chapter 4

electric field boundary value problems

The electric field distribution due to external sources is disturbed by the addition of a conducting or dielectric body because the resulting induced charges also contribute to the field. The complete solution must now also satisfy boundary conditions imposed by the materials.

4-1 THE UNIQUENESS THEOREM

Consider a linear dielectric material where the permittivity may vary with position:

$$\mathbf{D}=\varepsilon(r)\mathbf{E}=-\varepsilon(r)\nabla V \tag{1}$$

The special case of different constant permittivity media separated by an interface has $\varepsilon(r)$ as a step function. Using (1) in Gauss's law yields

$$\nabla\cdot[\varepsilon(r)\nabla V]=-\rho_f \tag{2}$$

which reduces to Poisson's equation in regions where $\varepsilon(r)$ is a constant. Let us call V_p a solution to (2).

The solution V_L to the homogeneous equation

$$\nabla\cdot[\varepsilon(r)\nabla V]=0 \tag{3}$$

which reduces to Laplace's equation when $\varepsilon(r)$ is constant, can be added to V_p and still satisfy (2) because (2) is linear in the potential:

$$\nabla\cdot[\varepsilon(r)\nabla(V_p+V_L)]=\nabla\cdot[\varepsilon(r)\nabla V_p]+\underbrace{\nabla\cdot[\varepsilon(r)\nabla V_L]}_{0}=-\rho_f \tag{4}$$

Any linear physical problem must only have one solution yet (3) and thus (2) have many solutions. We need to find what boundary conditions are necessary to uniquely specify this solution. Our method is to consider two different solutions V_1 and V_2 for the same charge distribution

$$\nabla\cdot(\varepsilon\nabla V_1)=-\rho_f,\qquad \nabla\cdot(\varepsilon\nabla V_2)=-\rho_f \tag{5}$$

so that we can determine what boundary conditions force these solutions to be identical, $V_1=V_2$.

The difference of these two solutions $V_T = V_1 - V_2$ obeys the homogeneous equation

$$\nabla \cdot (\varepsilon \nabla V_T) = 0 \tag{6}$$

We examine the vector expansion

$$\nabla \cdot (\varepsilon V_T \nabla V_T) = V_T \underbrace{\nabla \cdot (\varepsilon \nabla V_T)}_{0} + \varepsilon \nabla V_T \cdot \nabla V_T = \varepsilon |\nabla V_T|^2 \tag{7}$$

noting that the first term in the expansion is zero from (6) and that the second term is never negative.

We now integrate (7) over the volume of interest V, which may be of infinite extent and thus include all space

$$\int_V \nabla \cdot (\varepsilon V_T \nabla V_T)\, dV = \oint_S \varepsilon V_T \nabla V_T \cdot \mathbf{dS} = \int_V \varepsilon |\nabla V_T|^2\, dV \tag{8}$$

The volume integral is converted to a surface integral over the surface bounding the region using the divergence theorem. Since the integrand in the last volume integral of (8) is never negative, the integral itself can only be zero if V_T is zero at every point in the volume making the solution unique ($V_T = 0 \Rightarrow V_1 = V_2$). To force the volume integral to be zero, the surface integral term in (8) must be zero. This requires that on the surface S the two solutions must have the same value ($V_1 = V_2$) or their normal derivatives must be equal [$\nabla V_1 \cdot \mathbf{n} = \nabla V_2 \cdot \mathbf{n}$]. This last condition is equivalent to requiring that the normal components of the electric fields be equal ($\mathbf{E} = -\nabla V$).

Thus, a problem is uniquely posed when in addition to giving the charge distribution, the potential or the normal component of the electric field on the bounding surface surrounding the volume is specified. The bounding surface can be taken in sections with some sections having the potential specified and other sections having the normal field component specified.

If a particular solution satisfies (2) but it does not satisfy the boundary conditions, additional homogeneous solutions where $\rho_f = 0$, must be added so that the boundary conditions are met. No matter how a solution is obtained, even if guessed, if it satisfies (2) and all the boundary conditions, it is the only solution.

4-2 BOUNDARY VALUE PROBLEMS IN CARTESIAN GEOMETRIES

For most of the problems treated in Chapters 2 and 3 we restricted ourselves to one-dimensional problems where the electric field points in a single direction and only depends on that coördinate. For many cases, the volume is free of charge so that the system is described by Laplace's equation. Surface charge is present only on interfacial boundaries separating dissimilar conducting materials. We now consider such volume charge-free problems with two- and three dimensional variations.

4-2-1 Separation of Variables

Let us assume that within a region of space of constant permittivity with no volume charge, that solutions do not depend on the z coordinate. Then Laplace's equation reduces to

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0 \tag{1}$$

We try a solution that is a product of a function only of the x coordinate and a function only of y:

$$V(x, y)=X(x)\,Y(y) \tag{2}$$

This assumed solution is often convenient to use if the system boundaries lay in constant x or constant y planes. Then along a boundary, one of the functions in (2) is constant. When (2) is substituted into (1) we have

$$Y\frac{d^2X}{dx^2}+X\frac{d^2Y}{dy^2}=0\Rightarrow\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=0 \tag{3}$$

where the partial derivatives become total derivatives because each function only depends on a single coordinate. The second relation is obtained by dividing through by XY so that the first term is only a function of x while the second is only a function of y.

The only way the sum of these two terms can be zero for all values of x and y is if each term is separately equal to a constant so that (3) separates into two equations,

$$\frac{1}{X}\frac{d^2X}{dx^2}=k^2, \qquad \frac{1}{Y}\frac{d^2Y}{dy^2}=-k^2 \tag{4}$$

where k^2 is called the separation constant and in general can be a complex number. These equations can then be rewritten as the ordinary differential equations:

$$\frac{d^2X}{dx^2}-k^2X=0, \qquad \frac{d^2Y}{dy^2}+k^2Y=0 \tag{5}$$

4-2-2 Zero Separation Constant Solutions

When the separation constant is zero ($k^2=0$) the solutions to (5) are

$$X=a_1x+b_1, \qquad Y=c_1y+d_1 \tag{6}$$

where a_1, b_1, c_1, and d_1 are constants. The potential is given by the product of these terms which is of the form

$$V=a_2+b_2x+c_2y+d_2xy \tag{7}$$

The linear and constant terms we have seen before, as the potential distribution within a parallel plate capacitor with no fringing, so that the electric field is uniform. The last term we have not seen previously.

(a) Hyperbolic Electrodes

A hyperbolically shaped electrode whose surface shape obeys the equation $xy=ab$ is at potential V_0 and is placed above a grounded right-angle corner as in Figure 4-1. The

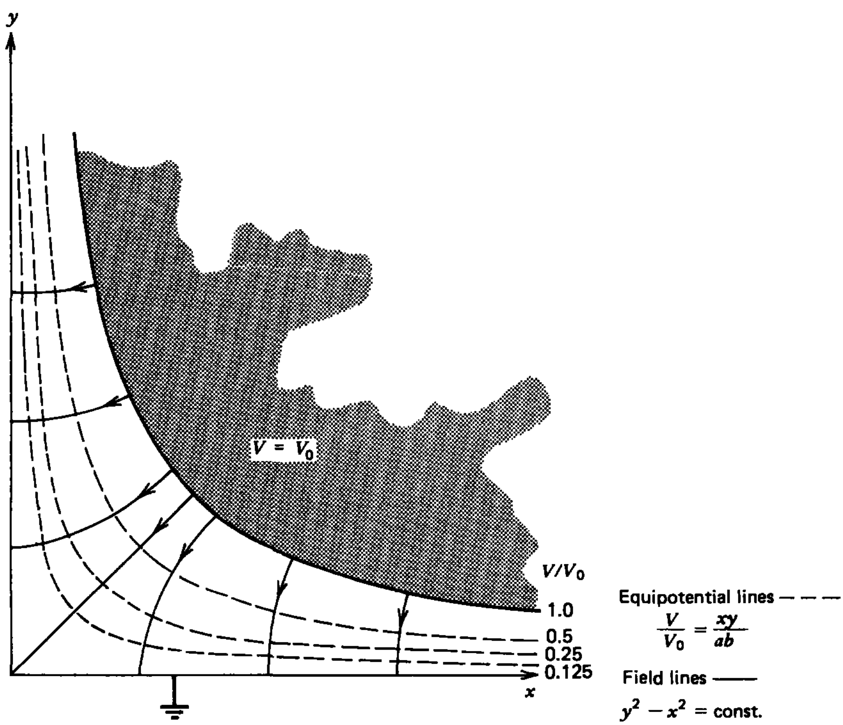


Figure 4-1 The equipotential and field lines for a hyperbolically shaped electrode at potential V_0 above a right-angle conducting corner are orthogonal hyperbolas.

boundary conditions are

$$V(x=0)=0, \qquad V(y=0)=0, \qquad V(xy=ab)=V_0 \tag{8}$$

so that the solution can be obtained from (7) as

$$V(x, y)=V_0xy/(ab) \tag{9}$$

The electric field is then

$$\mathbf{E}=-\nabla V=-\frac{V_0}{ab}[y\mathbf{i}_x+x\mathbf{i}_y] \tag{10}$$

The field lines drawn in Figure 4-1 are the perpendicular family of hyperbolas to the equipotential hyperbolas in (9):

$$\frac{dy}{dx}=\frac{E_y}{E_x}=\frac{x}{y}\Rightarrow y^2-x^2=\text{const} \tag{11}$$

(b) Resistor in an Open Box

A resistive medium is contained between two electrodes, one of which extends above and is bent through a right-angle corner as in Figure 4-2. We try zero separation constant

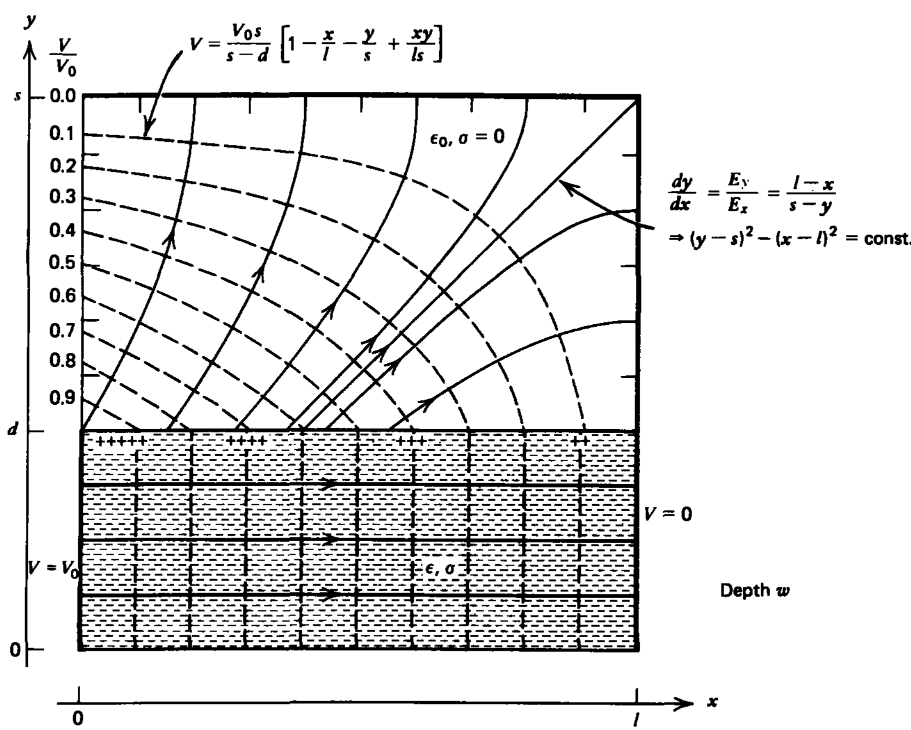


Figure 4-2 A resistive medium partially fills an open conducting box.

solutions given by (7) in each region enclosed by the electrodes:

$$V=\begin{cases} a_1+b_1x+c_1y+d_1xy, & o\le y\le d \\ a_2+b_2x+c_2y+d_2xy, & d\le y\le s \end{cases} \tag{12}$$

With the potential constrained on the electrodes and being continuous across the interface, the boundary conditions are

$$V(x=0)=V_0=a_1+c_1y \Rightarrow a_1=V_0,\quad c_1=0 \qquad (0\le y\le d)$$

$$V(x=l)=0=\begin{cases} \underset{V_0}{\underbrace{a_1}}+b_1l+\overset{0}{\cancel{c_1}}y+d_1ly \Rightarrow b_1=-V_0/l, \quad d_1=0 \\ \qquad\qquad\qquad\qquad\qquad\qquad (0\le y\le d) \\ a_2+b_2l+c_2y+d_2ly \Rightarrow a_2+b_2l=0, \quad c_2+d_2l=0 \\ \qquad\qquad\qquad\qquad\qquad\qquad (d\le y\le s) \end{cases}$$

$$V(y=s)=0=a_2+b_2x+c_2s+d_2xs \Rightarrow a_2+c_2s=0, \quad b_2+d_2s=0$$

$$\begin{aligned} V(y=d_+)=V(y=d_-)&=a_1+b_1x+\overset{0}{\cancel{c_1}}\,d+\overset{0}{\cancel{d_1}}\,xd \\ &=a_2+b_2x+c_2d+d_2xd \end{aligned} \tag{13}$$

$$\Rightarrow a_1=V_0=a_2+c_2d, \quad b_1=-V_0/l=b_2+d_2d$$

so that the constants in (12) are

$$\begin{aligned} &a_1=V_0, \quad b_1=-V_0/l, \quad c_1=0, \quad d_1=0 \\ &a_2=\frac{V_0}{(1-d/s)}, \quad b_2=-\frac{V_0}{l(1-d/s)}, \\ &c_2=-\frac{V_0}{s(1-d/s)}, \quad d_2=\frac{V_0}{ls(1-d/s)} \end{aligned} \tag{14}$$

The potential of (12) is then

$$V=\begin{cases} V_0(1-x/l), & 0\le y\le d \\ \dfrac{V_0 s}{s-d}\left(1-\dfrac{x}{l}-\dfrac{y}{s}+\dfrac{xy}{ls}\right), & d\le y\le s \end{cases} \tag{15}$$

with associated electric field

$$\mathbf{E}=-\nabla V=\begin{cases} \dfrac{V_0}{l}\mathbf{i}_x, & 0\le y\le d \\ \dfrac{V_0 s}{s-d}\left[\dfrac{\mathbf{i}_x}{l}\left(1-\dfrac{y}{s}\right)+\dfrac{\mathbf{i}_y}{s}\left(1-\dfrac{x}{l}\right)\right], & d<y<s \end{cases} \tag{16}$$

Note that in the dc steady state, the conservation of charge boundary condition of Section 3-3-5 requires that no current cross the interfaces at $y=0$ and $y=d$ because of the surrounding zero conductivity regions. The current and, thus, the electric field within the resistive medium must be purely tangential to the interfaces, $E_y(y=d_-)=E_y(y=0_+)=0$. The surface charge density on the interface at $y=d$ is then due only to the normal electric field above, as below, the field is purely tangential:

$$\sigma_f(y=d)=\varepsilon_0 E_y(y=d_+)-\varepsilon \overset{0}{\cancel{E_y}}(y=d_-)=\frac{\varepsilon_0 V_0}{s-d}\left(1-\frac{x}{l}\right) \tag{17}$$

The interfacial shear force is then

$$F_x=\int_0^l \sigma_f E_x(y=d)w\,dx=\frac{\varepsilon_0 V_0^2}{2(s-d)}w \tag{18}$$

If the resistive material is liquid, this shear force can be used to pump the fluid.*

4-2-3 Nonzero Separation Constant Solutions

Further solutions to (5) with nonzero separation constant ($k^2\neq 0$) are

$$\begin{aligned} X&=A_1\sinh kx+A_2\cosh kx=B_1 e^{kx}+B_2 e^{-kx}\\ Y&=C_1\sin ky+C_2\cos ky=D_1 e^{jky}+D_2 e^{-jky}\end{aligned} \tag{19}$$

When k is real, the solutions of X are hyperbolic or equivalently exponential, as drawn in Figure 4-3, while those of Y are trigonometric. If k is pure imaginary, then X becomes trigonometric and Y is hyperbolic (or exponential).

The solution to the potential is then given by the product of X and Y:

$$\begin{aligned} V=&E_1\sin ky\sinh kx+E_2\sin ky\cosh kx\\ &+E_3\cos ky\sinh kx+E_4\cos ky\cosh kx\end{aligned} \tag{20}$$

or equivalently

$$V=F_1\sin ky\,e^{kx}+F_2\sin ky\,e^{-kx}+F_3\cos ky\,e^{kx}+F_4\cos ky\,e^{-kx} \tag{21}$$

We can always add the solutions of (7) or any other Laplacian solutions to (20) and (21) to obtain a more general

* *See J. R. Melcher and G. I. Taylor, Electrohydrodynamics: A Review of the Role of Interfacial Shear Stresses,* Annual Rev. Fluid Mech., *Vol. 1, Annual Reviews, Inc., Palo Alto, Calif., 1969, ed. by Sears and Van Dyke, pp. 111–146. See also J. R. Melcher, "Electric Fields and Moving Media", film produced for the National Committee on Electrical Engineering Films by the Educational Development Center, 39 Chapel St., Newton, Mass. 02160. This film is described in* IEEE Trans. Education ***E-17***, *(1974) pp. 100–110.*

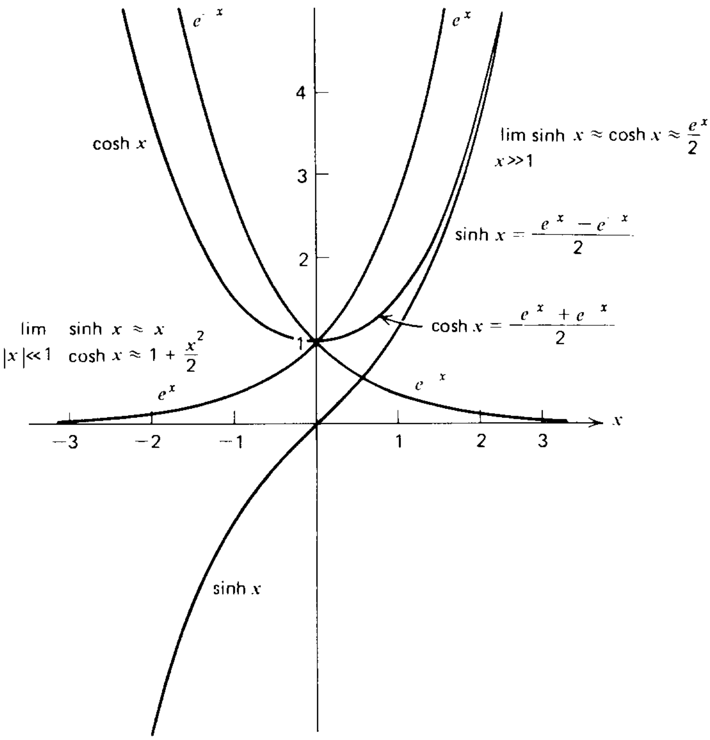


Figure 4-3 The exponential and hyperbolic functions for positive and negative arguments.

solution because Laplace's equation is linear. The values of the coefficients and of k are determined by boundary conditions.

When regions of space are of infinite extent in the x direction, it is often convenient to use the exponential solutions in (21) as it is obvious which solutions decay as x approaches $\pm\infty$. For regions of finite extent, it is usually more convenient to use the hyperbolic expressions of (20). A general property of Laplace solutions are that they are oscillatory in one direction and decay in the perpendicular direction.

4-2-4 Spatially Periodic Excitation

A sheet in the $x=0$ plane has the imposed periodic potential, $V=V_0 \sin ay$ shown in Figure 4-4. In order to meet this boundary condition we use the solution of (21) with $k=a$. The potential must remain finite far away from the source so

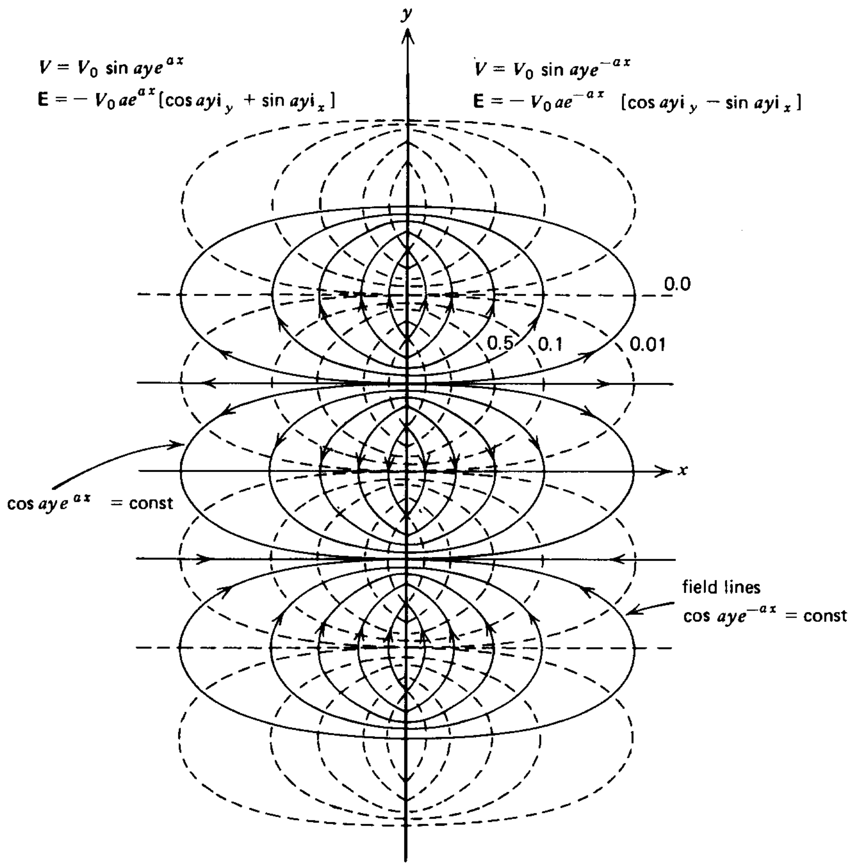


Figure 4-4 The potential and electric field decay away from an infinite sheet with imposed spatially periodic voltage. The field lines emanate from positive surface charge on the sheet and terminate on negative surface charge.

we write the solution separately for positive and negative x as

$$V=\begin{cases} V_0 \sin ay\, e^{-ax}, & x\geq 0 \\ V_0 \sin ay\, e^{ax}, & x\leq 0 \end{cases} \tag{22}$$

where we picked the amplitude coefficients to be continuous and match the excitation at $x=0$. The electric field is then

$$\mathbf{E}=-\nabla V=\begin{cases} -V_0 a\, e^{-ax}[\cos ay\mathbf{i}_y-\sin ay\mathbf{i}_x], & x>0 \\ -V_0 a\, e^{ax}[\cos ay\mathbf{i}_y+\sin ay\mathbf{i}_x], & x<0 \end{cases} \tag{23}$$

The surface charge density on the sheet is given by the discontinuity in normal component of $\mathbf{D}$ across the sheet:

$$\begin{aligned} \sigma_f(x=0) &= \varepsilon[E_x(x=0_+)-E_x(x=0_-)] \\ &= 2\varepsilon V_0 a \sin ay \end{aligned} \tag{24}$$

The field lines drawn in Figure 4-4 obey the equation

$$\frac{dy}{dx}=\frac{E_y}{E_x}=\mp\cot ay \Rightarrow \cos ay\, e^{\mp ax}=\text{const}\quad \begin{cases} x>0 \\ x<0 \end{cases} \tag{25}$$

4-2-5 Rectangular Harmonics

When excitations are not sinusoidally periodic in space, they can be made so by expressing them in terms of a trigonometric Fourier series. Any periodic function of y can be expressed as an infinite sum of sinusoidal terms as

$$f(y)=\tfrac{1}{2}b_0+\sum_{n=1}^{\infty}\left(a_n \sin\frac{2n\pi y}{\lambda}+b_n\cos\frac{2n\pi y}{\lambda}\right) \tag{26}$$

where λ is the fundamental period of $f(y)$.

The Fourier coefficients a_n are obtained by multiplying both sides of the equation by $\sin(2p\pi y/\lambda)$ and integrating over a period. Since the parameter p is independent of the index n, we may bring the term inside the summation on the right hand side. Because the trigonometric functions are orthogonal to one another, they integrate to zero except when the function multiplies itself:

$$\begin{aligned} &\int_0^{\lambda}\sin\frac{2p\pi y}{\lambda}\sin\frac{2n\pi y}{\lambda}\,dy=\begin{cases}0, & p\neq n\\ \lambda/2, & p=n\end{cases}\\ &\int_0^{\lambda}\sin\frac{2p\pi y}{\lambda}\cos\frac{2n\pi y}{\lambda}\,dy=0\end{aligned} \tag{27}$$

Every term in the series for $n\neq p$ integrates to zero. Only the term for $n=p$ is nonzero so that

$$a_p=\frac{2}{\lambda}\int_0^{\lambda} f(y)\sin\frac{2p\pi y}{\lambda}\,dy \tag{28}$$

To obtain the coefficients b_n, we similarly multiply by $\cos(2p\pi y/\lambda)$ and integrate over a period:

$$b_p=\frac{2}{\lambda}\int_0^{\lambda} f(y)\cos\frac{2p\pi y}{\lambda}\,dy \tag{29}$$

Consider the conducting rectangular box of infinite extent in the x and z directions and of width d in the y direction shown in Figure 4-5. The potential along the $x=0$ edge is V_0 while all other surfaces are grounded at zero potential. Any periodic function can be used for $f(y)$ if over the interval $0\leq y\leq d, f(y)$ has the properties

$$f(y)=V_0,\ 0<y<d;\ f(y=0)=f(y=d)=0 \tag{30}$$

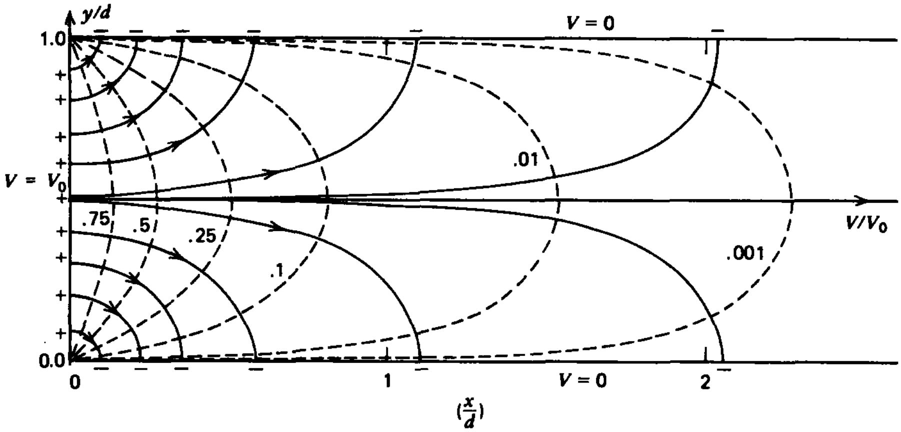


Figure 4-5 An open conducting box of infinite extent in the x and z directions and of finite width d in the y direction, has zero potential on all surfaces except the closed end at $x=0$, where $V=V_0$.

In particular, we choose the periodic square wave function with $\lambda = 2d$ shown in Figure 4-6 so that performing the integrations in (28) and (29) yields

$$a_p = -\frac{2V_0}{p\pi}(\cos p\pi - 1)$$
$$= \begin{cases} 0, & p \text{ even} \\ 4V_0/p\pi, & p \text{ odd} \end{cases} \tag{31}$$
$$b_p = 0$$

Thus the constant potential at $x=0$ can be written as the Fourier sine series

$$V(x=0) = V_0 = \frac{4V_0}{\pi}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin(n\pi y/d)}{n} \tag{32}$$

In Figure 4-6 we plot various partial sums of the Fourier series to show that as the number of terms taken becomes large, the series approaches the constant value V_0 except for the Gibbs overshoot of about 18% at $y=0$ and $y=d$ where the function is discontinuous.

The advantage in writing V_0 in a Fourier sine series is that each term in the series has a similar solution as found in (22) where the separation constant for each term is $k_n = n\pi/d$ with associated amplitude $4V_0/(n\pi)$.

The solution is only nonzero for $x>0$ so we immediately write down the total potential solution as

$$V(x, y) = \frac{4V_0}{\pi}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n}\sin\frac{n\pi y}{d}\, e^{-n\pi x/d} \tag{33}$$

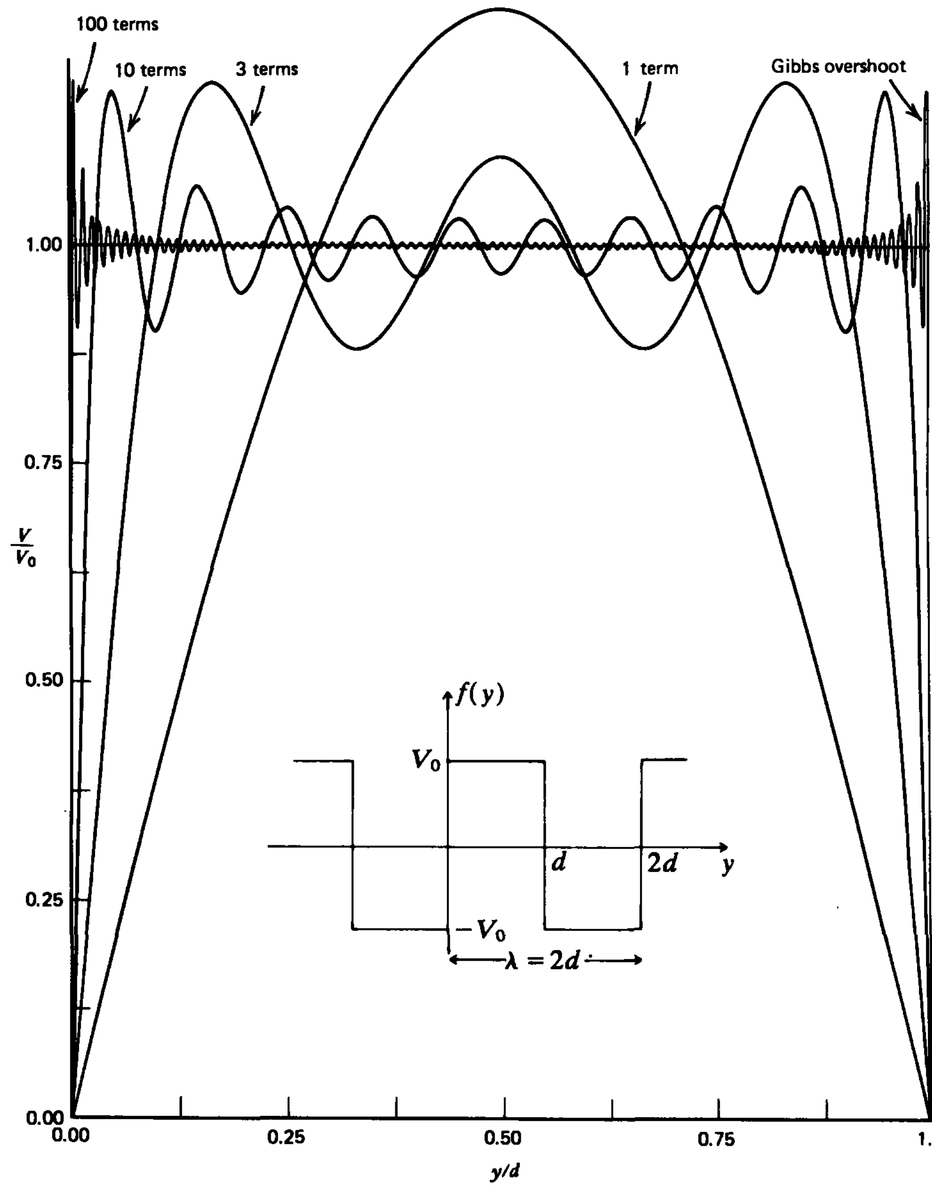


Figure 4-6 Fourier series expansion of the imposed constant potential along the $x = 0$ edge in Figure 4-5 for various partial sums. As the number of terms increases, the series approaches a constant except at the boundaries where the discontinuity in potential gives rise to the Gibbs phenomenon of an 18% overshoot with narrow width.

The electric field is then

$$\mathbf{E} = -\nabla V = -\frac{4V_0}{d} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left(-\sin\frac{n\pi y}{d}\mathbf{i}_x + \cos\frac{n\pi y}{d}\mathbf{i}_y\right) e^{-n\pi x/d} \tag{34}$$

The field and equipotential lines are sketched in Figure 4-5. Note that for $x \gg d$, the solution is dominated by the first harmonic. Far from a source, Laplacian solutions are insensitive to the details of the source geometry.

4-2-6 Three-Dimensional Solutions

If the potential depends on the three coordinates (x, y, z), we generalize our approach by trying a product solution of the form

$$V(x, y, z) = X(x)\, Y(y)\, Z(z) \tag{35}$$

which, when substituted into Laplace's equation, yields after division through by XYZ

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0 \tag{36}$$

three terms each wholly a function of a single coordinate so that each term again must separately equal a constant:

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k_x^2, \qquad \frac{1}{Y}\frac{d^2Y}{dy^2} = -k_y^2, \qquad \frac{1}{Z}\frac{d^2Z}{dz^2} = k_z^2 = k_x^2 + k_y^2 \tag{37}$$

We change the sign of the separation constant for the z dependence as the sum of separation constants must be zero. The solutions for nonzero separation constants are

$$\begin{aligned} X &= A_1 \sin k_x x + A_2 \cos k_x x \\ Y &= B_1 \sin k_y y + B_2 \cos k_y y \\ Z &= C_1 \sinh k_z z + C_2 \cosh k_z z = D_1\, e^{k_z z} + D_2\, e^{-k_z z} \end{aligned} \tag{38}$$

The solutions are written as if k_x, k_y, and k_z are real so that the x and y dependence is trigonometric while the z dependence is hyperbolic or equivalently exponential. However, k_x, k_y, or k_z may be imaginary converting hyperbolic functions to trigonometric and vice versa. Because the squares of the separation constants must sum to zero at least one of the solutions in (38) must be trigonometric and one must be hyperbolic. The remaining solution may be either trigonometric or hyperbolic depending on the boundary conditions. If the separation constants are all zero, in addition to the solutions of (6) we have the similar addition

$$Z = e_1 z + f_1 \tag{39}$$

4-3 SEPARATION OF VARIABLES IN CYLINDRICAL GEOMETRY

Product solutions to Laplace's equation in cylindrical coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{\partial^2 V}{\partial z^2}=0 \tag{1}$$

also separate into solvable ordinary differential equations.

4-3-1 Polar Solutions

If the system geometry does not vary with z, we try a solution that is a product of functions which only depend on the radius r and angle ϕ:

$$V(r,\phi)=R(r)\Phi(\phi) \tag{2}$$

which when substituted into (1) yields

$$\frac{\Phi}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+\frac{R}{r^2}\frac{d^2\Phi}{d\phi^2}=0 \tag{3}$$

This assumed solution is convenient when boundaries lay at a constant angle of ϕ or have a constant radius, as one of the functions in (2) is then constant along the boundary.

For (3) to separate, each term must only be a function of a single variable, so we multiply through by $r^2/R\Phi$ and set each term equal to a constant, which we write as n^2:

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)=n^2,\qquad \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-n^2 \tag{4}$$

The solution for Φ is easily solved as

$$\Phi=\begin{cases}A_1\sin n\phi+A_2\cos n\phi, & n\neq 0\\ B_1\phi+B_2, & n=0\end{cases} \tag{5}$$

The solution for the radial dependence is not as obvious. However, if we can find two independent solutions by any means, including guessing, the total solution is uniquely given as a linear combination of the two solutions. So, let us try a power-law solution of the form

$$R=A\,r^p \tag{6}$$

which when substituted into (4) yields

$$p^2=n^2\Rightarrow p=\pm n \tag{7}$$

For $n \neq 0$, (7) gives us two independent solutions. When $n = 0$ we refer back to (4) to solve

$$r\frac{dR}{dr} = \text{const} \Rightarrow R = D_1 \ln r + D_2 \tag{8}$$

so that the solutions are

$$R = \begin{cases} C_1 r^n + C_2 r^{-n}, & n \neq 0 \\ D_1 \ln r + D_2, & n = 0 \end{cases} \tag{9}$$

We recognize the $n = 0$ solution for the radial dependence as the potential due to a line charge. The $n = 0$ solution for the ϕ dependence shows that the potential increases linearly with angle. Generally n can be any complex number, although in usual situations where the domain is periodic and extends over the whole range $0 \leq \phi \leq 2\pi$, the potential at $\phi = 2\pi$ must equal that at $\phi = 0$ since they are the same point. This requires that n be an integer.

EXAMPLE 4-1 SLANTED CONDUCTING PLANES

Two planes of infinite extent in the z direction at an angle α to one another, as shown in Figure 4-7, are at a potential difference v. The planes do not intersect but come sufficiently close to one another that fringing fields at the electrode ends may be neglected. The electrodes extend from $r = a$ to $r = b$. What is the approximate capacitance per unit length of the structure?

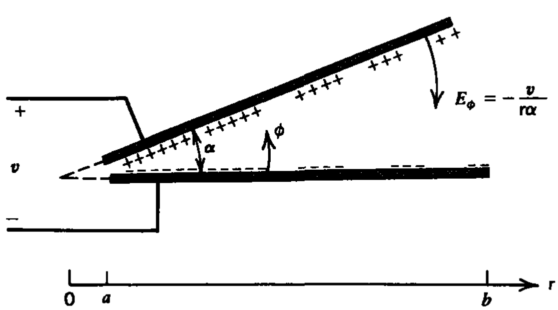


Figure 4-7 Two conducting planes at angle α stressed by a voltage v have a ϕ-directed electric field.

SOLUTION

We try the $n=0$ solution of (5) with no radial dependence as

$$V=B_1\phi+B_2$$

The boundary conditions impose the constraints

$$V(\phi=0)=0, \qquad V(\phi=\alpha)=v \Rightarrow V=v\phi/\alpha$$

The electric field is

$$E_\phi=-\frac{1}{r}\frac{dV}{d\phi}=-\frac{v}{r\alpha}$$

The surface charge density on the upper electrode is then

$$\sigma_f(\phi=\alpha)=-\varepsilon E_\phi(\phi=\alpha)=\frac{\varepsilon v}{r\alpha}$$

with total charge per unit length

$$\lambda(\phi=\alpha)=\int_{r=a}^{b}\sigma_f(\phi=\alpha)\,dr=\frac{\varepsilon v}{\alpha}\ln\frac{b}{a}$$

so that the capacitance per unit length is

$$C=\frac{\lambda}{v}=\frac{\varepsilon\ln(b/a)}{\alpha}$$

4-3-2 Cylinder in a Uniform Electric Field

(a) Field Solutions

An infinitely long cylinder of radius a, permittivity ε_2, and Ohmic conductivity σ_2 is placed within an infinite medium of permittivity ε_1 and conductivity σ_1. A uniform electric field at infinity $\mathbf{E}=E_0\mathbf{i}_x$ is suddenly turned on at $t=0$. This problem is analogous to the series lossy capacitor treated in Section 3-6-3. As there, we will similarly find that:

(i) At $t=0$ the solution is the same as for two lossless dielectrics, independent of the conductivities, with no interfacial surface charge, described by the boundary condition

$$\sigma_f(r=a)=D_r(r=a_+)-D_r(r=a_-)=0$$
$$\Rightarrow \varepsilon_1E_r(r=a_+)=\varepsilon_2E_r(r=a_-) \quad (10)$$

(ii) As $t\to\infty$, the steady-state solution depends only on the conductivities, with continuity of normal current at the cylinder interface,

$$J_r(r=a_+)=J_r(r=a_-)\Rightarrow \sigma_1 E_r(r=a_+)=\sigma_2 E_r(r=a_-) \tag{11}$$

(iii) The time constant describing the transition from the initial to steady-state solutions will depend on some weighted average of the ratio of permittivities to conductivities.

To solve the general transient problem we must find the potential both inside and outside the cylinder, joining the solutions in each region via the boundary conditions at $r=a$.

Trying the nonzero n solutions of (5) and (9), n must be an integer as the potential at $\phi=0$ and $\phi=2\pi$ must be equal, since they are the same point. For the most general case, an infinite series of terms is necessary, superposing solutions with $n=1,2,3,4,\cdots$. However, because of the form of the uniform electric field applied at infinity, expressed in cylindrical coordinates as

$$\mathbf{E}(r\to\infty)=E_0\mathbf{i}_x=E_0[\mathbf{i}_r\cos\phi-\mathbf{i}_\phi\sin\phi] \tag{12}$$

we can meet all the boundary conditions using only the $n=1$ solution.

