag{3}$$

If we take the curl of Ampere's law and perform similar operations, we also obtain the vector wave equation in  $\mathbf{H}$ :

$$\nabla^2 \mathbf{H} = \frac{1}{c^2} \frac{\partial^2 \mathbf{H}}{\partial t^2}\tag{4}$$

The solutions for  $\mathbf{E}$  and  $\mathbf{H}$  in (3) and (4) are not independent. If we solve for either  $\mathbf{E}$  or  $\mathbf{H}$ , the other field is obtained from (1). The vector wave equations in (3) and (4) are valid for any shaped waveguide. In particular, we limit ourselves in this text to waveguides whose cross-sectional shape is rectangular, as shown in Figure 8-27.

### 8-6-2 Transverse Magnetic (TM) Modes

We first consider TM modes where the magnetic field has  $x$  and  $y$  components but no  $z$  component. It is simplest to solve (3) for the  $z$  component of electric field and then obtain the other electric and magnetic field components in terms of  $E_z$  directly from Maxwell's equations in (1).

We thus assume solutions of the form

$$E_z = \text{Re} [\hat{E}_z(x, y) e^{j(\omega t - k_z z)}] \quad (5)$$

where an exponential  $z$  dependence is assumed because the cross-sectional area of the waveguide is assumed to be uniform in  $z$  so that none of the coefficients in (1) depends on  $z$ . Then substituting into (3) yields the Helmholtz equation:

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} - \left( k_z^2 - \frac{\omega^2}{c^2} \right) \hat{E}_z = 0 \quad (6)$$

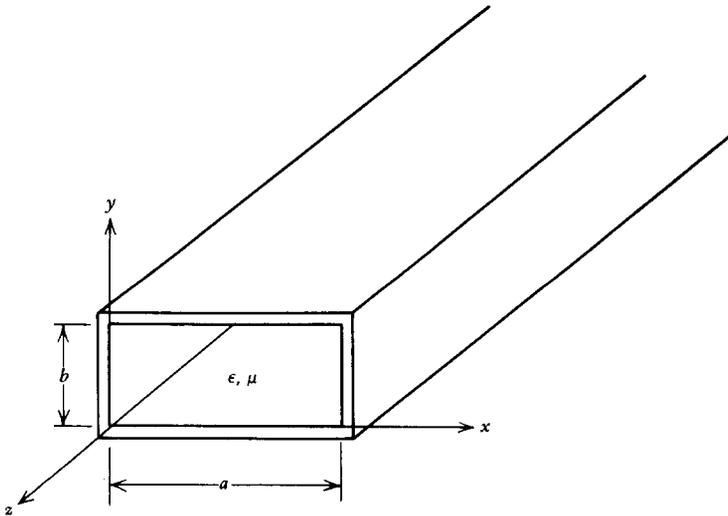


Figure 8-27 A lossless waveguide with rectangular cross section.

This equation can be solved by assuming the same product solution as used for solving Laplace's equation in Section 4-2-1, of the form

$$\hat{E}_z(x, y) = X(x)Y(y) \quad (7)$$

where  $X(x)$  is only a function of the  $x$  coordinate and  $Y(y)$  is only a function of  $y$ . Substituting this assumed form of solution into (6) and dividing through by  $X(x)Y(y)$  yields

$$\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} = k_z^2 - \frac{\omega^2}{c^2} \quad (8)$$

When solving Laplace's equation in Section 4-2-1 the right-hand side was zero. Here the reasoning is the same. The first term on the left-hand side in (8) is only a function of  $x$  while the second term is only a function of  $y$ . The only way a function of  $x$  and a function of  $y$  can add up to a constant for all  $x$  and  $y$  is if each function alone is a constant,

$$\begin{aligned} \frac{1}{X} \frac{d^2 X}{dx^2} &= -k_x^2 \\ \frac{1}{Y} \frac{d^2 Y}{dy^2} &= -k_y^2 \end{aligned} \quad (9)$$

where the separation constants must obey the relation

$$k_x^2 + k_y^2 + k_z^2 = k^2 = \omega^2/c^2 \quad (10)$$

When we solved Laplace's equation in Section 4-2-6, there was no time dependence so that  $\omega = 0$ . Then we found that at least one of the wavenumbers was imaginary, yielding decaying solutions. For finite frequencies it is possible for all three wavenumbers to be real for pure propagation. The values of these wavenumbers will be determined by the dimensions of the waveguide through the boundary conditions.

The solutions to (9) are sinusoids so that the transverse dependence of the axial electric field  $\hat{E}_z(x, y)$  is

$$\hat{E}_z(x, y) = (A_1 \sin k_x x + A_2 \cos k_x x)(B_1 \sin k_y y + B_2 \cos k_y y) \quad (11)$$

Because the rectangular waveguide in Figure 8-27 is composed of perfectly conducting walls, the tangential component of electric field at the walls is zero:

$$\begin{aligned} \hat{E}_z(x, y=0) &= 0, & \hat{E}_z(x=0, y) &= 0 \\ \hat{E}_z(x, y=b) &= 0, & \hat{E}_z(x=a, y) &= 0 \end{aligned} \quad (12)$$

These boundary conditions then require that  $A_2$  and  $B_2$  are zero so that (11) simplifies to

$$\hat{E}_z(x, y) = E_0 \sin k_x x \sin k_y y \quad (13)$$

where  $E_0$  is a field amplitude related to a source strength and the transverse wavenumbers must obey the equalities

$$\begin{aligned} k_x &= m\pi/a, & m &= 1, 2, 3, \dots \\ k_y &= n\pi/b, & n &= 1, 2, 3, \dots \end{aligned} \quad (14)$$

Note that if either  $m$  or  $n$  is zero in (13), the axial electric field is zero. The waveguide solutions are thus described as  $\text{TM}_{mn}$  modes where both  $m$  and  $n$  are integers greater than zero.

The other electric field components are found from the  $z$  component of Faraday's law, where  $\mathbf{H}_z = 0$  and the charge-free Gauss's law in (1):

$$\begin{aligned} \frac{\partial E_y}{\partial x} &= \frac{\partial E_x}{\partial y} \\ \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} &= 0 \end{aligned} \quad (15)$$

By taking  $\partial/\partial x$  of the top equation and  $\partial/\partial y$  of the lower equation, we eliminate  $E_x$  to obtain

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} = -\frac{\partial^2 E_z}{\partial y \partial z} \quad (16)$$

where the right-hand side is known from (13). The general solution for  $E_y$  must be of the same form as (11), again requiring the tangential component of electric field to be zero at the waveguide walls,

$$\hat{E}_y(x=0, y) = 0, \quad \hat{E}_y(x=a, y) = 0 \quad (17)$$

so that the solution to (16) is

$$\hat{E}_y = -\frac{jk_y k_z E_0}{k_x^2 + k_y^2} \sin k_x x \cos k_y y \quad (18)$$

We then solve for  $E_x$  using the upper equation in (15):

$$\hat{E}_x = -\frac{jk_x k_z E_0}{k_x^2 + k_y^2} \cos k_x x \sin k_y y \quad (19)$$

where we see that the boundary conditions

$$\hat{E}_x(x, y=0) = 0, \quad \hat{E}_x(x, y=b) = 0 \quad (20)$$

are satisfied.

The magnetic field is most easily found from Faraday's law

$$\hat{\mathbf{H}}(\mathbf{x}, y) = -\frac{1}{j\omega\mu} \nabla \times \hat{\mathbf{E}}(\mathbf{x}, y) \quad (21)$$

to yield

$$\begin{aligned} \hat{H}_x &= -\frac{1}{j\omega\mu} \left( \frac{\partial \hat{E}_z}{\partial y} - \frac{\partial \hat{E}_y}{\partial z} \right) \\ &= -\frac{k_y k^2}{j\omega\mu (k_x^2 + k_y^2)} E_0 \sin k_x x \cos k_y y \\ &= \frac{j\omega\epsilon k_y}{k_x^2 + k_y^2} E_0 \sin k_x x \cos k_y y \\ \hat{H}_y &= -\frac{1}{j\omega\mu} \left( \frac{\partial \hat{E}_z}{\partial x} - \frac{\partial \hat{E}_x}{\partial z} \right) \\ &= \frac{k_x k^2 E_0}{j\omega\mu (k_x^2 + k_y^2)} \cos k_x x \sin k_y y \\ &= -\frac{j\omega\epsilon k_x}{k_x^2 + k_y^2} E_0 \cos k_x x \sin k_y y \\ \hat{H}_z &= 0 \end{aligned} \quad (22)$$

Note the boundary conditions of the normal component of  $\mathbf{H}$  being zero at the waveguide walls are automatically satisfied:

$$\begin{aligned} \hat{H}_y(x, y=0) &= 0, & \hat{H}_y(x, y=b) &= 0 \\ \hat{H}_x(x=0, y) &= 0, & \hat{H}_x(x=a, y) &= 0 \end{aligned} \quad (23)$$

The surface charge distribution on the waveguide walls is found from the discontinuity of normal  $\mathbf{D}$  fields:

$$\begin{aligned} \hat{\sigma}_f(x=0, y) &= \epsilon \hat{E}_x(x=0, y) = -\frac{jk_x k_x \epsilon}{k_x^2 + k_y^2} E_0 \sin k_y y \\ \hat{\sigma}_f(x=a, y) &= -\epsilon \hat{E}_x(x=a, y) = \frac{jk_x k_x \epsilon}{k_x^2 + k_y^2} E_0 \cos m\pi \sin k_y y \\ \hat{\sigma}_f(x, y=0) &= \epsilon \hat{E}_y(x, y=0) = -\frac{jk_x k_y \epsilon}{k_x^2 + k_y^2} E_0 \sin k_x x \\ \hat{\sigma}_f(x, y=b) &= -\epsilon \hat{E}_y(x, y=b) = \frac{jk_x k_y \epsilon}{k_x^2 + k_y^2} E_0 \cos n\pi \sin k_x x \end{aligned} \quad (24)$$

Similarly, the surface currents are found by the discontinuity in the tangential components of  $\mathbf{H}$  to be purely  $z$  directed:

$$\begin{aligned}\hat{K}_z(x, y = 0) &= -\hat{H}_x(x, y = 0) = \frac{k_y k^2 E_0 \sin k_x x}{j\omega\mu(k_x^2 + k_y^2)} \\ \hat{K}_z(x, y = b) &= \hat{H}_x(x, y = b) = -\frac{k_y k^2 E_0}{j\omega\mu(k_x^2 + k_y^2)} \sin k_x x \cos n\pi \\ \hat{K}_z(x = 0, y) &= \hat{H}_y(x = 0, y) = \frac{k_x k^2 E_0}{j\omega\mu(k_x^2 + k_y^2)} \sin k_y y \\ \hat{K}_z(x = a, y) &= -\hat{H}_y(x = a, y) = -\frac{k_x k^2 E_0 \cos m\pi \sin k_y y}{j\omega\mu(k_x^2 + k_y^2)}\end{aligned}\quad (25)$$

We see that if  $m$  or  $n$  are even, the surface charges and surface currents on opposite walls are of opposite sign, while if  $m$  or  $n$  are odd, they are of the same sign. This helps us in plotting the field lines for the various  $\text{TM}_{mn}$  modes shown in Figure 8-28. The electric field is always normal and the magnetic field tangential to the waveguide walls. Where the surface charge is positive, the electric field points out of the wall, while it points in where the surface charge is negative. For higher order modes the field patterns shown in Figure 8-28 repeat within the waveguide.