Keeping the solution finite at $r=0$, we try solutions of the form

$$V(r,\phi)=\begin{cases}A(t)r\cos\phi, & r\le a\\ [B(t)r+C(t)/r]\cos\phi, & r\ge a\end{cases} \tag{13}$$

with associated electric field

$$\mathbf{E}=-\nabla V=\begin{cases}-A(t)[\cos\phi\,\mathbf{i}_r-\sin\phi\,\mathbf{i}_\phi]=-A(t)\mathbf{i}_x, & r<a\\ -[B(t)-C(t)/r^2]\cos\phi\,\mathbf{i}_r & \\ \quad +[B(t)+C(t)/r^2]\sin\phi\,\mathbf{i}_\phi, & r>a\end{cases} \tag{14}$$

We do not consider the $\sin\phi$ solution of (5) in (13) because at infinity the electric field would have to be y directed:

$$V=Dr\sin\phi\Rightarrow\mathbf{E}=-\nabla V=-D[\mathbf{i}_r\sin\phi+\mathbf{i}_\phi\cos\phi]=-D\mathbf{i}_y \tag{15}$$

The electric field within the cylinder is x directed. The solution outside is in part due to the imposed x-directed uniform field, so that as $r\to\infty$ the field of (14) must approach (12), requiring that $B(t)=-E_0$. The remaining contribution to the external field is equivalent to a two-dimensional line dipole (see Problem 3.1), with dipole moment per unit length:

$$p_x=\lambda d=2\pi\varepsilon C(t) \tag{16}$$

The other time-dependent amplitudes $A(t)$ and $C(t)$ are found from the following additional boundary conditions:

(i) the potential is continuous at $\mathrm{r}=a$, which is the same as requiring continuity of the tangential component of $\mathbf{E}$:

$$V(\mathrm{r}=a_+)=V(\mathrm{r}=a_-)\Rightarrow E_\phi(\mathrm{r}=a_-)=E_\phi(\mathrm{r}=a_+)$$
$$\Rightarrow Aa=Ba+C/a \tag{17}$$

(ii) charge must be conserved on the interface:

$$J_\mathrm{r}(\mathrm{r}=a_+)-J_\mathrm{r}(\mathrm{r}=a_-)+\frac{\partial\sigma_f}{\partial t}=0$$
$$\Rightarrow \sigma_1E_\mathrm{r}(\mathrm{r}=a_+)-\sigma_2E_\mathrm{r}(\mathrm{r}=a_-)$$
$$+\frac{\partial}{\partial t}[\varepsilon_1E_\mathrm{r}(\mathrm{r}=a_+)-\varepsilon_2E_\mathrm{r}(\mathrm{r}=a_-)]=0 \tag{18}$$

In the steady state, (18) reduces to (11) for the continuity of normal current, while for $t=0$ the time derivative must be noninfinite so σ_f is continuous and thus zero as given by (10).

Using (17) in (18) we obtain a single equation in $C(t)$:

$$\frac{dC}{dt}+\left(\frac{\sigma_1+\sigma_2}{\varepsilon_1+\varepsilon_2}\right)C=\frac{-a^2}{\varepsilon_1+\varepsilon_2}\left(E_0(\sigma_1-\sigma_2)+(\varepsilon_1-\varepsilon_2)\frac{dE_0}{dt}\right) \tag{19}$$

Since E_0 is a step function in time, the last term on the right-hand side is an impulse function, which imposes the initial condition

$$C(t=0)=-a^2\frac{(\varepsilon_1-\varepsilon_2)}{\varepsilon_1+\varepsilon_2}E_0 \tag{20}$$

so that the total solution to (19) is

$$C(t)=a^2E_0\left(\frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2}+\frac{2(\sigma_1\varepsilon_2-\sigma_2\varepsilon_1)}{(\sigma_1+\sigma_2)(\varepsilon_1+\varepsilon_2)}e^{-t/\tau}\right),\quad \tau=\frac{\varepsilon_1+\varepsilon_2}{\sigma_1+\sigma_2} \tag{21}$$

The interfacial surface charge is

$$\sigma_f(\mathrm{r}=\mathrm{a},t)=\varepsilon_1E_\mathrm{r}(\mathrm{r}=\mathrm{a}_+)-\varepsilon_2E_\mathrm{r}(\mathrm{r}=a_-)$$
$$=\left[-\varepsilon_1\left(B-\frac{C}{a^2}\right)+\varepsilon_2A\right]\cos\phi$$
$$=\left[(\varepsilon_1-\varepsilon_2)E_0+(\varepsilon_1+\varepsilon_2)\frac{C}{a^2}\right]\cos\phi$$
$$=\frac{2(\sigma_2\varepsilon_1-\sigma_1\varepsilon_2)}{\sigma_1+\sigma_2}E_0[1-e^{-t/\tau}]\cos\phi \tag{22}$$

The upper part of the cylinder $(-\pi/2 \leq \phi \leq \pi/2)$ is charged of one sign while the lower half $(\pi/2 \leq \phi \leq \frac{3}{2}\pi)$ is charged with the opposite sign, the net charge on the cylinder being zero. The cylinder is uncharged at each point on its surface if the relaxation times in each medium are the same, $\varepsilon_1/\sigma_1 = \varepsilon_2/\sigma_2$

The solution for the electric field at $t = 0$ is

$$\mathbf{E}(t=0) = \begin{cases} \dfrac{2\varepsilon_1 E_0}{\varepsilon_1+\varepsilon_2}[\cos\phi\mathbf{i}_r - \sin\phi\mathbf{i}_\phi] = \dfrac{2\varepsilon_1 E_0}{\varepsilon_1+\varepsilon_2}\mathbf{i}_x, & r<a \\ E_0\left[\left(1+\dfrac{a^2}{r^2}\dfrac{\varepsilon_2-\varepsilon_1}{\varepsilon_1+\varepsilon_2}\right)\cos\phi\mathbf{i}_r \right. & \\ \left. -\left(1-\dfrac{a^2}{r^2}\dfrac{\varepsilon_2-\varepsilon_1}{\varepsilon_1+\varepsilon_2}\right)\sin\phi\mathbf{i}_\phi\right], & r>a \end{cases} \tag{23}$$

The field inside the cylinder is in the same direction as the applied field, and is reduced in amplitude if $\varepsilon_2 > \varepsilon_1$ and increased in amplitude if $\varepsilon_2 < \varepsilon_1$, up to a limiting factor of two as ε_1 becomes large compared to ε_2. If $\varepsilon_2 = \varepsilon_1$, the solution reduces to the uniform applied field everywhere.

The dc steady-state solution is identical in form to (23) if we replace the permittivities in each region by their conductivities;

$$\mathbf{E}(t\to\infty) = \begin{cases} \dfrac{2\sigma_1 E_0}{\sigma_1+\sigma_2}[\cos\phi\mathbf{i}_r - \sin\phi\mathbf{i}_\phi] = \dfrac{2\sigma_1 E_0}{\sigma_1+\sigma_2}\mathbf{i}_x, & r<a \\ E_0\left[\left(1+\dfrac{a^2}{r^2}\dfrac{\sigma_2-\sigma_1}{\sigma_1+\sigma_2}\right)\cos\phi\mathbf{i}_r \right. & \\ \left. -\left(1-\dfrac{a^2}{r^2}\dfrac{\sigma_2-\sigma_1}{\sigma_1+\sigma_2}\right)\sin\phi\mathbf{i}_\phi\right], & r>a \end{cases} \tag{24}$$

(b) Field Line Plotting

Because the region outside the cylinder is charge free, we know that $\nabla \cdot \mathbf{E} = 0$. From the identity derived in Section 1-5-4*b*, that the divergence of the curl of a vector is zero, we thus know that the polar electric field with no z component can be expressed in the form

$$\begin{aligned} \mathbf{E}(r,\phi) &= \nabla \times \Sigma(r,\phi)\mathbf{i}_z \\ &= \frac{1}{r}\frac{\partial \Sigma}{\partial \phi}\mathbf{i}_r - \frac{\partial \Sigma}{\partial r}\mathbf{i}_\phi \end{aligned} \tag{25}$$

where Σ is called the stream function. Note that the stream function vector is in the direction perpendicular to the electric field so that its curl has components in the same direction as the field.

Along a field line, which is always perpendicular to the equipotential lines,

$$\frac{dr}{r\,d\phi}=\frac{E_r}{E_\phi}=-\frac{1}{r}\frac{\partial\Sigma/\partial\phi}{\partial\Sigma/\partial r} \tag{26}$$

By cross multiplying and grouping terms on one side of the equation, (26) reduces to

$$d\Sigma=\frac{\partial\Sigma}{\partial r}dr+\frac{\partial\Sigma}{\partial\phi}d\phi=0\Rightarrow\Sigma=\text{const} \tag{27}$$

Field lines are thus lines of constant Σ.

For the steady-state solution of (24), outside the cylinder

$$\begin{aligned}\frac{1}{r}\frac{\partial\Sigma}{\partial\phi}&=E_r=E_0\left(1+\frac{a^2}{r^2}\frac{\sigma_2-\sigma_1}{\sigma_1+\sigma_2}\right)\cos\phi\\-\frac{\partial\Sigma}{\partial r}&=E_\phi=-E_0\left(1-\frac{a^2}{r^2}\frac{\sigma_2-\sigma_1}{\sigma_1+\sigma_2}\right)\sin\phi\end{aligned} \tag{28}$$

we find by integration that

$$\Sigma=E_0\left(r+\frac{a^2}{r}\frac{\sigma_2-\sigma_1}{\sigma_1+\sigma_2}\right)\sin\phi \tag{29}$$

The steady-state field and equipotential lines are drawn in Figure 4-8 when the cylinder is perfectly conducting ($\sigma_2\to\infty$) or perfectly insulating ($\sigma_2=0$).

If the cylinder is highly conducting, the internal electric field is zero with the external electric field incident radially, as drawn in Figure 4-8*a*. In contrast, when the cylinder is perfectly insulating, the external field lines must be purely tangential to the cylinder as the incident normal current is zero, and the internal electric field has double the strength of the applied field, as drawn in Figure 4-8*b*.

4-3-3 Three-Dimensional Solutions

If the electric potential depends on all three coordinates, we try a product solution of the form

$$V(r,\phi,z)=R(r)\Phi(\phi)Z(z) \tag{30}$$

which when substituted into Laplace's equation yields

$$\frac{Z\Phi}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+\frac{RZ}{r^2}\frac{d^2\Phi}{d\phi^2}+R\Phi\frac{d^2Z}{dz^2}=0 \tag{31}$$

We now have a difficulty, as we cannot divide through by a factor to make each term a function only of a single variable.

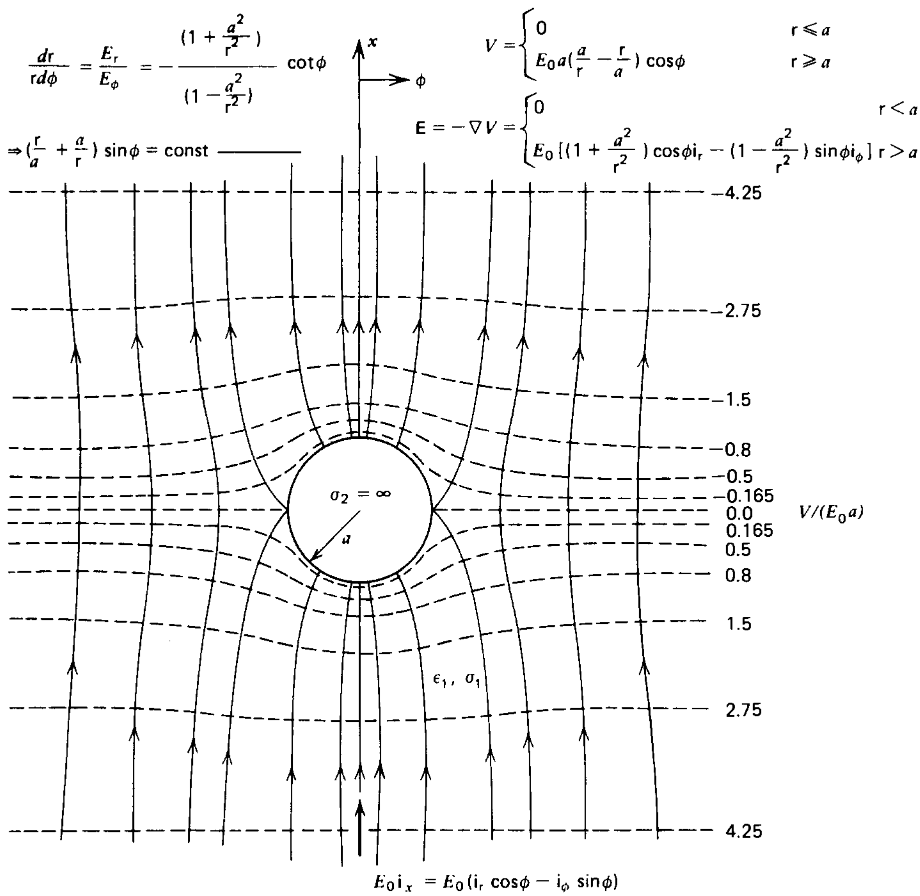


Figure 4-8 Steady-state field and equipotential lines about a (*a*) perfectly conducting or (*b*) perfectly insulating cylinder in a uniform electric field.

However, by dividing through by $V = R\Phi Z$,

$$\underbrace{\frac{1}{Rr}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{r^2\Phi}\frac{d^2\Phi}{d\phi^2}}_{-k^2} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{k^2} = 0 \tag{32}$$

we see that the first two terms are functions of r and ϕ while the last term is only a function of z. This last term must therefore equal a constant:

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = k^2 \Rightarrow Z = \begin{cases} A_1 \sinh kz + A_2 \cosh kz, & k \neq 0 \\ A_3 z + A_4, & k = 0 \end{cases} \tag{33}$$

$$V=\begin{cases}-2E_0 r\cos\phi & r\le a\\ -E_0 a\left(\frac{a}{r}+\frac{r}{a}\right)\cos\phi & r\ge a\end{cases}$$

$$\mathbf{E}=-\nabla V=\begin{cases}2E_0(\cos\phi\,\mathbf{i}_r-\sin\phi\,\mathbf{i}_\phi)=2E_0\,\mathbf{i}_x & r<a\\ E_0\left[\left(1-\frac{a^2}{r^2}\right)\cos\phi\,\mathbf{i}_r-\left(1+\frac{a^2}{r^2}\right)\sin\phi\,\mathbf{i}_\phi\right] & r>a\end{cases}$$

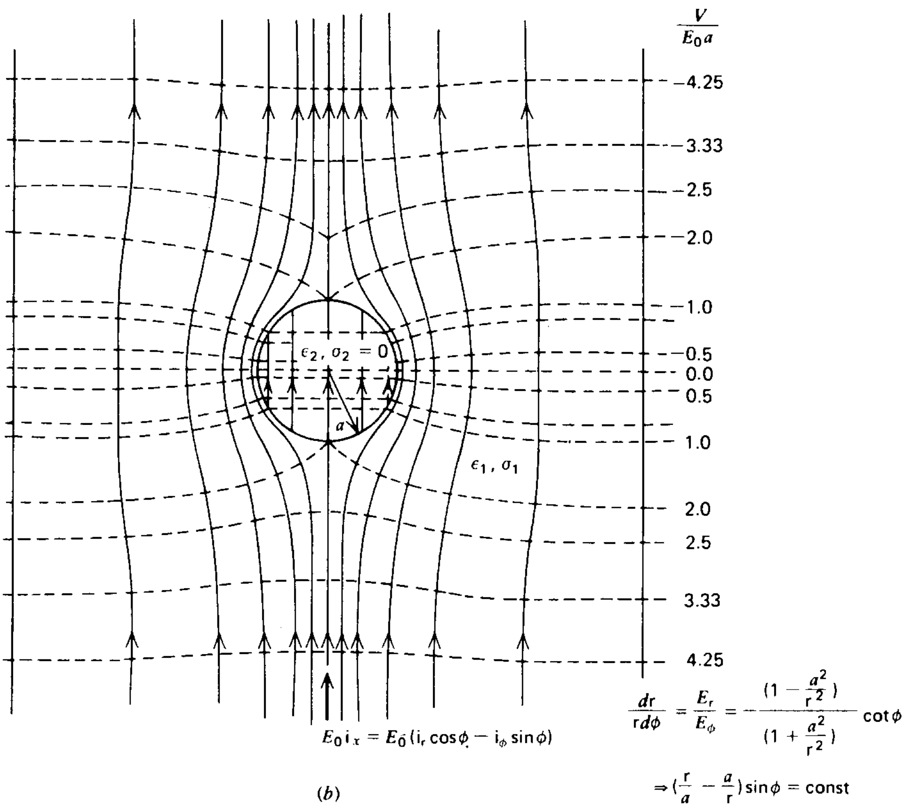


Figure 4-8*b*

The first two terms in (32) must now sum to $-k^2$ so that after multiplying through by r^2 we have

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+k^2r^2+\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=0 \tag{34}$$

Now again the first two terms are only a function of r, while the last term is only a function of ϕ so that (34) again separates:

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+k^2r^2=n^2,\qquad \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-n^2 \tag{35}$$

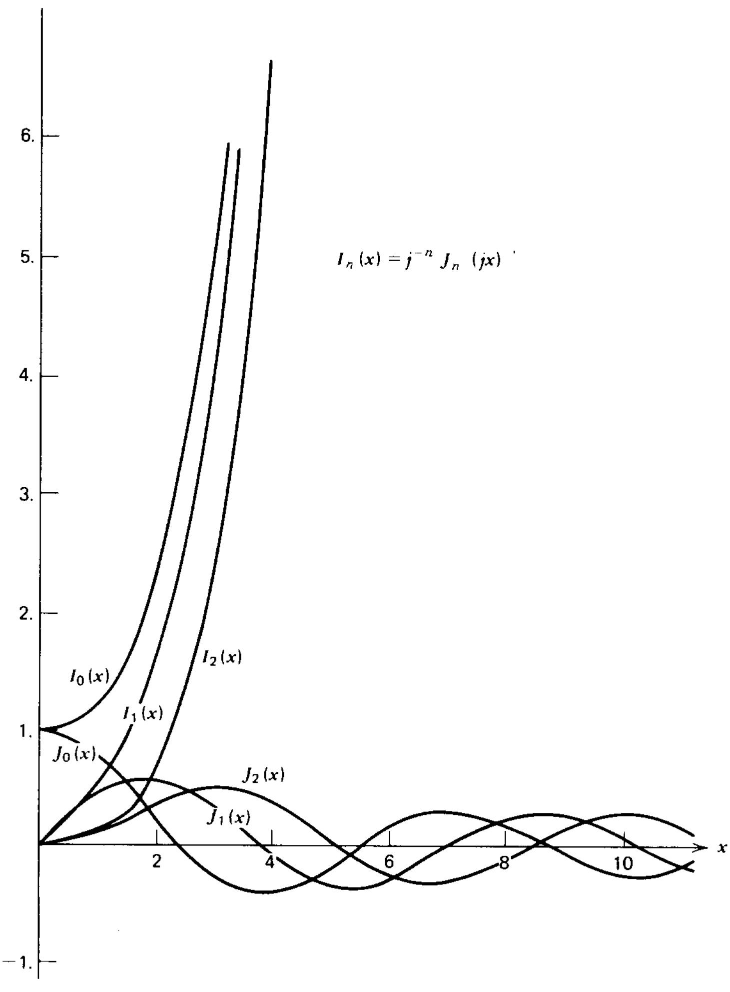
where n^2 is the second separation constant. The angular dependence thus has the same solutions as for the two-dimensional case

$$\Phi=\begin{cases} B_1 \sin n\phi + B_2 \cos n\phi, & n \neq 0 \\ B_3\phi + B_4, & n=0 \end{cases} \tag{36}$$

The resulting differential equation for the radial dependence

$$\mathrm{r}\frac{d}{d\mathrm{r}}\left(\mathrm{r}\frac{d\mathrm{R}}{d\mathrm{r}}\right)+(k^2\mathrm{r}^2-n^2)\mathrm{R}=0 \tag{37}$$

is Bessel's equation and for nonzero k has solutions in terms



(a)

Figure 4-9 The Bessel functions (a) $J_n(x)$ and $I_n(x)$, and (b) $Y_n(x)$ and $K_n(x)$.

of tabulated functions:

$$R=\begin{cases} C_1 J_n(kr)+C_2 Y_n(kr), & k\neq 0 \\ C_3 r^n + C_4 r^{-n}, & k=0, \quad n\neq 0 \\ C_5 \ln r + C_6, & k=0, \quad n=0 \end{cases} \tag{38}$$

where J_n is called a Bessel function of the first kind of order n and Y_n is called the nth-order Bessel function of the second kind. When $n=0$, the Bessel functions are of zero order while if $k=0$ the solutions reduce to the two-dimensional solutions of (9).

Some of the properties and limiting values of the Bessel functions are illustrated in Figure 4-9. Remember that k

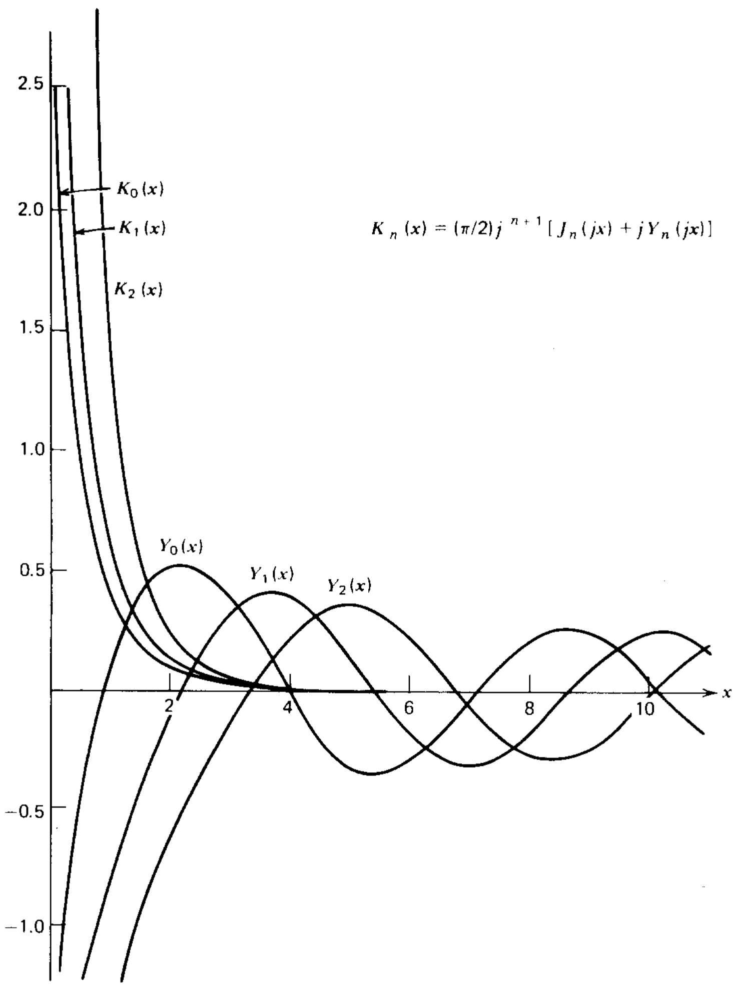


Figure 4-9*b*

can also be purely imaginary as well as real. When k is real so that the z dependence is hyperbolic or equivalently exponential, the Bessel functions are oscillatory while if k is imaginary so that the axial dependence on z is trigonometric, it is convenient to define the nonoscillatory modified Bessel functions as

$$\begin{aligned} I_n(k\mathrm{r}) &= j^{-n} J_n(jk\mathrm{r}) \\ K_n(k\mathrm{r}) &= \frac{\pi}{2} j^{n+1}[J_n(jk\mathrm{r}) + jY_n(jk\mathrm{r})] \end{aligned} \tag{39}$$

As in rectangular coordinates, if the solution to Laplace's equation decays in one direction, it is oscillatory in the perpendicular direction.

4-3-4 High Voltage Insulator Bushing

The high voltage insulator shown in Figure 4-10 consists of a cylindrical disk with Ohmic conductivity σ supported by a perfectly conducting cylindrical post above a ground plane.*

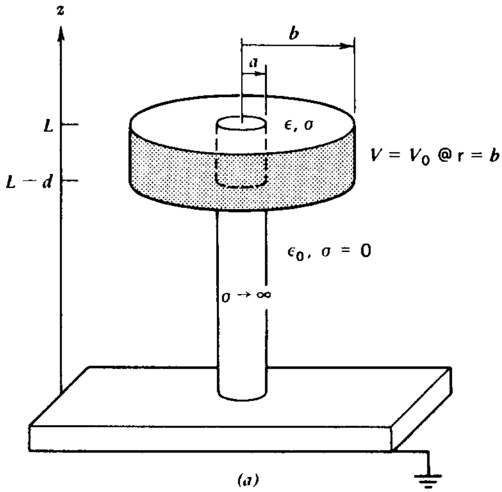
The plane at $z = 0$ and the post at $\mathrm{r} = a$ are at zero potential, while a constant potential is imposed along the circumference of the disk at $\mathrm{r} = b$. The region below the disk is free space so that no current can cross the surfaces at $z = L$ and $z = L - d$. Because the boundaries lie along surfaces at constant z or constant r we try the simple zero separation constant solutions in (33) and (38), which are independent of angle ϕ:

$$V(\mathrm{r}, z) = \begin{cases} A_1 z + B_1 z \ln \mathrm{r} + C_1 \ln \mathrm{r} + D_1, & L - d < z < L \\ A_2 z + B_2 z \ln \mathrm{r} + C_2 \ln \mathrm{r} + D_2, & 0 \le z \le L - d \end{cases} \tag{40}$$

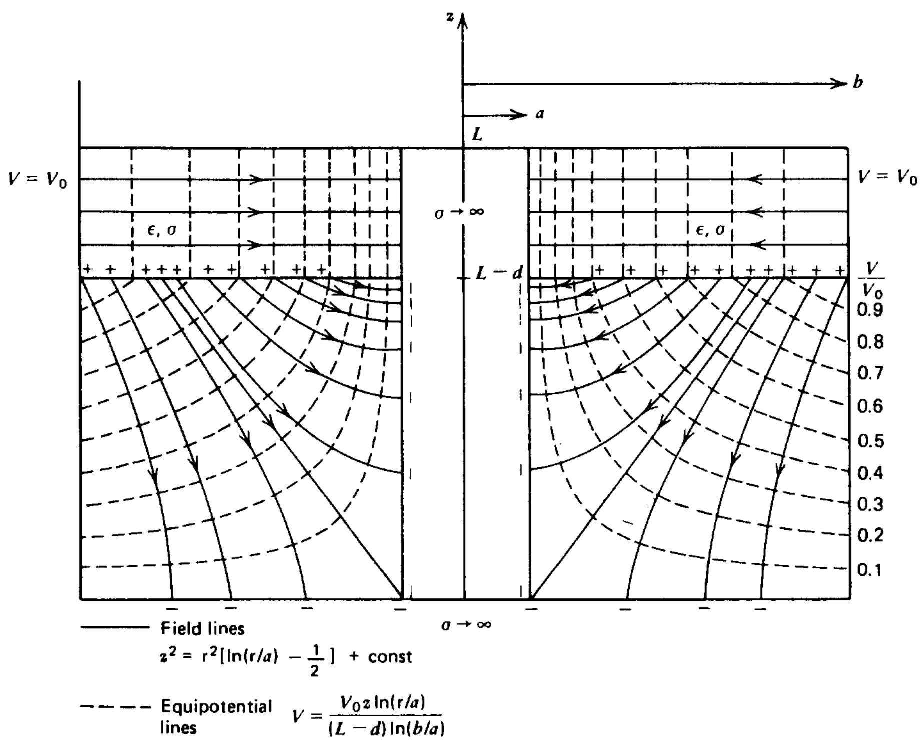
Applying the boundary conditions we relate the coefficients as

$$\begin{aligned} &V(z=0) = 0 \Rightarrow C_2 = D_2 = 0 \\ &V(\mathrm{r}=a) = 0 \Rightarrow \begin{cases} A_2 + B_2 \ln a = 0 \\ A_1 + B_1 \ln a = 0 \\ C_1 \ln a + D_1 = 0 \end{cases} \\ &V(\mathrm{r}=b, z > L-d) = V_0 \Rightarrow \begin{cases} A_1 + B_1 \ln b = 0 \\ C_1 \ln b + D_1 = V_0 \end{cases} \\ &V(z=(L-d)_-) = V(z=(L-d)_+) \Rightarrow (L-d)(A_2 + B_2 \ln \mathrm{r}) \\ &\qquad = (L-d)(A_1 + B_1 \ln \mathrm{r}) + C_1 \ln \mathrm{r} + D_1 \end{aligned} \tag{41}$$

* M. N. Horenstein, "Particle Contamination of High Voltage DC Insulators," PhD thesis, Massachusetts Institute of Technology, 1978.



(a)



(b)

Figure 4-10 (a) A finitely conducting disk is mounted upon a perfectly conducting cylindrical post and is placed on a perfectly conducting ground plane. (b) Field and equipotential lines.

which yields the values

$$A_1 = B_1 = 0, \qquad C_1 = \frac{V_0}{\ln(b/a)}, \qquad D_1 = -\frac{V_0 \ln a}{\ln(b/a)}$$
$$A_2 = -\frac{V_0 \ln a}{(L-d)\ln(b/a)}, \quad B_2 = \frac{V_0}{(L-d)\ln(b/a)}, \qquad C_2 = D_2 = 0 \tag{42}$$

The potential of (40) is then

$$V(\mathrm{r}, z) = \begin{cases} \dfrac{V_0 \ln(\mathrm{r}/a)}{\ln(b/a)}, & L-d \le z \le L \\ \dfrac{V_0 z \ln(\mathrm{r}/a)}{(L-d)\ln(b/a)}, & 0 \le z \le L-d \end{cases} \tag{43}$$

with associated electric field

$$\mathbf{E} = -\nabla V = \begin{cases} -\dfrac{V_0}{\mathrm{r}\ln(b/a)}\mathbf{i}_r, & L-d<z<L \\ -\dfrac{V_0}{(L-d)\ln(b/a)}\left(\ln\dfrac{\mathrm{r}}{a}\mathbf{i}_z + \dfrac{z}{\mathrm{r}}\mathbf{i}_r\right), & 0<z<L-d \end{cases} \tag{44}$$

The field lines in the free space region are

$$\frac{d\mathrm{r}}{dz} = \frac{E_r}{E_z} = \frac{z}{\mathrm{r}\ln(\mathrm{r}/a)} \Rightarrow z^2 = \mathrm{r}^2\left[\ln\frac{\mathrm{r}}{a} - \frac{1}{2}\right] + \text{const} \tag{45}$$

and are plotted with the equipotential lines in Figure 4-10b.

4-4 PRODUCT SOLUTIONS IN SPHERICAL GEOMETRY

In spherical coordinates, Laplace's equation is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0 \tag{1}$$

4-4-1 One-Dimensional Solutions

If the solution only depends on a single spatial coordinate, the governing equations and solutions for each of the three coordinates are

$$\text{(i)} \quad \frac{d}{dr}\left(r^2\frac{dV(r)}{dr}\right) = 0 \Rightarrow V(r) = \frac{A_1}{r} + A_2 \tag{2}$$

$$\text{(ii)}\quad \frac{d}{d\theta}\left(\sin\theta\frac{dV(\theta)}{d\theta}\right)=0 \Rightarrow V(\theta)=B_1\ln\left(\tan\frac{\theta}{2}\right)+B_2 \tag{3}$$

$$\text{(iii)}\quad \frac{d^2V(\phi)}{d\phi^2}=0 \Rightarrow V(\phi)=C_1\phi+C_2 \tag{4}$$

We recognize the radially dependent solution as the potential due to a point charge. The new solutions are those which only depend on θ or ϕ.

EXAMPLE 4-2 TWO CONES

Two identical cones with surfaces at angles $\theta=\alpha$ and $\theta=\pi-\alpha$ and with vertices meeting at the origin, are at a potential difference v, as shown in Figure 4-11. Find the potential and electric field.

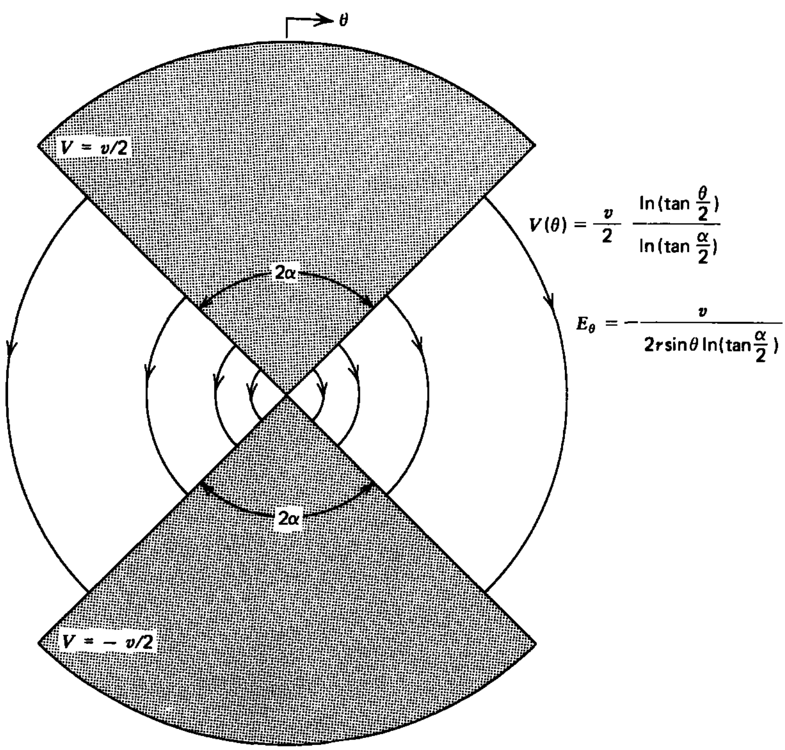


Figure 4-11 Two cones with vertices meeting at the origin are at a potential difference v.

SOLUTION

Because the boundaries are at constant values of θ, we try (3) as a solution:

$$V(\theta)=B_1 \ln [\tan (\theta/2)]+B_2$$

From the boundary conditions we have

$$V(\theta=\alpha)=\frac{v}{2}$$

$$V(\theta=\pi-\alpha)=\frac{-v}{2}\Rightarrow B_1=\frac{v}{2\ln [\tan (\alpha/2)]}, \qquad B_2=0$$

so that the potential is

$$V(\theta)=\frac{v \ln [\tan (\theta/2)]}{2 \ln [\tan (\alpha/2)]}$$

with electric field

$$\mathbf{E}=-\nabla V=\frac{-v}{2r \sin \theta \ln [\tan (\alpha/2)]}\mathbf{i}_\theta$$

4-4-2 Axisymmetric Solutions

If the solution has no dependence on the coordinate ϕ, we try a product solution

$$V(r, \theta)=R(r)\Theta(\theta) \qquad (5)$$

which when substituted into (1), after multiplying through by $r^2/R\Theta$, yields

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta \sin \theta}\frac{d}{d\theta}\left(\sin \theta \frac{d\Theta}{d\theta}\right)=0 \qquad (6)$$

Because each term is again only a function of a single variable, each term is equal to a constant. Anticipating the form of the solution, we choose the separation constant as $n(n+1)$ so that (6) separates to

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-n(n+1)R=0 \qquad (7)$$

$$\frac{d}{d\theta}\left(\sin \theta \frac{d\Theta}{d\theta}\right)+n(n+1)\Theta \sin \theta=0 \qquad (8)$$

For the radial dependence we try a power-law solution

$$R = Ar^p \tag{9}$$

which when substituted back into (7) requires

$$p(p+1) = n(n+1) \tag{10}$$

which has the two solutions

$$p = n, \qquad p = -(n+1) \tag{11}$$

When $n = 0$ we re-obtain the $1/r$ dependence due to a point charge.

To solve (8) for the θ dependence it is convenient to introduce the change of variable

$$\beta = \cos\theta \tag{12}$$

so that

$$\frac{d\Theta}{d\theta} = \frac{d\Theta}{d\beta}\frac{d\beta}{d\theta} = -\sin\theta\frac{d\Theta}{d\beta} = -(1-\beta^2)^{1/2}\frac{d\Theta}{d\beta} \tag{13}$$

Then (8) becomes

$$\frac{d}{d\beta}\left((1-\beta^2)\frac{d\Theta}{d\beta}\right) + n(n+1)\Theta = 0 \tag{14}$$

which is known as Legendre's equation. When n is an integer, the solutions are written in terms of new functions:

$$\Theta = B_n P_n(\beta) + C_n Q_n(\beta) \tag{15}$$

where the $P_n(\beta)$ are called Legendre polynomials of the first kind and are tabulated in Table 4-1. The Q_n solutions are called the Legendre functions of the second kind for which the first few are also tabulated in Table 4-1. Since all the Q_n are singular at $\theta = 0$ and $\theta = \pi$, where $\beta = \pm 1$, for all problems which include these values of angle, the coefficients C_n in (15) must be zero, so that many problems only involve the Legendre polynomials of first kind, $P_n(\cos\theta)$. Then using (9)–(11) and (15) in (5), the general solution for the potential with no ϕ dependence can be written as

$$V(r,\theta) = \sum_{n=0}^{\infty}(A_n r^n + B_n r^{-(n+1)})P_n(\cos\theta) \tag{16}$$

Table 4-1 Legendre polynomials of first and second kind

n	$P_n(\beta=\cos\theta)$	$Q_n(\beta=\cos\theta)$
0	1	$\frac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right)$
1	$\beta=\cos\theta$	$\frac{1}{2}\beta\ln\left(\frac{1+\beta}{1-\beta}\right)-1$
2	$\frac{1}{2}(3\beta^2-1)$ $=\frac{1}{2}(3\cos^2\theta-1)$	$\frac{1}{4}(3\beta^2-1)\ln\left(\frac{1+\beta}{1-\beta}\right)-\frac{3\beta}{2}$
3	$\frac{1}{2}(5\beta^3-3\beta)$ $=\frac{1}{2}(5\cos^3\theta-3\cos\theta)$	$\frac{1}{4}(5\beta^3-3\beta)\ln\left(\frac{1+\beta}{1-\beta}\right)-\frac{5}{2}\beta^2+\frac{2}{3}$
⋮		
m	$\frac{1}{2^m m!}\frac{d^m}{d\beta^m}(\beta^2-1)^m$	

4-4-3 Conducting Sphere in a Uniform Field

(a) Field Solution

A sphere of radius R, permittivity ε_2, and Ohmic conductivity σ_2 is placed within a medium of permittivity ε_1 and conductivity σ_1. A uniform dc electric field $E_0\mathbf{i}_z$ is applied at infinity. Although the general solution of (16) requires an infinite number of terms, the form of the uniform field at infinity in spherical coordinates,

$$\mathbf{E}(r\to\infty)=E_0\mathbf{i}_z=E_0(\mathbf{i}_r\cos\theta-\mathbf{i}_\theta\sin\theta) \tag{17}$$

suggests that all the boundary conditions can be met with just the $n=1$ solution:

$$V(r,\theta)=\begin{cases}Ar\cos\theta, & r\le R\\ (Br+C/r^2)\cos\theta, & r\ge R\end{cases} \tag{18}$$

We do not include the $1/r^2$ solution within the sphere ($r<R$) as the potential must remain finite at $r=0$. The associated

electric field is

$$\mathbf{E} = -\nabla V = \begin{cases} -A(\mathbf{i}_r \cos\theta - \mathbf{i}_\theta \sin\theta) = -A\mathbf{i}_z, & r<R \\ -(B-2C/r^3)\cos\theta\,\mathbf{i}_r + (B + C/r^3)\sin\theta\,\mathbf{i}_\theta, & r>R \end{cases} \tag{19}$$

The electric field within the sphere is uniform and z directed while the solution outside is composed of the uniform z-directed field, for as $r \to \infty$ the field must approach (17) so that $B = -E_0$, plus the field due to a point dipole at the origin, with dipole moment

$$p_z = 4\pi\varepsilon_1 C \tag{20}$$

Additional steady-state boundary conditions are the continuity of the potential at $r = R$ [equivalent to continuity of tangential $\mathbf{E}(r=R)$], and continuity of normal current at $r = R$,

$$\begin{aligned} &V(r=R_+) = V(r=R_-) \Rightarrow E_\theta(r=R_+) = E_\theta(r=R_-) \\ &\qquad \Rightarrow AR = BR + C/R^2 \\ &J_r(r=R_+) = J_r(r=R_-) \Rightarrow \sigma_1 E_r(r=R_+) = \sigma_2 E_r(r=R_-) \\ &\qquad \Rightarrow \sigma_1(B - 2C/R^3) = \sigma_2 A \end{aligned} \tag{21}$$

for which solutions are

$$A = -\frac{3\sigma_1}{2\sigma_1+\sigma_2}E_0, \qquad B = -E_0, \qquad C = \frac{(\sigma_2-\sigma_1)R^3}{2\sigma_1+\sigma_2}E_0 \tag{22}$$

The electric field of (19) is then

$$\mathbf{E} = \begin{cases} \dfrac{3\sigma_1 E_0}{2\sigma_1+\sigma_2}(\mathbf{i}_r\cos\theta - \mathbf{i}_\theta\sin\theta) = \dfrac{3\sigma_1 E_0}{2\sigma_1+\sigma_2}\mathbf{i}_z, & r<R \\ E_0\left[\left(1+\dfrac{2R^3(\sigma_2-\sigma_1)}{r^3(2\sigma_1+\sigma_2)}\right)\cos\theta\,\mathbf{i}_r \right. & \\ \qquad \left. -\left(1-\dfrac{R^3(\sigma_2-\sigma_1)}{r^3(2\sigma_1+\sigma_2)}\right)\sin\theta\,\mathbf{i}_\theta\right], & r>R \end{cases} \tag{23}$$

The interfacial surface charge is

$$\sigma_f(r=R) = \varepsilon_1 E_r(r=R_+) - \varepsilon_2 E_r(r=R_-) = \frac{3(\sigma_2\varepsilon_1 - \sigma_1\varepsilon_2)E_0}{2\sigma_1+\sigma_2}\cos\theta \tag{24}$$

which is of one sign on the upper part of the sphere and of opposite sign on the lower half of the sphere. The total charge on the entire sphere is zero. The charge is zero at every point on the sphere if the relaxation times in each region are equal:

$$\frac{\varepsilon_1}{\sigma_1} = \frac{\varepsilon_2}{\sigma_2} \tag{25}$$

The solution if both regions were lossless dielectrics with no interfacial surface charge, is similar in form to (23) if we replace the conductivities by their respective permittivities.

(b) Field Line Plotting

As we saw in Section 4-3-2*b* for a cylindrical geometry, the electric field in a volume charge-free region has no divergence, so that it can be expressed as the curl of a vector. For an axisymmetric field in spherical coordinates we write the electric field as

$$\begin{aligned}\mathbf{E}(r,\theta) &= \nabla \times \left(\frac{\Sigma(r,\theta)}{r\sin\theta}\mathbf{i}_\phi\right)\\ &= \frac{1}{r^2\sin\theta}\frac{\partial\Sigma}{\partial\theta}\mathbf{i}_r - \frac{1}{r\sin\theta}\frac{\partial\Sigma}{\partial r}\mathbf{i}_\theta\end{aligned} \tag{26}$$

Note again, that for a two-dimensional electric field, the stream function vector points in the direction orthogonal to both field components so that its curl has components in the same direction as the field. The stream function Σ is divided by $r\sin\theta$ so that the partial derivatives in (26) only operate on Σ.

The field lines are tangent to the electric field

$$\frac{dr}{r\,d\theta} = \frac{E_r}{E_\theta} = -\frac{1}{r}\frac{\partial\Sigma/\partial\theta}{\partial\Sigma/\partial r} \tag{27}$$

which after cross multiplication yields

$$d\Sigma = \frac{\partial\Sigma}{\partial r}dr + \frac{\partial\Sigma}{\partial\theta}d\theta = 0 \Rightarrow \Sigma = \text{const} \tag{28}$$

so that again Σ is constant along a field line.

For the solution of (23) outside the sphere, we relate the field components to the stream function using (26) as

$$\begin{aligned}E_r &= \frac{1}{r^2\sin\theta}\frac{\partial\Sigma}{\partial\theta} = E_0\left(1+\frac{2R^3(\sigma_2-\sigma_1)}{r^3(2\sigma_1+\sigma_2)}\right)\cos\theta\\ E_\theta &= -\frac{1}{r\sin\theta}\frac{\partial\Sigma}{\partial r} = -E_0\left(1-\frac{R^3(\sigma_2-\sigma_1)}{r^3(2\sigma_1+\sigma_2)}\right)\sin\theta\end{aligned} \tag{29}$$

so that by integration the stream function is

$$\Sigma = E_0\left(\frac{r^2}{2}+\frac{R^3(\sigma_2-\sigma_1)}{r(2\sigma_1+\sigma_2)}\right)\sin^2\theta \tag{30}$$

The steady-state field and equipotential lines are drawn in Figure 4-12 when the sphere is perfectly insulating ($\sigma_2=0$) or perfectly conducting ($\sigma_2\to\infty$).

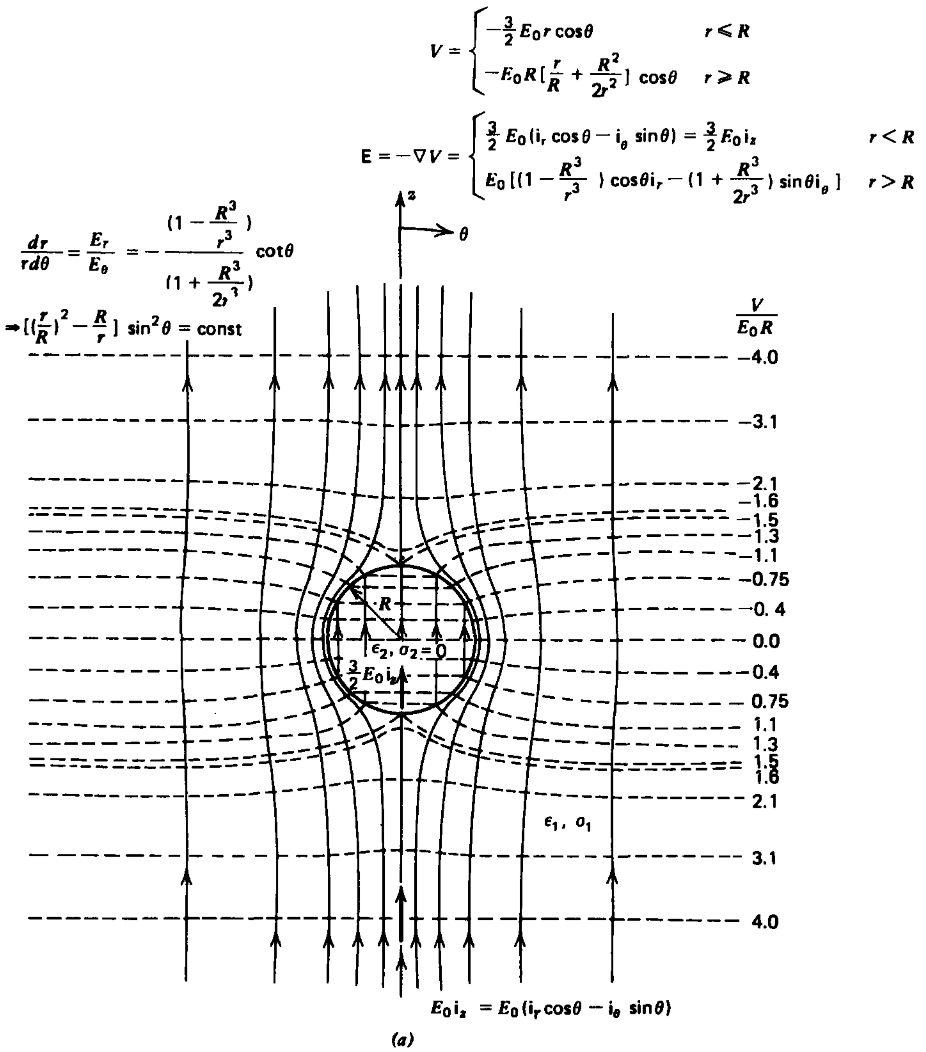


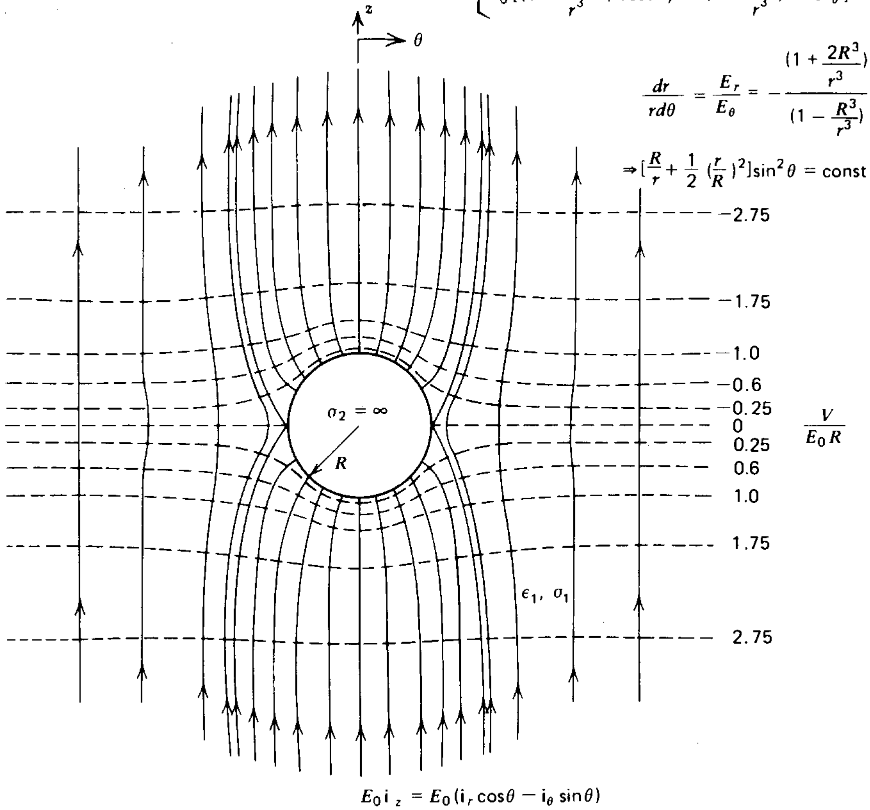
Figure 4-12 Steady-state field and equipotential lines about a (a) perfectly insulating or (b) perfectly conducting sphere in a uniform electric field.

$$V = \begin{cases} 0 & r \leq R \\ -E_0 R\left(\dfrac{r}{R} - \dfrac{R^2}{r^2}\right)\cos\theta & r \geq R \end{cases}$$

$$\mathbf{E} = -\nabla V = \begin{cases} 0 & r < R \\ E_0\left[\left(1 + \dfrac{2R^3}{r^3}\right)\cos\theta\, \mathbf{i}_r - \left(1 - \dfrac{R^3}{r^3}\right)\sin\theta\, \mathbf{i}_\theta\right] & r > R \end{cases}$$

$$\frac{dr}{r\,d\theta} = \frac{E_r}{E_\theta} = -\frac{\left(1 + \dfrac{2R^3}{r^3}\right)}{\left(1 - \dfrac{R^3}{r^3}\right)}\cot\theta$$

$$\Rightarrow \left[\frac{R}{r} + \frac{1}{2}\left(\frac{r}{R}\right)^2\right]\sin^2\theta = \text{const}$$



(*b*)

Figure 4-12*b*

If the conductivity of the sphere is less than that of the surrounding medium ($\sigma_2 < \sigma_1$), the electric field within the sphere is larger than the applied field. The opposite is true for ($\sigma_2 > \sigma_1$). For the insulating sphere in Figure 4-12*a*, the field lines go around the sphere as no current can pass through.

For the conducting sphere in Figure 4-12*b*, the electric field lines must be incident perpendicularly. This case is used as a polarization model, for as we see from (23) with $\sigma_2 \to \infty$, the external field is the imposed field plus the field of a point dipole with moment,

$$p_z = 4\pi\varepsilon_1 R^3 E_0 \tag{31}$$

If a dielectric is modeled as a dilute suspension of noninteracting, perfectly conducting spheres in free space with number density N, the dielectric constant is

$$\varepsilon = \frac{\varepsilon_0 E_0 + P}{E_0} = \frac{\varepsilon_0 E_0 + N p_z}{E_0} = \varepsilon_0(1 + 4\pi R^3 N) \tag{32}$$

4-4-4 Charged Particle Precipitation Onto a Sphere

The solution for a perfectly conducting sphere surrounded by free space in a uniform electric field has been used as a model for the charging of rain drops.* This same model has also been applied to a new type of electrostatic precipitator where small charged particulates are collected on larger spheres.†

Then, in addition to the uniform field $E_0\mathbf{i}_z$ applied at infinity, a uniform flux of charged particulate with charge density ρ_0, which we take to be positive, is also injected, which travels along the field lines with mobility μ. Those field lines that start at infinity where the charge is injected and that approach the sphere with negative radial electric field, deposit charged particulate, as in Figure 4-13. The charge then redistributes itself uniformly on the equipotential surface so that the total charge on the sphere increases with time. Those field lines that do not intersect the sphere or those that start on the sphere do not deposit any charge.

We assume that the self-field due to the injected charge is very much less than the applied field E_0. Then the solution of (23) with $\sigma_2 = \infty$ is correct here, with the addition of the radial field of a uniformly charged sphere with total charge $Q(t)$:

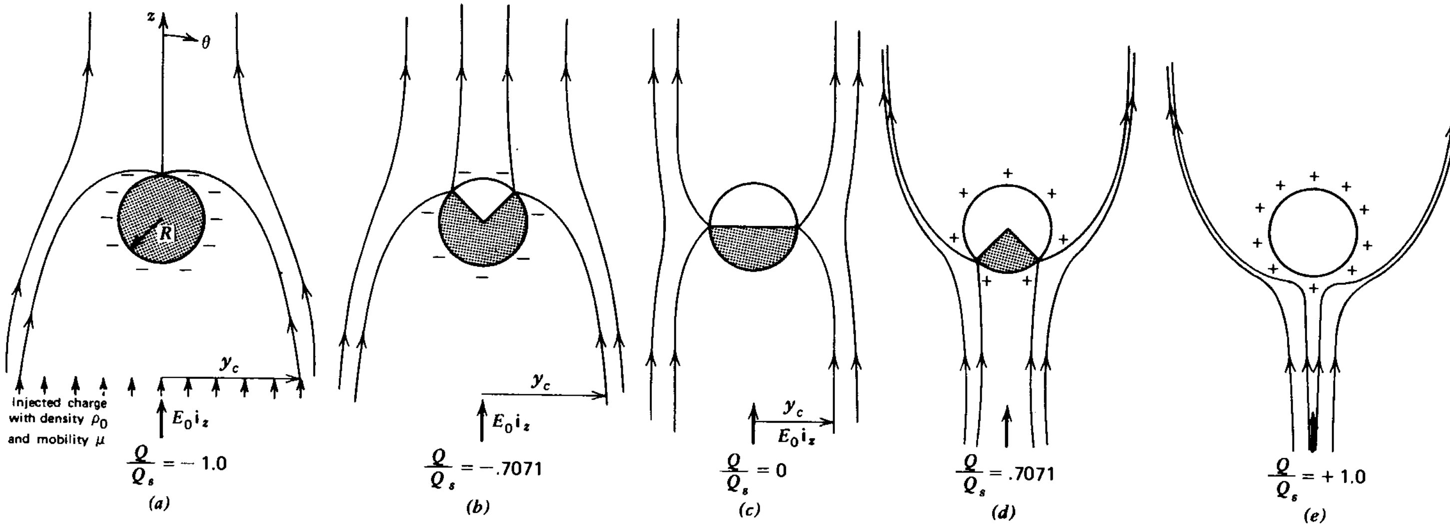
$$\mathbf{E} = \left[E_0\left(1 + \frac{2R^3}{r^3}\right)\cos\theta + \frac{Q}{4\pi\varepsilon r^2}\right]\mathbf{i}_r - E_0\left(1 - \frac{R^3}{r^3}\right)\sin\theta\,\mathbf{i}_\theta, \quad r > R \tag{33}$$

Charge only impacts the sphere where $E_r(r = R)$ is negative:

$$E_r(r = R) = 3E_0\cos\theta + \frac{Q}{4\pi\varepsilon R^2} < 0 \tag{34}$$

* *See*: *F. J. W. Whipple and J. A. Chalmers, On Wilson's Theory of the Collection of Charge by Falling Drops*, Quart. J. Roy. Met. Soc. ***70***, *(1944), p. 103.*

† *See*: *H. J. White,* Industrial Electrostatic Precipitation *Addison-Wesley, Reading. Mass. 1963, pp. 126–137.*



$$E_0R^2\left[\frac{R}{r}+\frac{1}{2}\left(\frac{r}{R}\right)^2\right]\sin^2\theta-\frac{Q\cos\theta}{4\pi\epsilon}=\text{constant}$$

Figure 4-13 Electric field lines around a uniformly charged perfectly conducting sphere in a uniform electric field with continuous positive charge injection from $z=-\infty$. Only those field lines that impact on the sphere with the electric field radially inward $[E_r(R)<0]$ deposit charge. (a) If the total charge on the sphere starts out as negative charge with magnitude greater or equal to the critical charge, the field lines within the distance y_c of the z axis impact over the entire sphere. (b)–(d) As the sphere charges up it tends to repel some of the incident charge and only part of the sphere collects charge. With increasing charge the angular window for charge collection decreases as does y_c. (e) For $Q\geq Q_s$ no further charge collects on the sphere so that the charge remains constant thereafter. The angular window and y_c have shrunk to zero.

which gives us a window for charge collection over the range of angle, where

$$\cos\theta \leq -\frac{Q}{12\pi\varepsilon E_0 R^2} \tag{35}$$

Since the magnitude of the cosine must be less than unity, the maximum amount of charge that can be collected on the sphere is

$$Q_s = 12\pi\varepsilon E_0 R^2 \tag{36}$$

As soon as this saturation charge is reached, all field lines emanate radially outward from the sphere so that no more charge can be collected. We define the critical angle θ_c as the angle where the radial electric field is zero, defined when (35) is an equality $\cos\theta_c = -Q/Q_s$. The current density charging the sphere is

$$\begin{aligned} J_r &= \rho_0\mu E_r(r=R) \\ &= 3\rho_0\mu E_0(\cos\theta + Q/Q_s), \qquad \theta_c < \theta < \pi \end{aligned} \tag{37}$$

The total charging current is then

$$\begin{aligned} \frac{dQ}{dt} &= -\int_{\theta=\theta_c}^{\pi} J_r 2\pi R^2 \sin\theta\, d\theta \\ &= -6\pi\rho_0\mu E_0 R^2 \int_{\theta=\theta_c}^{\pi} (\cos\theta + Q/Q_s)\sin\theta\, d\theta \\ &= -6\pi\rho_0\mu E_0 R^2 \left(-\tfrac{1}{4}\cos 2\theta - (Q/Q_s)\cos\theta\right)\Big|_{\theta=\theta_c}^{\pi} \\ &= -6\pi\rho_0\mu E_0 R^2 \left(-\tfrac{1}{4}(1-\cos 2\theta_c) + (Q/Q_s)(1+\cos\theta_c)\right) \end{aligned} \tag{38}$$

As long as $|Q| < Q_s$, θ_c is defined by the equality in (35). If Q exceeds Q_s, which can only occur if the sphere is intentionally overcharged, then $\theta_c = \pi$ and no further charging can occur as dQ/dt in (38) is zero. If Q is negative and exceeds Q_s in magnitude, $Q < -Q_s$, then the whole sphere collects charge as $\theta_c = 0$. Then for these conditions we have

$$\cos\theta_c = \begin{cases} -1, & Q > Q_s \\ -Q/Q_s & -Q_s < Q < Q_s \\ 1, & Q < -Q_s \end{cases} \tag{39}$$

$$\cos 2\theta_c = 2\cos^2\theta_c - 1 = \begin{cases} 1, & |Q| > Q_s \\ 2(Q/Q_s)^2 - 1, & |Q| < Q_s \end{cases} \tag{40}$$

so that (38) becomes

$$\frac{d\dfrac{Q}{Q_s}}{dt}=\begin{cases}0, & Q>Q_s\\ \dfrac{\rho_0\mu}{4\varepsilon}\left(1-\dfrac{Q}{Q_s}\right)^2, & -Q_s<Q<Q_s\\ -\dfrac{\rho_0\mu}{\varepsilon}\dfrac{Q}{Q_s}, & Q<-Q_s\end{cases} \tag{41}$$

with integrated solutions

$$\frac{Q}{Q_s}=\begin{cases}\dfrac{Q_0}{Q_s}, & Q>Q_s\\ \dfrac{\dfrac{Q_0}{Q_s}+\dfrac{(t-t_0)}{4\tau}\left(1-\dfrac{Q_0}{Q_s}\right)}{1+\dfrac{(t-t_0)}{4\tau}\left(1-\dfrac{Q_0}{Q_s}\right)}, & -Q_s<Q<Q_s\\ \dfrac{Q_0}{Q_s}e^{-t/\tau}, & Q<-Q_s\end{cases} \tag{42}$$

where Q_0 is the initial charge at $t=0$ and the characteristic charging time is

$$\tau=\varepsilon/(\rho_0\mu) \tag{43}$$

If the initial charge Q_0 is less than $-Q_s$, the charge magnitude decreases with the exponential law in (42) until the total charge reaches $-Q_s$ at $t=t_0$. Then the charging law switches to the next regime with $Q_0=-Q_s$, where the charge passes through zero and asymptotically slowly approaches $Q=Q_s$. The charge can never exceed Q_s unless externally charged. It then remains constant at this value repelling any additional charge. If the initial charge Q_0 has magnitude less than Q_s, then $t_0=0$. The time dependence of the charge is plotted in Figure 4-14 for various initial charge values Q_0. No matter the initial value of Q_0 for $Q<Q_s$, it takes many time constants for the charge to closely approach the saturation value Q_s. The force of repulsion on the injected charge increases as the charge on the sphere increases so that the charging current decreases.

The field lines sketched in Figure 4-13 show how the fields change as the sphere charges up. The window for charge collection decreases with increasing charge. The field lines are found by adding the stream function of a uniformly charged sphere with total charge Q to the solution of (30)

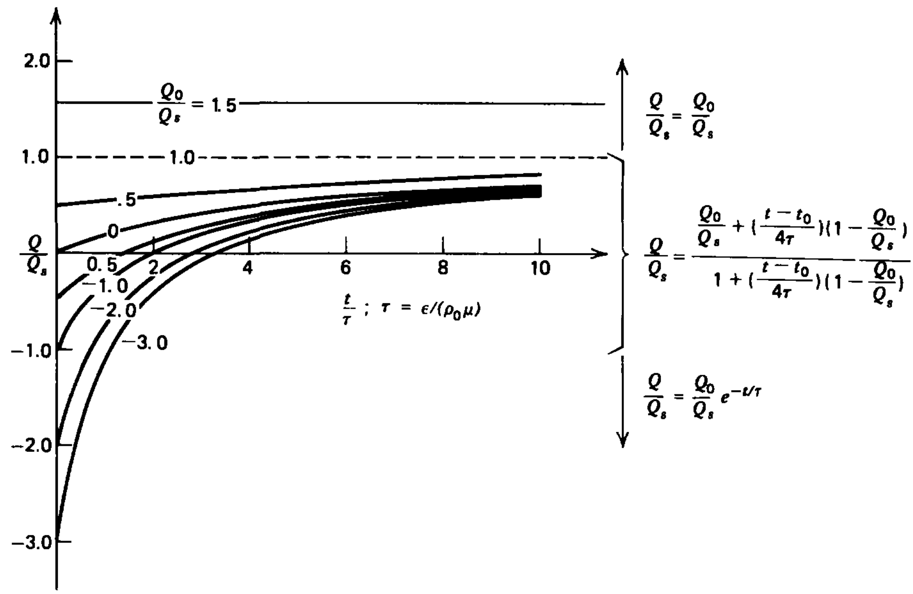


Figure 4-14 There are three regimes describing the charge build-up on the sphere. It takes many time constants $[\tau = \varepsilon/(\rho_0\mu)]$ for the charge to approach the saturation value Q_s, because as the sphere charges up the Coulombic repulsive force increases so that most of the charge goes around the sphere. If the sphere is externally charged to a value in excess of the saturation charge, it remains constant as all additional charge is completely repelled.

with $\sigma_2 \to \infty$:

$$\Sigma = E_0 R^2 \left[\frac{R}{r} + \frac{1}{2}\left(\frac{r}{R}\right)^2\right] \sin^2\theta - \frac{Q\cos\theta}{4\pi\varepsilon} \tag{44}$$

The streamline intersecting the sphere at $r = R, \theta = \theta_c$ separates those streamlines that deposit charge onto the sphere from those that travel past.

4-5 A NUMERICAL METHOD—SUCCESSIVE RELAXATION

In many cases, the geometry and boundary conditions are irregular so that closed form solutions are not possible. It then becomes necessary to solve Poisson's equation by a computational procedure. In this section we limit ourselves to dependence on only two Cartesian coordinates.

4-5-1 Finite Difference Expansions

The Taylor series expansion to second order of the potential V, at points a distance Δx on either side of the coordinate

(x, y), is

$$V(x+\Delta x, y) \approx V(x, y) + \left.\frac{\partial V}{\partial x}\right|_{x,y} \Delta x + \frac{1}{2}\left.\frac{\partial^2 V}{\partial x^2}\right|_{x,y} (\Delta x)^2$$
$$V(x-\Delta x, y) \approx V(x, y) - \left.\frac{\partial V}{\partial x}\right|_{x,y} \Delta x + \frac{1}{2}\left.\frac{\partial^2 V}{\partial x^2}\right|_{x,y} (\Delta x)^2 \quad (1)$$

If we add these two equations and solve for the second derivative, we have

$$\frac{\partial^2 V}{\partial x^2} \approx \frac{V(x+\Delta x, y) + V(x-\Delta x, y) - 2V(x, y)}{(\Delta x)^2} \quad (2)$$

Performing similar operations for small variations from y yields

$$\frac{\partial^2 V}{\partial y^2} \approx \frac{V(x, y+\Delta y) + V(x, y-\Delta y) - 2V(x, y)}{(\Delta y)^2} \quad (3)$$

If we add (2) and (3) and furthermore let $\Delta x = \Delta y$, Poisson's equation can be approximated as

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \approx \frac{1}{(\Delta x)^2}[V(x+\Delta x, y) + V(x-\Delta x, y)$$
$$+ V(x, y+\Delta y) + V(x, y-\Delta y) - 4V(x, y)] = -\frac{\rho_f(x, y)}{\varepsilon} \quad (4)$$

so that the potential at (x, y) is equal to the average potential of its four nearest neighbors plus a contribution due to any volume charge located at (x, y):

$$V(x, y) = \tfrac{1}{4}[V(x+\Delta x, y) + V(x-\Delta x, y)$$
$$+ V(x, y+\Delta y) + V(x, y-\Delta y)] + \frac{\rho_f(x, y)(\Delta x)^2}{4\varepsilon} \quad (5)$$

The components of the electric field are obtained by taking the difference of the two expressions in (1)

$$E_x(x, y) = -\left.\frac{\partial V}{\partial x}\right|_{x,y} \approx -\frac{1}{2\Delta x}[V(x+\Delta x, y) - V(x-\Delta x, y)]$$
$$E_y(x, y) = -\left.\frac{\partial V}{\partial y}\right|_{x,y} \approx -\frac{1}{2\Delta y}[V(x, y+\Delta y) - V(x, y-\Delta y)] \quad (6)$$

4-5-2 Potential Inside a Square Box

Consider the square conducting box whose sides are constrained to different potentials, as shown in Figure (4-15). We discretize the system by drawing a square grid with four

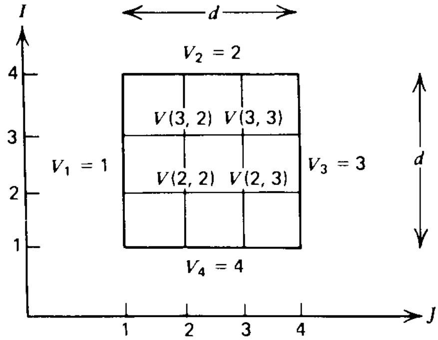


Figure 4-15 The potentials at the four interior points of a square conducting box with imposed potentials on its surfaces are found by successive numerical relaxation. The potential at any charge free interior grid point is equal to the average potential of the four adjacent points.

interior points. We must supply the potentials along the boundaries as proved in Section 4-1:

$$\begin{aligned} V_1 &= \sum_{I=1}^{4} V(I, J=1) = 1, \qquad & V_3 &= \sum_{I=1}^{4} V(I, J=4) = 3 \\ V_2 &= \sum_{J=1}^{4} V(I=4, J) = 2, \qquad & V_4 &= \sum_{J=1}^{4} V(I=1, J) = 4 \end{aligned} \tag{7}$$

Note the discontinuity in the potential at the corners.

We can write the charge-free discretized version of (5) as

$$V(I, J) = \tfrac{1}{4}[V(I+1, J) + V(I-1, J) + V(I, J+1) + V(I, J-1)] \tag{8}$$

We then guess any initial value of potential for all interior grid points not on the boundary. The boundary potentials must remain unchanged. Taking the interior points one at a time, we then improve our initial guess by computing the average potential of the four surrounding points.

We take our initial guess for all interior points to be zero inside the box:

$$\begin{aligned} V(2, 2) &= 0, \qquad & V(3, 3) &= 0 \\ V(3, 2) &= 0, \qquad & V(2, 3) &= 0 \end{aligned} \tag{9}$$

Then our first improved estimate for $V(2, 2)$ is

$$\begin{aligned} V(2, 2) &= \tfrac{1}{4}[V(2, 1) + V(2, 3) + V(1, 2) + V(3, 2)] \\ &= \tfrac{1}{4}[1 + 0 + 4 + 0] = 1.25 \end{aligned} \tag{10}$$

Using this value of $V(2,2)$ we improve our estimate for $V(3,2)$ as

$$\begin{aligned} V(3,2) &= \tfrac{1}{4}[V(2,2)+V(4,2)+V(3,1)+V(3,3)] \\ &= \tfrac{1}{4}[1.25+2+1+0] = 1.0625 \end{aligned} \quad (11)$$

Similarly for $V(3,3)$,

$$\begin{aligned} V(3,3) &= \tfrac{1}{4}[V(3,2)+V(3,4)+V(2,3)+V(4,3)] \\ &= \tfrac{1}{4}[1.0625+3+0+2] = 1.5156 \end{aligned} \quad (12)$$

and $V(2,3)$

$$\begin{aligned} V(2,3) &= \tfrac{1}{4}[V(2,2)+V(2,4)+V(1,3)+V(3,3)] \\ &= \tfrac{1}{4}[1.25+3+4+1.5156] = 2.4414 \end{aligned} \quad (13)$$

We then continue and repeat the procedure for the four interior points, always using the latest values of potential. As the number of iterations increase, the interior potential values approach the correct solutions. Table 4-2 shows the first ten iterations and should be compared to the exact solution to four decimal places, obtained by superposition of the rectangular harmonic solution in Section 4-2-5 (see problem 4-4):

$$\begin{aligned} V(x,y) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi \sinh n\pi} \Bigg[&\sin \frac{n\pi y}{d} \left(V_3 \sinh \frac{n\pi x}{d} - V_1 \sinh \frac{n\pi (x-d)}{d} \right) \\ &+ \sin \frac{n\pi x}{d} \left(V_2 \sinh \frac{n\pi y}{d} - V_4 \sinh \frac{n\pi (y-d)}{d} \right) \Bigg] \end{aligned} \quad (14)$$

where V_1, V_2, V_3 and V_4 are the boundary potentials that for this case are

$$V_1 = 1, \qquad V_2 = 2, \qquad V_3 = 3, \qquad V_4 = 4 \quad (15)$$

To four decimal places the numerical solutions remain unchanged for further iterations past ten.

Table 4-2 Potential values for the four interior points in Figure 4-15 obtained by successive relaxation for the first ten iterations

	0	1	2	3	4	5
V_1	0	1.2500	2.1260	2.3777	2.4670	2.4911
V_2	0	1.0625	1.6604	1.9133	1.9770	1.9935
V_3	0	1.5156	2.2755	2.4409	2.4829	2.4952
V_4	0	2.4414	2.8504	2.9546	2.9875	2.9966

	6	7	8	9	10	Exact
V_1	2.4975	2.4993	2.4998	2.4999	2.5000	2.5000
V_2	1.9982	1.9995	1.9999	2.0000	2.0000	1.9771
V_3	2.4987	2.4996	2.4999	2.5000	2.5000	2.5000
V_4	2.9991	2.9997	2.9999	3.0000	3.0000	3.0229

The results are surprisingly good considering the coarse grid of only four interior points. This relaxation procedure can be used for any values of boundary potentials, for any number of interior grid points, and can be applied to other boundary shapes. The more points used, the greater the accuracy. The method is easily implemented as a computer algorithm to do the repetitive operations.

PROBLEMS

Section 4.2

1. The hyperbolic electrode system of Section 4-2-2*a* only extends over the range $0 \leq x \leq x_0$, $0 \leq y \leq y_0$ and has a depth D.

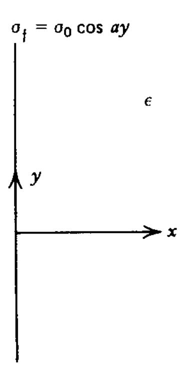
(a) Neglecting fringing field effects what is the approximate capacitance?

(b) A small positive test charge q (image charge effects are negligible) with mass m is released from rest from the surface of the hyperbolic electrode at $x = x_0$, $y = ab/x_0$. What is the velocity of the charge as a function of its position?

(c) What is the velocity of the charge when it hits the opposite electrode?

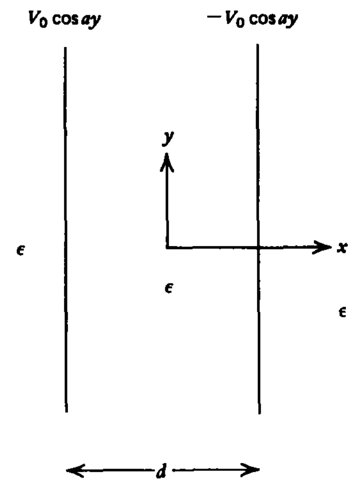
2. A sheet of free surface charge at $x = 0$ has charge distribution

$$\sigma_f = \sigma_0 \cos ay$$



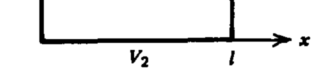
(a) What are the potential and electric field distributions?
(b) What is the equation of the field lines?

3. Two sheets of opposite polarity with their potential distributions constrained are a distance d apart.



(a) What are the potential and electric field distributions everywhere?
(b) What are the surface charge distributions on each sheet?

4. A conducting rectangular box of width d and length l is of infinite extent in the z direction. The potential along the $x=0$ edge is V_1 while all other surfaces are grounded ($V_2 = V_3 = V_4 = 0$).

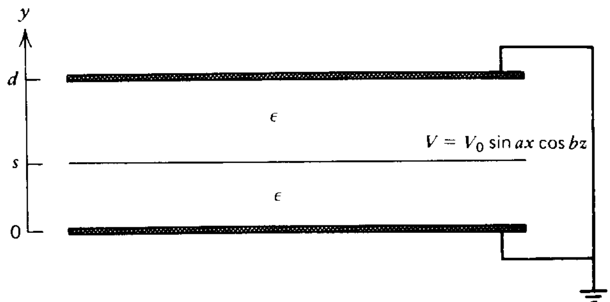


(a) What are the potential and electric field distributions?
(b) The potential at $y=0$ is now raised to V_2 while the surface at $x=0$ remains at potential V_1. The other two surfaces remain at zero potential ($V_3 = V_4 = 0$). What are the potential and electric field distributions? (**Hint:** Use superposition.)
(c) What is the potential distribution if each side is respectively at nonzero potentials V_1, V_2, V_3, and V_4?

5. A sheet with potential distribution

$$V = V_0 \sin ax \cos bz$$

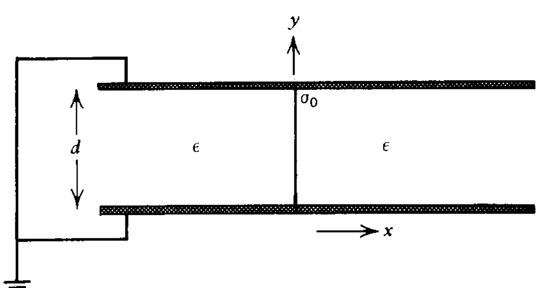
is placed parallel and between two parallel grounded conductors a distance d apart. It is a distance s above the lower plane.



(a) What are the potential and electric field distributions? (**Hint:** You can write the potential distribution by inspection using a spatially shifted hyperbolic function $\sinh c(y-d)$.)

(b) What is the surface charge distribution on each plane at $y=0$, $y=s$, and $y=d$?

6. A uniformly distributed surface charge σ_0 of width d and of infinite extent in the z direction is placed at $x=0$ perpendicular to two parallel grounded planes of spacing d.



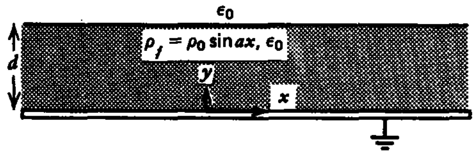
(a) What are the potential and electric field distributions? (**Hint:** Write σ_0 as a Fourier series.)

(b) What is the induced surface charge distribution on each plane?

(c) What is the total induced charge per unit length on each plane? **Hint:**

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$$

7. A slab of volume charge of thickness d with volume charge density $\rho_f = \rho_0 \sin ax$ is placed upon a conducting ground plane.



(a) Find a particular solution to Poisson's equation. Are the boundary conditions satisfied?

(b) If the solution to (a) does not satisfy all the boundary conditions, add a Laplacian solution which does.

(c) What is the electric field distribution everywhere and the surface charge distribution on the ground plane?

(d) What is the force per unit length on the volume charge and on the ground plane for a section of width $2\pi/a$? Are these forces equal?

(e) Repeat (a)–(c), if rather than free charge, the slab is a permanently polarized medium with polarization

$$\mathbf{P} = P_0 \sin ax \mathbf{i}_y$$

8. Consider the Cartesian coordinates (x, y) and define the complex quantity

$$z = x + jy, \qquad j = \sqrt{-1}$$

where z is not to be confused with the Cartesian coordinate. Any function of z also has real and imaginary parts

$$w(z) = u(x, y) + jv(x, y)$$

(a) Find u and v for the following functions:

(i) z^2
(ii) $\sin z$
(iii) $\cos z$
(iv) e^z
(v) $\ln z$

(b) Realizing that the partial derivatives of w are

$$\frac{\partial w}{\partial x} = \frac{dw}{dz}\frac{\partial z}{\partial x} = \frac{dw}{dz} = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{dw}{dz}\frac{\partial z}{\partial y} = j\frac{dw}{dz} = \frac{\partial u}{\partial y} + j\frac{\partial v}{\partial y}$$

show that u and v must be related as

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

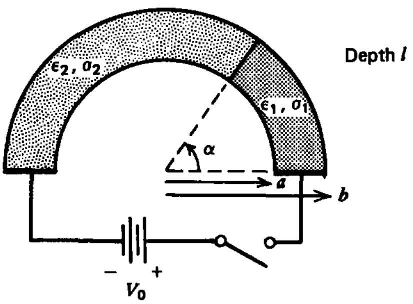
These relations are known as the Cauchy–Riemann equations and u and v are called conjugate functions.

(c) Show that both u and v obey Laplace's equation.

(d) Show that lines of constant u and v are perpendicular to each other in the xy plane. (**Hint:** Are ∇u and ∇v perpendicular vectors?)

Section 4.3

9. A half cylindrical shell of length l having inner radius a and outer radius b is composed of two different lossy dielectric materials $(\varepsilon_1, \sigma_1)$ for $0<\phi<\alpha$ and $(\varepsilon_2, \sigma_2)$ for $\alpha<\phi<\pi$. A step voltage V_0 is applied at $t=0$. Neglect variations with z.



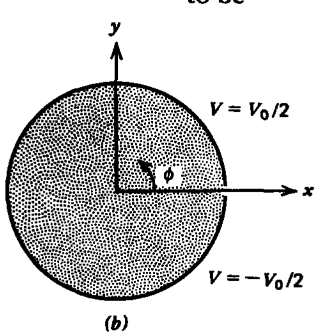
(a) What are the potential and electric field distributions within the shell at times $t=0$, $t=\infty$, and during the transient interval? (**Hint:** Assume potentials of the form $V(\phi)=A(t)\phi + B(t)$ and neglect effects of the region outside the half cylindrical shell.)