Slots are often cut in waveguide walls to allow the insertion of a small sliding probe that measures the electric field. These slots must be placed at positions of zero surface current so that the field distributions of a particular mode are only negligibly disturbed. If a slot is cut along the  $z$  direction on the  $y = b$  surface at  $x = a/2$ , the surface current given in (25) is zero for  $\text{TM}$  modes if  $\sin(k_x a/2) = 0$ , which is true for the  $m = \text{even}$  modes.

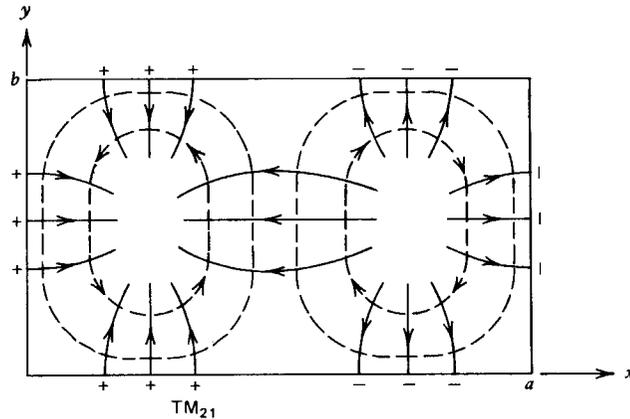
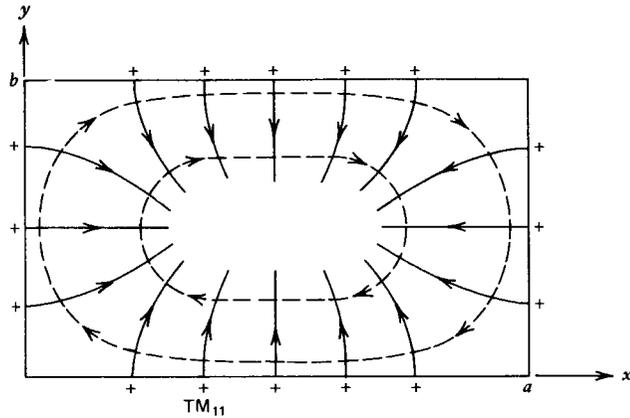
### 8-6-3 Transverse Electric (TE) Modes

When the electric field lies entirely in the  $xy$  plane, it is most convenient to first solve (4) for  $H_z$ . Then as for  $\text{TM}$  modes we assume a solution of the form

$$H_z = \text{Re} [\hat{H}_z(x, y) e^{j(\omega t - k_z z)}] \quad (26)$$

which when substituted into (4) yields

$$\frac{\partial^2 \hat{H}_z}{\partial x^2} + \frac{\partial^2 \hat{H}_z}{\partial y^2} - \left(k_z^2 - \frac{\omega^2}{c^2}\right) \hat{H}_z = 0 \quad (27)$$



Electric field (—)

$$\hat{E}_x = \frac{-jk_x k_y E_0}{k_x^2 + k_y^2} \cos k_x x \sin k_y y$$

$$\hat{E}_y = \frac{-jk_y k_x E_0}{k_x^2 + k_y^2} \sin k_x x \cos k_y y$$

$$\hat{E}_z = E_0 \sin k_x x \sin k_y y$$

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{k_y \tan k_x x}{k_x \tan k_y y}$$

$$\Rightarrow \frac{[\cos k_x x]^{(k_y/k_x)^2}}{\cos k_y y} = \text{const}$$

Magnetic field (----)

$$\hat{H}_x = \frac{j\omega\epsilon k_y}{k_x^2 + k_y^2} E_0 \sin k_x x \cos k_y y$$

$$\hat{H}_y = \frac{-j\omega\epsilon k_x}{k_x^2 + k_y^2} E_0 \cos k_x x \sin k_y y$$

$$\frac{dy}{dx} = \frac{H_y}{H_x} = \frac{-k_x \cot k_x x}{k_y \cot k_y y}$$

$$\Rightarrow \sin k_x x \sin k_y y = \text{const}$$

$$k_x = \frac{m\pi}{a}, \quad k_y = \frac{n\pi}{b}, \quad k_z = \left[ \frac{\omega^2}{c^2} - k_x^2 - k_y^2 \right]^{1/2}$$

Figure 8-28 The transverse electric and magnetic field lines for the  $TM_{11}$  and  $TM_{21}$  modes. The electric field is purely  $z$  directed where the field lines converge.

Again this equation is solved by assuming a product solution and separating to yield a solution of the same form as (11):

$$\hat{H}_z(x, y) = (A_1 \sin k_x x + A_2 \cos k_x x)(B_1 \sin k_y y + B_2 \cos k_y y) \quad (28)$$

The boundary conditions of zero normal components of  $\mathbf{H}$  at the waveguide walls require that

$$\begin{aligned} \hat{H}_x(x=0, y) = 0, \quad \hat{H}_x(x=a, y) = 0 \\ \hat{H}_y(x, y=0) = 0, \quad \hat{H}_y(x, y=b) = 0 \end{aligned} \quad (29)$$

Using identical operations as in (15)–(20) for the TM modes the magnetic field solutions are

$$\begin{aligned} \hat{H}_x &= \frac{jk_x k_y H_0}{k_x^2 + k_y^2} \sin k_x x \cos k_y y, \quad k_x = \frac{m\pi}{a}, \quad k_y = \frac{n\pi}{b} \\ \hat{H}_y &= \frac{jk_x k_y H_0}{k_x^2 + k_y^2} \cos k_x x \sin k_y y \\ \hat{H}_z &= H_0 \cos k_x x \cos k_y y \end{aligned} \quad (30)$$

The electric field is then most easily obtained from Ampere's law in (1),

$$\hat{\mathbf{E}} = \frac{1}{j\omega\epsilon} \nabla \times \hat{\mathbf{H}} \quad (31)$$

to yield

$$\begin{aligned} \hat{E}_x &= \frac{1}{j\omega\epsilon} \left( \frac{\partial}{\partial y} \hat{H}_z - \frac{\partial}{\partial z} \hat{H}_y \right) \\ &= -\frac{k_y k^2 H_0}{j\omega\epsilon (k_x^2 + k_y^2)} \cos k_x x \sin k_y y \\ &= \frac{j\omega\mu k_y}{k_x^2 + k_y^2} H_0 \cos k_x x \sin k_y y \\ \hat{E}_y &= \frac{1}{j\omega\epsilon} \left( \frac{\partial \hat{H}_x}{\partial z} - \frac{\partial \hat{H}_z}{\partial x} \right) \\ &= \frac{k_x k^2 H_0}{j\omega\epsilon (k_x^2 + k_y^2)} \sin k_x x \cos k_y y \\ &= -\frac{j\omega\mu k_x}{k_x^2 + k_y^2} H_0 \sin k_x x \cos k_y y \\ \hat{E}_z &= 0 \end{aligned} \quad (32)$$

We see in (32) that as required the tangential components of the electric field at the waveguide walls are zero. The

surface charge densities on each of the walls are:

$$\begin{aligned}
 \hat{\sigma}_f(x=0, y) &= \varepsilon \hat{E}_x(x=0, y) = \frac{-k_y k^2 H_0}{j\omega(k_x^2 + k_y^2)} \sin k_y y \\
 \hat{\sigma}_f(x=a, y) &= -\varepsilon \hat{E}_x(x=a, y) = \frac{k_y k^2 H_0}{j\omega(k_x^2 + k_y^2)} \cos m\pi \sin k_y y \\
 \hat{\sigma}_f(x, y=0) &= \varepsilon \hat{E}_y(x, y=0) = \frac{k_x k^2 H_0}{j\omega(k_x^2 + k_y^2)} \sin k_x x \\
 \hat{\sigma}_f(x, y=b) &= -\varepsilon \hat{E}_y(x, y=b) = -\frac{k_x k^2 H_0}{j\omega(k_x^2 + k_y^2)} \cos n\pi \sin k_x x
 \end{aligned} \tag{33}$$

For TE modes, the surface currents determined from the discontinuity of tangential  $\mathbf{H}$  now flow in closed paths on the waveguide walls:

$$\begin{aligned}
 \hat{\mathbf{K}}(x=0, y) &= \mathbf{i}_x \times \hat{\mathbf{H}}(x=0, y) \\
 &= \mathbf{i}_z \hat{H}_y(x=0, y) - \mathbf{i}_y \hat{H}_z(x=0, y) \\
 \hat{\mathbf{K}}(x=a, y) &= -\mathbf{i}_x \times \hat{\mathbf{H}}(x=a, y) \\
 &= -\mathbf{i}_z \hat{H}_y(x=a, y) + \mathbf{i}_y \hat{H}_z(x=a, y) \\
 \hat{\mathbf{K}}(x, y=0) &= \mathbf{i}_y \times \hat{\mathbf{H}}(x, y=0) \\
 &= -\mathbf{i}_z \hat{H}_x(x, y=0) + \mathbf{i}_x \hat{H}_z(x, y=0) \\
 \hat{\mathbf{K}}(x, y=b) &= -\mathbf{i}_y \times \hat{\mathbf{H}}(x, y=b) \\
 &= \mathbf{i}_z \hat{H}_x(x, y=b) - \mathbf{i}_x \hat{H}_z(x, y=b)
 \end{aligned} \tag{34}$$

Note that for TE modes either  $n$  or  $m$  (but not both) can be zero and still yield a nontrivial set of solutions. As shown in Figure 8-29, when  $n$  is zero there is no variation in the fields in the  $y$  direction and the electric field is purely  $y$  directed while the magnetic field has no  $y$  component. The  $\text{TE}_{11}$  and  $\text{TE}_{21}$  field patterns are representative of the higher order modes.