(b) What is the time dependence of the surface charge at $\phi=\alpha$?

(c) What is the resistance and capacitance?

10. The potential on an infinitely long cylinder is constrained to be

$$V(r=a)=V_0 \sin n\phi$$



(a) Find the potential and electric field everywhere.

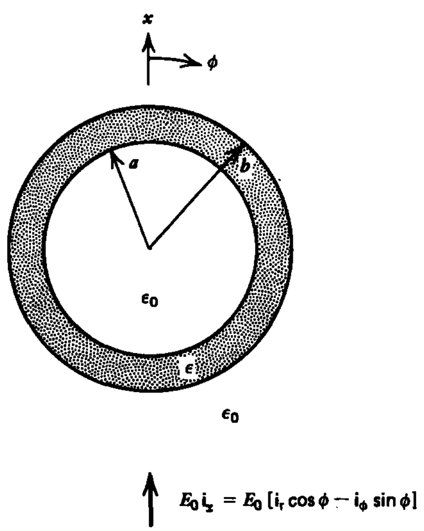
(b) The potential is now changed so that it is constant on each half of the cylinder:

$$V(r=a,\phi)=\begin{cases} V_0/2, & 0<\phi<\pi \\ -V_0/2, & \pi<\phi<2\pi \end{cases}$$

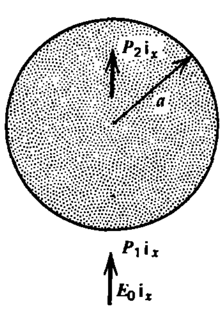
Write this square wave of potential in a Fourier series.

(c) Use the results of (a) and (b) to find the potential and electric field due to this square wave of potential.

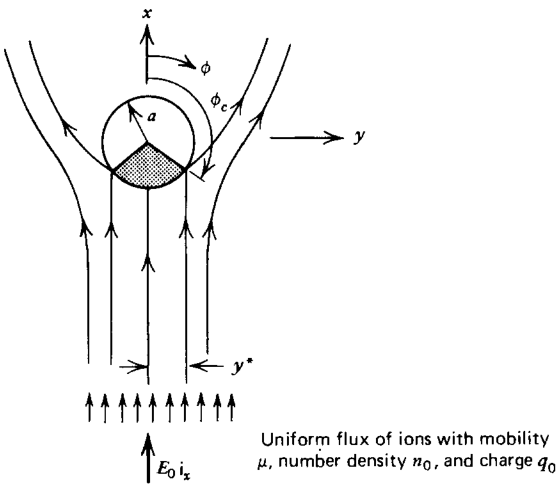
11. A cylindrical dielectric shell of inner radius a and outer radius b is placed in free space within a uniform electric field $E_0\mathbf{i}_x$. What are the potential and electric field distributions everywhere?



12. A permanently polarized cylinder $P_2\mathbf{i}_x$ of radius a is placed within a polarized medium $P_1\mathbf{i}_x$ of infinite extent. A uniform electric field $E_0\mathbf{i}_x$ is applied at infinity. There is no free charge on the cylinder. What are the potential and electric field distributions?



13. One type of electrostatic precipitator has a perfectly conducting cylinder of radius a placed within a uniform electric field $E_0\mathbf{i}_x$. A uniform flux of positive ions with charge q_0 and number density n_0 are injected at infinity and travel along the field lines with mobility μ. Those field lines that approach the cylinder with $E_r < 0$ deposit ions, which redistribute themselves uniformly on the surface of the cylinder. The self-field due to the injected charge is negligible compared to E_0.



(a) If the uniformly distributed charge per unit length on the cylinder is $\lambda(t)$, what is the field distribution? Where is the electric field zero? This point is called a critical point because ions flowing past one side of this point miss the cylinder while those on the other side are collected. What equation do the field lines obey? (**Hint:** To the field solution of Section 4-3-2*a*, add the field due to a line charge λ.)

(b) Over what range of angle ϕ, $\phi_c < \phi < 2\pi - \phi_c$, is there a window (shaded region in figure) for charge collection as a function of $\lambda(t)$? (**Hint:** $E_r < 0$ for charge collection.)

(c) What is the maximum amount of charge per unit length that can be collected on the cylinder?

(d) What is the cylinder charging current per unit length? (**Hint:** $dI = -q_0 n_0 \mu E_r a\, d\phi$)

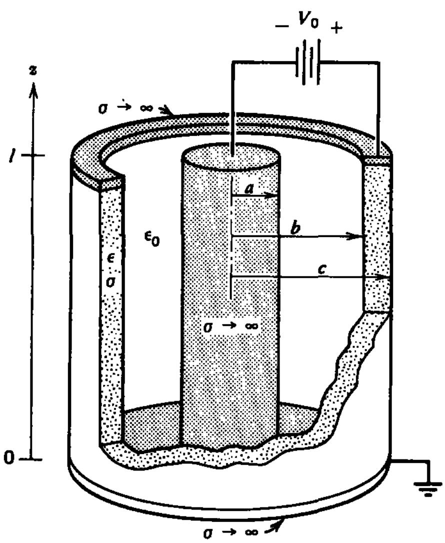
(e) Over what range of $y = y^*$ at $r = \infty, \phi = \pi$ do the injected ions impact on the cylinder as a function of $\lambda(t)$? What is this charging current per unit length? Compare to (d).

14. The cylinder of Section 4-3-2 placed within a lossy medium is allowed to reach the steady state.

(a) At $t = 0$ the imposed electric field at infinity is suddenly set to zero. What is the time dependence of the surface charge distribution at $r = a$?

(b) Find the surface charge distribution if the field at infinity is a sinusoidal function of time $E_0 \cos \omega t$.

15. A perfectly conducting cylindrical can of radius c open at one end has its inside surface coated with a resistive layer. The bottom at $z = 0$ and a perfectly conducting center post of radius a are at zero potential, while a constant potential V_0 is imposed at the top of the can.



(a) What are the potential and electric field distributions within the structure ($a < r < c$, $0 < z < l$)? (**Hint:** Try the zero separation constant solutions $n = 0$, $k = 0$.)

(b) What is the surface charge distribution and the total charge at $r = a$, $r = b$, and $z = 0$?

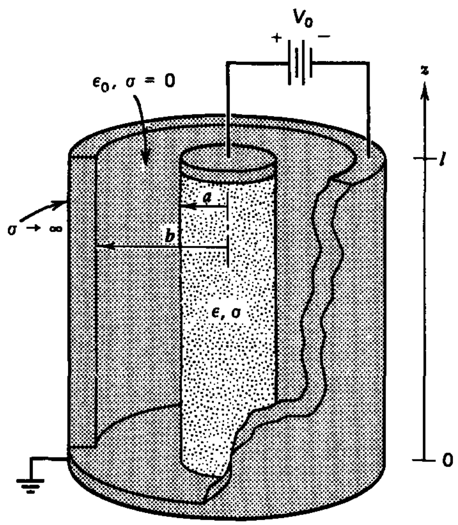
(c) What is the equation of the field lines in the free space region?

16. An Ohmic conducting cylinder of radius a is surrounded by a grounded perfectly conducting cylindrical can of radius b open at one end. A voltage V_0 is applied at the top of the resistive cylinder. Neglect variations with ϕ.

(a) What are the potential and electric field distributions within the structure, $0 < z < l$, $0 < r < b$? (**Hint:** Try the zero separation constant solutions $n = 0$, $k = 0$ in each region.)

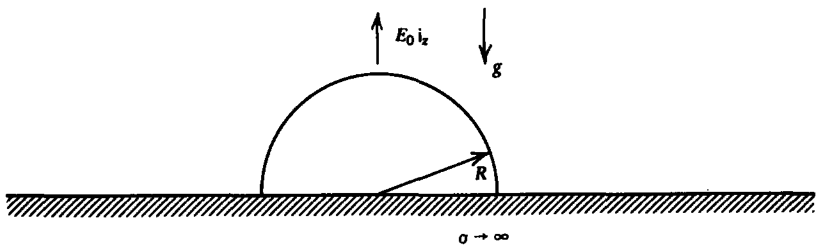
(b) What is the surface charge distribution and total charge on the interface at $r = a$?

(c) What is the equation of the field lines in the free space region?



Section 4.4

17. A perfectly conducting hemisphere of radius R is placed upon a ground plane of infinite extent. A uniform field $E_0\mathbf{i}_z$ is applied at infinity.



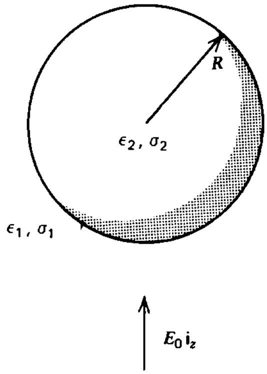
(a) How much more charge is on the hemisphere than would be on the plane over the area occupied by the hemisphere.

(b) If the hemisphere has mass density ρ_m and is in a gravity field $-g\mathbf{i}_z$, how large must E_0 be to lift the hemisphere? **Hint:**

$$\int \sin\theta \cos^m\theta \, d\theta = -\frac{\cos^{m+1}\theta}{m+1}$$

18. A sphere of radius R, permittivity ε_2, and Ohmic conductivity σ_2 is placed within a medium of permittivity ε_1 and conductivity σ_1. A uniform electric field $E_0\mathbf{i}_z$ is suddenly turned on at $t=0$.

(a) What are the necessary boundary and initial conditions?



(b) What are the potential and electric field distributions as a function of time?

(c) What is the surface charge at $r=R$?

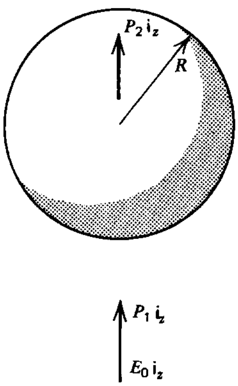
(d) Repeat (b) and (c) if the applied field varies sinusoidally with time as $E_0 \cos \omega t$ and has been on a long time.

19. The surface charge distribution on a dielectric sphere with permittivity ε_2 and radius R is

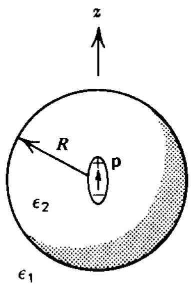
$$\sigma_f=\sigma_0(3\cos^2\theta-1)$$

The surrounding medium has permittivity ε_1. What are the potential and electric field distributions? (**Hint:** Try the $n=2$ solutions.)

20. A permanently polarized sphere $P_2\mathbf{i}_z$ of radius R is placed within a polarized medium $P_1\mathbf{i}_z$. A uniform electric field $E_0\mathbf{i}_z$ is applied at infinity. There is no free charge at $r=R$. What are the potential and electric field distributions?



21. A point dipole $\mathbf{p}=p\mathbf{i}_z$ is placed at the center of a dielectric sphere that is surrounded by a different dielectric medium. There is no free surface charge on the interface.



What are the potential and electric field distributions? **Hint:**

$$\lim_{r\to 0} V(r, \theta) = \frac{p\cos\theta}{4\pi\varepsilon_2 r^2}$$

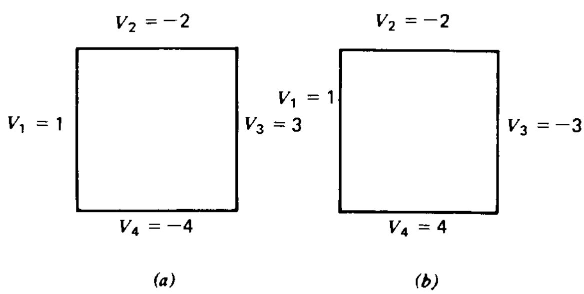
Section 4.5

22. The conducting box with sides of length d in Section 4-5-2 is filled with a uniform distribution of volume charge with density

$$\rho_0 = -\frac{72\varepsilon_0}{d^2}\,[\text{coul-m}^{-3}]$$

What are the potentials at the four interior points when the outside of the box is grounded?

23. Repeat the relaxation procedure of Section 4-5-2 if the boundary potentials are:



(a) $V_1 = 1,\ V_2 = -2,\ V_3 = 3,\ V_4 = -4$

(b) $V_1 = 1,\ V_2 = -2,\ V_3 = -3,\ V_4 = 4$

(c) Compare to four decimal places with the exact solution.

chapter 5

the magnetic field

The ancient Chinese knew that the iron oxide magnetite (Fe_3O_4) attracted small pieces of iron. The first application of this effect was the navigation compass, which was not developed until the thirteenth century. No major advances were made again until the early nineteenth century when precise experiments discovered the properties of the magnetic field.

5-1 FORCES ON MOVING CHARGES

5-1-1 The Lorentz Force Law

It was well known that magnets exert forces on each other, but in 1820 Oersted discovered that a magnet placed near a current carrying wire will align itself perpendicular to the wire. Each charge q in the wire, moving with velocity $\mathbf{v}$ in the magnetic field $\mathbf{B}$ [teslas, (kg-s^{-2}-A^{-1})], felt the empirically determined Lorentz force perpendicular to both $\mathbf{v}$ and $\mathbf{B}$

$$\mathbf{f} = q(\mathbf{v}\times\mathbf{B}) \tag{1}$$

as illustrated in Figure 5-1. A distribution of charge feels a differential force $d\mathbf{f}$ on each moving incremental charge element dq:

$$d\mathbf{f} = dq(\mathbf{v}\times\mathbf{B}) \tag{2}$$

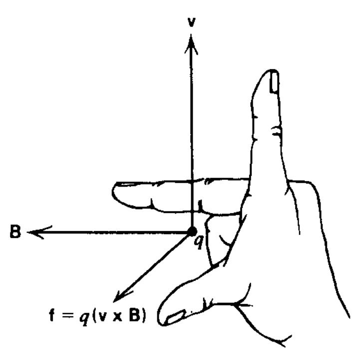


Figure 5-1 A charge moving through a magnetic field experiences the Lorentz force perpendicular to both its motion and the magnetic field.

Moving charges over a line, surface, or volume, respectively constitute line, surface, and volume currents, as in Figure 5-2, where (2) becomes

$$d\mathbf{f}=\begin{cases}\rho_f\mathbf{v}\times\mathbf{B}\,dV=\mathbf{J}\times\mathbf{B}\,dV & (\mathbf{J}=\rho_f\mathbf{v},\text{ volume current density})\\ \sigma_f\mathbf{v}\times\mathbf{B}\,dS=\mathbf{K}\times\mathbf{B}\,dS & \\ & (\mathbf{K}=\sigma_f\mathbf{v},\text{ surface current density})\\ \lambda_f\mathbf{v}\times\mathbf{B}\,dl=\mathbf{I}\times\mathbf{B}\,dl & (\mathbf{I}=\lambda_f\mathbf{v},\text{ line current})\end{cases} \quad (3)$$

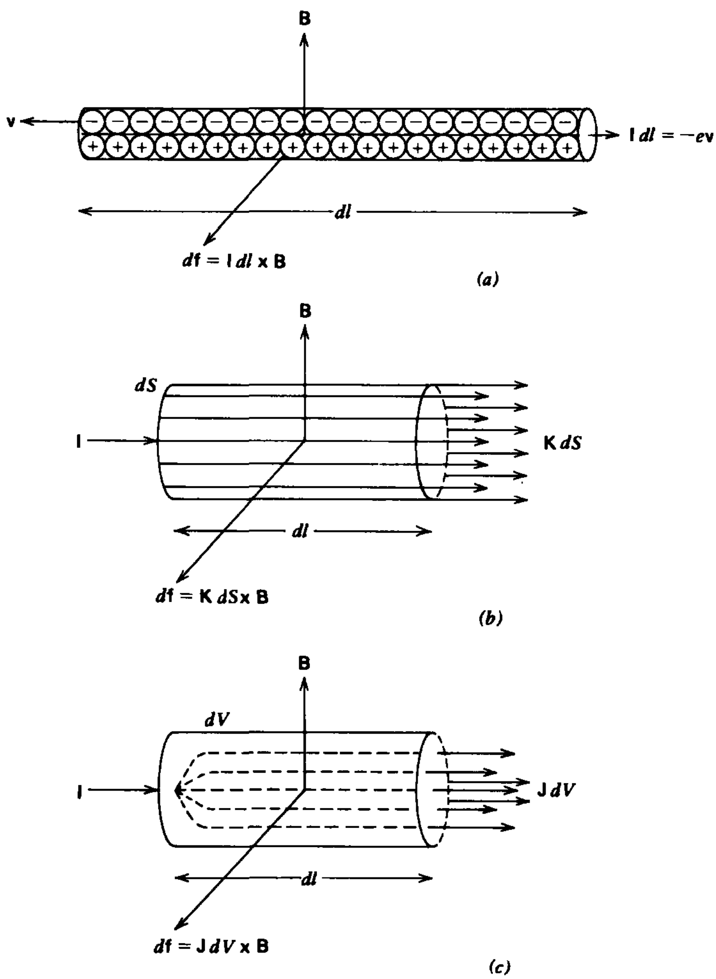


Figure 5-2 Moving line, surface, and volume charge distributions constitute currents. (*a*) In metallic wires the net charge is zero since there are equal amounts of negative and positive charges so that the Coulombic force is zero. Since the positive charge is essentially stationary, only the moving electrons contribute to the line current in the direction opposite to their motion. (*b*) Surface current. (*c*) Volume current.

The total magnetic force on a current distribution is then obtained by integrating (3) over the total volume, surface, or contour containing the current. If there is a net charge with its associated electric field $\mathbf{E}$, the total force densities include the Coulombic contribution:

$$\begin{aligned} \mathbf{f} &= q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \quad \text{Newton} \\ \mathbf{F}_L &= \lambda_f(\mathbf{E}+\mathbf{v}\times\mathbf{B}) = \lambda_f\mathbf{E}+\mathbf{I}\times\mathbf{B} \quad \text{N/m} \\ \mathbf{F}_S &= \sigma_f(\mathbf{E}+\mathbf{v}\times\mathbf{B}) = \sigma_f\mathbf{E}+\mathbf{K}\times\mathbf{B} \quad \text{N/m}^2 \\ \mathbf{F}_V &= \rho_f(\mathbf{E}+\mathbf{v}\times\mathbf{B}) = \rho_f\mathbf{E}+\mathbf{J}\times\mathbf{B} \quad \text{N/m}^3 \end{aligned} \tag{4}$$

In many cases the net charge in a system is very small so that the Coulombic force is negligible. This is often true for conduction in metal wires. A net current still flows because of the difference in velocities of each charge carrier.

Unlike the electric field, the magnetic field cannot change the kinetic energy of a moving charge as the force is perpendicular to the velocity. It can alter the charge's trajectory but not its velocity magnitude.

5-1-2 Charge Motions in a Uniform Magnetic Field

The three components of Newton's law for a charge q of mass m moving through a uniform magnetic field $B_z\mathbf{i}_z$ are

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v}\times\mathbf{B} \Rightarrow \begin{cases} m\dfrac{dv_x}{dt} = qv_yB_z \\ m\dfrac{dv_y}{dt} = -qv_xB_z \\ m\dfrac{dv_z}{dt} = 0 \Rightarrow v_z = \text{const} \end{cases} \tag{5}$$

The velocity component along the magnetic field is unaffected. Solving the first equation for v_y and substituting the result into the second equation gives us a single equation in v_x:

$$\frac{d^2v_x}{dt^2}+\omega_0^2v_x = 0, \qquad v_y = \frac{1}{\omega_0}\frac{dv_x}{dt}, \qquad \omega_0 = \frac{qB_z}{m} \tag{6}$$

where ω_0 is called the Larmor angular velocity or the cyclotron frequency (see Section 5-1-4). The solutions to (6) are

$$\begin{aligned} v_x &= A_1 \sin\omega_0 t + A_2\cos\omega_0 t \\ v_y &= \frac{1}{\omega_0}\frac{dv_x}{dt} = A_1\cos\omega_0 t - A_2\sin\omega_0 t \end{aligned} \tag{7}$$

where A_1 and A_2 are found from initial conditions. If at $t=0$,

$$\mathbf{v}(t=0)=v_0\mathbf{i}_x \tag{8}$$

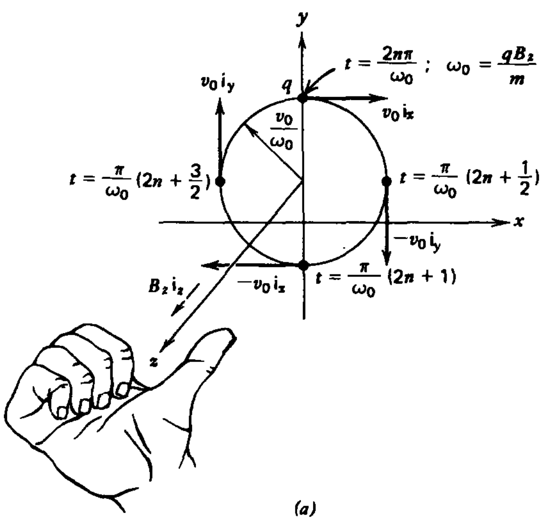
then (7) and Figure 5-3*a* show that the particle travels in a circle, with constant speed v_0 in the xy plane:

$$v=v_0(\cos\omega_0 t\,\mathbf{i}_x-\sin\omega_0 t\,\mathbf{i}_y) \tag{9}$$

with radius

$$R=v_0/\omega_0 \tag{10}$$

If the particle also has a velocity component along the magnetic field in the z direction, the charge trajectory becomes a helix, as shown in Figure 5-3*b*.



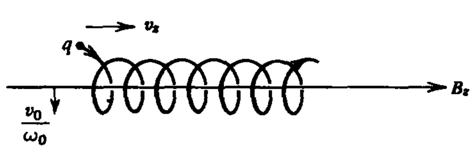


Figure 5-3 (*a*) A positive charge q, initially moving perpendicular to a magnetic field, feels an orthogonal force putting the charge into a circular motion about the magnetic field where the Lorentz force is balanced by the centrifugal force. Note that the charge travels in the direction (in this case clockwise) so that its self-field through the loop [see Section 5-2-1] is opposite in direction to the applied field. (*b*) A velocity component in the direction of the magnetic field is unaffected resulting in a helical trajectory.

5-1-3 The Mass Spectrograph

The mass spectrograph uses the circular motion derived in Section 5-1-2 to determine the masses of ions and to measure the relative proportions of isotopes, as shown in Figure 5-4. Charges enter between parallel plate electrodes with a y-directed velocity distribution. To pick out those charges with a particular magnitude of velocity, perpendicular electric and magnetic fields are imposed so that the net force on a charge is

$$f_x = q(E_x + v_y B_z) \tag{11}$$

For charges to pass through the narrow slit at the end of the channel, they must not be deflected by the fields so that the force in (11) is zero. For a selected velocity $v_y = v_0$ this requires a negatively x directed electric field

$$E_x = \frac{V}{s} = -v_0 B_0 \tag{12}$$

which is adjusted by fixing the applied voltage V. Once the charge passes through the slit, it no longer feels the electric field and is only under the influence of the magnetic field. It thus travels in a circle of radius

$$r = \frac{v_0}{\omega_0} = \frac{v_0 m}{qB_0} \tag{13}$$

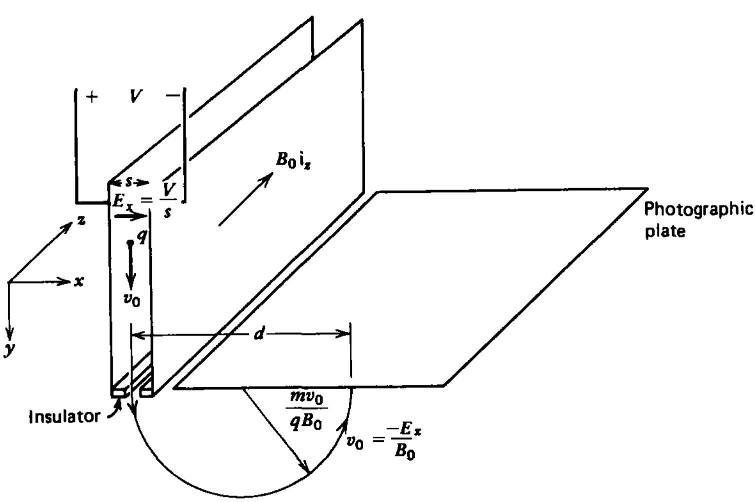


Figure 5-4 The mass spectrograph measures the mass of an ion by the radius of its trajectory when moving perpendicular to a magnetic field. The crossed uniform electric field selects the ion velocity that can pass through the slit.

which is directly proportional to the mass of the ion. By measuring the position of the charge when it hits the photographic plate, the mass of the ion can be calculated. Different isotopes that have the same number of protons but different amounts of neutrons will hit the plate at different positions.

For example, if the mass spectrograph has an applied voltage of $V=-100$ V across a 1-cm gap ($E_x=-10^4$ V/m) with a magnetic field of 1 tesla, only ions with velocity

$$v_y=-E_x/B_0=10^4 \text{ m/sec} \tag{14}$$

will pass through. The three isotopes of magnesium, ${}_{12}Mg^{24}$, ${}_{12}Mg^{25}$, ${}_{12}Mg^{26}$, each deficient of one electron, will hit the photographic plate at respective positions:

$$d=2r=\frac{2\times 10^4 N(1.67\times 10^{-27})}{1.6\times 10^{-19}(1)}\approx 2\times 10^{-4}\, N \Rightarrow 0.48, 0.50, 0.52 \text{ cm} \tag{15}$$

where N is the number of protons and neutrons ($m=1.67\times 10^{-27}$ kg) in the nucleus.

5-1-4 The Cyclotron

A cyclotron brings charged particles to very high speeds by many small repeated accelerations. Basically it is composed of a split hollow cylinder, as shown in Figure 5-5, where each half is called a "dee" because their shape is similar to the

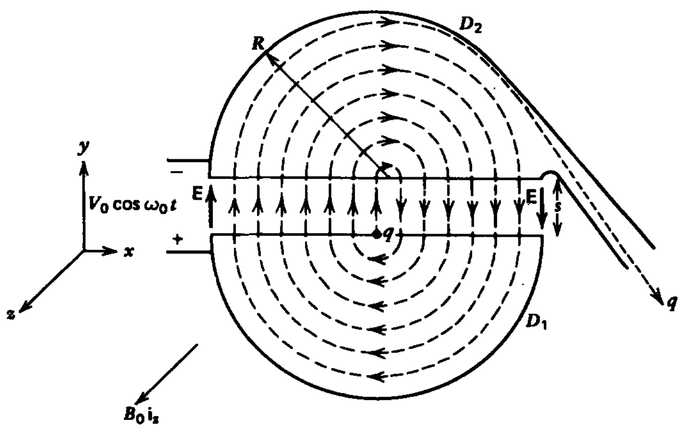


Figure 5-5 The cyclotron brings ions to high speed by many small repeated accelerations by the electric field in the gap between dees. Within the dees the electric field is negligible so that the ions move in increasingly larger circular orbits due to an applied magnetic field perpendicular to their motion.

fourth letter of the alphabet. The dees are put at a sinusoidally varying potential difference. A uniform magnetic field $B_0\mathbf{i}_z$ is applied along the axis of the cylinder. The electric field is essentially zero within the cylindrical volume and assumed uniform $E_y = v(t)/s$ in the small gap between dees. A charge source at the center of D_1 emits a charge q of mass m with zero velocity at the peak of the applied voltage at $t = 0$. The electric field in the gap accelerates the charge towards D_2. Because the gap is so small the voltage remains approximately constant at V_0 while the charge is traveling between dees so that its displacement and velocity are

$$m\frac{dv_y}{dt} = \frac{qV_0}{s} \Rightarrow v_y = \frac{qV_0}{sm}t$$
$$v_y = \frac{dy}{dt} \Rightarrow y = \frac{qV_0 t^2}{2ms} \tag{16}$$

The charge thus enters D_2 at time $t = [2ms^2/qV_0]^{1/2}$ later with velocity $v_y = \sqrt{2qV_0/m}$. Within D_2 the electric field is negligible so that the charge travels in a circular orbit of radius $r = v_y/\omega_0 = mv_y/qB_0$ due to the magnetic field alone. The frequency of the voltage is adjusted to just equal the angular velocity $\omega_0 = qB_0/m$ of the charge, so that when the charge re-enters the gap between dees the polarity has reversed accelerating the charge towards D_1 with increased velocity. This process is continually repeated, since every time the charge enters the gap the voltage polarity accelerates the charge towards the opposite dee, resulting in a larger radius of travel. Each time the charge crosses the gap its velocity is increased by the same amount so that after n gap traversals its velocity and orbit radius are

$$v_n = \left(\frac{2qnV_0}{m}\right)^{1/2}, \qquad R_n = \frac{v_n}{\omega_0} = \left(\frac{2nmV_0}{qB_0^2}\right)^{1/2} \tag{17}$$

If the outer radius of the dees is R, the maximum speed of the charge

$$v_{\max} = \omega_0 R = \frac{qB_0}{m}R \tag{18}$$

is reached after $2n = qB_0^2R^2/mV_0$ round trips when $R_n = R$. For a hydrogen ion ($q = 1.6\times 10^{-19}$ coul, $m = 1.67\times 10^{-27}$ kg), within a magnetic field of 1 tesla ($\omega_0 \approx 9.6\times 10^7$ radian/sec) and peak voltage of 100 volts with a cyclotron radius of one meter, we reach $v_{\max} = 9.6\times 10^7$ m/s (which is about 30% of the speed of light) in about $2n \approx 9.6\times 10^5$ round-trips, which takes a time $\tau = 4n\pi/\omega_o \approx 2\pi/100 \approx 0.06$ sec. To reach this speed with an electrostatic accelerator would require

$$\tfrac{1}{2}mv^2 = qV \Rightarrow V = \frac{mv_{max}^2}{2q} \approx 48 \times 10^6 \text{ Volts} \qquad (19)$$

The cyclotron works at much lower voltages because the angular velocity of the ions remains constant for fixed qB_0/m and thus arrives at the gap in phase with the peak of the applied voltage so that it is sequentially accelerated towards the opposite dee. It is not used with electrons because their small mass allows them to reach relativistic velocities close to the speed of light, which then greatly increases their mass, decreasing their angular velocity ω_0, putting them out of phase with the voltage.

5-1-5 Hall Effect

When charges flow perpendicular to a magnetic field, the transverse displacement due to the Lorentz force can give rise to an electric field. The geometry in Figure 5-6 has a uniform magnetic field $B_0\mathbf{i}_z$ applied to a material carrying a current in the y direction. For positive charges as for holes in a p-type semiconductor, the charge velocity is also in the positive y direction, while for negative charges as occur in metals or in n-type semiconductors, the charge velocity is in the negative y direction. In the steady state where the charge velocity does not vary with time, the net force on the charges must be zero,

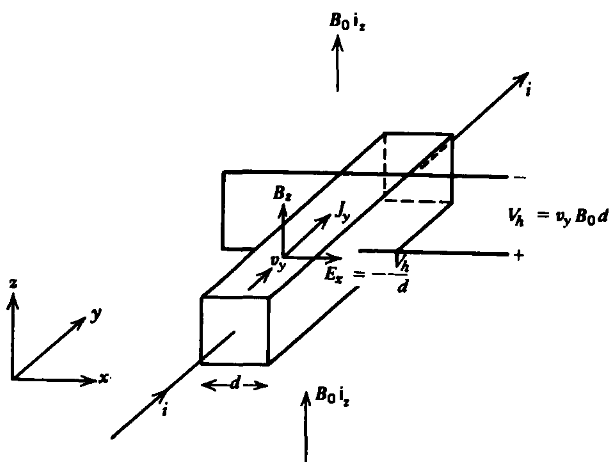


Figure 5-6 A magnetic field perpendicular to a current flow deflects the charges transversely giving rise to an electric field and the Hall voltage. The polarity of the voltage is the same as the sign of the charge carriers.

which requires the presence of an x-directed electric field

$$\mathbf{E}+\mathbf{v}\times\mathbf{B}=0\Rightarrow E_x=-v_yB_0 \tag{20}$$

A transverse potential difference then develops across the material called the Hall voltage:

$$V_h=-\int_0^d E_x\,dx=v_yB_0d \tag{21}$$

The Hall voltage has its polarity given by the sign of v_y; positive voltage for positive charge carriers and negative voltage for negative charges. This measurement provides an easy way to determine the sign of the predominant charge carrier for conduction.

5-2 MAGNETIC FIELD DUE TO CURRENTS

Once it was demonstrated that electric currents exert forces on magnets, Ampere immediately showed that electric currents also exert forces on each other and that a magnet could be replaced by an equivalent current with the same result. Now magnetic fields could be turned on and off at will with their strength easily controlled.

5-2-1 The Biot–Savart Law

Biot and Savart quantified Ampere's measurements by showing that the magnetic field **B** at a distance **r** from a moving charge is

$$\mathbf{B}=\frac{\mu_0 q\mathbf{v}\times\mathbf{i}_r}{4\pi r^2}\text{ teslas (kg-s}^{-2}\text{-A}^{-1}) \tag{1}$$

as in Figure 5-7a, where μ_0 is a constant called the permeability of free space and in SI units is defined as having the exact numerical value

$$\mu_0\equiv 4\pi\times 10^{-7}\text{ henry/m (kg-m-A}^{-2}\text{-s}^{-2}) \tag{2}$$

The 4π is introduced in (1) for the same reason it was introduced in Coulomb's law in Section 2-2-1. It will cancel out a 4π contribution in frequently used laws that we will soon derive from (1). As for Coulomb's law, the magnetic field drops off inversely as the square of the distance, but its direction is now perpendicular both to the direction of charge flow and to the line joining the charge to the field point.

In the experiments of Ampere and those of Biot and Savart, the charge flow was constrained as a line current within a wire. If the charge is distributed over a line with

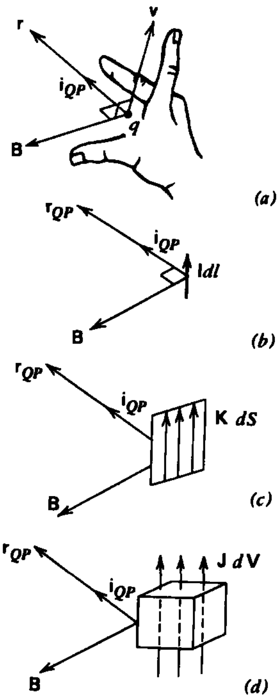


Figure 5-7 The magnetic field generated by a current is perpendicular to the current and the unit vector joining the current element to the field point; (*a*) point charge; (*b*) line current; (*c*) surface current; (*d*) volume current.

current **I**, or a surface with current per unit length **K**, or over a volume with current per unit area **J**, we use the differential-sized current elements, as in Figures 5-7*b*–5-7*d*:

$$dq\,\mathbf{v}=\begin{cases}\mathbf{I}\,dl & \text{(line current)}\\ \mathbf{K}\,dS & \text{(surface current)}\\ \mathbf{J}\,dV & \text{(volume current)}\end{cases} \tag{3}$$

The total magnetic field for a current distribution is then obtained by integrating the contributions from all the incremental elements:

$$\mathbf{B}=\begin{cases}\dfrac{\mu_0}{4\pi}\displaystyle\int_L \frac{\mathbf{I}\,dl\times\mathbf{i}_{QP}}{r_{QP}^2} & \text{(line current)}\\ \dfrac{\mu_0}{4\pi}\displaystyle\int_S \frac{\mathbf{K}\,dS\times\mathbf{i}_{QP}}{r_{QP}^2} & \text{(surface current)}\\ \dfrac{\mu_0}{4\pi}\displaystyle\int_V \frac{\mathbf{J}\,dV\times\mathbf{i}_{QP}}{r_{QP}^2} & \text{(volume current)}\end{cases} \tag{4}$$

The direction of the magnetic field due to a current element is found by the right-hand rule, where if the forefinger of the right hand points in the direction of current and the middle finger in the direction of the field point, then the thumb points in the direction of the magnetic field. This magnetic field **B** can then exert a force on other currents, as given in Section 5-1-1.

5-2-2 Line Currents

A constant current I_1 flows in the z direction along a wire of infinite extent, as in Figure 5-8a. Equivalently, the right-hand rule allows us to put our thumb in the direction of current. Then the fingers on the right hand curl in the direction of **B**, as shown in Figure 5-8a. The unit vector in the direction of the line joining an incremental current element $I_1\,dz$ at z to a field point P is

$$\mathbf{i}_{QP}=\mathbf{i}_r\cos\theta-\mathbf{i}_z\sin\theta=\mathbf{i}_r\frac{r}{r_{QP}}-\mathbf{i}_z\frac{z}{r_{QP}}\tag{5}$$

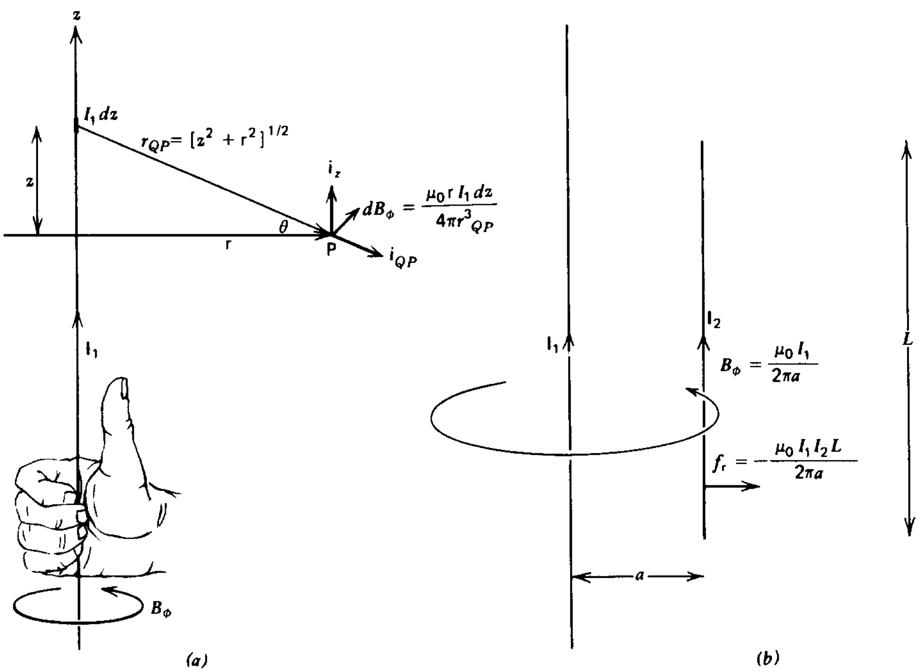


Figure 5-8 (a) The magnetic field due to an infinitely long z-directed line current is in the ϕ direction. (b) Two parallel line currents attract each other if flowing in the same direction and repel if oppositely directed.

with distance

$$r_{QP} = (z^2 + r^2)^{1/2} \tag{6}$$

The magnetic field due to this current element is given by (4) as

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I_1\, dz(\mathbf{i}_z \times \mathbf{i}_{QP})}{r_{QP}^2} = \frac{\mu_0 I_1 r\, dz}{4\pi(z^2 + r^2)^{3/2}} \mathbf{i}_\phi \tag{7}$$

The total magnetic field from the line current is obtained by integrating the contributions from all elements:

$$\begin{aligned} B_\phi &= \frac{\mu_0 I_1 r}{4\pi} \int_{-\infty}^{+\infty} \frac{dz}{(z^2 + r^2)^{3/2}} \\ &= \frac{\mu_0 I_1 r}{4\pi} \left. \frac{z}{r^2(z^2 + r^2)^{1/2}} \right|_{z=-\infty}^{+\infty} \\ &= \frac{\mu_0 I_1}{2\pi r} \end{aligned} \tag{8}$$

If a second line current I_2 of finite length L is placed at a distance a and parallel to I_1, as in Figure 5-8b, the force on I_2 due to the magnetic field of I_1 is

$$\begin{aligned} \mathbf{f} &= \int_{-L/2}^{+L/2} I_2\, dz\, \mathbf{i}_z \times \mathbf{B} \\ &= \int_{-L/2}^{+L/2} I_2\, dz \frac{\mu_0 I_1}{2\pi a} (\mathbf{i}_z \times \mathbf{i}_\phi) \\ &= -\frac{\mu_0 I_1 I_2 L}{2\pi a} \mathbf{i}_r \end{aligned} \tag{9}$$

If both currents flow in the same direction ($I_1 I_2 > 0$), the force is attractive, while if they flow in opposite directions ($I_1 I_2 < 0$), the force is repulsive. This is opposite in sense to the Coulombic force where opposite charges attract and like charges repel.