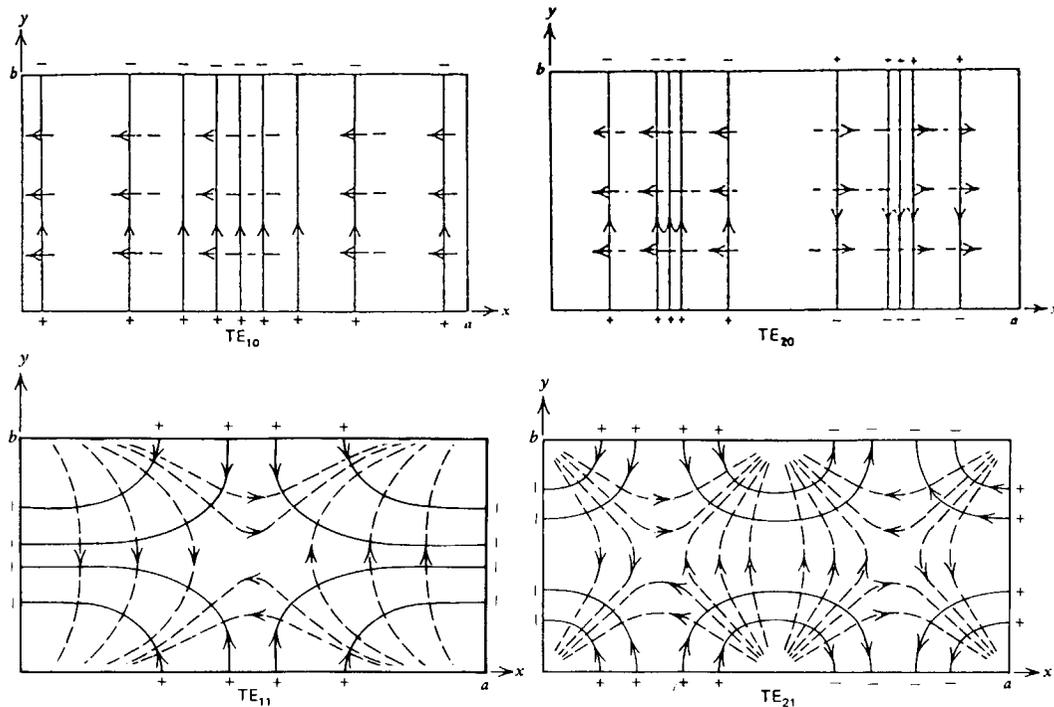
#### 8-6-4 Cut-Off

The transverse wavenumbers are

$$k_x = \frac{m\pi}{a}, \quad k_y = \frac{n\pi}{b} \tag{35}$$

so that the axial variation of the fields is obtained from (10) as

$$k_z = \left[ \frac{\omega^2}{c^2} - k_x^2 - k_y^2 \right]^{1/2} = \left[ \frac{\omega^2}{c^2} - \left( \frac{m\pi}{a} \right)^2 - \left( \frac{n\pi}{b} \right)^2 \right]^{1/2} \tag{36}$$



Electric field (—)

$$\hat{E}_x = \frac{j\omega\mu k_y}{k_x^2 + k_y^2} H_0 \cos k_x x \sin k_y y$$

$$\hat{E}_y = \frac{-j\omega\mu k_x}{k_x^2 + k_y^2} H_0 \sin k_x x \cos k_y y$$

$$k_x = \frac{m\pi}{a}, \quad k_y = \frac{n\pi}{b}, \quad k_z = \left[ \frac{\omega^2}{c^2} - k_x^2 - k_y^2 \right]^{1/2}$$

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{-k_x \tan k_x x}{k_y \tan k_y y}$$

$$\Rightarrow \cos k_x x \cos k_y y = \text{const}$$

Magnetic field (---)

$$\hat{H}_x = \frac{jk_z k_x H_0}{k_x^2 + k_y^2} \sin k_x x \cos k_y y$$

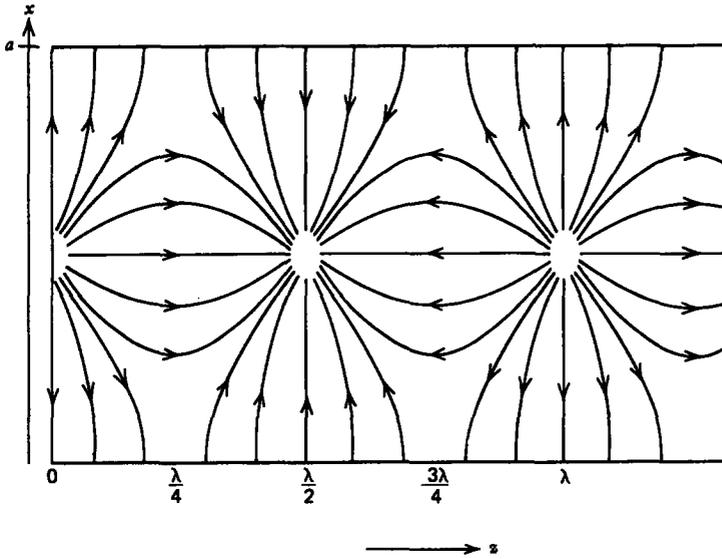
$$\hat{H}_y = \frac{jk_z k_y H_0}{k_x^2 + k_y^2} \cos k_x x \sin k_y y$$

$$\hat{H}_z = H_0 \cos k_x x \cos k_y y$$

$$\frac{dy}{dx} = \frac{H_y}{H_x} = \frac{k_y \cot k_x x}{k_x \cot k_y y}$$

$$\Rightarrow \frac{[\sin k_x x]^{(k_y/k_x)^2}}{\sin k_y y} = \text{const}$$

Figure 8-29 (a) The transverse electric and magnetic field lines for various TE modes. The magnetic field is purely z directed where the field lines converge. The TE<sub>10</sub> mode is called the dominant mode since it has the lowest cut-off frequency. (b) Surface current lines for the TE<sub>10</sub> mode.



(b)

Figure 8-29

Thus, although  $k_x$  and  $k_y$  are real,  $k_z$  can be either pure real or pure imaginary. A real value of  $k_z$  represents power flow down the waveguide in the  $z$  direction. An imaginary value of  $k_z$  means exponential decay with no time-average power flow. The transition from propagating waves ( $k_z$  real) to evanescence ( $k_z$  imaginary) occurs for  $k_z = 0$ . The frequency when  $k_z$  is zero is called the cut-off frequency  $\omega_c$ :

$$\omega_c = c \left[ \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \right]^{1/2} \quad (37)$$

This frequency varies for each mode with the mode parameters  $m$  and  $n$ . If we assume that  $a$  is greater than  $b$ , the lowest cut-off frequency occurs for the  $TE_{10}$  mode, which is called the dominant or fundamental mode. No modes can propagate below this lowest critical frequency  $\omega_{c0}$ :

$$\omega_{c0} = \frac{\pi c}{a} \Rightarrow f_{c0} = \frac{\omega_{c0}}{2\pi} = \frac{c}{2a} \text{ Hz} \quad (38)$$

If an air-filled waveguide has  $a = 1 \text{ cm}$ , then  $f_{c0} = 1.5 \times 10^{10} \text{ Hz}$ , while if  $a = 10 \text{ m}$ , then  $f_{c0} = 15 \text{ MHz}$ . This explains why we usually cannot hear the radio when driving through a tunnel. As the frequency is raised above  $\omega_{c0}$ , further modes can propagate.

The phase and group velocity of the waves are

$$v_p = \frac{\omega}{k_z} = \frac{\omega}{\left[ \frac{\omega^2}{c^2} - \left( \frac{m\pi}{a} \right)^2 - \left( \frac{n\pi}{b} \right)^2 \right]^{1/2}} \tag{39}$$

$$v_g = \frac{d\omega}{dk_z} = \frac{k_z c^2}{\omega} = \frac{c^2}{v_p} \Rightarrow v_g v_p = c^2$$

At cut-off,  $v_g = 0$  and  $v_p = \infty$  with their product always a constant.

**8-6-5 Waveguide Power Flow**

The time-averaged power flow per unit area through the waveguide is found from the Poynting vector:

$$\langle \mathbf{S} \rangle = \frac{1}{2} \text{Re} (\hat{\mathbf{E}} \times \hat{\mathbf{H}}^*) \tag{40}$$

**(a) Power Flow for the TM Modes**

Substituting the field solutions found in Section 8-6-2 into (40) yields

$$\begin{aligned} \langle \mathbf{S} \rangle &= \frac{1}{2} \text{Re} [(\hat{E}_x \mathbf{i}_x + \hat{E}_y \mathbf{i}_y + \hat{E}_z \mathbf{i}_z) e^{-jk_z z} \times (\hat{H}_x^* \mathbf{i}_x + \hat{H}_y^* \mathbf{i}_y) e^{+jk_z^* z}] \\ &= \frac{1}{2} \text{Re} [(\hat{E}_x \hat{H}_y^* - \hat{E}_y \hat{H}_x^*) \mathbf{i}_z + \hat{E}_z (\hat{H}_x^* \mathbf{i}_y - \hat{H}_y^* \mathbf{i}_x)] e^{-j(k_z - k_z^*)z} \end{aligned} \tag{41}$$

where we remember that  $k_z$  may be imaginary for a particular mode if the frequency is below cut-off. For propagating modes where  $k_z$  is real so that  $k_z = k_z^*$ , there is no  $z$  dependence in (41). For evanescent modes where  $k_z$  is pure imaginary, the  $z$  dependence of the Poynting vector is a real decaying exponential of the form  $e^{-2|k_z|z}$ . For either case we see from (13) and (22) that the product of  $\hat{E}_z$  with  $\hat{H}_x$  and  $\hat{H}_y$  is pure imaginary so that the real parts of the  $x$ - and  $y$ -directed time average power flow are zero in (41). Only the  $z$ -directed power flow can have a time average:

$$\begin{aligned} \langle \mathbf{S} \rangle &= \frac{\omega \epsilon |E_0|^2}{2(k_x^2 + k_y^2)} \text{Re} [k_z e^{-j(k_z - k_z^*)z} (k_x^2 \cos^2 k_x x \sin^2 k_y y \\ &\quad + k_y^2 \sin^2 k_x x \cos^2 k_y y)] \mathbf{i}_z \end{aligned} \tag{42}$$