5-2-3 Current Sheets

(a) Single Sheet of Surface Current

A constant current $K_0\mathbf{i}_z$ flows in the $y = 0$ plane, as in Figure 5-9a. We break the sheet into incremental line currents $K_0\, dx$, each of which gives rise to a magnetic field as given by (8). From Table 1-2, the unit vector $\mathbf{i}_\phi$ is equivalent to the Cartesian components

$$\mathbf{i}_\phi = -\sin\phi\, \mathbf{i}_x + \cos\phi\, \mathbf{i}_y \tag{10}$$

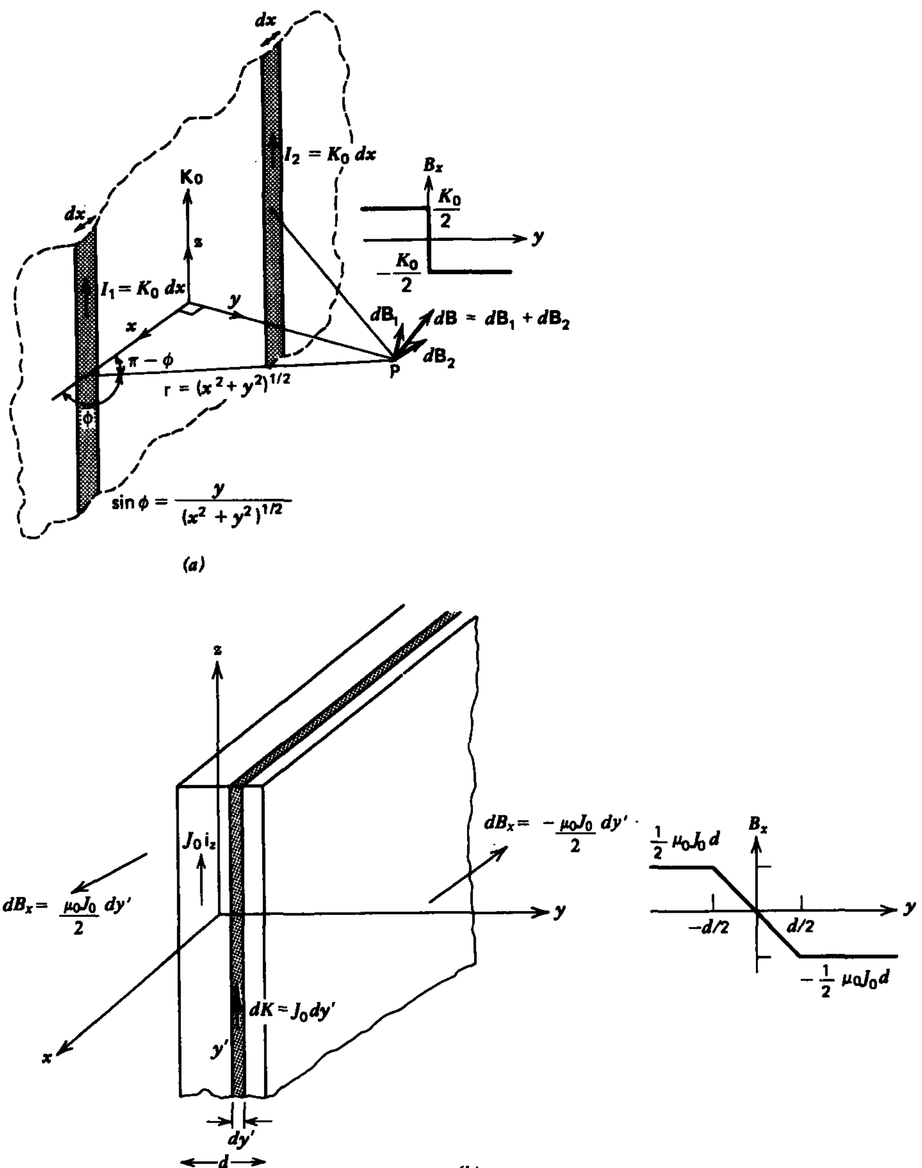


Figure 5-9 (*a*) A uniform surface current of infinite extent generates a uniform magnetic field oppositely directed on each side of the sheet. The magnetic field is perpendicular to the surface current but parallel to the plane of the sheet. (*b*) The magnetic field due to a slab of volume current is found by superimposing the fields due to incremental surface currents. (*c*) Two parallel but oppositely directed surface current sheets have fields that add in the region between the sheets but cancel outside the sheet. (*d*) The force on a current sheet is due to the average field on each side of the sheet as found by modeling the sheet as a uniform volume current distributed over an infinitesimal thickness Δ.

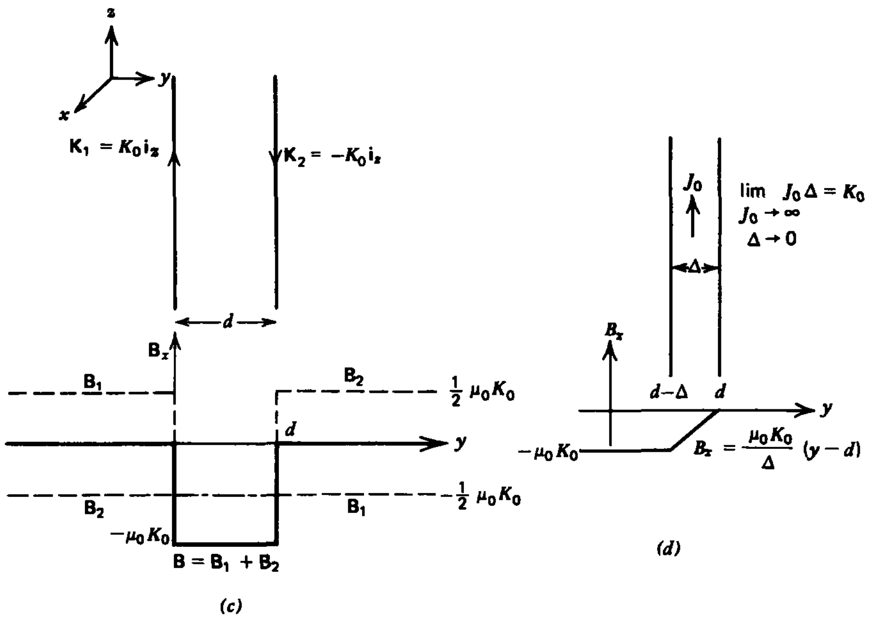


Figure 5-9

The symmetrically located line charge elements a distance x on either side of a point P have y magnetic field components that cancel but x components that add. The total magnetic field is then

$$\begin{aligned} B_x &= -\int_{-\infty}^{+\infty} \frac{\mu_0 K_0 \sin\phi}{2\pi(x^2+y^2)^{1/2}}\,dx \\ &= \frac{-\mu_0 K_0 y}{2\pi}\int_{-\infty}^{+\infty}\frac{dx}{(x^2+y^2)} \\ &= \frac{-\mu_0 K_0}{2\pi}\tan^{-1}\frac{x}{y}\Big|_{-\infty}^{+\infty} \\ &= \begin{cases} -\mu_0 K_0/2, & y>0 \\ \mu_0 K_0/2, & y<0 \end{cases} \end{aligned} \tag{11}$$

The field is constant and oppositely directed on each side of the sheet.

(b) Slab of Volume Current

If the z-directed current $J_0 \mathbf{i}_z$ is uniform over a thickness d, as in Figure 5-9b, we break the slab into incremental current sheets $J_0\,dy'$. The magnetic field from each current sheet is given by (11). When adding the contributions of all the differential-sized sheets, those to the left of a field point give a negatively x directed magnetic field while those to the right contribute a positively x-directed field:

$$B_x = \begin{cases} \displaystyle\int_{-d/2}^{+d/2} \frac{-\mu_0 J_0\, dy'}{2} = \frac{-\mu_0 J_0 d}{2}, & y > \dfrac{d}{2} \\ \displaystyle\int_{-d/2}^{+d/2} \frac{\mu_0 J_0\, dy'}{2} = \frac{\mu_0 J_0 d}{2}, & y < -\dfrac{d}{2} \\ \displaystyle\int_{-d/2}^{y} \frac{-\mu_0 J_0\, dy'}{2} + \int_{y}^{d/2} \frac{\mu_0 J_0\, dy'}{2} = -\mu_0 J_0 y, & -\dfrac{d}{2} \le y \le \dfrac{d}{2} \end{cases} \quad (12)$$

The total force per unit area on the slab is zero:

$$\begin{aligned} F_{S_y} &= \int_{-d/2}^{+d/2} J_0 B_x\, dy = -\mu_0 J_0^2 \int_{-d/2}^{+d/2} y\, dy \\ &= -\mu_0 J_0^2 \frac{y^2}{2}\bigg|_{-d/2}^{+d/2} = 0 \end{aligned} \quad (13)$$

A current distribution cannot exert a net force on itself.

(c) Two Parallel Current Sheets

If a second current sheet with current flowing in the opposite direction $-K_0\mathbf{i}_z$ is placed at $y=d$ parallel to a current sheet $K_0\mathbf{i}_z$ at $y=0$, as in Figure 5-9c, the magnetic field due to each sheet alone is

$$\mathbf{B}_1 = \begin{cases} \dfrac{-\mu_0 K_0}{2}\mathbf{i}_x, & y > 0 \\ \dfrac{\mu_0 K_0}{2}\mathbf{i}_x, & y < 0 \end{cases} \qquad \mathbf{B}_2 = \begin{cases} \dfrac{\mu_0 K_0}{2}\mathbf{i}_x, & y > d \\ \dfrac{-\mu_0 K_0}{2}\mathbf{i}_x, & y < d \end{cases} \quad (14)$$

Thus in the region outside the sheets, the fields cancel while they add in the region between:

$$\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 = \begin{cases} -\mu_0 K_0 \mathbf{i}_x, & 0 < y < d \\ 0, & y < 0, y > d \end{cases} \quad (15)$$

The force on a surface current element on the second sheet is

$$\mathbf{df} = -K_0 \mathbf{i}_z\, dS \times \mathbf{B} \quad (16)$$

However, since the magnetic field is discontinuous at the current sheet, it is not clear which value of magnetic field to use. Tp take the limit properly, we model the current sheet at $y=d$ as a thin volume current with density J_0 and thickness Δ, as in Figure 5-9d, where $K_0 = J_0\Delta$.

The results of (12) show that in a slab of uniform volume current, the magnetic field changes linearly to its values at the surfaces

$$B_x(y=d-\Delta) = -\mu_0 K_0$$
$$B_x(y=d) = 0 \tag{17}$$

so that the magnetic field within the slab is

$$B_x = \frac{\mu_0 K_0}{\Delta}(y-d) \tag{18}$$

The force per unit area on the slab is then

$$\begin{aligned} \mathbf{F}_S &= -\int_{d-\Delta}^{d} \frac{\mu_0 K_0}{\Delta} J_0 (y-d)\mathbf{i}_y\, dy \\ &= \frac{-\mu_0 K_0 J_0}{\Delta}\frac{(y-d)^2}{2}\mathbf{i}_y\Big|_{d-\Delta}^{d} \\ &= \frac{\mu_0 K_0 J_0 \Delta}{2}\mathbf{i}_y = \frac{\mu_0 K_0^2}{2}\mathbf{i}_y \end{aligned} \tag{19}$$

The force acts to separate the sheets because the currents are in opposite directions and thus repel one another.

Just as we found for the electric field on either side of a sheet of surface charge in Section 3-9-1, when the magnetic field is discontinuous on either side of a current sheet $\mathbf{K}$, being $\mathbf{B}_1$ on one side and $\mathbf{B}_2$ on the other, the average magnetic field is used to compute the force on the sheet:

$$\mathbf{df} = \mathbf{K}\, dS \times \frac{(\mathbf{B}_1 + \mathbf{B}_2)}{2} \tag{20}$$

In our case

$$\mathbf{B}_1 = -\mu_0 K_0 \mathbf{i}_x, \qquad \mathbf{B}_2 = 0 \tag{21}$$

5-2-4 Hoops of Line Current

(a) Single hoop

A circular hoop of radius a centered about the origin in the xy plane carries a constant current I, as in Figure 5-10a. The distance from any point on the hoop to a point at z along the z axis is

$$r_{QP} = (z^2 + a^2)^{1/2} \tag{22}$$

in the direction

$$\mathbf{i}_{QP} = \frac{(-a\mathbf{i}_r + z\mathbf{i}_z)}{(z^2+a^2)^{1/2}} \tag{23}$$

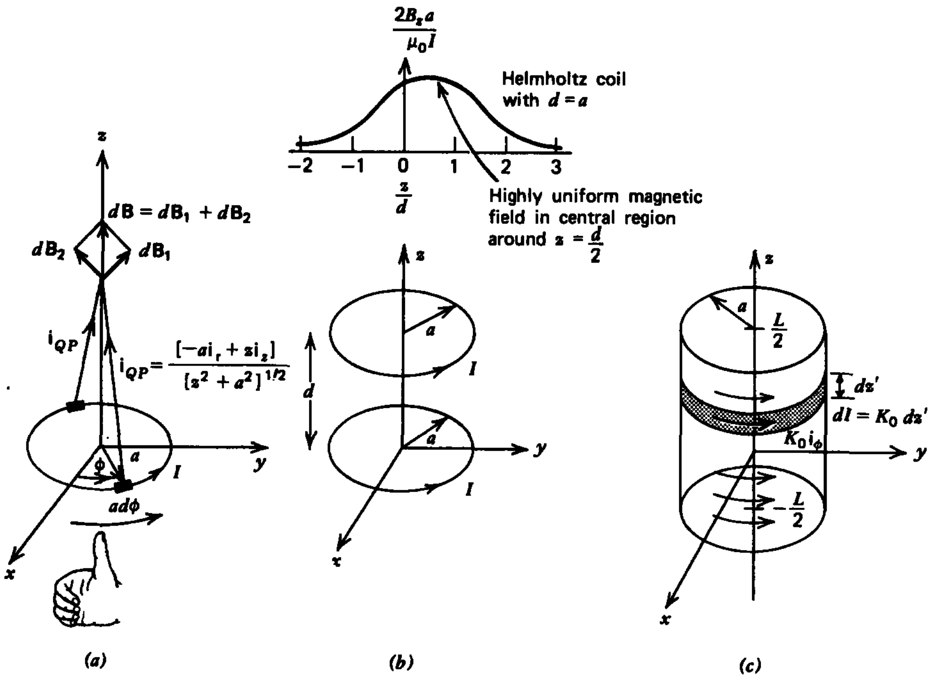


Figure 5-10 (*a*) The magnetic field due to a circular current loop is z directed along the axis of the hoop. (*b*) A Helmholtz coil, formed by two such hoops at a distance apart d equal to their radius, has an essentially uniform field near the center at $z = d/2$. (*c*) The magnetic field on the axis of a cylinder with a ϕ-directed surface current is found by integrating the fields due to incremental current loops.

so that the incremental magnetic field due to a current element of differential size is

$$\begin{aligned} d\mathbf{B} &= \frac{\mu_0}{4\pi r_{QP}^2} Ia\, d\phi\, \mathbf{i}_\phi \times \mathbf{i}_{QP} \\ &= \frac{\mu_0 Ia\, d\phi}{4\pi(z^2+a^2)^{3/2}}(a\mathbf{i}_z + z\mathbf{i}_r) \end{aligned} \tag{24}$$

The radial unit vector changes direction as a function of ϕ, being oppositely directed at $-\phi$, so that the total magnetic field due to the whole hoop is purely z directed:

$$\begin{aligned} B_z &= \frac{\mu_0 Ia^2}{4\pi(z^2+a^2)^{3/2}} \int_0^{2\pi} d\phi \\ &= \frac{\mu_0 Ia^2}{2(z^2+a^2)^{3/2}} \end{aligned} \tag{25}$$

The direction of the magnetic field can be checked using the right-hand rule. Curling the fingers on the right hand in the direction of the current puts the thumb in the direction of the magnetic field. Note that the magnetic field along the z axis is positively z directed both above and below the hoop.

(b) Two Hoops (Helmholtz Coil)

Often it is desired to have an accessible region in space with an essentially uniform magnetic field. This can be arıanged by placing another coil at $z=d$, as in Figure 5-10*b*. Then the total magnetic field along the z axis is found by superposing the field of (25) for each hoop:

$$B_z=\frac{\mu_0 I a^2}{2}\left(\frac{1}{(z^2+a^2)^{3/2}}+\frac{1}{((z-d)^2+a^2)^{3/2}}\right) \tag{26}$$

We see then that the slope of B_z,

$$\frac{\partial B_z}{\partial z}=\frac{3\mu_0 I a^2}{2}\left(\frac{-z}{(z^2+a^2)^{5/2}}-\frac{(z-d)}{((z-d)^2+a^2)^{5/2}}\right) \tag{27}$$

is zero at $z=d/2$. The second derivative,

$$\frac{\partial^2 B_z}{\partial z^2}=\frac{3\mu_0 I a^2}{2}\left(\frac{5z^2}{(z^2+a^2)^{7/2}}-\frac{1}{(z^2+a^2)^{5/2}}+\frac{5(z-d)^2}{((z-d)^2+a^2)^{7/2}}-\frac{1}{((z-d)^2+a^2)^{5/2}}\right) \tag{28}$$

can also be set to zero at $z=d/2$, if $d=a$, giving a highly uniform field around the center of the system, as plotted in Figure 5-10*b*. Such a configuration is called a Helmholtz coil.

(c) Hollow Cylinder of Surface Current

A hollow cylinder of length L and radius a has a uniform surface current $K_0\mathbf{i}_\phi$ as in Figure 5-10*c*. Such a configuration is arranged in practice by tightly winding N turns of a wire around a cylinder and imposing a current I through the wire. Then the current per unit length is

$$K_0=NI/L \tag{29}$$

The magnetic field along the z axis at the position z due to each incremental hoop at z' is found from (25) by replacing z by $(z-z')$ and I by $K_0\,dz'$:

$$dB_z=\frac{\mu_0 a^2 K_0\,dz'}{2[(z-z')^2+a^2]^{3/2}} \tag{30}$$

The total axial magnetic field is then

$$B_z = \int_{z'=-L/2}^{+L/2} \frac{\mu_0 a^2 K_0}{2} \frac{dz'}{[(z-z')^2+a^2]^{3/2}}$$
$$= \frac{\mu_0 a^2 K_0}{2} \frac{(z'-z)}{a^2[(z-z')^2+a^2]^{1/2}}\Bigg|_{z'=-L/2}^{+L/2}$$
$$= \frac{\mu_0 K_0}{2}\left(\frac{-z+L/2}{[(z-L/2)^2+a^2]^{1/2}} + \frac{z+L/2}{[(z+L/2)^2+a^2]^{1/2}}\right) \quad (31)$$

As the cylinder becomes very long, the magnetic field far from the ends becomes approximately constant

$$\lim_{L\to\infty} B_z = \mu_0 K_0 \quad (32)$$

5-3 DIVERGENCE AND CURL OF THE MAGNETIC FIELD

Because of our success in examining various vector operations on the electric field, it is worthwhile to perform similar operations on the magnetic field. We will need to use the following vector identities from Section 1-5-4, Problem 1-24 and Sections 2-4-1 and 2-4-2:

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \quad (1)$$

$$\nabla \times (\nabla f) = 0 \quad (2)$$

$$\nabla\left(\frac{1}{r_{QP}}\right) = -\frac{\mathbf{i}_{QP}}{r_{QP}^2} \quad (3)$$

$$\int_V \nabla^2\left(\frac{1}{r_{QP}}\right) dV = \begin{cases} 0, & r_{QP} \neq 0 \\ -4\pi, & r_{QP} = 0 \end{cases} \quad (4)$$

$$\nabla \cdot (\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot\nabla\times\mathbf{B} \quad (5)$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + (\nabla\cdot\mathbf{B})\mathbf{A} - (\nabla\cdot\mathbf{A})\mathbf{B} \quad (6)$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) \quad (7)$$

5-3-1 Gauss's Law for the Magnetic Field

Using (3) the magnetic field due to a volume distribution of current $\mathbf{J}$ is rewritten as

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}\times\mathbf{i}_{QP}}{r_{QP}^2} dV$$
$$= \frac{-\mu_0}{4\pi}\int_V \mathbf{J}\times\nabla\left(\frac{1}{r_{QP}}\right) dV \quad (8)$$

If we take the divergence of the magnetic field with respect to field coordinates, the del operator can be brought inside the integral as the integral is only over the source coordinates:

$$\nabla\cdot\mathbf{B}=\frac{-\mu_0}{4\pi}\int_V \nabla\cdot\left[\mathbf{J}\times\nabla\left(\frac{1}{r_{QP}}\right)\right]dV \tag{9}$$

The integrand can be expanded using (5)

$$\nabla\cdot\left[\mathbf{J}\times\nabla\left(\frac{1}{r_{QP}}\right)\right]=\nabla\left(\frac{1}{r_{QP}}\right)\cdot\underbrace{(\nabla\times\mathbf{J})}_{0}-\mathbf{J}\cdot\underbrace{\nabla\times\left[\nabla\left(\frac{1}{r_{QP}}\right)\right]}_{0}=0 \tag{10}$$

The first term on the right-hand side in (10) is zero because **J** is not a function of field coordinates, while the second term is zero from (2), the curl of the gradient is always zero. Then (9) reduces to

$$\nabla\cdot\mathbf{B}=0 \tag{11}$$

This contrasts with Gauss's law for the displacement field where the right-hand side is equal to the electric charge density. Since nobody has yet discovered any net magnetic charge, there is no source term on the right-hand side of (11).

The divergence theorem gives us the equivalent integral representation

$$\int_V \nabla\cdot\mathbf{B}\,dV=\oint_S \mathbf{B}\cdot d\mathbf{S}=0 \tag{12}$$

which tells us that the net magnetic flux through a closed surface is always zero. As much flux enters a surface as leaves it. Since there are no magnetic charges to terminate the magnetic field, the field lines are always closed.

5-3-2 Ampere's Circuital Law

We similarly take the curl of (8) to obtain

$$\nabla\times\mathbf{B}=\frac{-\mu_0}{4\pi}\int_V \nabla\times\left[\mathbf{J}\times\nabla\left(\frac{1}{r_{QP}}\right)\right]dV \tag{13}$$

where again the del operator can be brought inside the integral and only operates on r_{QP}.

We expand the integrand using (6):

$$\nabla\times\left[\mathbf{J}\times\nabla\left(\frac{1}{r_{QP}}\right)\right]=\underbrace{\left[\nabla\left(\frac{1}{r_{QP}}\right)\cdot\nabla\right]\mathbf{J}}_{0}-(\mathbf{J}\cdot\nabla)\nabla\left(\frac{1}{r_{QP}}\right)+\left[\nabla^2\left(\frac{1}{r_{QP}}\right)\right]\mathbf{J}-\underbrace{(\nabla\cdot\mathbf{J})}_{0}\nabla\left(\frac{1}{r_{QP}}\right) \tag{14}$$

where two terms on the right-hand side are zero because **J** is not a function of the field coordinates. Using the identity of (7),

$$\nabla\left[\mathbf{J}\cdot\nabla\left(\frac{1}{r_{QP}}\right)\right]=\underbrace{\left[\nabla\left(\frac{1}{r_{QP}}\right)\cdot\nabla\right]\mathbf{J}}_{0}+(\mathbf{J}\cdot\nabla)\nabla\left(\frac{1}{r_{QP}}\right)$$
$$+\nabla\left(\frac{1}{r_{QP}}\right)\times\underbrace{(\nabla\times\mathbf{J})}_{0}+\mathbf{J}\times\underbrace{\left[\nabla\times\nabla\left(\frac{1}{r_{QP}}\right)\right]}_{0} \tag{15}$$

the second term on the right-hand side of (14) can be related to a pure gradient of a quantity because the first and third terms on the right of (15) are zero since **J** is not a function of field coordinates. The last term in (15) is zero because the curl of a gradient is always zero. Using (14) and (15), (13) can be rewritten as

$$\nabla\times\mathbf{B}=\frac{\mu_0}{4\pi}\int_V\left\{\nabla\left[\mathbf{J}\cdot\nabla\left(\frac{1}{r_{QP}}\right)\right]-\mathbf{J}\nabla^2\left(\frac{1}{r_{QP}}\right)\right\}dV \tag{16}$$

Using the gradient theorem, a corollary to the divergence theorem, (see Problem 1-15*a*), the first volume integral is converted to a surface integral

$$\nabla\times\mathbf{B}=\frac{\mu_0}{4\pi}\left[\underbrace{\int_S\mathbf{J}\cdot\nabla\left(\frac{1}{r_{QP}}\right)\mathbf{dS}}_{0}-\int_V\mathbf{J}\nabla^2\left(\frac{1}{r_{QP}}\right)dV\right] \tag{17}$$

This surface completely surrounds the current distribution so that S is outside in a zero current region where $\mathbf{J}=0$ so that the surface integral is zero. The remaining volume integral is nonzero only when $r_{QP}=0$, so that using (4) we finally obtain

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J} \tag{18}$$

which is known as Ampere's law.

Stokes' theorem applied to (18) results in Ampere's circuital law:

$$\int_S\nabla\times\frac{\mathbf{B}}{\mu_0}\cdot\mathbf{dS}=\oint_L\frac{\mathbf{B}}{\mu_0}\cdot\mathbf{dl}=\int_S\mathbf{J}\cdot\mathbf{dS} \tag{19}$$

Like Gauss's law, choosing the right contour based on symmetry arguments often allows easy solutions for **B**.

If we take the divergence of both sides of (18), the left-hand side is zero because the divergence of the curl of a vector is always zero. This requires that magnetic field systems have divergence-free currents so that charge cannot accumulate. Currents must always flow in closed loops.

5-3-3 Currents With Cylindrical Symmetry

(a) Surface Current

A surface current $K_0\mathbf{i}_z$ flows on the surface of an infinitely long hollow cylinder of radius a. Consider the two symmetrically located line charge elements $dI = K_0\, a d\phi$ and their effective fields at a point P in Figure 5-11a. The magnetic field due to both current elements cancel in the radial direction but add in the ϕ direction. The total magnetic field can be found by doing a difficult integration over ϕ. However,

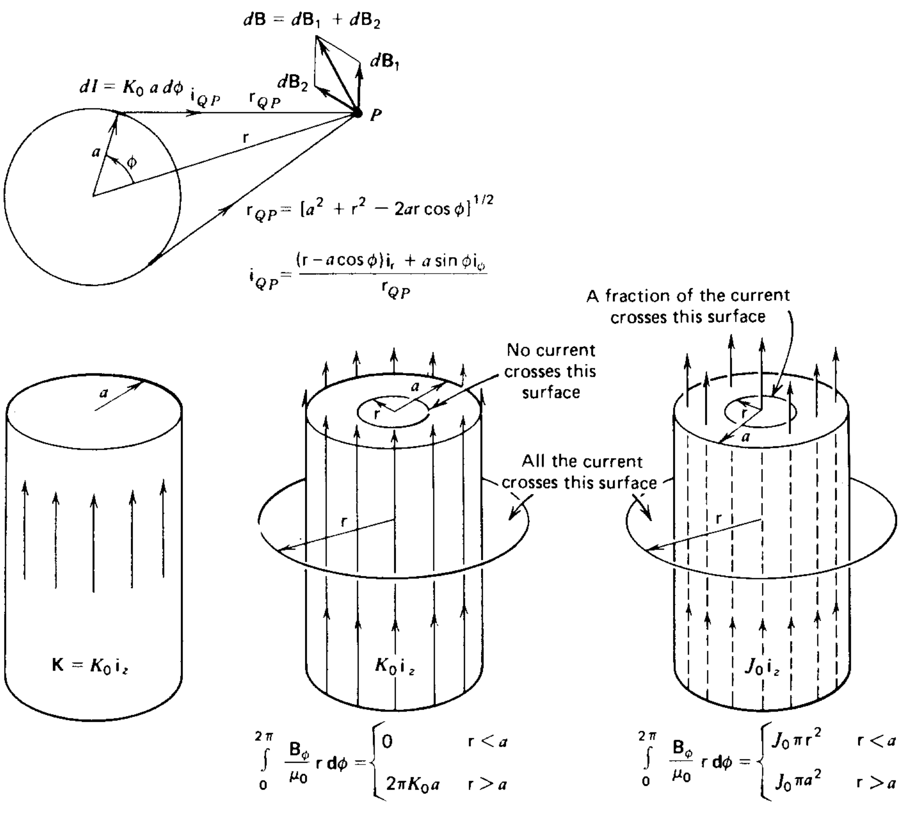


Figure 5-11 (a) The magnetic field of an infinitely long cylinder carrying a surface current parallel to its axis can be found using the Biot–Savart law for each incremental line current element. Symmetrically located elements have radial field components that cancel but ϕ field components that add. (b) Now that we know that the field is purely ϕ directed, it is easier to use Ampere's circuital law for a circular contour concentric with the cylinder. For $r < a$ no current passes through the contour while for $r > a$ all the current passes through the contour. (c) If the current is uniformly distributed over the cylinder the smaller contour now encloses a fraction of the current.

using Ampere's circuital law of (19) is much easier. Since we know the magnetic field is ϕ directed and by symmetry can only depend on r and not ϕ or z, we pick a circular contour of constant radius r as in Figure 5-11b. Since $\mathbf{dl} = r\,d\phi\,\mathbf{i}_\phi$ is in the same direction as $\mathbf{B}$, the dot product between the magnetic field and $\mathbf{dl}$ becomes a pure multiplication. For $r<a$ no current passes through the surface enclosed by the contour, while for $r>a$ all the current is purely perpendicular to the normal to the surface of the contour:

$$\oint_L \frac{\mathbf{B}}{\mu_0}\cdot \mathbf{dl} = \int_0^{2\pi} \frac{B_\phi}{\mu_0} r\,d\phi = \frac{2\pi r B_\phi}{\mu_0} = \begin{cases} K_0 2\pi a = I, & r>a \\ 0, & r<a \end{cases} \tag{20}$$

where I is the total current on the cylinder.

The magnetic field is thus

$$B_\phi = \begin{cases} \mu_0 K_0 a/r = \mu_0 I/(2\pi r), & r>a \\ 0, & r<a \end{cases} \tag{21}$$

Outside the cylinder, the magnetic field is the same as if all the current was concentrated along the axis as a line current.

(b) Volume Current

If the cylinder has the current uniformly distributed over the volume as $J_0\mathbf{i}_z$, the contour surrounding the whole cylinder still has the total current $I = J_0\pi a^2$ passing through it. If the contour has a radius smaller than that of the cylinder, only the fraction of current proportional to the enclosed area passes through the surface as shown in Figure 5-11c:

$$\oint_L \frac{B_\phi}{\mu_0} r\,d\phi = \frac{2\pi r B_\phi}{\mu_0} = \begin{cases} J_0\pi a^2 = I, & r>a \\ J_0\pi r^2 = I r^2/a^2, & r<a \end{cases} \tag{22}$$

so that the magnetic field is

$$B_\phi = \begin{cases} \dfrac{\mu_0 J_0 a^2}{2r} = \dfrac{\mu_0 I}{2\pi r}, & r>a \\[2ex] \dfrac{\mu_0 J_0 r}{2} = \dfrac{\mu_0 I r}{2\pi a^2}, & r<a \end{cases} \tag{23}$$

5-4 THE VECTOR POTENTIAL

5-4-1 Uniqueness

Since the divergence of the magnetic field is zero, we may write the magnetic field as the curl of a vector,

$$\nabla\cdot\mathbf{B} = 0 \Rightarrow \mathbf{B} = \nabla\times\mathbf{A} \tag{1}$$

where **A** is called the vector potential, as the divergence of the curl of any vector is always zero. Often it is easier to calculate **A** and then obtain the magnetic field from (1).

From Ampere's law, the vector potential is related to the current density as

$$\nabla\times\mathbf{B}=\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}=\mu_0\mathbf{J} \tag{2}$$

We see that (1) does not uniquely define **A**, as we can add the gradient of any term to **A** and not change the value of the magnetic field, since the curl of the gradient of any function is always zero:

$$\mathbf{A}\rightarrow\mathbf{A}+\nabla f\Rightarrow\mathbf{B}=\nabla\times(\mathbf{A}+\nabla f)=\nabla\times\mathbf{A} \tag{3}$$

Helmholtz's theorem states that to uniquely specify a vector, both its curl and divergence must be specified and that far from the sources, the fields must approach zero. To prove this theorem, let's say that we are given the curl and divergence of **A** and we are to determine what **A** is. Is there any other vector **C**, different from **A** that has the same curl and divergence? We try **C** of the form

$$\mathbf{C}=\mathbf{A}+\mathbf{a} \tag{4}$$

and we will prove that **a** is zero.

By definition, the curl of **C** must equal the curl of **A** so that the curl of **a** must be zero:

$$\nabla\times\mathbf{C}=\nabla\times(\mathbf{A}+\mathbf{a})=\nabla\times\mathbf{A}\Rightarrow\nabla\times\mathbf{a}=0 \tag{5}$$

This requires that **a** be derivable from the gradient of a scalar function f:

$$\nabla\times\mathbf{a}=0\Rightarrow\mathbf{a}=\nabla f \tag{6}$$

Similarly, the divergence condition requires that the divergence of **a** be zero,

$$\nabla\cdot\mathbf{C}=\nabla\cdot(\mathbf{A}+\mathbf{a})=\nabla\cdot\mathbf{A}\Rightarrow\nabla\cdot\mathbf{a}=0 \tag{7}$$

so that the Laplacian of f must be zero,

$$\nabla\cdot\mathbf{a}=\nabla^2 f=0 \tag{8}$$

In Chapter 2 we obtained a similar equation and solution for the electric potential that goes to zero far from the charge distribution:

$$\nabla^2 V = -\frac{\rho}{\varepsilon} \Rightarrow V = \int_V \frac{\rho\, dV}{4\pi\varepsilon r_{QP}} \tag{9}$$

If we equate f to V, then ρ must be zero giving us that the scalar function f is also zero. That is, the solution to Laplace's equation of (8) for zero sources everywhere is zero, even though Laplace's equation in a region does have nonzero solutions if there are sources in other regions of space. With f zero, from (6) we have that the vector $\mathbf{a}$ is also zero and then $\mathbf{C} = \mathbf{A}$, thereby proving Helmholtz's theorem.

5-4-2 The Vector Potential of a Current Distribution

Since we are free to specify the divergence of the vector potential, we take the simplest case and set it to zero:

$$\nabla \cdot \mathbf{A} = 0 \tag{10}$$

Then (2) reduces to

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \tag{11}$$

Each vector component of (11) is just Poisson's equation so that the solution is also analogous to (9)

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}\, dV}{r_{QP}} \tag{12}$$

The vector potential is often easier to use since it is in the same direction as the current, and we can avoid the often complicated cross product in the Biot–Savart law. For moving point charges, as well as for surface and line currents, we use (12) with the appropriate current elements:

$$\mathbf{J}\, dV \to \mathbf{K}\, dS \to \mathbf{I}\, dL \to q\mathbf{v} \tag{13}$$

5-4-3 The Vector Potential and Magnetic Flux

Using Stokes' theorem, the magnetic flux through a surface can be expressed in terms of a line integral of the vector potential:

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_S \nabla \times \mathbf{A} \cdot d\mathbf{S} = \oint_L \mathbf{A} \cdot d\mathbf{l} \tag{14}$$

(a) Finite Length Line Current

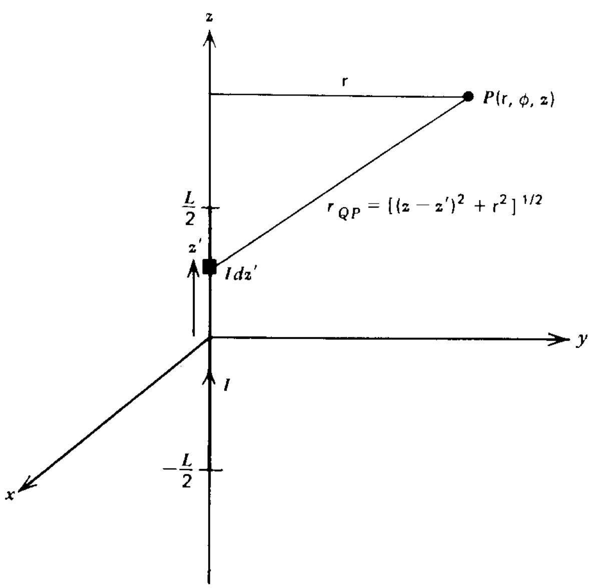
The problem of a line current I of length L, as in Figure 5-12a appears to be nonphysical as the current must be continuous. However, we can imagine this line current to be part of a closed loop and we calculate the vector potential and magnetic field from this part of the loop.

The distance r_{QP} from the current element $I\,dz'$ to the field point at coordinate (r, ϕ, z) is

$$r_{QP} = [(z-z')^2 + \mathrm{r}^2]^{1/2} \tag{15}$$

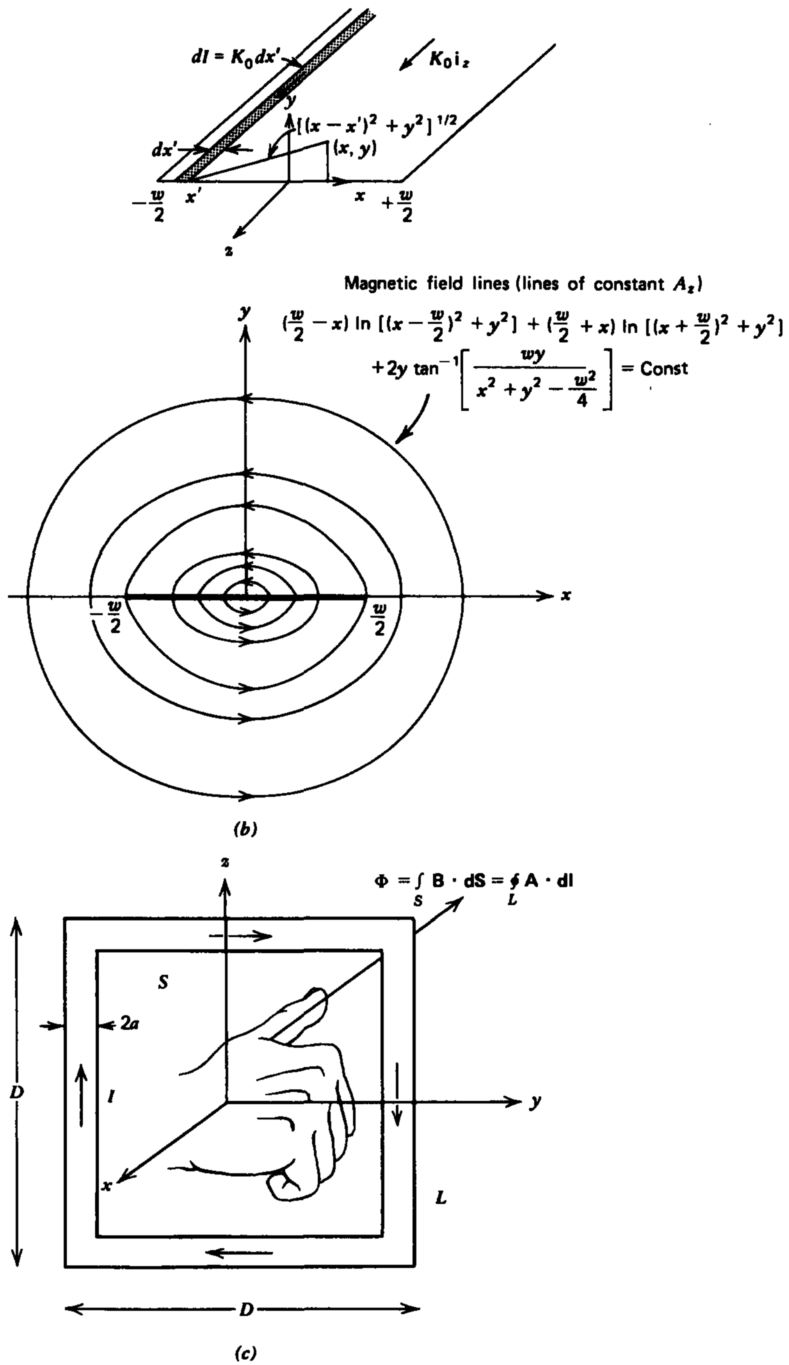
The vector potential is then

$$\begin{aligned} A_z &= \frac{\mu_0 I}{4\pi}\int_{-L/2}^{L/2} \frac{dz'}{[(z-z')^2+\mathrm{r}^2]^{1/2}} \\ &= \frac{\mu_0 I}{4\pi} \ln\left(\frac{-z+L/2+[(z-L/2)^2+\mathrm{r}^2]^{1/2}}{-(z+L/2)+[(z+L/2)^2+\mathrm{r}^2]^{1/2}}\right) \\ &= \frac{\mu_0 I}{4\pi}\left(\sinh^{-1}\frac{-z+L/2}{\mathrm{r}} + \sinh^{-1}\frac{z+L/2}{\mathrm{r}}\right) \end{aligned} \tag{16}$$



(a)

Figure 5-12 (a) The magnetic field due to a finite length line current is most easily found using the vector potential, which is in the direction of the current. This problem is physical only if the line current is considered to be part of a closed loop. (b) The magnetic field from a length w of surface current is found by superposing the vector potential of (a) with $L \to \infty$. The field lines are lines of constant A_z. (c) The magnetic flux through a square current loop is in the $-x$ direction by the right-hand rule.



dI = K0dx'
K0iz
y
[(x − x')² + y²]^1/2
(x, y)
dx'
−w/2
x'
x
+w/2
z
Magnetic field lines (lines of constant Az)
(w/2 − x) ln [(x − w/2)² + y²] + (w/2 + x) ln [(x + w/2)² + y²]
+2y tan⁻¹[wy / (x² + y² − w²/4)] = Const
y
x
−w/2
w/2
(b)
z
Φ = ∫S B · dS = ∮L A · dl
S
2a
D
I
y
x
L
D
(c)

Figure 5-12

with associated magnetic field

$$\begin{aligned}
\mathbf{B} &= \nabla \times \mathbf{A} \\
&= \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\mathbf{i}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\mathbf{i}_\phi + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right)\mathbf{i}_z \\
&= -\frac{\partial A_z}{\partial r}\mathbf{i}_\phi \\
&= \frac{-\mu_0 I r}{4\pi}\left(\frac{1}{[(z-L/2)^2+r^2]^{1/2}\{-z+L/2+[(z-L/2)^2+r^2]^{1/2}\}}\right. \\
&\quad \left. - \frac{1}{[(z+L/2)^2+r^2]^{1/2}\{-(z+L/2)+[(z+L/2)^2+r^2]^{1/2}\}}\right)\mathbf{i}_\phi \\
&= \frac{\mu_0 I}{4\pi r}\left(\frac{-z+L/2}{[r^2+(z-L/2)^2]^{1/2}} + \frac{z+L/2}{[r^2+(z+L/2)^2]^{1/2}}\right)\mathbf{i}_\phi
\end{aligned} \tag{17}$$

For large L, (17) approaches the field of an infinitely long line current as given in Section 5-2-2:

$$\lim_{L\to\infty}\begin{cases} A_z = \dfrac{-\mu_0 I}{2\pi}\ln r + \text{const} \\[2mm] B_\phi = -\dfrac{\partial A_z}{\partial r} = \dfrac{\mu_0 I}{2\pi r} \end{cases} \tag{18}$$

Note that the vector potential constant in (18) is infinite, but this is unimportant as this constant has no contribution to the magnetic field.

(b) Finite Width Surface Current

If a surface current $K_0\mathbf{i}_z$, of width w, is formed by laying together many line current elements, as in Figure 5-12b, the vector potential at (x, y) from the line current element $K_0\,dx'$ at position x' is given by (18):

$$dA_z = \frac{-\mu_0 K_0\,dx'}{4\pi}\ln[(x-x')^2+y^2] \tag{19}$$

The total vector potential is found by integrating over all elements:

$$A_z = \frac{-\mu_0 K_0}{4\pi} \int_{-w/2}^{+w/2} \ln\,[(x-x')^2+y^2]\,dx'$$

$$= \frac{-\mu_0 K_0}{4\pi}\left((x'-x)\ln\,[(x-x')^2+y^2] - 2(x'-x)\right.$$

$$\left.+2y\tan^{-1}\frac{(x'-x)}{y}\right)\Bigg|_{-w/2}^{+w/2}$$

$$= \frac{-\mu_0 K_0}{4\pi}\left\{\left(\frac{w}{2}-x\right)\ln\left[\left(x-\frac{w}{2}\right)^2+y^2\right]\right.$$

$$+\left(\frac{w}{2}+x\right)\ln\left[\left(x+\frac{w}{2}\right)^2+y^2\right]$$

$$\left.-2w+2y\tan^{-1}\frac{wy}{y^2+x^2-w^2/4}\right\} \qquad (20)^*$$

The magnetic field is then

$$\mathbf{B} = \mathbf{i}_x \frac{\partial A_z}{\partial y} - \mathbf{i}_y \frac{\partial A_z}{\partial x}$$

$$= \frac{-\mu_0 K_0}{4\pi}\left(2\tan^{-1}\frac{wy}{y^2+x^2-w^2/4}\mathbf{i}_x + \ln\frac{(x+w/2)^2+y^2}{(x-w/2)^2+y^2}\mathbf{i}_y\right) \qquad (21)$$

The vector potential in two-dimensional geometries is also useful in plotting field lines,

$$\frac{dy}{dx} = \frac{B_y}{B_x} = \frac{-\partial A_z/\partial x}{\partial A_z/\partial y} \qquad (22)$$

for if we cross multiply (22),

$$\frac{\partial A_z}{\partial x}dx + \frac{\partial A_z}{\partial y}dy = dA_z = 0 \Rightarrow A_z = \text{const} \qquad (23)$$

we see that it is constant on a field line. The field lines in Figure 5-12*b* are just lines of constant A_z. The vector potential thus plays the same role as the electric stream function in Sections 4.3.2*b* and 4.4.3*b*.

(c) Flux Through a Square Loop

The vector potential for the square loop in Figure 5-12*c* with very small radius *a* is found by superposing (16) for each side with each component of **A** in the same direction as the current in each leg. The resulting magnetic field is then given by four

$^*\tan^{-1}(a-b)+\tan^{-1}(a+b)=\tan^{-1}\dfrac{2a}{1-a^2+b^2}$

terms like that in (17) so that the flux can be directly computed by integrating the normal component of **B** over the loop area. This method is straightforward but the algebra is cumbersome.

An easier method is to use (14) since we already know the vector potential along each leg. We pick a contour that runs along the inside wire boundary at small radius a. Since each leg is identical, we only have to integrate over one leg, then multiply the result by 4:

$$\begin{aligned}
\Phi &= 4 \int_{\substack{a-D/2 \\ r=a}}^{-a+D/2} A_z\, dz \\
&= \frac{\mu_0 I}{\pi} \int_{a-D/2}^{-a+D/2} \left(\sinh^{-1} \frac{-z+D/2}{a} + \sinh^{-1} \frac{z+D/2}{a}\right) dz \\
&= \frac{\mu_0 I}{\pi} \left\{ -\left(\frac{D}{2} - z\right) \sinh^{-1} \frac{-z+D/2}{a} + \left[\left(\frac{D}{2} - z\right)^2 + a^2\right]^{1/2} \right. \\
&\quad \left. + \left(\frac{D}{2} + z\right) \sinh^{-1} \frac{z+D/2}{a} - \left[\left(\frac{D}{2} + z\right)^2 + a^2\right]^{1/2} \right\} \Bigg|_{a-D/2}^{-a+D/2} \\
&= 2 \frac{\mu_0 I}{\pi} \left(-a \sinh^{-1} 1 + a\sqrt{2} + (D-a) \sinh^{-1} \frac{D-a}{a} \right. \\
&\quad \left. - [(D-a)^2 + a^2]^{1/2} \right)
\end{aligned} \tag{24}$$

As a becomes very small, (24) reduces to

$$\lim_{a \to 0} \Phi = 2 \frac{\mu_0 I}{\pi} D \left(\sinh^{-1} \left(\frac{D}{a}\right) - 1\right) \tag{25}$$

We see that the flux through the loop is proportional to the current. This proportionality constant is called the self-inductance and is only a function of the geometry:

$$L = \frac{\Phi}{I} = 2 \frac{\mu_0 D}{\pi} \left(\sinh^{-1} \left(\frac{D}{a}\right) - 1\right) \tag{26}$$

Inductance is more fully developed in Chapter 6.

5-5 MAGNETIZATION

Our development thus far has been restricted to magnetic fields in free space arising from imposed current distributions. Just as small charge displacements in dielectric materials contributed to the electric field, atomic motions constitute microscopic currents, which also contribute to the magnetic field. There is a direct analogy between polarization and magnetization, so our development will parallel that of Section 3-1.

5-5-1 The Magnetic Dipole

Classical atomic models describe an atom as orbiting electrons about a positively charged nucleus, as in Figure 5-13.

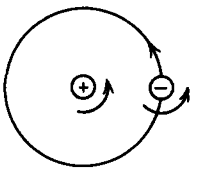


Figure 5-13 Atomic currents arise from orbiting electrons in addition to the spin contributions from the electron and nucleus.

The nucleus and electron can also be imagined to be spinning. The simplest model for these atomic currents is analogous to the electric dipole and consists of a small current loop of area dS carrying a current I, as in Figure 5-14. Because atomic dimensions are so small, we are only interested in the magnetic field far from this magnetic dipole. Then the shape of the loop is not important, thus for simplicity we take it to be rectangular.

The vector potential for this loop is then

$$\mathbf{A}=\frac{\mu_0 I}{4\pi}\left[dx\left(\frac{1}{r_3}-\frac{1}{r_1}\right)\mathbf{i}_x+dy\left(\frac{1}{r_4}-\frac{1}{r_2}\right)\mathbf{i}_y\right] \tag{1}$$

where we assume that the distance from any point on each side of the loop to the field point P is approximately constant.

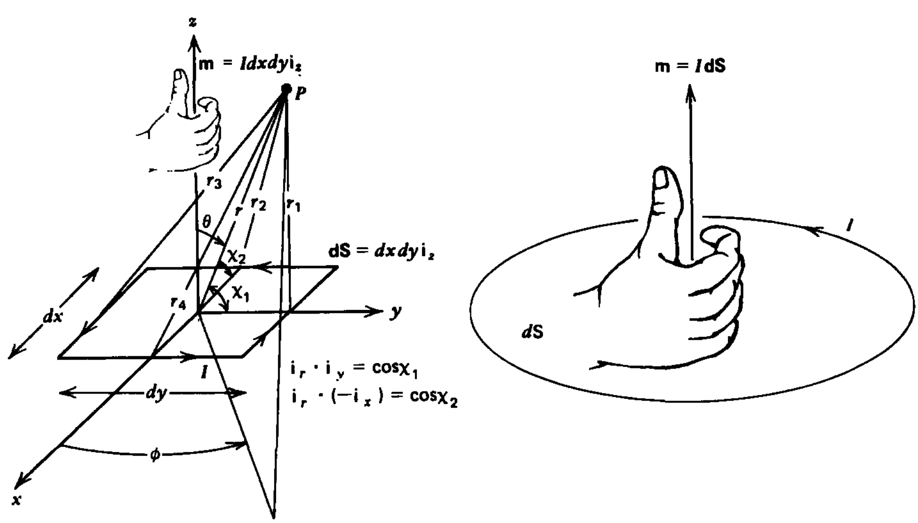


Figure 5-14 A magnetic dipole consists of a small circulating current loop. The magnetic moment is in the direction normal to the loop by the right-hand rule.

Using the law of cosines, these distances are related as

$$r_1^2 = r^2 + \left(\frac{dy}{2}\right)^2 - r\,dy\cos\chi_1; \qquad r_2^2 = r^2 + \left(\frac{dx}{2}\right)^2 - r\,dx\cos\chi_2$$
$$r_3^2 = r^2 + \left(\frac{dy}{2}\right)^2 + r\,dy\cos\chi_1, \qquad r_4^2 = r^2 + \left(\frac{dx}{2}\right)^2 + r\,dx\cos\chi_2 \tag{2}$$

where the angles χ_1 and χ_2 are related to the spherical coordinates from Table 1-2 as

$$\mathbf{i}_r \cdot \mathbf{i}_y = \cos\chi_1 = \sin\theta\sin\phi$$
$$-\mathbf{i}_r \cdot \mathbf{i}_x = \cos\chi_2 = -\sin\theta\cos\phi \tag{3}$$

In the far field limit (1) becomes

$$\lim_{\substack{r\gg dx\\ r\gg dy}} \mathbf{A} = \frac{\mu_0 I}{4\pi}\left[\frac{dx}{r}\left(\frac{1}{\left[1+\frac{dy}{2r}\left(\frac{dy}{2r}+2\cos\chi_1\right)\right]^{1/2}} - \frac{1}{\left[1+\frac{dy}{2r}\left(\frac{dy}{2r}-2\cos\chi_1\right)\right]^{1/2}}\right)\right.$$
$$\left.+\frac{dy}{r}\left(\frac{1}{\left[1+\frac{dx}{2r}\left(\frac{dx}{2r}+2\cos\chi_2\right)\right]^{1/2}} - \frac{1}{\left[1+\frac{dx}{2r}\left(\frac{dx}{2r}-2\cos\chi_2\right)\right]^{1/2}}\right)\right]$$
$$\approx \frac{-\mu_0 I}{4\pi r^2}\,dx\,dy[\cos\chi_1\mathbf{i}_x + \cos\chi_2\mathbf{i}_y] \tag{4}$$

Using (3), (4) further reduces to

$$\mathbf{A} = \frac{\mu_0 I\,dS}{4\pi r^2}\sin\theta[-\sin\phi\,\mathbf{i}_x + \cos\phi\,\mathbf{i}_y]$$
$$= \frac{\mu_0 I\,dS}{4\pi r^2}\sin\theta\,\mathbf{i}_\phi \tag{5}$$

where we again used Table 1-2 to write the bracketed Cartesian unit vector term as $\mathbf{i}_\phi$. The magnetic dipole moment $\mathbf{m}$ is defined as the vector in the direction perpendicular to the loop (in this case $\mathbf{i}_z$) by the right-hand rule with magnitude equal to the product of the current and loop area:

$$\mathbf{m} = I\,dS\,\mathbf{i}_z = I\,\mathbf{dS} \tag{6}$$

Then the vector potential can be more generally written as

$$\mathbf{A}=\frac{\mu_0 m}{4\pi r^2}\sin\theta\,\mathbf{i}_\phi=\frac{\mu_0 \mathbf{m}}{4\pi r^2}\times\mathbf{i}_r \tag{7}$$

with associated magnetic field

$$\begin{aligned}\mathbf{B}=\nabla\times\mathbf{A}&=\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\phi\sin\theta)\mathbf{i}_r-\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\mathbf{i}_\theta\\&=\frac{\mu_0 m}{4\pi r^3}[2\cos\theta\,\mathbf{i}_r+\sin\theta\,\mathbf{i}_\theta]\end{aligned} \tag{8}$$

This field is identical in form to the electric dipole field of Section 3-1-1 if we replace p/ε_0 by $\mu_0 m$.

5-5-2 Magnetization Currents

Ampere modeled magnetic materials as having the volume filled with such infinitesimal circulating current loops with number density N, as illustrated in Figure 5-15. The magnetization vector $\mathbf{M}$ is then defined as the magnetic dipole density:

$$\mathbf{M}=N\mathbf{m}=NI\,\mathbf{dS}\text{ amp/m} \tag{9}$$

For the differential sized contour in the xy plane shown in Figure 5-15, only those dipoles with moments in the x or y directions (thus z components of currents) will give rise to currents crossing perpendicularly through the surface bounded by the contour. Those dipoles completely within the contour give no net current as the current passes through the contour twice, once in the positive z direction and on its return in the negative z direction. Only those dipoles on either side of the edges—so that the current only passes through the contour once, with the return outside the contour—give a net current through the loop.

Because the length of the contour sides Δx and Δy are of differential size, we assume that the dipoles along each edge do not change magnitude or direction. Then the net total current linked by the contour near each side is equal to the product of the current per dipole I and the number of dipoles that just pass through the contour once. If the normal vector to the dipole loop (in the direction of $\mathbf{m}$) makes an angle θ with respect to the direction of the contour side at position x, the net current linked along the line at x is

$$-IN\,dS\,\Delta y\cos\theta|_x=-M_y(x)\,\Delta y \tag{10}$$

The minus sign arises because the current within the contour adjacent to the line at coordinate x flows in the $-z$ direction.

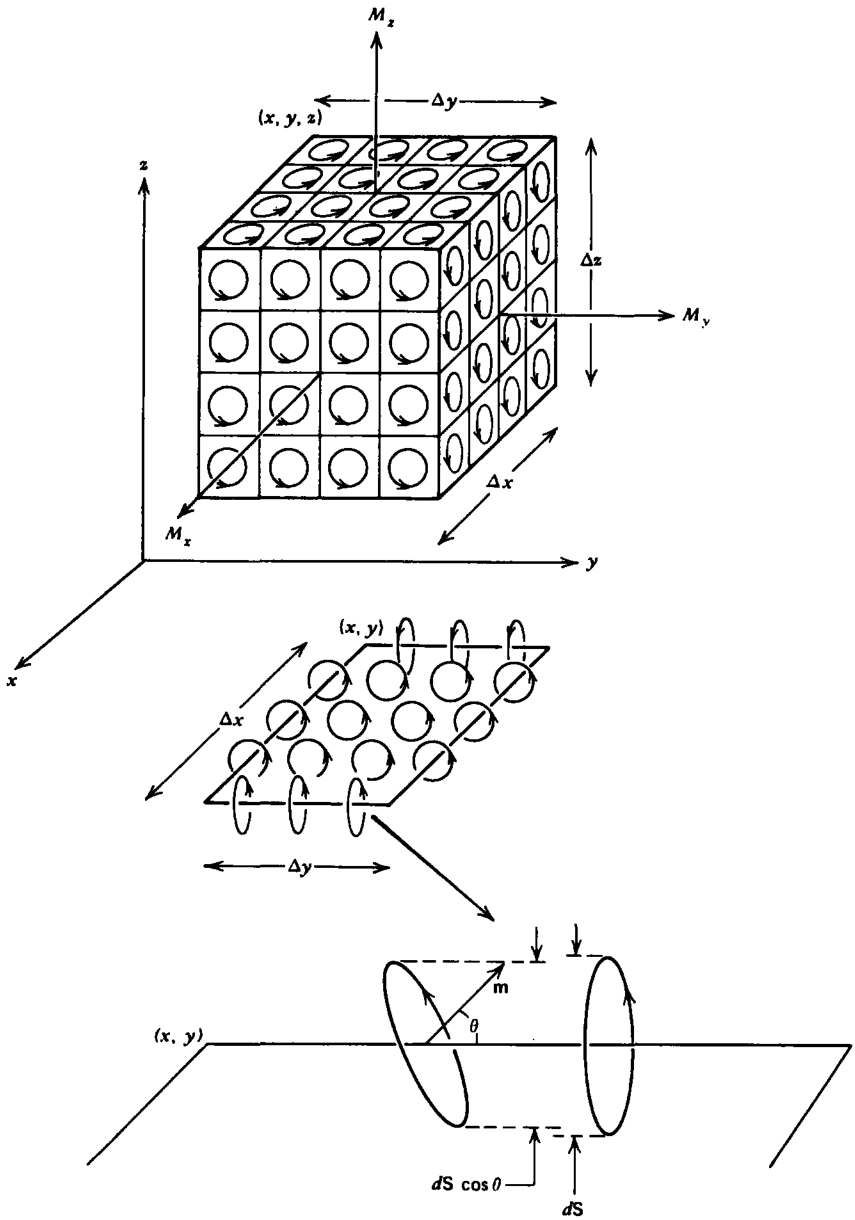


Figure 5-15 Many such magnetic dipoles within a material linking a closed contour gives rise to an effective magnetization current that is also a source of the magnetic field.

Similarly, near the edge at coordinate $x+\Delta x$, the net current linked perpendicular to the contour is

$$IN\,dS\,\Delta y\cos\theta|_{x+\Delta x}=M_y(x+\Delta x)\,\Delta y \qquad (11)$$

Along the edges at y and $y+\Delta y$, the current contributions are

$$IN\,dS\,\Delta x\cos\theta|_y = M_x(y)\,\Delta x$$
$$-IN\,dS\,\Delta x\cos\theta|_{y+\Delta y} = -M_x(y+\Delta y)\,\Delta x \tag{12}$$

The total current in the z direction linked by this contour is thus the sum of contributions in (10)–(12):

$$I_{z\,\text{tot}} = \Delta x\,\Delta y\left(\frac{M_y(x+\Delta x)-M_y(x)}{\Delta x} - \frac{M_x(y+\Delta y)-M_x(y)}{\Delta y}\right) \tag{13}$$

If the magnetization is uniform, the net total current is zero as the current passing through the loop at one side is canceled by the current flowing in the opposite direction at the other side. Only if the magnetization changes with position can there be a net current through the loop's surface. This can be accomplished if either the current per dipole, area per dipole, density of dipoles, or angle of orientation of the dipoles is a function of position.

In the limit as Δx and Δy become small, terms on the right-hand side in (13) define partial derivatives so that the current per unit area in the z direction is

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} J_z = \frac{I_{z\,\text{tot}}}{\Delta x\,\Delta y} = \left(\frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y}\right) = (\nabla\times\mathbf{M})_z \tag{14}$$

which we recognize as the z component of the curl of the magnetization. If we had orientated our loop in the xz or yz planes, the current density components would similarly obey the relations

$$J_y = \left(\frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x}\right) = (\nabla\times\mathbf{M})_y$$
$$J_x = \left(\frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}\right) = (\nabla\times\mathbf{M})_x \tag{15}$$

so that in general

$$\mathbf{J}_m = \nabla\times\mathbf{M} \tag{16}$$

where we subscript the current density with an m to represent the magnetization current density, often called the Amperian current density.

These currents are also sources of the magnetic field and can be used in Ampere's law as

$$\nabla\times\frac{\mathbf{B}}{\mu_0} = \mathbf{J}_m + \mathbf{J}_f = \mathbf{J}_f + \nabla\times\mathbf{M} \tag{17}$$

where $\mathbf{J}_f$ is the free current due to the motion of free charges as contrasted to the magnetization current $\mathbf{J}_m$, which is due to the motion of bound charges in materials.

As we can only impose free currents, it is convenient to define the vector **H** as the magnetic field intensity to be distinguished from **B**, which we will now call the magnetic flux density:

$$\mathbf{H}=\frac{\mathbf{B}}{\mu_0}-\mathbf{M} \Rightarrow \mathbf{B}=\mu_0(\mathbf{H}+\mathbf{M}) \tag{18}$$

Then (17) can be recast as

$$\nabla\times\left(\frac{\mathbf{B}}{\mu_0}-\mathbf{M}\right)=\nabla\times\mathbf{H}=\mathbf{J}_f \tag{19}$$

The divergence and flux relations of Section 5-3-1 are unchanged and are in terms of the magnetic flux density **B**. In free space, where $\mathbf{M}=0$, the relation of (19) between **B** and **H** reduces to

$$\mathbf{B}=\mu_0\mathbf{H} \tag{20}$$

This is analogous to the development of the polarization with the relationships of **D**, **E**, and **P**. Note that in (18), the constant parameter μ_0 multiplies both **H** and **M**, unlike the permittivity ε_0 which only multiplies **E**.

Equation (19) can be put into an equivalent integral form using Stokes' theorem:

$$\int_S (\nabla\times\mathbf{H})\cdot d\mathbf{S}=\oint_L \mathbf{H}\cdot d\mathbf{l}=\int_S \mathbf{J}_f\cdot d\mathbf{S} \tag{21}$$

The free current density $\mathbf{J}_f$ is the source of the **H** field, the magnetization current density $\mathbf{J}_m$ is the source of the **M** field, while the total current, $\mathbf{J}_f+\mathbf{J}_m$, is the source of the **B** field.

5-5-3 Magnetic Materials

There are direct analogies between the polarization processes found in dielectrics and magnetic effects. The constitutive law relating the magnetization **M** to an applied magnetic field **H** is found by applying the Lorentz force to our atomic models.

(a) Diamagnetism

The orbiting electrons as atomic current loops is analogous to electronic polarization, with the current in the direction opposite to their velocity. If the electron ($e=1.6\times10^{-19}$ coul) rotates at angular speed ω at radius R, as in Figure 5-16, the current and dipole moment are

$$I=\frac{e\omega}{2\pi},\qquad m=I\pi R^2=\frac{e\omega}{2}R^2 \tag{22}$$

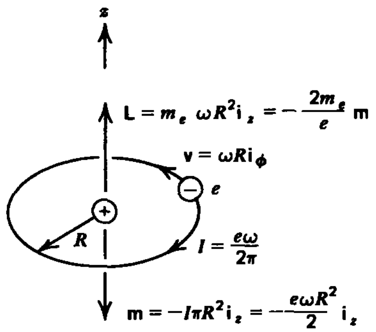


Figure 5-16 The orbiting electron has its magnetic moment **m** in the direction opposite to its angular momentum **L** because the current is opposite to the electron's velocity.

Note that the angular momentum **L** and magnetic moment **m** are oppositely directed and are related as

$$\mathbf{L} = m_e R \mathbf{i}_r \times \mathbf{v} = m_e \omega R^2 \mathbf{i}_z = -\frac{2m_e}{e}\mathbf{m} \tag{23}$$

where $m_e = 9.1 \times 10^{-31}$ kg is the electron mass.

Since quantum theory requires the angular momentum to be quantized in units of $h/2\pi$, where Planck's constant is $h = 6.62 \times 10^{-34}$ joule-sec, the smallest unit of magnetic moment, known as the Bohr magneton, is

$$m_B = \frac{eh}{4\pi m_e} \approx 9.3 \times 10^{-24} \text{ amp-m}^2 \tag{24}$$

Within a homogeneous material these dipoles are randomly distributed so that for every electron orbiting in one direction, another electron nearby is orbiting in the opposite direction so that in the absence of an applied magnetic field there is no net magnetization.

The Coulombic attractive force on the orbiting electron towards the nucleus with atomic number Z is balanced by the centrifugal force:

$$m_e \omega^2 R = \frac{Ze^2}{4\pi\varepsilon_0 R^2} \tag{25}$$

Since the left-hand side is just proportional to the square of the quantized angular momentum, the orbit radius R is also quantized for which the smallest value is

$$R = \frac{4\pi\varepsilon_0}{m_e Z e^2}\left(\frac{h}{2\pi}\right)^2 \approx \frac{5 \times 10^{-11}}{Z} \text{ m} \tag{26}$$

with resulting angular speed

$$\omega = \frac{Z^2 e^4 m_e}{(4\pi\varepsilon_0)^2 (h/2\pi)^3} \approx 1.3 \times 10^{16} Z^2 \tag{27}$$

When a magnetic field $H_0\mathbf{i}_z$ is applied, as in Figure 5-17, electron loops with magnetic moment opposite to the field feel an additional radial force inwards, while loops with colinear moment and field feel a radial force outwards. Since the orbital radius R cannot change because it is quantized, this magnetic force results in a change of orbital speed $\Delta\omega$:

$$m_e(\omega + \Delta\omega_1)^2 R = e\left(\frac{Ze}{4\pi\varepsilon_0 R^2} + (\omega + \Delta\omega_1) R\mu_0 H_0\right)$$

$$m_e(\omega + \Delta\omega_2)^2 R = e\left(\frac{Ze}{4\pi\varepsilon_0 R^2} - (\omega + \Delta\omega_2) R\mu_0 H_0\right) \tag{28}$$

where the first electron speeds up while the second one slows down.

Because the change in speed $\Delta\omega$ is much less than the natural speed ω, we solve (28) approximately as

$$\Delta\omega_1 = \frac{e\omega\mu_0 H_0}{2m_e\omega - e\mu_0 H_0}$$
$$\Delta\omega_2 = \frac{-e\omega\mu_0 H_0}{2m_e\omega + e\mu_0 H_0} \tag{29}$$

where we neglect quantities of order $(\Delta\omega)^2$. However, even with very high magnetic field strengths of $H_0 = 10^6$ amp/m we see that usually

$$e\mu_0 H_0 \ll 2m_e\omega_0$$
$$(1.6 \times 10^{-19})(4\pi \times 10^{-7})10^6 \ll 2(9.1 \times 10^{-31})(1.3 \times 10^{16}) \tag{30}$$

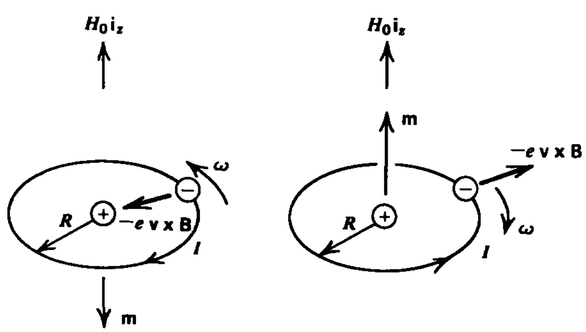


Figure 5-17 Diamagnetic effects, although usually small, arise in all materials because dipoles with moments parallel to the magnetic field have an increase in the orbiting electron speed while those dipoles with moments opposite to the field have a decrease in speed. The loop radius remains constant because it is quantized.

so that (29) further reduces to

$$\Delta\omega_1 \approx -\Delta\omega_2 \approx \frac{e\mu_0 H_0}{2m_e} \approx 1.1\times 10^5 H_0 \tag{31}$$

The net magnetic moment for this pair of loops,

$$m = \frac{eR^2}{2}(\omega_2-\omega_1) = -eR^2\,\Delta\omega_1 = \frac{-e^2\mu_0 R^2}{2m_e} H_0 \tag{32}$$

is opposite in direction to the applied magnetic field.

If we have N such loop pairs per unit volume, the magnetization field is

$$\mathbf{M} = N\mathbf{m} = -\frac{Ne^2\mu_0 R^2}{2m_e} H_0 \mathbf{i}_z \tag{33}$$

which is also oppositely directed to the applied magnetic field.

Since the magnetization is linearly related to the field, we define the magnetic susceptibility χ_m as

$$\mathbf{M} = \chi_m \mathbf{H}, \qquad \chi_m = -\frac{Ne^2\mu_0 R^2}{2m_e} \tag{34}$$

where χ_m is negative. The magnetic flux density is then

$$\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M}) = \mu_0(1+\chi_m)\mathbf{H} = \mu_0\mu_r\mathbf{H} = \mu\mathbf{H} \tag{35}$$

where $\mu_r = 1+\chi_m$ is called the relative permeability and μ is the permeability. In free space $\chi_m = 0$ so that $\mu_r = 1$ and $\mu = \mu_0$. The last relation in (35) is usually convenient to use, as all the results in free space are still correct within linear permeable material if we replace μ_0 by μ. In diamagnetic materials, where the susceptibility is negative, we have that $\mu_r < 1$, $\mu < \mu_0$. However, substituting in our typical values

$$\chi_m = -\frac{Ne^2\mu_0 R^2}{2m_e} \approx \frac{4.4\times 10^{-35}}{Z^2} N \tag{36}$$

we see that even with $N \approx 10^{30}$ atoms/m^3, χ_m is much less than unity so that diamagnetic effects are very small.

(b) Paramagnetism

As for orientation polarization, an applied magnetic field exerts a torque on each dipole tending to align its moment with the field, as illustrated for the rectangular magnetic dipole with moment at an angle θ to a uniform magnetic field **B** in Figure 5-18*a*. The force on each leg is

$$\begin{aligned} \mathbf{df}_1 &= -\mathbf{df}_2 = I\,\Delta x\,\mathbf{i}_x\times\mathbf{B} = I\,\Delta x[B_y\mathbf{i}_z - B_z\mathbf{i}_y] \\ \mathbf{df}_3 &= -\mathbf{df}_4 = I\,\Delta y\,\mathbf{i}_y\times\mathbf{B} = I\,\Delta y(-B_x\mathbf{i}_z + B_z\mathbf{i}_x) \end{aligned} \tag{37}$$

In a uniform magnetic field, the forces on opposite legs are equal in magnitude but opposite in direction so that the net

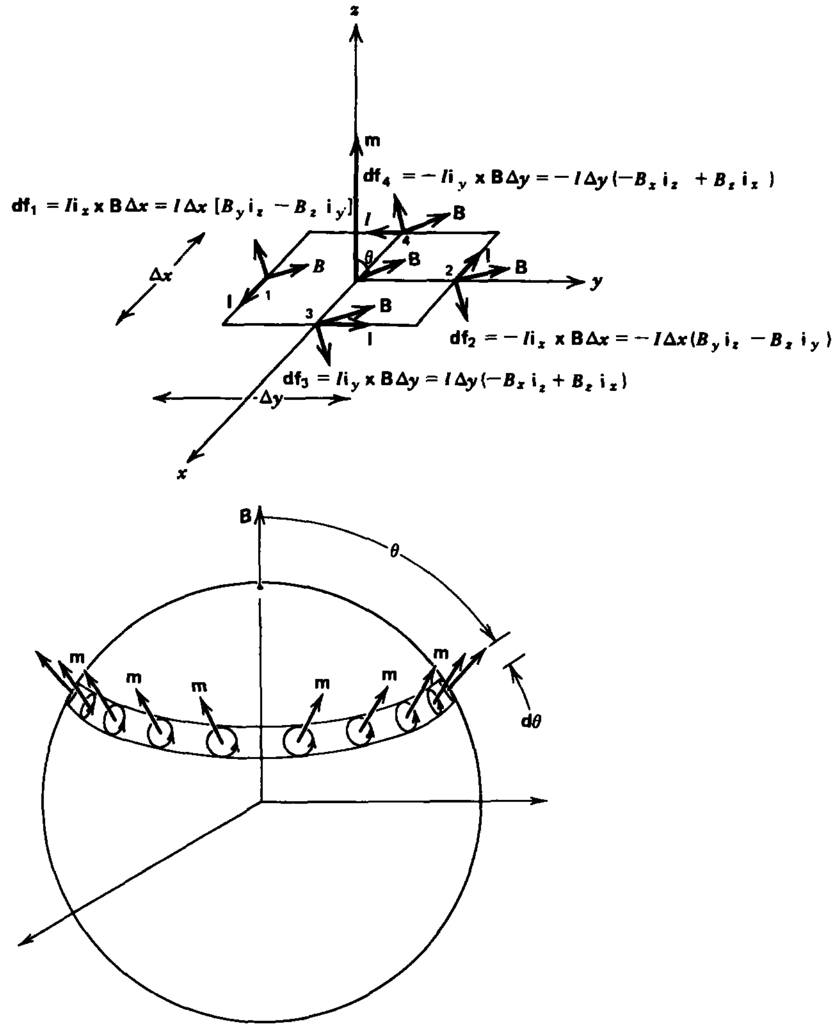


Figure 5-18 (*a*) A torque is exerted on a magnetic dipole with moment at an angle θ to an applied magnetic field. (*b*) From Boltzmann statistics, thermal agitation opposes the alignment of magnetic dipoles. All the dipoles at an angle θ, together have a net magnetization in the direction of the applied field.

force on the loop is zero. However, there is a torque:

$$\begin{aligned}\mathbf{T} &= \sum_{n=1}^{4} \mathbf{r}\times d\mathbf{f}_n \\ &= \frac{\Delta y}{2}(-\mathbf{i}_y\times d\mathbf{f}_1+\mathbf{i}_y\times d\mathbf{f}_2)+\frac{\Delta x}{2}(\mathbf{i}_x\times d\mathbf{f}_3-\mathbf{i}_x\times d\mathbf{f}_4) \\ &= I\,\Delta x\,\Delta y(B_x\mathbf{i}_y-B_y\mathbf{i}_x)=\mathbf{m}\times\mathbf{B}\end{aligned} \tag{38}$$

The incremental amount of work necessary to turn the dipole by a small angle $d\theta$ is

$$dW = T\,d\theta = m\mu_0 H_0 \sin\theta\, d\theta \tag{39}$$

so that the total amount of work necessary to turn the dipole from $\theta = 0$ to any value of θ is

$$W = \int_0^\theta T\,d\theta = -m\mu_0 H_0 \cos\theta\big|_0^\theta = m\mu_0 H_0(1-\cos\theta) \tag{40}$$

This work is stored as potential energy, for if the dipole is released it will try to orient itself with its moment parallel to the field. Thermal agitation opposes this alignment where Boltzmann statistics describes the number density of dipoles having energy W as

$$n = n_1 e^{-W/kT} = n_1 e^{-m\mu_0 H_0(1-\cos\theta)/kT} = n_0 e^{m\mu_0 H_0 \cos\theta/kT} \tag{41}$$

where we lump the constant energy contribution in (40) within the amplitude n_0, which is found by specifying the average number density of dipoles N within a sphere of radius R:

$$\begin{aligned} N &= \frac{1}{\frac{4}{3}\pi R^3}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\int_{r=0}^{R} n_0\, e^{a\cos\theta}\, r^2 \sin\theta\, dr\, d\theta\, d\phi \\ &= \frac{n_0}{2}\int_{\theta=0}^{\pi} \sin\theta\, e^{a\cos\theta}\, d\theta \end{aligned} \tag{42}$$

where we let

$$a = m\mu_0 H_0/kT \tag{43}$$

With the change of variable

$$u = a\cos\theta, \qquad du = -a\sin\theta\, d\theta \tag{44}$$

the integration in (42) becomes

$$N = \frac{-n_0}{2a}\int_a^{-a} e^u\, du = \frac{n_0}{a}\sinh a \tag{45}$$

so that (41) becomes

$$n = \frac{Na}{\sinh a}\, e^{a\cos\theta} \tag{46}$$

From Figure 5-18*b* we see that all the dipoles in the shell over the interval θ to $\theta + d\theta$ contribute to a net magnetization. which is in the direction of the applied magnetic field:

$$dM = \frac{mn}{\frac{4}{3}\pi R^3}\cos\theta\, r^2 \sin\theta\, dr\, d\theta\, d\phi \tag{47}$$

so that the total magnetization due to all the dipoles within the sphere is

$$M=\frac{maN}{2\sinh a}\int_{\theta=0}^{\pi}\sin\theta\cos\theta\, e^{a\cos\theta}\,d\theta \tag{48}$$

Again using the change of variable in (44), (48) integrates to

$$\begin{aligned} M&=\frac{-mN}{2a\sinh a}\int_{a}^{-a}ue^{u}\,du\\ &=\frac{-mN}{2a\sinh a}e^{u}(u-1)\Big|_{a}^{-a}\\ &=\frac{-mN}{2a\sinh a}[e^{-a}(-a-1)-e^{a}(a-1)]\\ &=\frac{-mN}{a\sinh a}[-a\cosh a+\sinh a]\\ &=mN[\coth a-1/a] \end{aligned} \tag{49}$$

which is known as the Langevin equation and is plotted as a function of reciprocal temperature in Figure 5-19. At low temperatures (high a) the magnetization saturates at $M=mN$ as all the dipoles have their moments aligned with the field. At room temperature, a is typically very small. Using the parameters in (26) and (27) in a strong magnetic field of $H_0=10^6$ amps/m, a is much less than unity:

$$a=\frac{m\mu_0H_0}{kT}=\frac{e\omega}{2}R^2\frac{\mu_0H_0}{kT}\approx 8\times10^{-4} \tag{50}$$

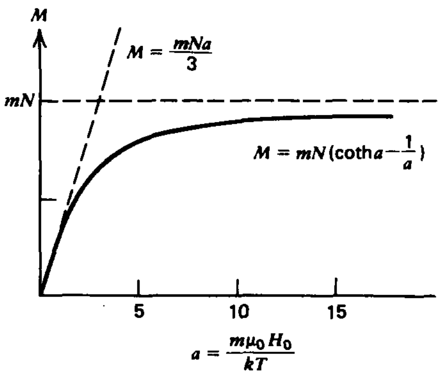


Figure 5-19 The Langevin equation describes the net magnetization. At low temperatures (high a) all the dipoles align with the field causing saturation. At high temperatures ($a \ll 1$) the magnetization increases linearly with field.

In this limit, Langevin's equation simplifies to

$$\lim_{a\ll 1} M \approx mN\left[\frac{1+a^2/2}{a+a^3/6}-\frac{1}{a}\right]$$

$$\approx mN\left(\frac{(1+a^2/2)(1-a^3/6)}{a}-\frac{1}{a}\right]$$

$$\approx \frac{mNa}{3}\approx\frac{\mu_0 m^2 N}{3kT}H_0 \tag{51}$$

In this limit the magnetic susceptibility χ_m is positive:

$$\mathbf{M}=\chi_m\mathbf{H},\qquad \chi_m=\frac{\mu_0 m^2 N}{3kT} \tag{52}$$

but even with $N\approx 10^{30}$ atoms/m^3, it is still very small:

$$\chi_m\approx 7\times 10^{-4} \tag{53}$$

(c) Ferromagnetism

As for ferroelectrics (see Section 3-1-5), sufficiently high coupling between adjacent magnetic dipoles in some iron alloys causes them to spontaneously align even in the absence of an applied magnetic field. Each of these microscopic domains act like a permanent magnet, but they are randomly distributed throughout the material so that the macroscopic magnetization is zero. When a magnetic field is applied, the dipoles tend to align with the field so that domains with a magnetization along the field grow at the expense of non-aligned domains.