If  $k_z$  is imaginary, we have that  $\langle \mathbf{S} \rangle = 0$  while a real  $k_z$  results in a nonzero time-average power flow. The total  $z$ -directed

power flow is found by integrating (42) over the cross-sectional area of the waveguide:

$$\begin{aligned} \langle P \rangle &= \int_{x=0}^a \int_{y=0}^b \langle S_z \rangle dx dy \\ &= \frac{\omega \epsilon k_z ab E_0^2}{8(k_x^2 + k_y^2)} \end{aligned} \quad (43)$$

where it is assumed that  $k_z$  is real, and we used the following identities:

$$\begin{aligned} \int_0^a \sin^2 \frac{m\pi x}{a} dx &= \frac{a}{m\pi} \left( \frac{1}{2} \frac{m\pi x}{a} - \frac{1}{4} \sin \frac{2m\pi x}{a} \right) \Big|_0^a \\ &= \begin{cases} a/2, & m \neq 0 \\ 0, & m = 0 \end{cases} \\ \int_0^a \cos^2 \frac{m\pi x}{a} dx &= \frac{a}{m\pi} \left( \frac{1}{2} \frac{m\pi x}{a} + \frac{1}{4} \sin \frac{2m\pi x}{a} \right) \Big|_0^a \\ &= \begin{cases} a/2, & m \neq 0 \\ a, & m = 0 \end{cases} \end{aligned} \quad (44)$$

For the TM modes, both  $m$  and  $n$  must be nonzero.

#### (b) Power Flow for the TE Modes

The same reasoning is used for the electromagnetic fields found in Section 8-6-3 substituted into (40):

$$\begin{aligned} \langle S \rangle &= \frac{1}{2} \text{Re} [(\hat{E}_x \mathbf{i}_x + \hat{E}_y \mathbf{i}_y) e^{-jk_z z} \times (\hat{H}_x^* \mathbf{i}_x + \hat{H}_y^* \mathbf{i}_y + \hat{H}_z^* \mathbf{i}_z) e^{+jk_z^* z}] \\ &= \frac{1}{2} \text{Re} [(\hat{E}_x \hat{H}_y^* - \hat{E}_y \hat{H}_x^*) \mathbf{i}_z - \hat{H}_z^* (\hat{E}_x \mathbf{i}_y - \hat{E}_y \mathbf{i}_x)] e^{-j(k_z - k_z^*)z} \end{aligned} \quad (45)$$

Similarly, again we have that the product of  $H_z^*$  with  $\hat{E}_x$  and  $\hat{E}_y$  is pure imaginary so that there are no x- and y-directed time average power flows. The z-directed power flow reduces to

$$\begin{aligned} \langle S_z \rangle &= \frac{1}{2} \frac{\omega \mu H_0^2}{(k_x^2 + k_y^2)} (k_y^2 \cos^2 k_x x \sin^2 k_y y \\ &\quad + k_x^2 \sin^2 k_x x \cos^2 k_y y) \text{Re} (k_z e^{-j(k_z - k_z^*)z}) \end{aligned} \quad (46)$$

Again we have nonzero z-directed time average power flow only if  $k_z$  is real. Then the total z-directed power is

$$\langle P \rangle = \int_{x=0}^a \int_{y=0}^b \langle S_z \rangle dx dy = \begin{cases} \frac{\omega \mu k_z ab H_0^2}{8(k_x^2 + k_y^2)}, & m, n \neq 0 \\ \frac{\omega \mu k_z ab H_0^2}{4(k_x^2 + k_y^2)}, & m \text{ or } n = 0 \end{cases} \quad (47)$$

where we again used the identities of (44). Note the factor of 2 differences in (47) for either the  $TE_{10}$  or  $TE_{01}$  modes. Both  $m$  and  $n$  cannot be zero as the  $TE_{00}$  mode reduces to the trivial spatially constant uncoupled  $z$ -directed magnetic field.

### 8-6-6 Wall Losses

If the waveguide walls have a high but noninfinite Ohmic conductivity  $\sigma_w$ , we can calculate the spatial attenuation rate using the approximate perturbation approach described in Section 8-3-4b. The fields decay as  $e^{-\alpha z}$ , where

$$\alpha = \frac{1}{2} \frac{\langle P_{dL} \rangle}{\langle P \rangle} \quad (48)$$

where  $\langle P_{dL} \rangle$  is the time-average dissipated power per unit length and  $\langle P \rangle$  is the electromagnetic power flow in the lossless waveguide derived in Section 8-6-5 for each of the modes.

In particular, we calculate  $\alpha$  for the  $TE_{10}$  mode ( $k_x = \pi/a$ ,  $k_y = 0$ ). The waveguide fields are then

$$\begin{aligned} \hat{\mathbf{H}} &= H_0 \left( \mathbf{i}_x \frac{jk_z a}{\pi} \sin \frac{\pi x}{a} + \cos \frac{\pi x}{a} \mathbf{i}_z \right) \\ \hat{\mathbf{E}} &= -\frac{j\omega\mu a}{\pi} H_0 \sin \frac{\pi x}{a} \mathbf{i}_y \end{aligned} \quad (49)$$

The surface current on each wall is found from (34) as

$$\begin{aligned} \hat{\mathbf{K}}(x=0, y) &= \hat{\mathbf{K}}(x=a, y) = -H_0 \mathbf{i}_y, \\ \hat{\mathbf{K}}(x, y=0) &= -\hat{\mathbf{K}}(x, y=b) = H_0 \left( -\mathbf{i}_z \frac{jk_z a}{\pi} \sin \frac{\pi x}{a} + \mathbf{i}_x \cos \frac{\pi x}{a} \right) \end{aligned} \quad (50)$$

With lossy walls the electric field component  $\mathbf{E}_w$  within the walls is in the same direction as the surface current proportional by a surface conductivity  $\sigma_w \delta$ , where  $\delta$  is the skin depth as found in Section 8-3-4b. The time-average dissipated power density per unit area in the walls is then:

$$\begin{aligned} \langle P_d(x=0, y) \rangle &= \langle P_d(x=a, y) \rangle \\ &= \frac{1}{2} \text{Re} (\hat{\mathbf{E}}_w \cdot \hat{\mathbf{K}}^*) = \frac{1}{2} \frac{H_0^2}{\sigma_w \delta} \\ \langle P_d(x, y=0) \rangle &= \langle P_d(x, y=b) \rangle \\ &= \frac{1}{2} \frac{H_0^2}{\sigma_w \delta} \left[ \left( \frac{k_z a}{\pi} \right)^2 \sin^2 \frac{\pi x}{a} + \cos^2 \frac{\pi x}{a} \right] \end{aligned} \quad (51)$$

The total time average dissipated power per unit length  $\langle P_{dL} \rangle$  required in (48) is obtained by integrating each of the

terms in (51) along the waveguide walls:

$$\begin{aligned}
 \langle P_{dL} \rangle &= \int_0^b [\langle P_d(x=0, y) \rangle + \langle P_d(x=a, y) \rangle] dy \\
 &\quad + \int_0^a [\langle P_d(x, y=0) \rangle + \langle P_d(x, y=b) \rangle] dx \\
 &= \frac{H_0^2 b}{\sigma_w \delta} + \frac{H_0^2}{\sigma_w \delta} \int_0^a \left[ \left( \frac{k_z a}{\pi} \right)^2 \sin^2 \frac{\pi x}{a} + \cos^2 \frac{\pi x}{a} \right] dx \\
 &= \frac{H_0^2}{\sigma_w \delta} \left\{ b + \frac{a}{2} \left[ \left( \frac{k_z a}{\pi} \right)^2 + 1 \right] \right\} = \frac{H_0^2}{\sigma_w \delta} \left[ b + \frac{a}{2} \left( \frac{\omega^2 a^2}{\pi^2 c^2} \right) \right] \quad (52)
 \end{aligned}$$

while the electromagnetic power above cut-off for the TE<sub>10</sub> mode is given by (47),

$$\langle P \rangle = \frac{\omega \mu k_z a b H_0^2}{4(\pi/a)^2} \quad (53)$$

so that

$$\alpha = \frac{1}{2} \frac{\langle P_{dL} \rangle}{\langle P \rangle} = \frac{2 \left( \frac{\pi}{a} \right)^2 \left[ b + \frac{a}{2} \left( \frac{\omega^2 a^2}{\pi^2 c^2} \right) \right]}{\omega \mu a b k_z \sigma_w \delta} \quad (54)$$

where

$$k_z = \left[ \frac{\omega^2}{c^2} - \left( \frac{\pi}{a} \right)^2 \right]^{1/2}; \quad \frac{\omega}{c} > \frac{\pi}{a} \quad (55)$$

## 8-7 DIELECTRIC WAVEGUIDE

We found in Section 7-10-6 for fiber optics that electromagnetic waves can also be guided by dielectric structures if the wave travels from the dielectric to free space at an angle of incidence greater than the critical angle. Waves propagating along the dielectric of thickness  $2d$  in Figure 8-30 are still described by the vector wave equations derived in Section 8-6-1.

### 8-7-1 TM Solutions

We wish to find solutions where the fields are essentially confined within the dielectric. We neglect variations with  $y$  so that for TM waves propagating in the  $z$  direction the  $z$  component of electric field is given in Section 8-6-2 as

$$E_z(x, t) = \begin{cases} \operatorname{Re} [A_2 e^{-\alpha(x-d)} e^{j(\omega t - k_z z)}], & x \geq d \\ \operatorname{Re} [(A_1 \sin k_x x + B_1 \cos k_x x) e^{j(\omega t - k_z z)}], & |x| \leq d \\ \operatorname{Re} [A_3 e^{\alpha(x+d)} e^{j(\omega t - k_z z)}], & x \leq -d \end{cases} \quad (1)$$

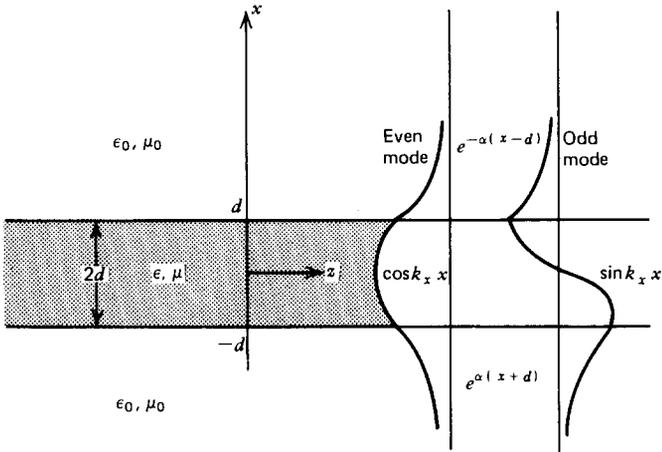


Figure 8-30 TE and TM modes can also propagate along dielectric structures. The fields can be essentially confined to the dielectric over a frequency range if the speed of the wave in the dielectric is less than that outside. It is convenient to separate the solutions into even and odd modes.

where we choose to write the solution outside the dielectric in the decaying wave form so that the fields are predominantly localized around the dielectric.