The friction-like behavior of domain wall motion is a lossy process so that the magnetization varies with the magnetic field in a nonlinear way, as described by the hysteresis loop in Figure 5-20. A strong field aligns all the domains to saturation. Upon decreasing **H**, the magnetization lags behind so that a remanent magnetization M_r exists even with zero field. In this condition we have a permanent magnet. To bring the magnetization to zero requires a negative coercive field $-H_c$.

Although nonlinear, the main engineering importance of ferromagnetic materials is that the relative permeability μ_r is often in the thousands:

$$\mu=\mu_r\mu_0=\mathbf{B}/\mathbf{H} \tag{54}$$

This value is often so high that in engineering applications we idealize it to be infinity. In this limit

$$\lim_{\mu\to\infty}\mathbf{B}=\mu\mathbf{H}\Rightarrow\mathbf{H}=0,\qquad \mathbf{B}\text{ finite} \tag{55}$$

the **H** field becomes zero to keep the **B** field finite.

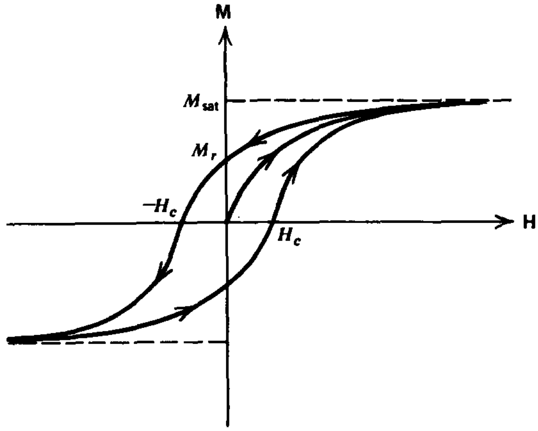


Figure 5-20 Ferromagnetic materials exhibit hysteresis where the magnetization saturates at high field strengths and retains a net remanent magnetization $\mathbf{M}_r$ even when $\mathbf{H}$ is zero. A coercive field $-\mathbf{H}_c$ is required to bring the magnetization back to zero.

EXAMPLE 5-1 INFINITE LINE CURRENT WITHIN A MAGNETICALLY PERMEABLE CYLINDER

A line current I of infinite extent is within a cylinder of radius a that has permeability μ, as in Figure 5-21. The cylinder is surrounded by free space. What are the **B**, **H**, and **M** fields everywhere? What is the magnetization current?

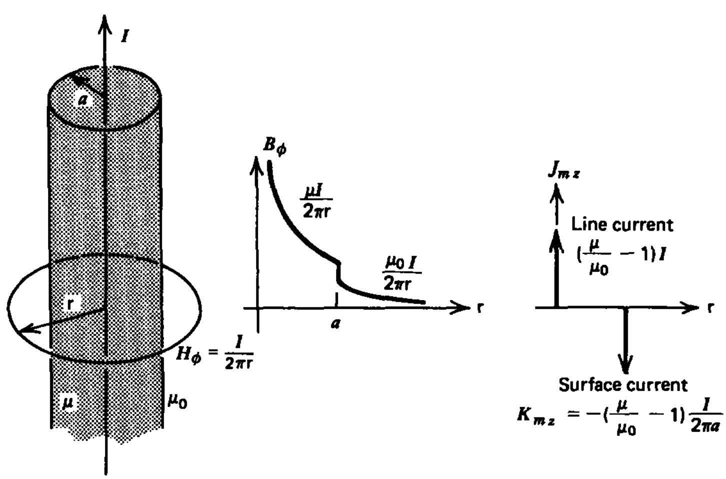


Figure 5-21 A free line current of infinite extent placed within a permeable cylinder gives rise to a line magnetization current along the axis and an oppositely directed surface magnetization current on the cylinder surface.

SOLUTION

Pick a circular contour of radius r around the current. Using the integral form of Ampere's law, (21), the **H** field is of the same form whether inside or outside the cylinder:

$$\oint_L \mathbf{H}\cdot d\mathbf{l} = H_\phi 2\pi r = I \Rightarrow H_\phi = \frac{I}{2\pi r}$$

The magnetic flux density differs in each region because the permeability differs:

$$B_\phi = \begin{cases} \mu H_\phi = \dfrac{\mu I}{2\pi r}, & 0<r<a \\ \mu_0 H_\phi = \dfrac{\mu_0 I}{2\pi r}, & r>a \end{cases}$$

The magnetization is obtained from the relation

$$\mathbf{M} = \frac{\mathbf{B}}{\mu_0} - \mathbf{H}$$

as

$$M_\phi = \begin{cases} \left(\dfrac{\mu}{\mu_0} - 1\right) H_\phi = \dfrac{\mu-\mu_0}{\mu_0}\dfrac{I}{2\pi r}, & 0<r<a \\ 0, & r>a \end{cases}$$

The volume magnetization current can be found using (16):

$$\mathbf{J}_m = \nabla\times\mathbf{M} = -\frac{\partial M_\phi}{\partial z}\mathbf{i}_r + \frac{1}{r}\frac{\partial}{\partial r}(rM_\phi)\mathbf{i}_z = 0, \qquad 0<r<a$$

There is no bulk magnetization current because there are no bulk free currents. However, there is a line magnetization current at $r=0$ and a surface magnetization current at $r=a$. They are easily found using the integral form of (16) from Stokes' theorem:

$$\int_S \nabla\times\mathbf{M}\cdot d\mathbf{S} = \oint_L \mathbf{M}\cdot d\mathbf{l} = \int_S \mathbf{J}_m\cdot d\mathbf{S}$$

Pick a contour around the center of the cylinder with $r<a$:

$$M_\phi 2\pi r = \left(\frac{\mu-\mu_0}{\mu_0}\right) I = I_m$$

where I_m is the magnetization line current. The result remains unchanged for any radius $r<a$ as no more current is enclosed since $\mathbf{J}_m = 0$ for $0<r<a$. As soon as $r>a$, M_ϕ becomes zero so that the total magnetization current becomes zero. Therefore, at $r=a$ a surface magnetization current must flow whose total current is equal in magnitude but opposite in sign to the line magnetization current:

$$K_{zm}=\frac{-I_m}{2\pi a}=-\frac{(\mu-\mu_0)I}{\mu_0 2\pi a}$$

5-6 BOUNDARY CONDITIONS

At interfacial boundaries separating materials of differing properties, the magnetic fields on either side of the boundary must obey certain conditions. The procedure is to use the integral form of the field laws for differential sized contours, surfaces, and volumes in the same way as was performed for electric fields in Section 3-3.

To summarize our development thus far, the field laws for magnetic fields in differential and integral form are

$$\nabla\times\mathbf{H}=\mathbf{J}_f,\qquad \oint_L \mathbf{H}\cdot d\mathbf{l}=\int_S \mathbf{J}_f\cdot d\mathbf{S} \tag{1}$$

$$\nabla\times\mathbf{M}=\mathbf{J}_m,\qquad \oint_L \mathbf{M}\cdot d\mathbf{l}=\int_S \mathbf{J}_m\cdot d\mathbf{S} \tag{2}$$

$$\nabla\cdot\mathbf{B}=0,\qquad \oint_S \mathbf{B}\cdot d\mathbf{S}=0 \tag{3}$$

5-6-1 Tangential Component of H

We apply Ampere's circuital law of (1) to the contour of differential size enclosing the interface, as shown in Figure 5-22*a*. Because the interface is assumed to be infinitely thin, the short sides labelled *c* and *d* are of zero length and so offer

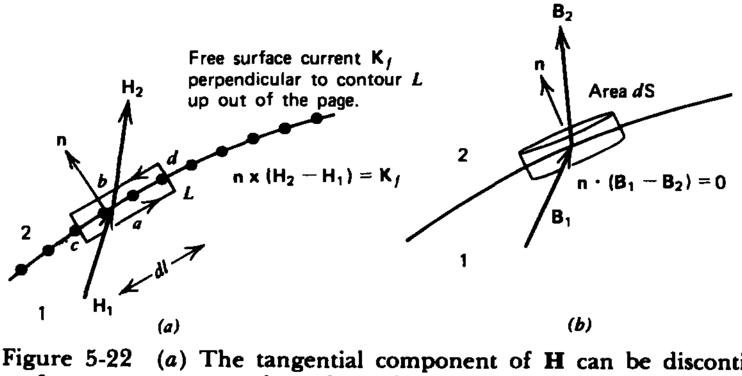


Figure 5-22 (*a*) The tangential component of **H** can be discontinuous in a free surface current across a boundary. (*b*) The normal component of **B** is always continuous across an interface.

no contribution to the line integral. The remaining two sides yield

$$\oint_L \mathbf{H}\cdot d\mathbf{l}=(H_{1t}-H_{2t})\,dl=K_{fn}\,dl \tag{4}$$

where K_{fn} is the component of free surface current perpendicular to the contour by the right-hand rule in this case up out of the page. Thus, the tangential component of magnetic field can be discontinuous by a free surface current,

$$(H_{1t}-H_{2t})=K_{fn}\Rightarrow \mathbf{n}\times(\mathbf{H}_2-\mathbf{H}_1)=\mathbf{K}_f \tag{5}$$

where the unit normal points from region 1 towards region 2. If there is no surface current, the tangential component of **H** is continuous.

5-6-2 Tangential Component of M

Equation (2) is of the same form as (6) so we may use the results of (5) replacing **H** by **M** and $\mathbf{K}_f$ by $\mathbf{K}_m$, the surface magnetization current:

$$(M_{1t}-M_{2t})=K_{mn},\qquad \mathbf{n}\times(\mathbf{M}_2-\mathbf{M}_1)=\mathbf{K}_m \tag{6}$$

This boundary condition confirms the result for surface magnetization current found in Example 5-1.

5-6-3 Normal Component of B

Figure 5-22*b* shows a small volume whose upper and lower surfaces are parallel and are on either side of the interface. The short cylindrical side, being of zero length, offers no contribution to (3), which thus reduces to

$$\oint_S \mathbf{B}\cdot d\mathbf{S}=(B_{2n}-B_{1n})\,dS=0 \tag{7}$$

yielding the boundary condition that the component of **B** normal to an interface of discontinuity is always continuous:

$$B_{1n}-B_{2n}=0\Rightarrow \mathbf{n}\cdot(\mathbf{B}_1-\mathbf{B}_2)=0 \tag{8}$$

EXAMPLE 5-2 MAGNETIC SLAB WITHIN A UNIFORM MAGNETIC FIELD

A slab of infinite extent in the x and y directions is placed within a uniform magnetic field $H_0\mathbf{i}_z$ as shown in Figure 5-23.

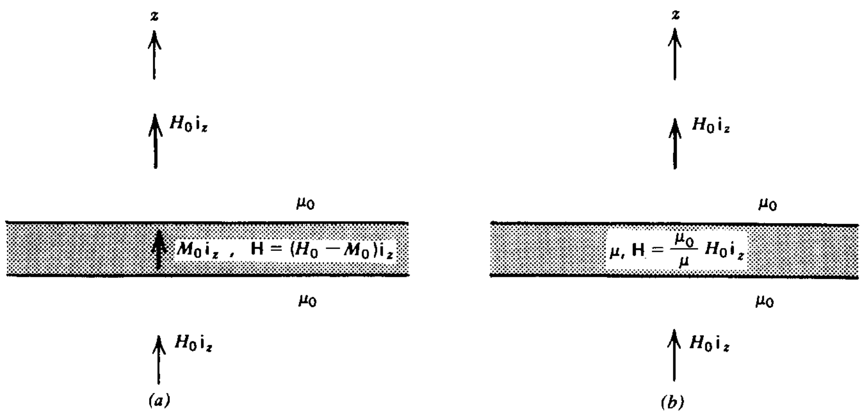


Figure 5-23 A (*a*) permanently magnetized or (*b*) linear magnetizable material is placed within a uniform magnetic field.

Find the **H** field within the slab when it is
(a) permanently magnetized with magnetization $M_0\mathbf{i}_z$,
(b) a linear permeable material with permeability μ.

SOLUTION

For both cases, (8) requires that the **B** field across the boundaries be continuous as it is normally incident.

(a) For the permanently magnetized slab, this requires that

$$\mu_0 H_0 = \mu_0(H + M_0) \Rightarrow H = H_0 - M_0$$

Note that when there is no externally applied field ($H_0 = 0$), the resulting field within the slab is oppositely directed to the magnetization so that $\mathbf{B} = 0$.

(b) For a linear permeable medium (8) requires

$$\mu_0 H_0 = \mu H \Rightarrow H = \frac{\mu_0}{\mu} H_0$$

For $\mu > \mu_0$ the internal magnetic field is reduced. If H_0 is set to zero, the magnetic field within the slab is also zero.

5-7 MAGNETIC FIELD BOUNDARY VALUE PROBLEMS

5-7-1 The Method of Images

A line current I of infinite extent in the z direction is a distance d above a plane that is either perfectly conducting or infinitely permeable, as shown in Figure 5-24. For both cases

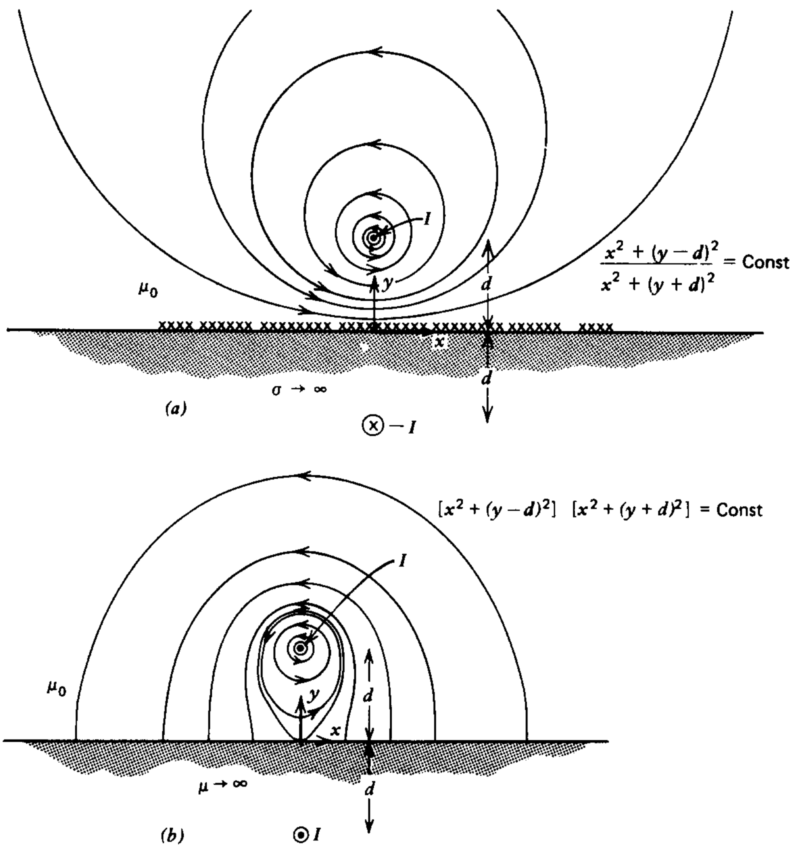


Figure 5-24 (*a*) A line current above a perfect conductor induces an oppositely directed surface current that is equivalent to a symmetrically located image line current. (*b*) The field due to a line current above an infinitely permeable medium is the same as if the medium were replaced by an image current now in the same direction as the original line current.

the **H** field within the material must be zero but the boundary conditions at the interface are different. In the perfect conductor both **B** and **H** must be zero, so that at the interface the normal component of **B** and thus **H** must be continuous and thus zero. The tangential component of **H** is discontinuous in a surface current.

In the infinitely permeable material **H** is zero but **B** is finite. No surface current can flow because the material is not a conductor, so the tangential component of **H** is continuous and thus zero. The **B** field must be normally incident.

Both sets of boundary conditions can be met by placing an image current I at $y=-d$ flowing in the opposite direction for the conductor and in the same direction for the permeable material.

Using the upper sign for the conductor and the lower sign for the infinitely permeable material, the vector potential due to both currents is found by superposing the vector potential found in Section 5-4-3*a*, Eq. (18), for each infinitely long line current:

$$A_z = \frac{-\mu_0 I}{2\pi}\{\ln [x^2+(y-d)^2]^{1/2} \mp \ln [x^2+(y+d)^2]^{1/2}\}$$

$$= \frac{-\mu_0 I}{4\pi}\{\ln [x^2+(y-d)^2] \mp \ln [x^2+(y+d)^2]\} \qquad (1)$$

with resultant magnetic field

$$\mathbf{H} = \frac{1}{\mu_0}\nabla\times\mathbf{A} = \frac{1}{\mu_0}\left(\mathbf{i}_x\frac{\partial A_z}{\partial y} - \mathbf{i}_y\frac{\partial A_z}{\partial x}\right)$$

$$= \frac{-I}{2\pi}\left\{\frac{(y-d)\mathbf{i}_x - x\mathbf{i}_y}{[x^2+(y-d)^2]} \mp \frac{(y+d)\mathbf{i}_x - x\mathbf{i}_y}{[x^2+(y+d)^2]}\right\} \qquad (2)$$

The surface current distribution for the conducting case is given by the discontinuity in tangential **H**,

$$K_z = -H_x(y=0) = -\frac{Id}{\pi[d^2+x^2]} \qquad (3)$$

which has total current

$$I_T = \int_{-\infty}^{+\infty} K_z\, dx = -\frac{Id}{\pi}\int_{-\infty}^{+\infty}\frac{dx}{(x^2+d^2)}$$

$$= -\frac{Id}{\pi}\frac{1}{d}\tan^{-1}\frac{x}{d}\bigg|_{-\infty}^{+\infty} = -I \qquad (4)$$

just equal to the image current.

The force per unit length on the current for each case is just due to the magnetic field from its image:

$$\mathbf{f} = \pm\frac{\mu_0 I^2}{4\pi d}\mathbf{i}_y \qquad (5)$$

being repulsive for the conductor and attractive for the permeable material.

The magnetic field lines plotted in Figure 5-24 are just lines of constant A_z as derived in Section 5-4-3*b*. Right next to the line current the self-field term dominates and the field lines are circles. The far field in Figure 5-24*b*, when the line and image current are in the same direction, is the same as if we had a single line current of $2I$.

5-7-2 Sphere in a Uniform Magnetic Field

A sphere of radius R is placed within a uniform magnetic field $H_0\mathbf{i}_z$. The sphere and surrounding medium may have any of the following properties illustrated in Figure 5-25:

(i) Sphere has permeability μ_2 and surrounding medium has permeability μ_1.
(ii) Perfectly conducting sphere in free space.
(iii) Uniformly magnetized sphere $M_2\mathbf{i}_z$ in a uniformly magnetized medium $M_1\mathbf{i}_z$.

For each of these three cases, there are no free currents in either region so that the governing equations in each region are

$$\nabla\cdot\mathbf{B}=0$$
$$\nabla\times\mathbf{H}=0 \tag{5}$$

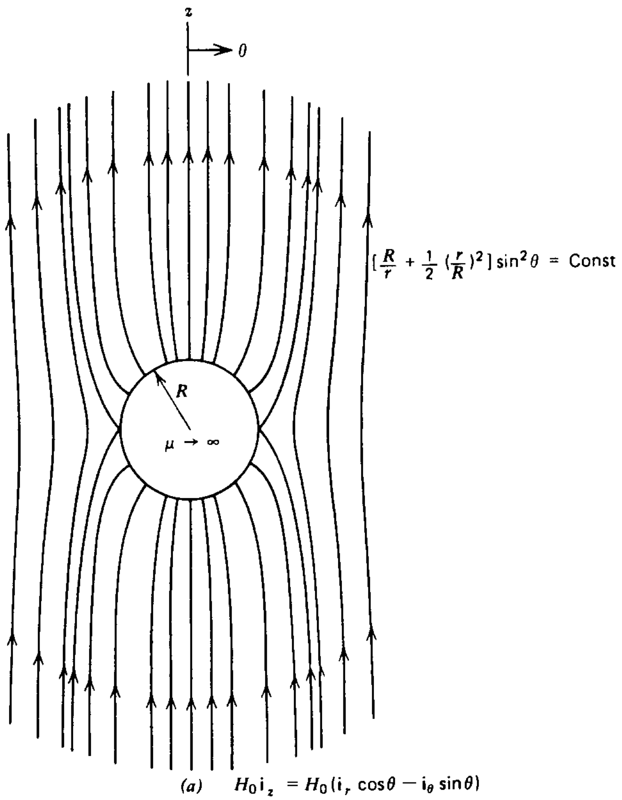


Figure 5-25 Magnetic field lines about an (a) infinitely permeable and (b) perfectly conducting sphere in a uniform magnetic field.

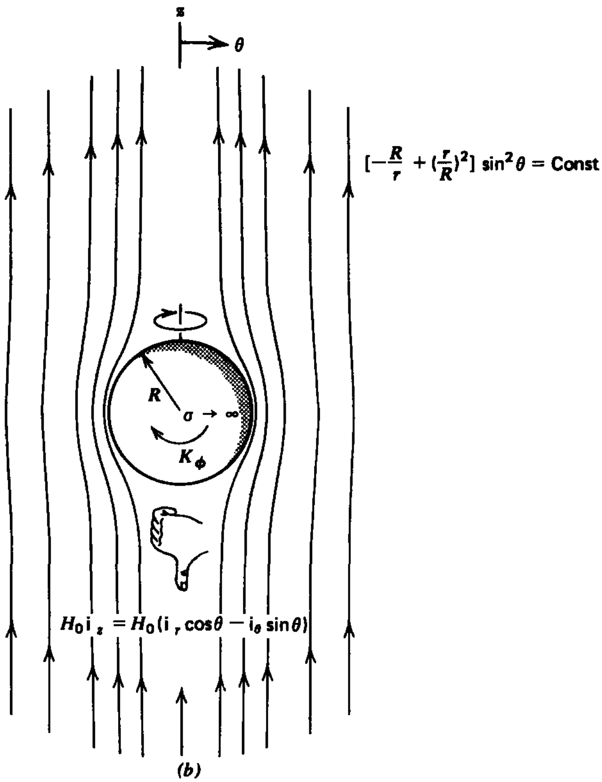


Figure 5-25

Because the curl of **H** is zero, we can define a scalar magnetic potential

$$\mathbf{H} = \nabla \chi \tag{6}$$

where we avoid the use of a negative sign as is used with the electric field since the potential χ is only introduced as a mathematical convenience and has no physical significance. With **B** proportional to **H** or for uniform magnetization, the divergence of **H** is also zero so that the scalar magnetic potential obeys Laplace's equation in each region:

$$\nabla^2 \chi = 0 \tag{7}$$

We can then use the same techniques developed for the electric field in Section 4-4 by trying a scalar potential in each region as

$$\chi = \begin{cases} Ar\cos\theta, & r < R \\ (Dr + C/r^2)\cos\theta & r > R \end{cases} \tag{8}$$

The associated magnetic field is then

$$\mathbf{H}=\nabla\chi=\frac{\partial\chi}{\partial r}\mathbf{i}_r+\frac{1}{r}\frac{\partial\chi}{\partial\theta}\mathbf{i}_\theta+\frac{1}{r\sin\theta}\frac{\partial\chi}{\partial\phi}\mathbf{i}_\phi$$

$$=\begin{cases}A(\mathbf{i}_r\cos\theta-\mathbf{i}_\theta\sin\theta)=A\mathbf{i}_z, & r<R\\(D-2C/r^3)\cos\theta\mathbf{i}_r-(D+C/r^3)\sin\theta\mathbf{i}_\theta, & r>R\end{cases}\tag{9}$$

For the three cases, the magnetic field far from the sphere must approach the uniform applied field:

$$\mathbf{H}(r=\infty)=H_0\mathbf{i}_z=H_0(\mathbf{i}_r\cos\theta-\mathbf{i}_\theta\sin\theta)\Rightarrow D=H_0\tag{10}$$

The other constants, A and C, are found from the boundary conditions at $r=R$. The field within the sphere is uniform, in the same direction as the applied field. The solution outside the sphere is the imposed field plus a contribution as if there were a magnetic dipole at the center of the sphere with moment $m_z=4\pi C$.

(i) If the sphere has a different permeability from the surrounding region, both the tangential components of $\mathbf{H}$ and the normal components of $\mathbf{B}$ are continuous across the spherical surface:

$$\begin{aligned}&H_\theta(r=R_+)=H_\theta(r=R_-)\Rightarrow A=D+C/R^3\\&B_r(r=R_+)=B_r(r=R_-)\Rightarrow\mu_1H_r(r=R_+)=\mu_2H_r(r=R_-)\end{aligned}\tag{11}$$

which yields solutions

$$A=\frac{3\mu_1H_0}{\mu_2+2\mu_1},\qquad C=-\frac{\mu_2-\mu_1}{\mu_2+2\mu_1}R^3H_0\tag{12}$$

The magnetic field distribution is then

$$\mathbf{H}=\begin{cases}\dfrac{3\mu_1H_0}{\mu_2+2\mu_1}(\mathbf{i}_r\cos\theta-\mathbf{i}_\theta\sin\theta)=\dfrac{3\mu_1H_0\mathbf{i}_z}{\mu_2+2\mu_1}, & r<R\\ H_0\left\{\left[1+\dfrac{2R^3}{r^3}\left(\dfrac{\mu_2-\mu_1}{\mu_2+2\mu_1}\right)\right]\cos\theta\mathbf{i}_r\right.\\ \left.\quad-\left[1-\dfrac{R^3}{r^3}\left(\dfrac{\mu_2-\mu_1}{\mu_2+2\mu_1}\right)\right]\sin\theta\mathbf{i}_\theta\right\}, & r>R\end{cases}\tag{13}$$

The magnetic field lines are plotted in Figure 5-25a when $\mu_2\to\infty$. In this limit, $\mathbf{H}$ within the sphere is zero, so that the field lines incident on the sphere are purely radial. The field lines plotted are just lines of constant stream function Σ, found in the same way as for the analogous electric field problem in Section 4-4-3b.

(ii) If the sphere is perfectly conducting, the internal magnetic field is zero so that $A=0$. The normal component of **B** right outside the sphere is then also zero:

$$H_r(r=R_+)=0 \Rightarrow C=H_0R^3/2 \tag{14}$$

yielding the solution

$$\mathbf{H}=H_0\left[\left(1-\frac{R^3}{r^3}\right)\cos\theta\,\mathbf{i}_r-\left(1+\frac{R^3}{2r^3}\right)\sin\theta\,\mathbf{i}_\theta\right],\quad r>R \tag{15}$$

The interfacial surface current at $r=R$ is obtained from the discontinuity in the tangential component of **H**:

$$K_\phi=H_\theta(r=R)=-\tfrac{3}{2}H_0\sin\theta \tag{16}$$

The current flows in the negative ϕ direction around the sphere. The right-hand rule, illustrated in Figure 5-25*b*, shows that the resulting field from the induced current acts in the direction opposite to the imposed field. This opposition results in the zero magnetic field inside the sphere.

The field lines plotted in Figure 5-25*b* are purely tangential to the perfectly conducting sphere as required by (14).

(iii) If both regions are uniformly magnetized, the boundary conditions are

$$\begin{aligned}
H_\theta(r=R_+)&=H_\theta(r=R_-)\Rightarrow A=D+C/R^3\\
B_r(r=R_+)&=B_r(r=R_-)\Rightarrow H_r(r=R_+)+M_1\cos\theta\\
&\qquad=H_r(r=R_-)+M_2\cos\theta
\end{aligned} \tag{17}$$

with solutions

$$\begin{aligned}
A&=H_0+\tfrac{1}{3}(M_1-M_2)\\
C&=\frac{R^3}{3}(M_1-M_2)
\end{aligned} \tag{18}$$

so that the magnetic field is

$$\mathbf{H}=\begin{cases}
[H_0+\frac{1}{3}(M_1-M_2)][\cos\theta\,\mathbf{i}_r-\sin\theta\,\mathbf{i}_\theta] \\
\qquad =[H_0+\frac{1}{3}(M_1-M_2)]\mathbf{i}_z \quad r<R\\
\left(H_0-\frac{2R^3}{3r^3}(M_1-M_2)\right)\cos\theta\,\mathbf{i}_r\\
\qquad -\left(H_0+\frac{R^3}{3r^3}(M_1-M_2)\right)\sin\theta\,\mathbf{i}_\theta,\quad r>R
\end{cases} \tag{19}$$

Because the magnetization is uniform in each region, the curl of **M** is zero everywhere but at the surface of the sphere, so that the volume magnetization current is zero with a surface magnetization current at $r=R$ given by

$$\begin{aligned}\mathbf{K}_m &= \mathbf{n}\times(\mathbf{M}_1-\mathbf{M}_2)\\ &= \mathbf{i}_r\times(M_1-M_2)\mathbf{i}_z\\ &= \mathbf{i}_r\times(M_1-M_2)(\mathbf{i}_r\cos\theta-\sin\theta\,\mathbf{i}_\theta)\\ &= -(M_1-M_2)\sin\theta\,\mathbf{i}_\phi\end{aligned}\tag{20}$$

5-8 MAGNETIC FIELDS AND FORCES

5-8-1 Magnetizable Media

A magnetizable medium carrying a free current $\mathbf{J}_f$ is placed within a magnetic field $\mathbf{B}$, which is a function of position. In addition to the Lorentz force, the medium feels the forces on all its magnetic dipoles. Focus attention on the rectangular magnetic dipole shown in Figure 5-26. The force on each current carrying leg is

$$\begin{aligned}\mathbf{f} &= \mathbf{i}\,dl\times(B_x\mathbf{i}_x+B_y\mathbf{i}_y+B_z\mathbf{i}_z)\\ \Rightarrow \mathbf{f}(x) &= -i\,\Delta y[-B_x\mathbf{i}_z+B_z\mathbf{i}_x]|_x\\ \mathbf{f}(x+\Delta x) &= i\,\Delta y[-B_x\mathbf{i}_z+B_z\mathbf{i}_x]|_{x+\Delta x}\\ \mathbf{f}(y) &= i\,\Delta x[B_y\mathbf{i}_z-B_z\mathbf{i}_y]|_y\\ \mathbf{f}(y+\Delta y) &= -i\,\Delta x[B_y\mathbf{i}_z-B_z\mathbf{i}_y]|_{y+\Delta y}\end{aligned}\tag{1}$$

so that the total force on the dipole is

$$\begin{aligned}\mathbf{f} &= \mathbf{f}(x)+\mathbf{f}(x+\Delta x)+\mathbf{f}(y)+\mathbf{f}(y+\Delta y)\\ &= i\,\Delta x\,\Delta y\left[\frac{B_z(x+\Delta x)-B_z(x)}{\Delta x}\mathbf{i}_x-\frac{B_x(x+\Delta x)-B_x(x)}{\Delta x}\mathbf{i}_z\right.\\ &\quad\left.+\frac{B_z(y+\Delta y)-B_z(y)}{\Delta y}\mathbf{i}_y-\frac{B_y(y+\Delta y)-B_y(y)}{\Delta y}\mathbf{i}_z\right]\end{aligned}\tag{2}$$

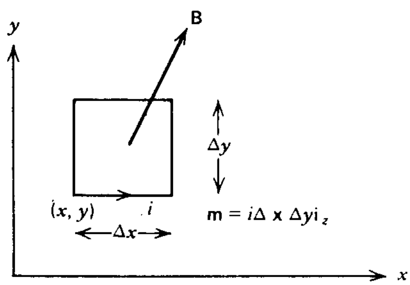


Figure 5-26 A magnetic dipole in a magnetic field **B**.

In the limit of infinitesimal Δx and Δy the bracketed terms define partial derivatives while the coefficient is just the magnetic dipole moment $\mathbf{m} = i\,\Delta x\,\Delta y\,\mathbf{i}_z$:

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \mathbf{f} = m_z\left[\frac{\partial B_z}{\partial x}\mathbf{i}_x - \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right)\mathbf{i}_z + \frac{\partial B_z}{\partial y}\mathbf{i}_y\right] \tag{3}$$

Ampere's and Gauss's law for the magnetic field relate the field components as

$$\nabla \cdot \mathbf{B} = 0 \Rightarrow \frac{\partial B_z}{\partial z} = -\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right) \tag{4}$$

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J}_f + \nabla \times \mathbf{M}) = \mu_0 \mathbf{J}_T \Rightarrow \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 J_{Tx}$$

$$\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \mu_0 J_{Ty}$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \mu_0 J_{Tz} \tag{5}$$

which puts (3) in the form

$$\mathbf{f} = m_z\left(\frac{\partial B_x}{\partial z}\mathbf{i}_x + \frac{\partial B_y}{\partial z}\mathbf{i}_y + \frac{\partial B_z}{\partial z}\mathbf{i}_z - \mu_0(J_{Ty}\mathbf{i}_x - J_{Tx}\mathbf{i}_y)\right)$$
$$= (\mathbf{m} \cdot \nabla)\mathbf{B} + \mu_0 \mathbf{m} \times \mathbf{J}_T \tag{6}$$

where $\mathbf{J}_T$ is the sum of free and magnetization currents.

If there are N such dipoles per unit volume, the force density on the dipoles and on the free current is

$$\begin{aligned} \mathbf{F} = N\mathbf{f} &= (\mathbf{M} \cdot \nabla)\mathbf{B} + \mu_0 \mathbf{M} \times \mathbf{J}_T + \mathbf{J}_f \times \mathbf{B} \\ &= \mu_0(\mathbf{M} \cdot \nabla)(\mathbf{H} + \mathbf{M}) + \mu_0 \mathbf{M} \times (\mathbf{J}_f + \nabla \times \mathbf{M}) + \mu_0 \mathbf{J}_f \times (\mathbf{H} + \mathbf{M}) \\ &= \mu_0(\mathbf{M} \cdot \nabla)(\mathbf{H} + \mathbf{M}) + \mu_0 \mathbf{M} \times (\nabla \times \mathbf{M}) + \mu_0 \mathbf{J}_f \times \mathbf{H} \end{aligned} \tag{7}$$

Using the vector identity

$$\mathbf{M} \times (\nabla \times \mathbf{M}) = -(\mathbf{M} \cdot \nabla)\mathbf{M} + \tfrac{1}{2}\nabla(\mathbf{M} \cdot \mathbf{M}) \tag{8}$$

(7) can be reduced to

$$\mathbf{F} = \mu_0(\mathbf{M} \cdot \nabla)\mathbf{H} + \mu_0 \mathbf{J}_f \times \mathbf{H} + \nabla\left(\frac{\mu_0}{2}\mathbf{M} \cdot \mathbf{M}\right) \tag{9}$$

The total force on the body is just the volume integral of $\mathbf{F}$:

$$\mathbf{f} = \int_V \mathbf{F}\, dV \tag{10}$$

In particular, the last contribution in (9) can be converted to a surface integral using the gradient theorem, a corollary to the divergence theorem (see Problem 1-15*a*):

$$\int_V \nabla\left(\frac{\mu_0}{2}\mathbf{M}\cdot\mathbf{M}\right) dV = \oint_S \frac{\mu_0}{2}\mathbf{M}\cdot\mathbf{M}\, d\mathbf{S} \tag{11}$$

Since this surface S surrounds the magnetizable medium, it is in a region where $\mathbf{M}=0$ so that the integrals in (11) are zero. For this reason the force density of (9) is written as

$$\mathbf{F} = \mu_0(\mathbf{M}\cdot\nabla)\mathbf{H} + \mu_0\mathbf{J}_f\times\mathbf{H} \tag{12}$$

It is the first term on the right-hand side in (12) that accounts for an iron object to be drawn towards a magnet. Magnetizable materials are attracted towards regions of higher $\mathbf{H}$.

5-8-2 Force on a Current Loop

(a) Lorentz Force Only

Two parallel wires are connected together by a wire that is free to move, as shown in Figure 5-27*a*. A current I is imposed and the whole loop is placed in a uniform magnetic field $B_0\mathbf{i}_x$. The Lorentz force on the moveable wire is

$$f_y = IB_0 l \tag{13}$$

where we neglect the magnetic field generated by the current, assuming it to be much smaller than the imposed field B_0.