The wavenumbers and decay rate obey the relations

$$\begin{aligned} k_x^2 + k_z^2 &= \omega^2 \epsilon \mu \\ -\alpha^2 + k_z^2 &= \omega^2 \epsilon_0 \mu_0 \end{aligned} \quad (2)$$

The  $z$  component of the wavenumber must be the same in all regions so that the boundary conditions can be met at each interface. For propagation in the dielectric and evanescence in free space, we must have that

$$\omega^2 \epsilon_0 \mu_0 < k_z^2 < \omega^2 \epsilon \mu \quad (3)$$

All the other electric and magnetic field components can be found from (1) in the same fashion as for metal waveguides in Section 8-6-2. However, it is convenient to separately consider each of the solutions for  $E_z$  within the dielectric.

### (a) Odd Solutions

If  $E_z$  in each half-plane above and below the centerline are oppositely directed, the field within the dielectric must vary solely as  $\sin k_x x$ :

$$\hat{E}_z = \begin{cases} A_2 e^{-\alpha(x-d)}, & x \geq d \\ A_1 \sin k_x x, & |x| \leq d \\ A_3 e^{\alpha(x+d)}, & x \leq -d \end{cases} \quad (4)$$

Then because in the absence of volume charge the electric field has no divergence,

$$\frac{\partial \hat{E}_x}{\partial x} - jk_z \hat{E}_z \Rightarrow \hat{E}_x = \begin{cases} -\frac{jk_z}{\alpha} A_2 e^{-\alpha(x-d)}, & x \geq d \\ -\frac{jk_z}{k_x} A_1 \cos k_x x, & |x| \leq d \\ \frac{jk_z}{\alpha} A_3 e^{\alpha(x+d)}, & x \leq -d \end{cases} \quad (5)$$

while from Faraday's law the magnetic field is

$$\hat{H}_y = -\frac{1}{j\omega\mu} \left( -jk_z \hat{E}_x - \frac{\partial \hat{E}_z}{\partial x} \right)$$

$$\Rightarrow \hat{H}_y = \begin{cases} -\frac{j\omega\epsilon_0 A_2}{\alpha} e^{-\alpha(x-d)}, & x \geq d \\ -\frac{j\omega\epsilon A_1}{k_x} \cos k_x x, & |x| \leq d \\ \frac{j\omega\epsilon_0 A_3}{\alpha} e^{\alpha(x+d)}, & x \leq -d \end{cases} \quad (6)$$

At the boundaries where  $x = \pm d$  the tangential electric and magnetic fields are continuous:

$$E_z(x = \pm d_-) = E_z(x = \pm d_+) \Rightarrow A_1 \sin k_x d = A_2$$

$$-A_1 \sin k_x d = A_3$$

$$H_y(x = \pm d_-) = H_y(x = \pm d_+) \Rightarrow \frac{-j\omega\epsilon A_1}{k_x} \cos k_x d = \frac{-j\omega\epsilon_0 A_2}{\alpha} \quad (7)$$

$$\frac{-j\omega\epsilon A_1}{k_x} \cos k_x d = \frac{j\omega\epsilon_0 A_3}{\alpha}$$

which when simultaneously solved yields

$$\left. \begin{aligned} \frac{A_2}{A_1} = \sin k_x d = \frac{\epsilon\alpha}{\epsilon_0 k_x} \cos k_x d \\ \frac{A_3}{A_1} = -\sin k_x d = -\frac{\epsilon\alpha}{\epsilon_0 k_x} \cos k_x d \end{aligned} \right\} \Rightarrow \alpha = \frac{\epsilon_0}{\epsilon} k_x \tan k_x d \quad (8)$$

The allowed values of  $\alpha$  and  $k_x$  are obtained by self-consistently solving (8) and (2), which in general requires a numerical method. The critical condition for a guided wave occurs when  $\alpha = 0$ , which requires that  $k_x d = n\pi$  and  $k_z^2 = \omega^2 \epsilon_0 \mu_0$ . The critical frequency is then obtained from (2) as

$$\omega^2 = \frac{k_x^2}{\epsilon\mu - \epsilon_0\mu_0} = \frac{(n\pi/d)^2}{\epsilon\mu - \epsilon_0\mu_0} \quad (9)$$

Note that this occurs for real frequencies only if  $\epsilon\mu > \epsilon_0\mu_0$ .

**(b) Even Solutions**

If  $E_z$  is in the same direction above and below the dielectric, solutions are similarly

$$\hat{E}_z = \begin{cases} B_2 e^{-\alpha(x-d)}, & x \geq d \\ B_1 \cos k_x x, & |x| \leq d \\ B_3 e^{\alpha(x+d)}, & x \leq -d \end{cases} \quad (10)$$

$$\hat{E}_x = \begin{cases} -\frac{jk_x}{\alpha} B_2 e^{-\alpha(x-d)}, & x \geq d \\ \frac{jk_x}{k_x} B_1 \sin k_x x, & |x| \leq d \\ \frac{jk_x}{\alpha} B_3 e^{\alpha(x+d)}, & x \leq -d \end{cases} \quad (11)$$

$$\hat{H}_y = \begin{cases} -\frac{j\omega\epsilon_0}{\alpha} B_2 e^{-\alpha(x-d)}, & x \geq d \\ \frac{j\omega\epsilon}{k_x} B_1 \sin k_x x, & |x| \leq d \\ \frac{j\omega\epsilon_0}{\alpha} B_3 e^{\alpha(x+d)}, & x \leq -d \end{cases} \quad (12)$$

Continuity of tangential electric and magnetic fields at  $x = \pm d$  requires

$$\begin{aligned} B_1 \cos k_x d &= B_2, & B_1 \cos k_x d &= B_3 \\ \frac{j\omega\epsilon}{k_x} B_1 \sin k_x d &= -\frac{j\omega\epsilon_0}{\alpha} B_2, & -\frac{j\omega\epsilon B_1}{k_x} \sin k_x d &= \frac{j\omega\epsilon_0 B_3}{\alpha} \end{aligned} \quad (13)$$

or

$$\left. \begin{aligned} \frac{B_2}{B_1} &= \cos k_x d = -\frac{\epsilon\alpha}{\epsilon_0 k_x} \sin k_x d \\ \frac{B_3}{B_1} &= \cos k_x d = -\frac{\epsilon\alpha}{\epsilon_0 k_x} \sin k_x d \end{aligned} \right\} \Rightarrow \alpha = -\frac{\epsilon_0 k_x}{\epsilon} \cot k_x d \quad (14)$$

**8-7-2 TE Solutions**

The same procedure is performed for the TE solutions by first solving for  $H_z$ .

**(a) Odd Solutions**

$$\hat{H}_z = \begin{cases} A_2 e^{-\alpha(x-d)}, & x \geq d \\ A_1 \sin k_x x, & |x| \leq d \\ A_3 e^{\alpha(x+d)}, & x \leq -d \end{cases} \quad (15)$$

$$\hat{H}_x = \begin{cases} -\frac{jk_z}{\alpha} A_2 e^{-\alpha(x-d)}, & x \geq d \\ -\frac{jk_z}{k_x} A_1 \cos k_x x, & |x| \leq d \\ \frac{jk_z}{\alpha} A_3 e^{\alpha(x+d)}, & x \leq -d \end{cases} \quad (16)$$

$$\hat{E}_y = \begin{cases} \frac{j\omega\mu_0}{\alpha} A_2 e^{-\alpha(x-d)}, & x \geq d \\ \frac{j\omega\mu}{k_x} A_1 \cos k_x x, & |x| \leq d \\ -\frac{j\omega\mu_0}{\alpha} A_3 e^{\alpha(x+d)}, & x \leq -d \end{cases} \quad (17)$$

where continuity of tangential  $\mathbf{E}$  and  $\mathbf{H}$  across the boundaries requires

$$\alpha = \frac{\mu_0}{\mu} k_x \tan k_x d \quad (18)$$

**(b) Even Solutions**

$$\hat{H}_z = \begin{cases} B_2 e^{-\alpha(x-d)}, & x \geq d \\ B_1 \cos k_x x, & |x| \leq d \\ B_3 e^{\alpha(x+d)}, & x \leq -d \end{cases} \quad (19)$$

$$\hat{H}_x = \begin{cases} -\frac{jk_z}{\alpha} B_2 e^{-\alpha(x-d)}, & x \geq d \\ \frac{jk_z}{k_x} B_1 \sin k_x x, & |x| \leq d \\ \frac{jk_z}{\alpha} B_3 e^{\alpha(x+d)}, & x \leq -d \end{cases} \quad (20)$$

$$\hat{E}_y = \begin{cases} \frac{j\omega\mu_0}{\alpha} B_2 e^{-\alpha(x-d)}, & x \geq d \\ -\frac{j\omega\mu}{k_x} B_1 \sin k_x x, & |x| \leq d \\ -\frac{j\omega\mu_0}{\alpha} B_3 e^{\alpha(x+d)}, & x \leq -d \end{cases} \quad (21)$$

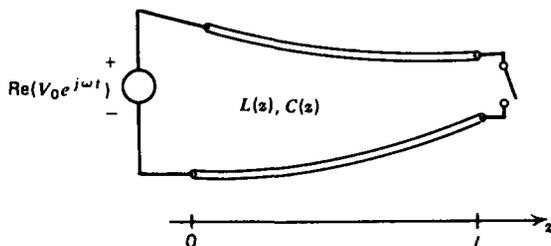
where  $\alpha$  and  $k_x$  are related as

$$\alpha = -\frac{\mu_0}{\mu} k_x \cot k_x d \quad (22)$$

**PROBLEMS**

Section 8-1

1. Find the inductance and capacitance per unit length and the characteristic impedance for the wire above plane and two wire line shown in Figure 8-3. (Hint: See Section 2-6-4c.)
2. The inductance and capacitance per unit length on a lossless transmission line is a weak function of  $z$  as the distance between electrodes changes slowly with  $z$ .



- (a) For this case write the transmission line equations as single equations in voltage and current.
- (b) Consider an exponential line, where

$$L(z) = L_0 e^{\alpha z}, \quad C(z) = C_0 e^{-\alpha z}$$

If the voltage and current vary sinusoidally with time as

$$v(z, t) = \text{Re} [\hat{v}(z) e^{j\omega t}], \quad i(z, t) = \text{Re} [\hat{i}(z) e^{j\omega t}]$$

find the general form of solution for the spatial distributions of  $\hat{v}(z)$  and  $\hat{i}(z)$ .

(c) The transmission line is excited by a voltage source  $V_0 \cos \omega t$  at  $z = 0$ . What are the voltage and current distributions if the line is short or open circuited at  $z = l$ ?

(d) For what range of frequency do the waves strictly decay with distance? What is the cut-off frequency for wave propagation?

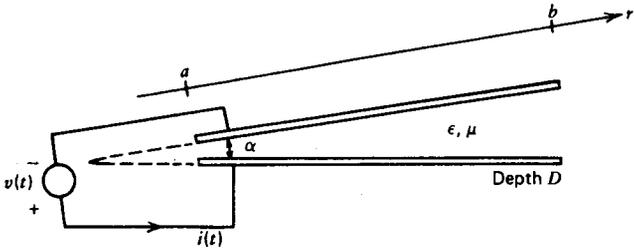
(e) What are the resonant frequencies of the short circuited line?

(f) What condition determines the resonant frequencies of the open circuited line.

3. Two conductors of length  $l$  extending over the radial distance  $a \leq r \leq b$  are at a constant angle  $\alpha$  apart.

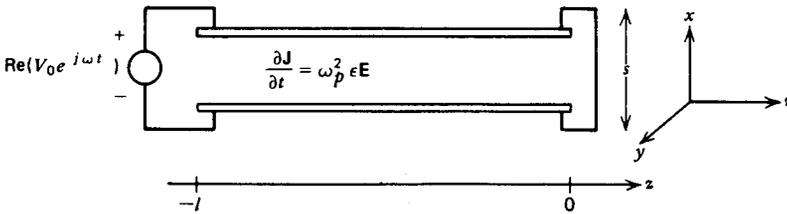
(a) What are the electric and magnetic fields in terms of the voltage and current?

(b) Find the inductance and capacitance per unit length. What is the characteristic impedance?



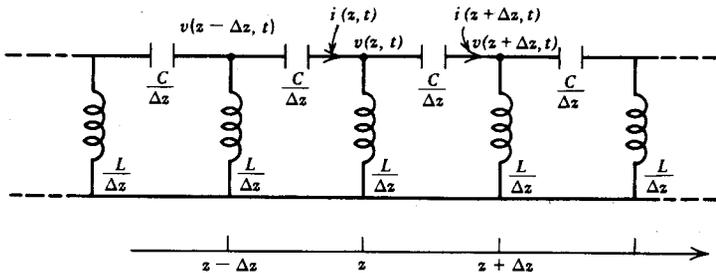
4. A parallel plate transmission line is filled with a conducting plasma with constitutive law

$$\frac{\partial \mathbf{J}}{\partial t} = \omega_p^2 \epsilon \mathbf{E}$$



- (a) How are the electric and magnetic fields related?
- (b) What are the transmission line equations for the voltage and current?
- (c) For sinusoidal signals of the form  $e^{j(\omega t - kz)}$ , how are  $\omega$  and  $k$  related? Over what frequency range do we have propagation or decay?
- (d) The transmission line is short circuited at  $z = 0$  and excited by a voltage source  $V_0 \cos \omega t$  at  $z = -l$ . What are the voltage and current distributions?
- (e) What are the resonant frequencies of the system?

5. An unusual type of distributed system is formed by series capacitors and shunt inductors.



- (a) What are the governing partial differential equations relating the voltage and current?

(b) What is the dispersion relation between  $\omega$  and  $k$  for signals of the form  $e^{j(\omega t - kz)}$ ?

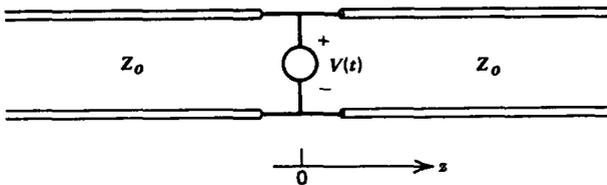
(c) What are the group and phase velocities of the waves? Why are such systems called "backward wave"?

(d) A voltage  $V_0 \cos \omega t$  is applied at  $z = -l$  with the  $z = 0$  end short circuited. What are the voltage and current distributions along the line?

(e) What are the resonant frequencies of the system?

### Section 8-2

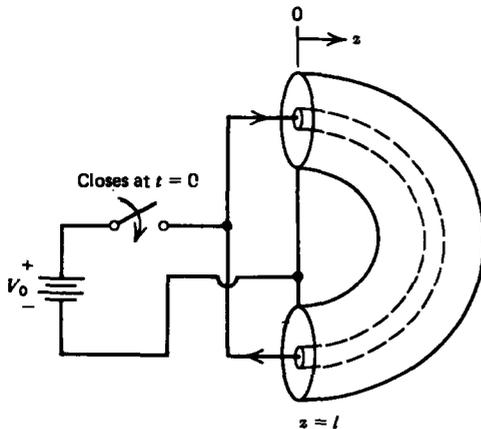
6. An infinitely long transmission line is excited at its center by a step voltage  $V_0$  turned on at  $t = 0$ . The line is initially at rest.



(a) Plot the voltage and current distributions at time  $T$ .

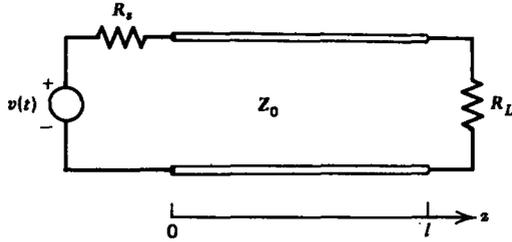
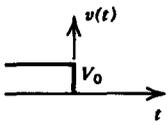
(b) At this time  $T$  the voltage is set to zero. Plot the voltage and current everywhere at time  $2T$ .

7. A transmission line of length  $l$  excited by a step voltage source has its ends connected together. Plot the voltage and current at  $z = l/4$ ,  $l/2$ , and  $3l/4$  as a function of time.



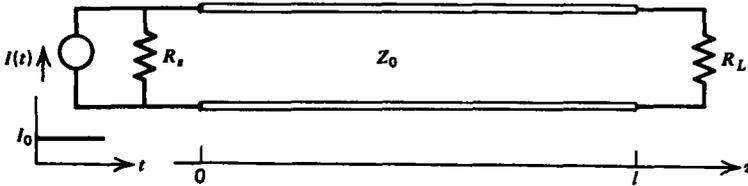
8. The dc steady state is reached for a transmission line loaded at  $z = l$  with a resistor  $R_L$  and excited at  $z = 0$  by a dc voltage  $V_0$  applied through a source resistor  $R_s$ . The voltage source is suddenly set to zero at  $t = 0$ .

(a) What is the initial voltage and current along the line?



(b) Find the voltage at the  $z = l$  end as a function of time. (Hint: Use difference equations.)

9. A step current source turned on at  $t = 0$  is connected to the  $z = 0$  end of a transmission line in parallel with a source resistance  $R_s$ . A load resistor  $R_L$  is connected at  $z = l$ .



(a) What is the load voltage and current as a function of time? (Hint: Use a Thevenin equivalent network at  $z = 0$  with the results of Section 8-2-3.)

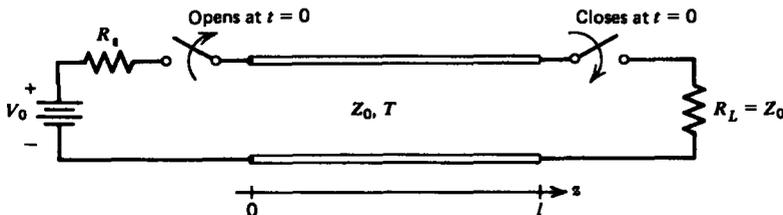
(b) With  $R_s = \infty$  plot versus time the load voltage when  $R_L = \infty$  and the load current when  $R_L = 0$ .

(c) If  $R_s = \infty$  and  $R_L = \infty$ , solve for the load voltage in the quasi-static limit assuming the transmission line is a capacitor. Compare with (b).

(d) If  $R_s$  is finite but  $R_L = 0$ , what is the time dependence of the load current?

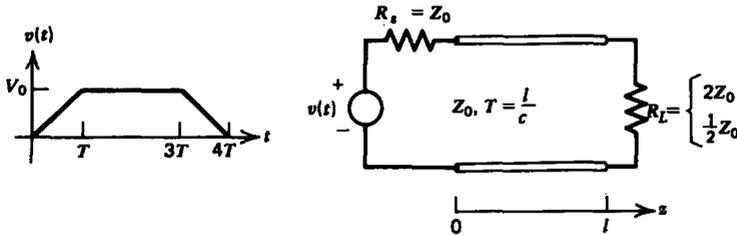
(e) Repeat (d) in the quasi-static limit where the transmission line behaves as an inductor. When are the results of (d) and (e) approximately equal?

10. Switched transmission line systems with an initial dc voltage can be used to generate high voltage pulses of short time duration. The line shown is charged up to a dc voltage  $V_0$  when at  $t = 0$  the load switch is closed and the source switch is opened.

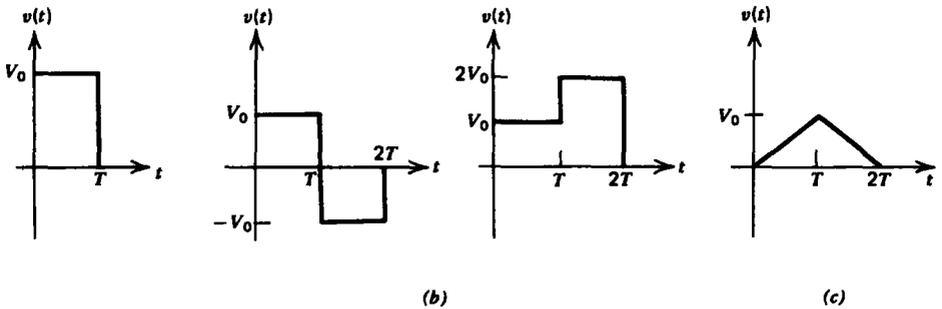
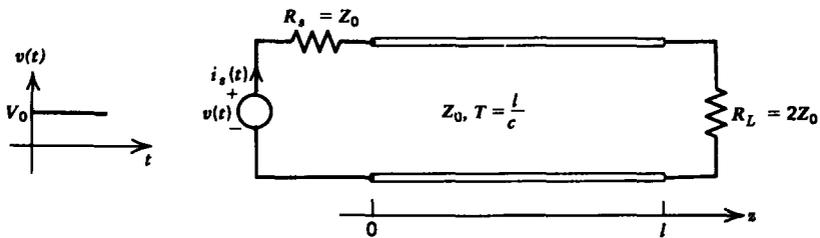


- (a) What are the initial line voltage and current? What are  $V_+$  and  $V_-$ ?  
 (b) Sketch the time dependence of the load voltage.

11. For the trapezoidal voltage excitation shown, plot versus time the current waveforms at  $z = 0$  and  $z = l$  for  $R_L = 2Z_0$  and  $R_L = \frac{1}{2}Z_0$ .



12. A step voltage is applied to a loaded transmission line with  $R_L = 2Z_0$  through a matching source resistor.

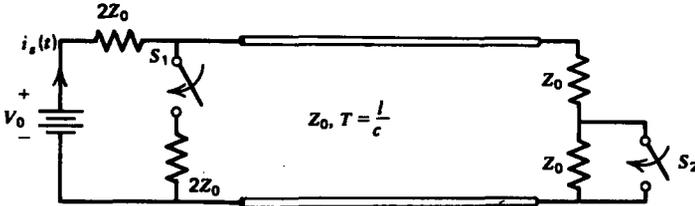


- (a) Sketch the source current  $i_s(t)$ .  
 (b) Using superposition of delayed step voltages find the time dependence of  $i_s(t)$  for the various pulse voltages shown.  
 (c) By integrating the appropriate solution of (b), find  $i_s(t)$  if the applied voltage is the triangle wave shown.

13. A dc voltage has been applied for a long time to the transmission line circuit shown with switches  $S_1$  and  $S_2$  open

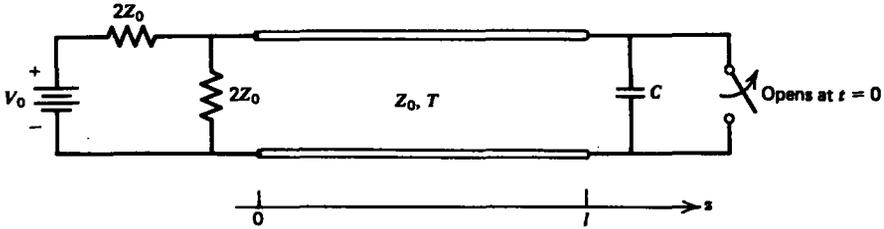
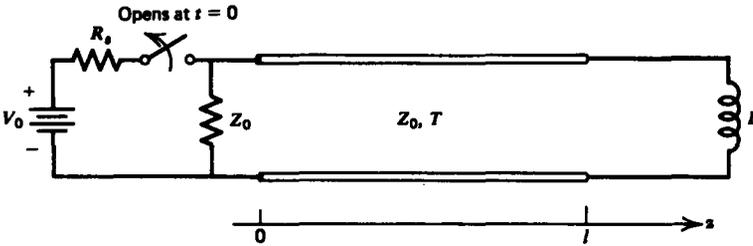
when at  $t = 0$ :

- (a)  $S_2$  is suddenly closed with  $S_1$  kept open;
- (b)  $S_1$  is suddenly closed with  $S_2$  kept open;
- (c) Both  $S_1$  and  $S_2$  are closed.



For each of these cases plot the source current  $i_s(t)$  versus time.

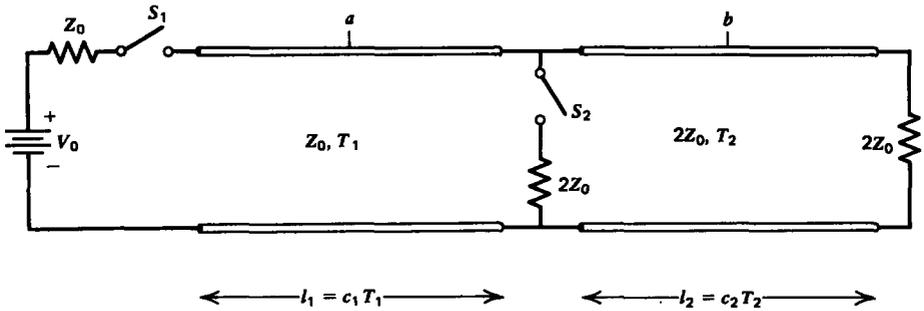
14. For each of the transmission line circuits shown, the switch opens at  $t = 0$  after the dc voltage has been applied for a long time.



- (a) What are the transmission line voltages and currents right before the switches open? What are  $V_+$  and  $V_-$  at  $t = 0$ ?
- (b) Plot the voltage and current as a function of time at  $z = l/2$ .

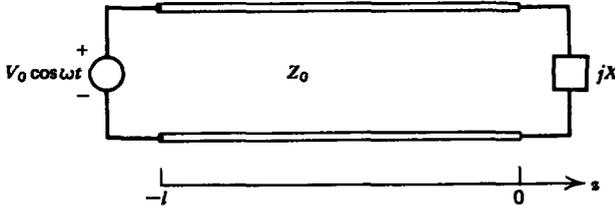
15. A transmission line is connected to another transmission line with double the characteristic impedance.

- (a) With switch  $S_2$  open, switch  $S_1$  is suddenly closed at  $t = 0$ . Plot the voltage and current as a function of time half-way down each line at points  $a$  and  $b$ .
- (b) Repeat (a) if  $S_2$  is closed.



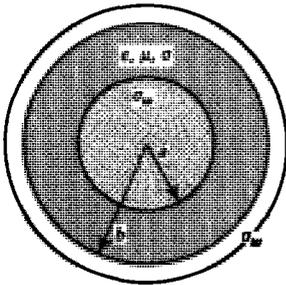
Section 8-3

16. A transmission line is excited by a voltage source  $V_0 \cos \omega t$  at  $z = -l$ . The transmission line is loaded with a purely reactive load with impedance  $jX$  at  $z = 0$ .



- (a) Find the voltage and current distribution along the line.
- (b) Find an expression for the resonant frequencies of the system if the load is capacitive or inductive. What is the solution if  $|X| = Z_0$ ?
- (c) Repeat (a) and (b) if the transmission line is excited by a current source  $I_0 \cos \omega t$  at  $z = -l$ .

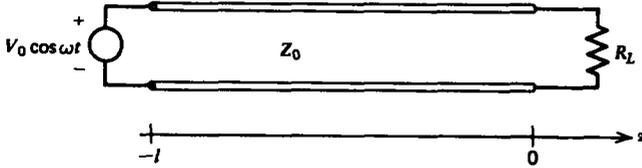
17. (a) Find the resistance and conductance per unit lengths for a coaxial cable whose dielectric has a small Ohmic conductivity  $\sigma$  and walls have a large conductivity  $\sigma_w$ . (Hint: The skin depth  $\delta$  is much smaller than the radii or thickness of either conductor.)



- (b) What is the decay rate of the fields due to the losses?
- (c) If the dielectric is lossless ( $\sigma = 0$ ) with a fixed value of

outer radius  $b$ , what value of inner radius  $a$  will minimize the decay rate? (Hint:  $1 + 1/3.6 \approx \ln 3.6$ .)

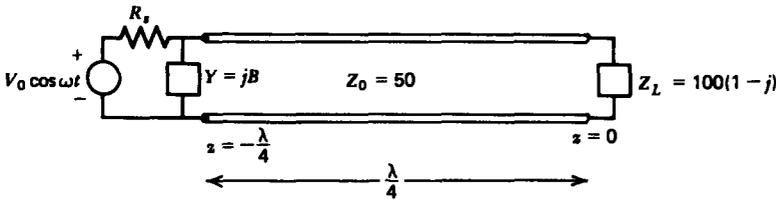
18. A transmission line of length  $l$  is loaded by a resistor  $R_L$ .



- (a) Find the voltage and current distributions along the line.
- (b) Reduce the solutions of (a) when the line is much shorter than a wavelength.
- (c) Find the approximate equivalent circuits in the long wavelength limit ( $kl \ll 1$ ) when  $R_L$  is very small ( $R_L \ll Z_0$ ) and when it is very large ( $R_L \gg Z_0$ ).

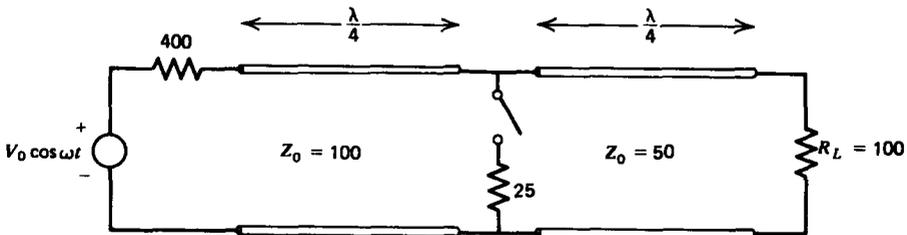
Section 8-4

19. For the transmission line shown:



- (a) Find the values of lumped reactive admittance  $Y = jB$  and non-zero source resistance  $R_s$  that maximizes the power delivered by the source. (Hint: Do not use the Smith chart.)
- (b) What is the time-average power dissipated in the load?

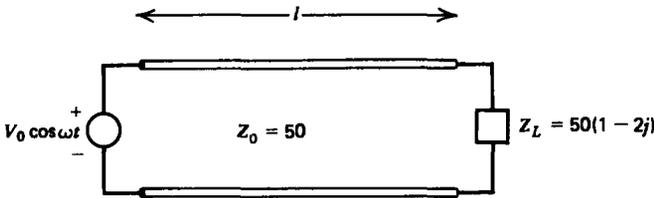
20. (a) Find the time-average power delivered by the source for the transmission line system shown when the switch is open or closed. (Hint: Do not use the Smith chart.)



- (b) For each switch position, what is the time average power dissipated in the load resistor  $R_L$ ?

(c) For each switch position what is the VSWR on each line?

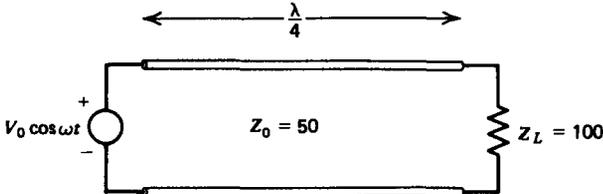
21. (a) Using the Smith chart find the source current delivered (magnitude and phase) for the transmission line system shown, for  $l = \lambda/8, \lambda/4, 3\lambda/8,$  and  $\lambda/2$ .



(b) For each value of  $l$ , what are the time-average powers delivered by the source and dissipated in the load impedance  $Z_L$ ?

(c) What is the VSWR?

22. (a) Without using the Smith chart find the voltage and current distributions for the transmission line system shown.



(b) What is the VSWR?

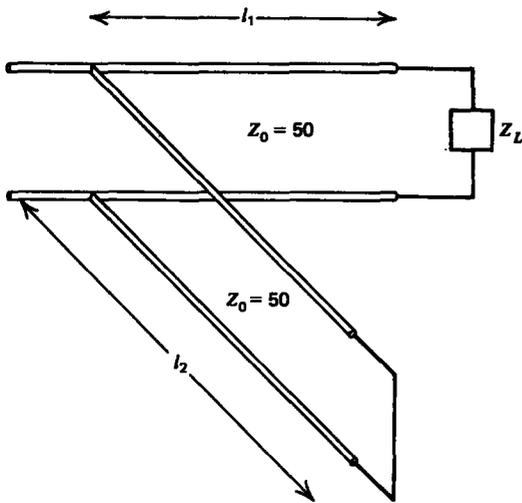
(c) At what positions are the voltages a maximum or a minimum? What is the voltage magnitude at these positions?

23. The VSWR on a 100-Ohm transmission line is 3. The distance between successive voltage minima is 50 cm while the distance from the load to the first minima is 20 cm. What are the reflection coefficient and load impedance?

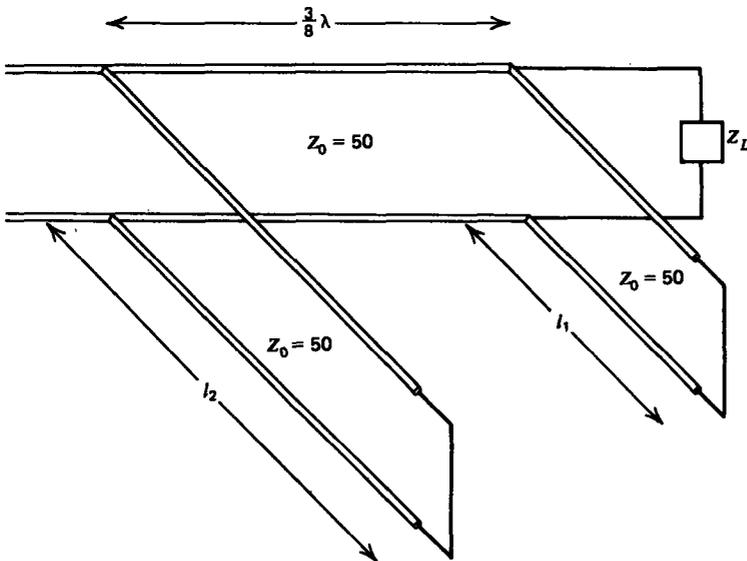
Section 8-5

24. For each of the following load impedances in the single-stub tuning transmission line system shown, find all values of the length of the line  $l_1$  and stub length  $l_2$  necessary to match the load to the line.

- (a)  $Z_L = 100(1 - j)$
- (b)  $Z_L = 50(1 + 2j)$
- (c)  $Z_L = 25(2 - j)$
- (d)  $Z_L = 25(1 + 2j)$

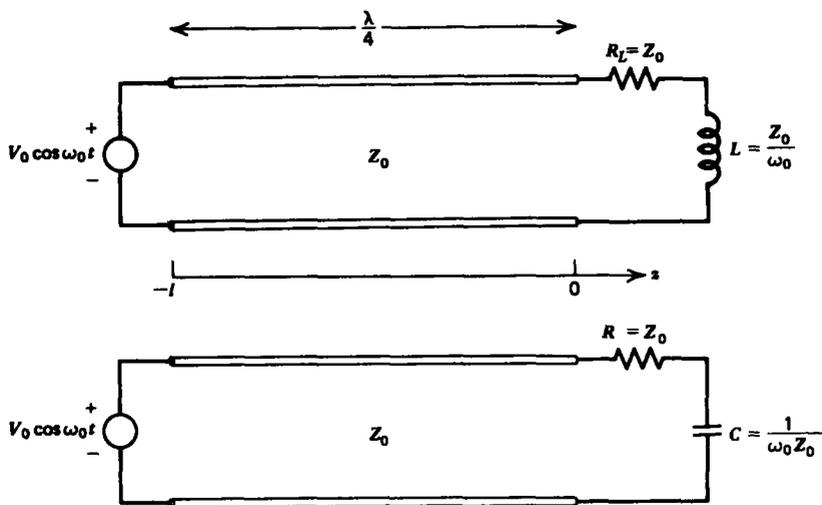


25. For each of the following load impedances in the double-stub tuning transmission line system shown, find stub lengths  $l_1$  and  $l_2$  to match the load to the line.



- (a)  $Z_L = 100(1-j)$       (c)  $Z_L = 25(2-j)$   
 (b)  $Z_L = 50(1+2j)$       (d)  $Z_L = 25(1+2j)$

26. (a) Without using the Smith chart, find the input impedance  $Z_{in}$  at  $z = -l = -\lambda/4$  for each of the loads shown.  
 (b) What is the input current  $i(z = -l, t)$  for each of the loads?



(c) The frequency of the source is doubled to  $2\omega_0$ . The line length  $l$  and loads  $L$  and  $C$  remain unchanged. Repeat (a) and (b).

(d) The frequency of the source is halved to  $\frac{1}{2}\omega_0$ . Repeat (a) and (b).

Section 8-6

27. A rectangular metal waveguide is filled with a plasma with constitutive law

$$\frac{\partial \mathbf{J}_f}{\partial t} = \omega_p^2 \epsilon \mathbf{E}$$

(a) Find the TE and TM solutions that satisfy the boundary conditions.

(b) What is the wavenumber  $k_z$  along the axis? What is the cut-off frequency?

(c) What are the phase and group velocities of the waves?

(d) What is the total electromagnetic power flowing down the waveguide for each of the modes?

(e) If the walls have a large but finite conductivity, what is the spatial decay rate for TE<sub>10</sub> propagating waves?

28. (a) Find the power dissipated in the walls of a waveguide with large but finite conductivity  $\sigma_w$  for the TM<sub>m,n</sub> modes (Hint: Use Equation (25).)

(b) What is the spatial decay rate for propagating waves?

29. (a) Find the equations of the electric and magnetic field lines in the  $xy$  plane for the TE and TM modes.

(b) Find the surface current field lines on each of the

waveguide surfaces for the  $TE_{mn}$  modes. **Hint:**

$$\int \tan x dx = -\ln \cos x$$

$$\int \cot x dx = \ln \sin x$$

(c) For all modes verify the conservation of charge relation on the  $x = 0$  surface:

$$\nabla_{\Sigma} \cdot \mathbf{K} + \frac{\partial \sigma_f}{\partial t} = 0$$

30. (a) Find the first ten lowest cut-off frequencies if  $a = b = 1$  cm in a free space waveguide.

(b) What are the necessary dimensions for a square free space waveguide to have a lowest cut-off frequency of  $10^{10}$ ,  $10^8$ ,  $10^6$ ,  $10^4$ , or  $10^2$  Hz?

31. A rectangular waveguide of height  $b$  and width  $a$  is short circuited by perfectly conducting planes at  $z = 0$  and  $z = l$ .

(a) Find the general form of the TE and TM electric and magnetic fields. (**Hint:** Remember to consider waves traveling in the  $\pm z$  directions.)

(b) What are the natural frequencies of this resonator?

(c) If the walls have a large conductivity  $\sigma_w$  find the total time-average power  $\langle P_d \rangle$  dissipated in the  $TE_{101}$  mode.

(d) What is the total time-average electromagnetic energy  $\langle W \rangle$  stored in the resonator?

(e) Find the  $Q$  of the resonator, defined as

$$Q = \frac{\omega_0 \langle W \rangle}{\langle P_d \rangle}$$

where  $\omega_0$  is the resonant frequency.

### Section 8.7

32. (a) Find the critical frequency where the spatial decay rate  $\alpha$  is zero for all the dielectric modes considered.

(b) Find approximate values of  $\alpha$ ,  $k_x$ , and  $k_z$  for a very thin dielectric, where  $k_x d \ll 1$ .

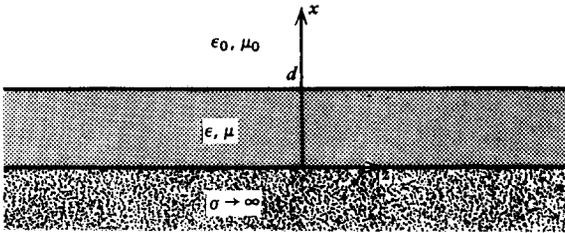
(c) For each of the solutions find the time-average power per unit length in each region.

(d) If the dielectric has a small Ohmic conductivity  $\sigma$ , what is the approximate attenuation rate of the fields.

33. A dielectric waveguide of thickness  $d$  is placed upon a perfect conductor.

(a) Which modes can propagate along the dielectric?

(b) For each of these modes, what are the surface current and charges on the conductor?



(c) Verify the conservation of charge relation:

$$\nabla_{\Sigma} \cdot \mathbf{K} + \frac{\partial \sigma_f}{\partial t} = 0$$

(d) If the conductor has a large but noninfinite Ohmic conductivity  $\sigma_w$ , what is the approximate power per unit area dissipated?

(e) What is the approximate attenuation rate of the fields?

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