(b) Magnetization Force Only

The sliding wire is now surrounded by an infinitely permeable hollow cylinder of inner radius a and outer radius b, both being small compared to the wire's length l, as in Figure 5-27*b*. For distances near the cylinder, the solution is approximately the same as if the wire were infinitely long. For $r>0$ there is no current, thus the magnetic field is curl and divergence free within each medium so that the magnetic scalar potential obeys Laplace's equation as in Section 5-7-2. In cylindrical geometry we use the results of Section 4-3 and try a scalar potential of the form

$$\chi = \left(Ar + \frac{C}{r}\right)\cos\phi \tag{14}$$

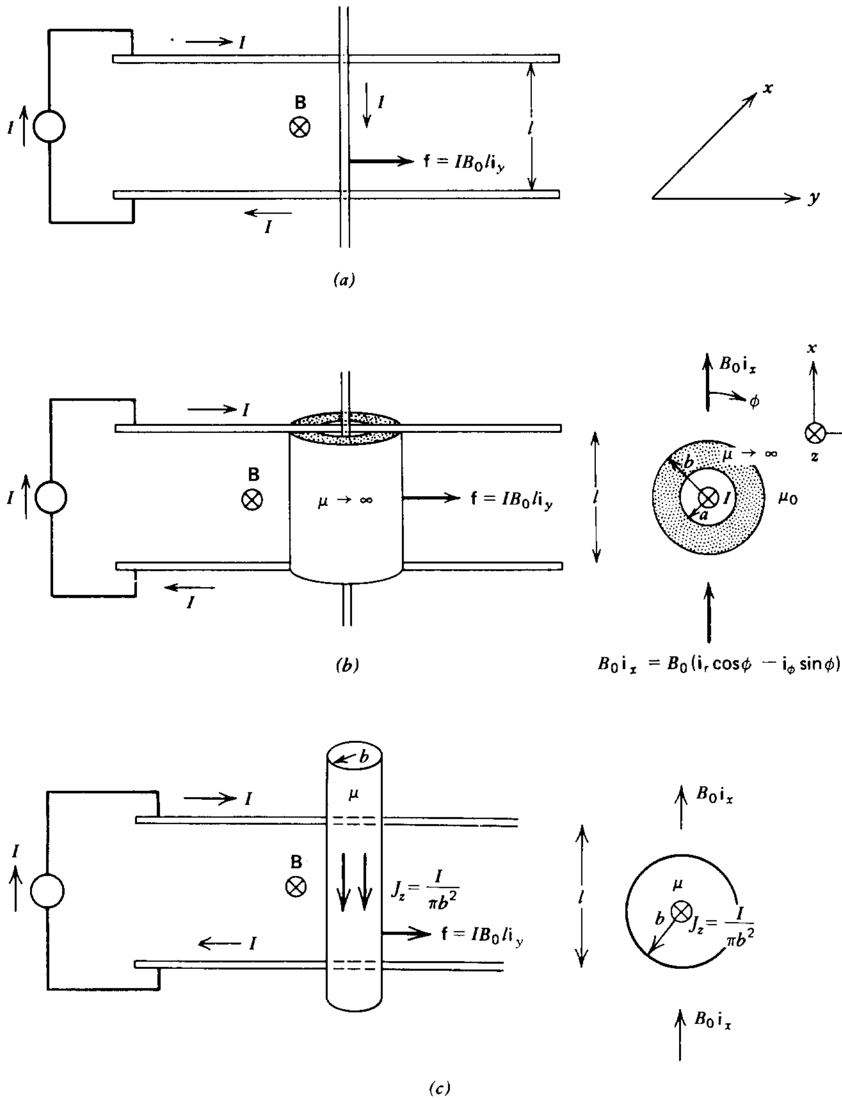


Figure 5-27 (*a*) The Lorentz force on a current carrying wire in a magnetic field. (*b*) If the current-carrying wire is surrounded by an infinitely permeable hollow cylinder, there is no Lorentz force as the imposed magnetic field is zero where the current is. However, the magnetization force on the cylinder is the same as in (*a*). (*c*) The total force on a current-carrying magnetically permeable wire is also unchanged.

in each region, where $\mathbf{B} = \nabla\chi$ because $\nabla \times \mathbf{B} = 0$. The constants are evaluated by requiring that the magnetic field approach the imposed field $B_0\mathbf{i}_x$ at $r = \infty$ and be normally incident onto the infinitely permeable cylinder at $r = a$ and $r = b$. In addition, we must add the magnetic field generated by the line current. The magnetic field in each region is then (see Problem 32*a*):

$$\mathbf{B}=\begin{cases}\dfrac{\mu_0 I}{2\pi r}\mathbf{i}_\phi, & 0<r<a\\[2ex] \dfrac{2B_0b^2}{b^2-a^2}\left[\left(1-\dfrac{a^2}{r^2}\right)\cos\phi\,\mathbf{i}_r-\left(1+\dfrac{a^2}{r^2}\right)\sin\phi\,\mathbf{i}_\phi\right]+\dfrac{\mu I}{2\pi r}\mathbf{i}_\phi, & \\ & a<r<b\\[2ex] B_0\left[\left(1+\dfrac{b^2}{r^2}\right)\cos\phi\,\mathbf{i}_r-\left(1-\dfrac{b^2}{r^2}\right)\sin\phi\,\mathbf{i}_\phi\right]+\dfrac{\mu_0 I}{2\pi r}\mathbf{i}_\phi, & \\ & r>b\end{cases}\tag{15}$$

Note the infinite flux density in the iron ($\mu\to\infty$) due to the line current that sets up the finite **H** field. However, we see that none of the imposed magnetic field is incident upon the current carrying wire because it is shielded by the infinitely permeable cylindrical shell so that the Lorentz force contribution on the wire is zero. There is, however, a magnetization force on the cylindrical shell where the internal magnetic field **H** is entirely due to the line current, $H_\phi = I/2\pi r$ because with $\mu\to\infty$, the contribution due to B_0 is negligibly small:

$$\begin{aligned}\mathbf{F}&=\mu_0(\mathbf{M}\cdot\nabla)\mathbf{H}\\ &=\mu_0\left(M_r\frac{\partial}{\partial r}(H_\phi\mathbf{i}_\phi)+\frac{M_\phi}{r}\frac{\partial}{\partial\phi}(H_\phi\mathbf{i}_\phi)\right)\end{aligned}\tag{16}$$

Within the infinitely permeable shell the magnetization and **H** fields are

$$\begin{aligned}H_\phi&=\frac{I}{2\pi r}\\ \mu_0 M_r&=B_r-\mu_0\overset{0}{\cancel{H_r}}=\frac{2B_0b^2}{b^2-a^2}\left(1-\frac{a^2}{r^2}\right)\cos\phi\\ \mu_0 M_\phi&=B_\phi-\mu_0 H_\phi=-\frac{2B_0b^2}{(b^2-a^2)}\left(1+\frac{a^2}{r^2}\right)\sin\phi+\frac{(\mu-\mu_0)I}{2\pi r}\end{aligned}\tag{17}$$

Although H_ϕ only depends on r, the unit vector $\mathbf{i}_\phi$ depends on ϕ:

$$\mathbf{i}_\phi=(-\sin\phi\,\mathbf{i}_x+\cos\phi\,\mathbf{i}_y)\tag{18}$$

so that the force density of (16) becomes

$$\begin{aligned}\mathbf{F}&=-\frac{B_rI}{2\pi r^2}\mathbf{i}_\phi+\frac{(B_\phi-\mu_0H_\phi)I}{2\pi r^2}\frac{d}{d\phi}(\mathbf{i}_\phi)\\ &=\frac{I}{2\pi r^2}[-B_r(-\sin\phi\,\mathbf{i}_x+\cos\phi\,\mathbf{i}_y)\\ &\quad+(B_\phi-\mu_0H_\phi)(-\cos\phi\,\mathbf{i}_x-\sin\phi\,\mathbf{i}_y)]\end{aligned}$$

$$
\begin{aligned}
&= \frac{I}{2\pi \mathrm{r}^2}\left\{ -\frac{2B_0 b^2}{b^2-a^2}\left[\left(1-\frac{a^2}{\mathrm{r}^2}\right)\cos\phi(-\sin\phi\,\mathbf{i}_x+\cos\phi\,\mathbf{i}_y)\right.\right.\\
&\quad \left. -\left(1+\frac{a^2}{\mathrm{r}^2}\right)\sin\phi(\cos\phi\,\mathbf{i}_x+\sin\phi\,\mathbf{i}_y)\right]\\
&\quad \left. +\frac{(\mu-\mu_0)I}{2\pi \mathrm{r}}(\cos\phi\,\mathbf{i}_x+\sin\phi\,\mathbf{i}_y)\right\}\\
&= \frac{I}{2\pi \mathrm{r}^2}\left[-\frac{2B_0 b^2}{b^2-a^2}\left(-2\sin\phi\cos\phi\,\mathbf{i}_x-\frac{2a^2}{\mathrm{r}^2}\mathbf{i}_y\right)\right.\\
&\quad \left. +\frac{(\mu-\mu_0)I}{2\pi \mathrm{r}}(\cos\phi\,\mathbf{i}_x+\sin\phi\,\mathbf{i}_y)\right]
\end{aligned}
\tag{19}
$$

The total force on the cylinder is obtained by integrating (19) over r and ϕ:

$$
\mathbf{f} = \int_{\phi=0}^{2\pi}\int_{\mathrm{r}=a}^{b} \mathbf{F}l\,\mathrm{r}\,d\mathrm{r}\,d\phi \tag{20}
$$

All the trigonometric terms in (19) integrate to zero over ϕ so that the total force is

$$
\begin{aligned}
f_y &= \frac{2B_0 b^2 Il}{(b^2-a^2)}\int_{\mathrm{r}=a}^{b}\frac{a^2}{\mathrm{r}^3}\,d\mathrm{r}\\
&= -\frac{B_0 b^2 Il}{(b^2-a^2)}\frac{a^2}{\mathrm{r}^2}\bigg|_a^b\\
&= IB_0 l
\end{aligned}
\tag{21}
$$

The force on the cylinder is the same as that of an unshielded current-carrying wire given by (13). If the iron core has a finite permeability, the total force on the wire (Lorentz force) and on the cylinder (magnetization force) is again equal to (13). This fact is used in rotating machinery where current-carrying wires are placed in slots surrounded by highly permeable iron material. Most of the force on the whole assembly is on the iron and not on the wire so that very little restraining force is necessary to hold the wire in place. The force on a current-carrying wire surrounded by iron is often calculated using only the Lorentz force, neglecting the presence of the iron. The correct answer is obtained but for the wrong reasons. Actually there is very little **B** field near the wire as it is almost surrounded by the high permeability iron so that the Lorentz force on the wire is very small. The force is actually on the iron core.

(c) Lorentz and Magnetization Forces

If the wire itself is highly permeable with a uniformly distributed current, as in Figure 5-27*c*, the magnetic field is (see Problem 32*a*)

$$\mathbf{H}=\begin{cases}\dfrac{2B_0}{\mu+\mu_0}(\mathbf{i}_r\cos\phi-\mathbf{i}_\phi\sin\phi)+\dfrac{Ir}{2\pi b^2}\mathbf{i}_\phi \\ \quad =\dfrac{2B_0}{\mu+\mu_0}\mathbf{i}_x+\dfrac{I}{2\pi b^2}(-y\mathbf{i}_x+x\mathbf{i}_y), \quad r<b \\ \dfrac{B_0}{\mu_0}\left[\left(1+\dfrac{b^2}{r^2}\dfrac{\mu-\mu_0}{\mu+\mu_0}\right)\cos\phi\,\mathbf{i}_r\right. \\ \quad \left.-\left(1-\dfrac{b^2}{r^2}\dfrac{\mu-\mu_0}{\mu+\mu_0}\right)\sin\phi\,\mathbf{i}_\phi\right]+\dfrac{I}{2\pi r}\mathbf{i}_\phi, \quad r>b\end{cases} \tag{22}$$

It is convenient to write the fields within the cylinder in Cartesian coordinates using (18) as then the force density given by (12) is

$$\begin{aligned}\mathbf{F}&=\mu_0(\mathbf{M}\cdot\nabla)\mathbf{H}+\mu_0\mathbf{J}_f\times\mathbf{H}\\ &=(\mu-\mu_0)(\mathbf{H}\cdot\nabla)\mathbf{H}+\frac{\mu_0 I}{\pi b^2}\mathbf{i}_z\times\mathbf{H}\\ &=(\mu-\mu_0)\left(H_x\frac{\partial}{\partial x}+H_y\frac{\partial}{\partial y}\right)(H_x\mathbf{i}_x+H_y\mathbf{i}_y)+\frac{\mu_0 I}{\pi b^2}(H_x\mathbf{i}_y-H_y\mathbf{i}_x)\end{aligned} \tag{23}$$

Since within the cylinder ($r<b$) the partial derivatives of **H** are

$$\begin{aligned}\frac{\partial H_x}{\partial x}&=\frac{\partial H_y}{\partial y}=0\\ \frac{\partial H_x}{\partial y}&=-\frac{\partial H_y}{\partial x}=-\frac{I}{2\pi b^2}\end{aligned} \tag{24}$$

(23) reduces to

$$\begin{aligned}\mathbf{F}&=(\mu-\mu_0)\left(H_x\frac{\partial H_y}{\partial x}\mathbf{i}_y+H_y\frac{\partial H_x}{\partial y}\mathbf{i}_x\right)+\frac{\mu_0 I}{\pi b^2}(H_x\mathbf{i}_y-H_y\mathbf{i}_x)\\ &=\frac{I}{2\pi b^2}(\mu+\mu_0)(H_x\mathbf{i}_y-H_y\mathbf{i}_x)\\ &=\frac{I(\mu+\mu_0)}{2\pi b^2}\left[\left(\frac{2B_0}{\mu+\mu_0}-\frac{Iy}{2\pi b^2}\right)\mathbf{i}_y-\frac{Ix}{2\pi b^2}\mathbf{i}_x\right]\end{aligned} \tag{25}$$

Realizing from Table 1-2 that

$$y\mathbf{i}_y+x\mathbf{i}_x=r[\sin\phi\,\mathbf{i}_y+\cos\phi\,\mathbf{i}_x]=r\mathbf{i}_r \tag{26}$$

the force density can be written as

$$\mathbf{F}=\frac{IB_0}{\pi b^2}\mathbf{i}_y-\frac{I^2(\mu+\mu_0)}{(2\pi b^2)^2}\mathrm{r}\,(\sin\phi\,\mathbf{i}_y+\cos\phi\,\mathbf{i}_x) \tag{27}$$

The total force on the permeable wire is

$$\mathbf{f}=\int_{\phi=0}^{2\pi}\int_{\mathrm{r}=0}^{b}\mathbf{F}l\mathrm{r}\,d\mathrm{r}\,d\phi \tag{28}$$

We see that the trigonometric terms in (27) integrate to zero so that only the first term contributes:

$$\begin{aligned} f_y &= \frac{IB_0 l}{\pi b^2}\int_{\phi=0}^{2\pi}\int_{\mathrm{r}=0}^{b}\mathrm{r}\,d\mathrm{r}\,d\phi \\ &= IB_0 l \end{aligned} \tag{29}$$

The total force on the wire is independent of its magnetic permeability.

PROBLEMS

Section 5-1

1. A charge q of mass m moves through a uniform magnetic field $B_o\mathbf{i}_z$. At $t=0$ its velocity and displacement are

$$\mathbf{v}(t=0)=v_{x0}\mathbf{i}_x+v_{y0}\mathbf{i}_y+v_{z0}\mathbf{i}_z$$

$$\mathbf{r}(t=0)=x_0\mathbf{i}_x+y_0\mathbf{i}_y+z_0\mathbf{i}_z$$

(a) What is the subsequent velocity and displacement?

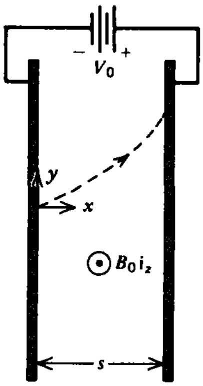
(b) Show that its motion projected onto the xy plane is a circle. What is the radius of this circle and where is its center?

(c) What is the time dependence of the kinetic energy of the charge $\frac{1}{2}m|\mathbf{v}|^2$?

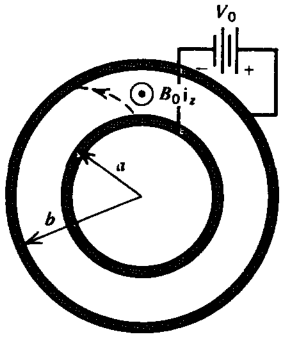
2. A magnetron is essentially a parallel plate capacitor stressed by constant voltage V_0 where electrons of charge $-e$ are emitted at $x=0$, $y=0$ with zero initial velocity. A transverse magnetic field $B_0\mathbf{i}_z$ is applied. Neglect the electric and magnetic fields due to the electrons in comparison to the applied field.

(a) What is the velocity and displacement of an electron, injected with zero initial velocity at $t=0$?

(b) What value of magnetic field will just prevent the electrons from reaching the other electrode? This is the cut-off magnetic field.



(a)



(c)

(c) A magnetron is built with coaxial electrodes where electrons are injected from $r = a$, $\phi = 0$ with zero initial velocity. Using the relations from Table 1-2,

$$\mathbf{i}_r = \cos\phi\,\mathbf{i}_x + \sin\phi\,\mathbf{i}_y$$

$$\mathbf{i}_\phi = -\sin\phi\,\mathbf{i}_x + \cos\phi\,\mathbf{i}_y$$

show that

$$\frac{d\mathbf{i}_r}{dt} = \mathbf{i}_\phi \frac{d\phi}{dt} = \frac{v_\phi}{r}\mathbf{i}_\phi$$

$$\frac{d\mathbf{i}_\phi}{dt} = -\mathbf{i}_r \frac{d\phi}{dt} = -\frac{v_\phi}{r}\mathbf{i}_r$$

What is the acceleration of a charge with velocity

$$\mathbf{v} = v_r\mathbf{i}_r + v_\phi\mathbf{i}_\phi?$$

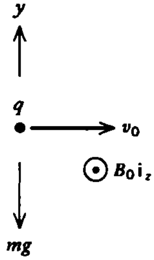
(d) Find the velocity of the electrons as a function of radial position.
Hint:

$$\frac{dv_r}{dt} = \frac{dv_r}{dr}\frac{dr}{dt} = v_r\frac{dv_r}{dr} = \frac{d}{dr}\left(\tfrac{1}{2}v_r^2\right)$$

$$\frac{dv_\phi}{dt} = \frac{dv_\phi}{dr}\frac{dr}{dt} = v_r\frac{dv_\phi}{dr}$$

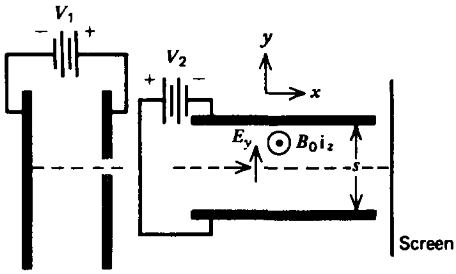
(e) What is the cutoff magnetic field? Check your answer with (b) in the limit $b = a + s$ where $s \ll a$.

3. A charge q of mass m within a gravity field $-g\mathbf{i}_y$ has an initial velocity $v_0\mathbf{i}_x$. A magnetic field $B_0\mathbf{i}_z$ is applied. What



value of B_0 will keep the particle moving at constant speed in the x direction?

4. The charge to mass ratio of an electron e/m was first measured by Sir J. J. Thomson in 1897 by the cathode-ray tube device shown. Electrons emitted by the cathode pass through a slit in the anode into a region with crossed electric and magnetic fields, both being perpendicular to the electrons velocity. The end of the tube is coated with a fluorescent material that produces a bright spot where the electron beam impacts.



(a) What is the velocity of the electrons when passing through the slit if their initial cathode velocity is v_0?

(b) The electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ are adjusted so that the vertical deflection of the beam is zero. What is the initial electron velocity? (Neglect gravity.)

(c) The voltage V_2 is now set to zero. What is the radius R of the electrons motion about the magnetic field?

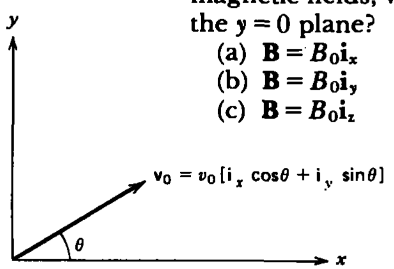
(d) What is e/m in terms of E, B, and R?

5. A charge q of mass m at $t=0$ crosses the origin with velocity $\mathbf{v}_0 = \mathbf{v}_{x0}\mathbf{i}_x + v_{y0}\mathbf{i}_y$. For each of the following applied magnetic fields, where and when does the charge again cross the $y=0$ plane?

(a) $\mathbf{B} = B_0 \mathbf{i}_x$

(b) $\mathbf{B} = B_0 \mathbf{i}_y$

(c) $\mathbf{B} = B_0 \mathbf{i}_z$



6. In 1896 Zeeman observed that an atom in a magnetic field had a fine splitting of its spectral lines. A classical theory of the Zeeman effect, developed by Lorentz, modeled the electron with mass m as being bound to the nucleus by a spring-like force with spring constant k so that in the absence of a magnetic field its natural frequency was $\omega_k = \sqrt{k/m}$.

(a) A magnetic field $B_0\mathbf{i}_z$ is applied. Write Newton's law for the x, y, and z displacements of the electron including the spring and Lorentz forces.

(b) Because these equations are linear, guess exponential solutions of the form e^{st}. What are the natural frequencies?

(c) Because ω_k is typically in the optical range ($\omega_k \approx 10^{15}$ radian/sec), show that the frequency splitting is small compared to ω_k even for a strong field of $B_0 = 1$ tesla. In this limit, find approximate expressions for the natural frequencies of (b).

7. A charge q moves through a region where there is an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$. The medium is very viscous so that inertial effects are negligible,

$$\beta \mathbf{v} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

where β is the viscous drag coefficient. What is the velocity of the charge? (**Hint:** $(\mathbf{v} \times \mathbf{B}) \times \mathbf{B} = -\mathbf{v}(\mathbf{B} \cdot \mathbf{B}) + \mathbf{B}(\mathbf{v} \cdot \mathbf{B})$ and $\mathbf{v} \cdot \mathbf{B} = (q/\beta)\mathbf{E} \cdot \mathbf{B}$.)

8. Charges of mass m, charge q, and number density n move through a conducting material and collide with the host medium with a collision frequency ν in the presence of an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$.

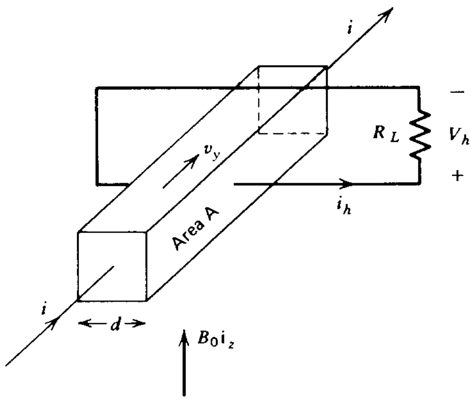
(a) Write Newton's first law for the charge carriers, along the same lines as developed in Section 3-2-2, with the addition of the Lorentz force.

(b) Neglecting particle inertia and diffusion, solve for the particle velocity $\mathbf{v}$.

(c) What is the constitutive law relating the current density $\mathbf{J} = qn\mathbf{v}$ to $\mathbf{E}$ and $\mathbf{B}$. This is the generalized Ohm's law in the presence of a magnetic field.

(d) What is the Ohmic conductivity σ? A current i is passed through this material in the presence of a perpendicular magnetic field. A resistor R_L is connected across the terminals. What is the Hall voltage? (See top of page 379).

(e) What value of R_L maximizes the power dissipated in the load?



Section 5.2

9. A point charge q is traveling within the magnetic field of an infinitely long line current I. At $r = r_0$ its velocity is

$$\mathbf{v}(t=0) = v_{r0}\mathbf{i}_r + v_{\phi 0}\mathbf{i}_\phi + v_{z0}\mathbf{i}_z$$

Its subsequent velocity is only a function of r.

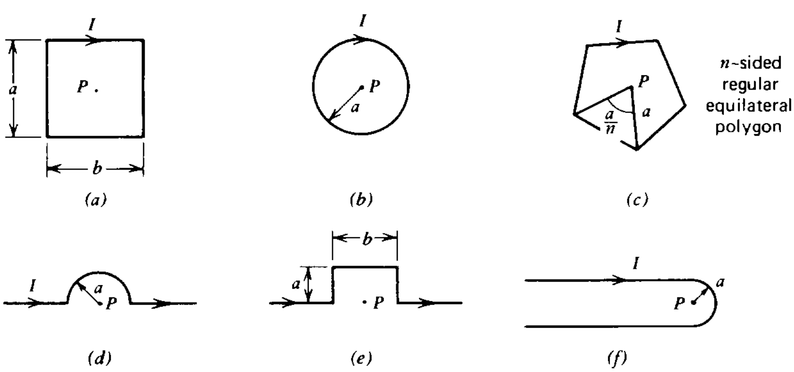
(a) What is the velocity of the charge as a function of position? **Hint:** See Problem $2c$ and $2d$,

$$\int \frac{\ln x}{x}\,dx = \tfrac{1}{2}(\ln x)^2$$

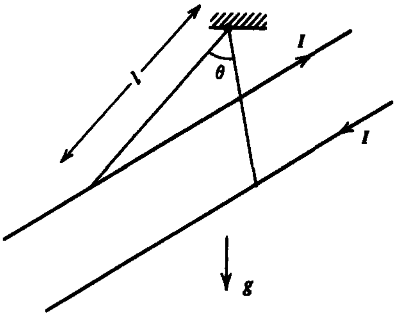
(b) What is the kinetic energy of the charge?

(c) What is the closest distance that the charge can approach the line current if $v_{\phi 0} = 0$?

10. Find the magnetic field at the point P shown for the following line currents:

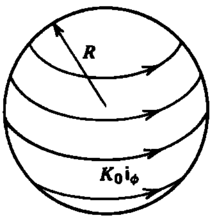


11. Two long parallel line currents of mass per unit length m in a gravity field g each carry a current I in opposite



directions. They are suspended by cords of length l. What is the angle θ between the cords?

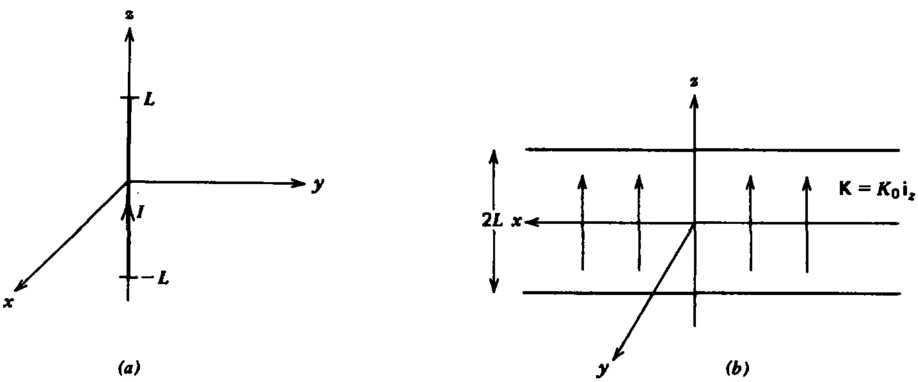
12. A constant current $K_0\mathbf{i}_\phi$ flows on the surface of a sphere of radius R.



(a) What is the magnetic field at the center of the sphere? (**HINT:** $\mathbf{i}_\phi \times \mathbf{i}_r = \mathbf{i}_\theta = \cos\theta\cos\phi\mathbf{i}_x + \cos\theta\sin\phi\mathbf{i}_y - \sin\theta\mathbf{i}_z$.)

(b) Use the results of (a) to find the magnetic field at the center of a spherical shell of inner radius R_1 and outer radius R_2 carrying a uniformly distributed volume current $J_0\mathbf{i}_\phi$.

13. A line current I of length $2L$ flows along the z axis.



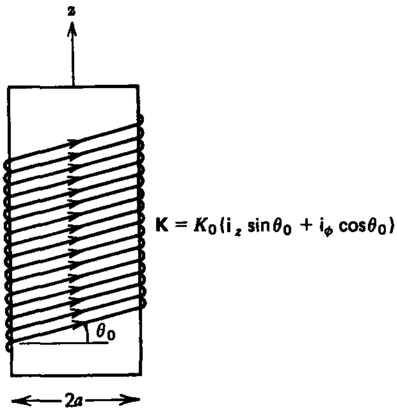
(a) What is the magnetic field everywhere in the $z=0$ plane?

(b) Use the results of (a) to find the magnetic field in the $z=0$ plane due to an infinitely long current sheet of height $2L$ and uniform current density $K_0\mathbf{i}_z$. **Hint:** Let $u = x^2+y^2$

$$\int \frac{du}{u(u^2+bu-a)^{1/2}} = \frac{1}{\sqrt{a}}\sin^{-1}\left(\frac{bu+2a}{u\sqrt{b^2+4a}}\right)$$

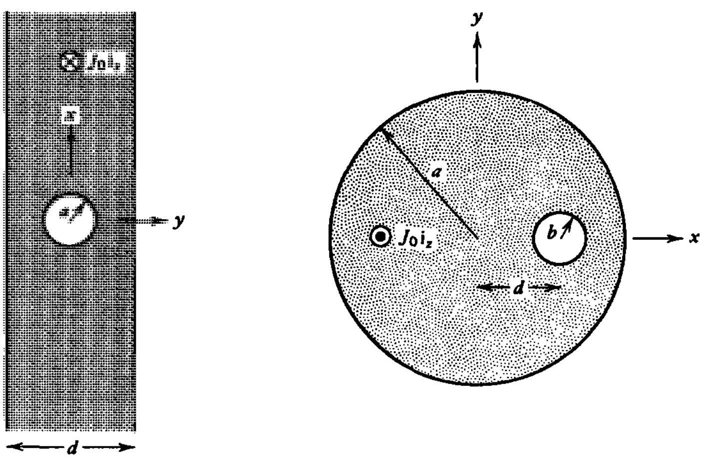
14. Closely spaced wires are wound about an infinitely long cylindrical core at pitch angle θ_0. A current flowing in the wires then approximates a surface current

$$\mathbf{K} = K_0(\mathbf{i}_z \sin\theta_0 + \mathbf{i}_\phi \cos\theta_0)$$



What is the magnetic field everywhere?

15. An infinite slab carries a uniform current $J_0\mathbf{i}_z$ except within a cylindrical hole of radius a centered within the slab.



(a) (b)

(a) Find the magnetic field everywhere? (**Hint:** Use superposition replacing the hole by two oppositely directed currents.)

(b) An infinitely long cylinder of radius a carrying a uniform current $J_0\mathbf{i}_z$ has an off-axis hole of radius b with center a distance d from the center of the cylinder. What is the magnetic field within the hole? (**Hint:** Convert to Cartesian coordinates $r\mathbf{i}_\phi = x\mathbf{i}_y - y\mathbf{i}_x$.)

Section 5.3

16. Which of the following vectors can be a magnetic field $\mathbf{B}$? If so, what is the current density $\mathbf{J}$?

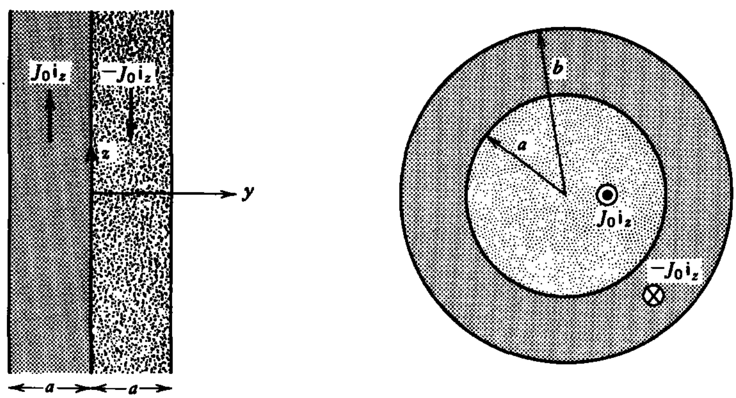
(a) $\mathbf{B} = ar\mathbf{i}_r$

(b) $\mathbf{B} = a(x\mathbf{i}_y - y\mathbf{i}_x)$

(c) $\mathbf{B} = a(x\mathbf{i}_x - y\mathbf{i}_y)$

(d) $\mathbf{B} = ar\mathbf{i}_\phi$

17. Find the magnetic field everywhere for each of the following current distributions:



(a) (c)

(a) $\mathbf{J} = \begin{cases} J_0\mathbf{i}_z, & -a<y<0 \\ -J_0\mathbf{i}_z, & 0<y<a \end{cases}$

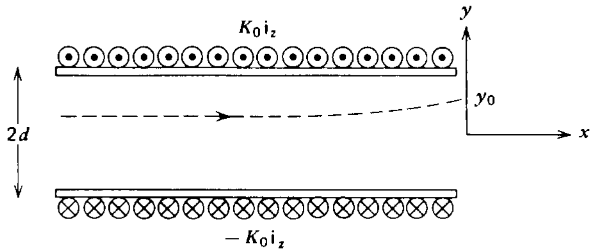
(b) $\mathbf{J} = \dfrac{J_0 y}{a}\mathbf{i}_z, \quad -a<y<a$

(c) $\mathbf{J} = \begin{cases} J_0\mathbf{i}_z, & 0<r<a \\ -J_0\mathbf{i}_z, & a<r<b \end{cases}$

(d) $\mathbf{J} = \begin{cases} \dfrac{J_0 r}{a}\mathbf{i}_z, & r<a \\ 0, & r>a \end{cases}$

Section 5.4

18. Two parallel semi-infinite current sheets a distance d apart have their currents flowing in opposite directions and extend over the interval $-\infty < x < 0$.



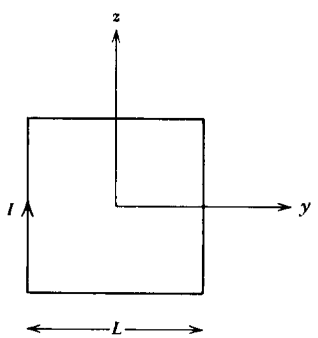
(a) What is the vector potential? (**Hint:** Use superposition of the results in Section 5-3-4*b*.)

(b) What is the magnetic field everywhere?

(c) How much magnetic flux per unit length emanates through the open face at $x = 0$? How much magnetic flux per unit length passes through each current sheet?

(d) A magnetic field line emanates at the position $y_0(0 < y_0 < d)$ in the $x = 0$ plane. At what value of y is this field line at $x = -\infty$?

19. (a) Show that $\nabla \cdot \mathbf{A} \neq 0$ for the finite length line current in Section 5-4-3a. Why is this so?

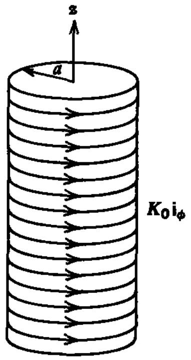


(b) Find the vector potential for a square loop.

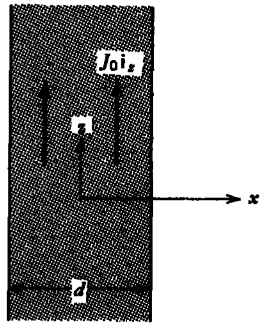
(c) What is $\nabla \cdot \mathbf{A}$ now?

20. Find the magnetic vector potential and magnetic field for the following current distributions: (**Hint:** $\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})$)

(i) Infinitely long cylinder of radius a carrying a
  (a) surface current $K_0 \mathbf{i}_\phi$
  (b) surface current $K_0 \mathbf{i}_z$
  (c) volume current $J_0 \mathbf{i}_z$



(a)



(d)

(ii) Infinitely long slab of thickness d carrying a

(d) volume current $J_0\mathbf{i}_z$

(e) volume current $\frac{J_0 x}{d}\mathbf{i}_z$

Section 5.5

21. A general definition for the magnetic dipole moment for any shaped current loop is

$$\mathbf{m} = \frac{1}{2}\oint \mathbf{r}\times\mathbf{I}\, dl$$

If the current is distributed over a surface or volume or is due to a moving point charge we use

$$\mathbf{I}\, dl \rightarrow q\mathbf{v} \rightarrow \mathbf{K}\, dS \rightarrow \mathbf{J}\, dV$$

What is the magnetic dipole moment for the following current distributions:

(a) a point charge q rotated at constant angular speed ω at radius a;

(b) a circular current loop of radius a carrying a current I;

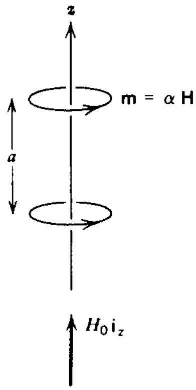
(c) a disk of radius a with surface current $K_0\mathbf{i}_\phi$;

(d) a uniformly distributed sphere of surface or volume charge with total charge Q and radius R rotating in the ϕ direction at constant angular speed ω. (**Hint:** $\mathbf{i}_r \times \mathbf{i}_\phi = -\mathbf{i}_\theta = -[\cos\theta\cos\phi\,\mathbf{i}_x + \cos\theta\sin\phi\,\mathbf{i}_y - \sin\theta\,\mathbf{i}_z]$)

22. Two identical point magnetic dipoles $\mathbf{m}$ with magnetic polarizability $\alpha(\mathbf{m} = \alpha\mathbf{H})$ are a distance a apart along the z axis. A macroscopic field $H_0\mathbf{i}_z$ is applied.

(a) What is the local magnetic field acting on each dipole?

(b) What is the force on each dipole?



(c) Repeat (a) and (b) if we have an infinite array of such dipoles. **Hint:**

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.2$$

(d) If we assume that there is one such dipole within each volume of a^3, what is the permeability of the medium?

23. An orbiting electron with magnetic moment $m_z \mathbf{i}_z$ is in a uniform magnetic field $B_0 \mathbf{i}_z$ when at $t = 0$ it is slightly displaced so that its angular momentum $\mathbf{L} = -(2m_e/e)\mathbf{m}$ now also has x and y components.

(a) Show that the torque equation can be put in terms of the magnetic moment

$$\frac{d\mathbf{m}}{dt} = -\gamma \mathbf{m} \times \mathbf{B}$$

where γ is called the gyromagnetic ratio. What is γ?

(b) Write out the three components of (a) and solve for the magnetic moment if at $t = 0$ the moment is initially

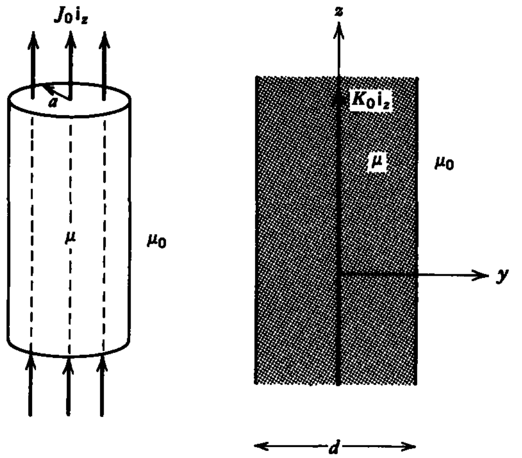
$$\mathbf{m}(t = 0) = m_{x0}\mathbf{i}_x + m_{y0}\mathbf{i}_y + m_{z0}\mathbf{i}_z$$

(c) Show that the magnetic moment precesses about the applied magnetic field. What is the precessional frequency?

24. What are the **B**, **H**, and **M** fields and the resulting magnetization currents for the following cases:

(a) A uniformly distributed volume current $J_0 \mathbf{i}_z$ through a cylinder of radius a and permeability μ surrounded by free space.

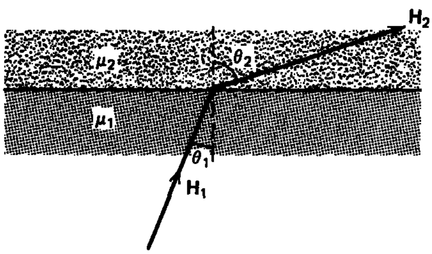
(b) A current sheet $K_0 \mathbf{i}_z$ centered within a permeable slab of thickness d surrounded by free space.



(a) (b)

Section 5.6

25. A magnetic field with magnitude H_1 is incident upon the flat interface separating two different linearly permeable materials at an angle θ_1 from the normal. There is no surface



current on the interface. What is the magnitude and angle of the magnetic field in region 2?

26. A cylinder of radius a and length L is permanently magnetized as $M_0\mathbf{i}_z$.

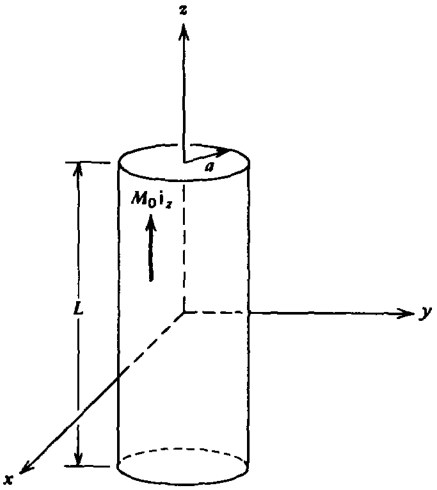
(a) What are the **B** and **H** fields everywhere along its axis?

(b) What are the fields far from the magnet ($r \gg a$, $r \gg L$)?

(c) Use the results of (a) to find the **B** and **H** fields everywhere due to a permanently magnetized slab $M_0\mathbf{i}_z$ of infinite xy extent and thickness L.

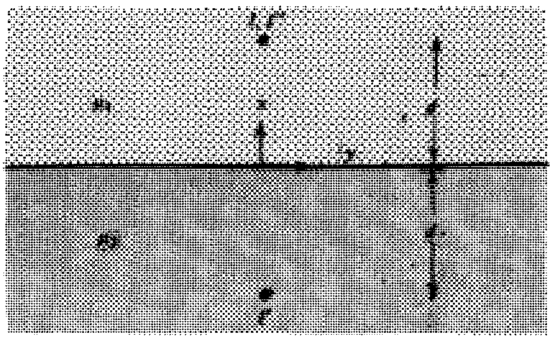
(d) Repeat (a) and (b) if the cylinder has magnetization $M_0(1 - r/a)\mathbf{i}_z$. **Hint:**

$$\int \frac{dr}{(a^2+r^2)^{1/2}} = \ln\left(r + \sqrt{a^2+r^2}\right)$$



Section 5.7

27. A z-directed line current I is a distance d above the interface separating two different magnetic materials with permeabilities μ_1 and μ_2.

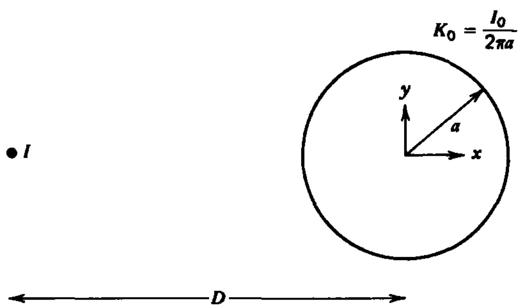


(a) Find the image currents I' at position $x=-d$ and I'' at $x=d$ that satisfy all the boundary conditions. The field in region 1 is due to I and I' while the field in region 2 is due to I''. (**Hint:** See the analogous dielectric problem in Section 3-3-3.)

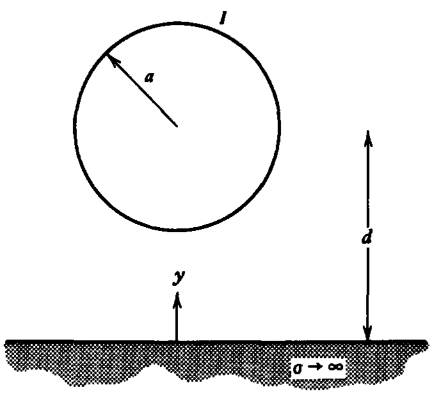
(b) What is the force per unit length on the line current I?

28. An infinitely long line current I is parallel to and a distance D from the axis of a perfectly conducting cylinder of radius a carrying a total surface current I_0.

(a) Find suitable image currents and verify that the boundary conditions are satisfied. (**Hint:** $x\mathbf{i}_y - y\mathbf{i}_x = r\mathbf{i}_\phi$; $\mathbf{i}_y = \sin\phi\,\mathbf{i}_r + \cos\phi\,\mathbf{i}_\phi$; $x = r\cos\phi$.)



(a)



(d)

(b) What is the surface current distribution on the cylinder? What total current flows on the cylinder? **Hint:**

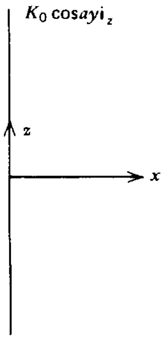
$$\int \frac{d\phi}{a + b\cos\phi} = \frac{2}{[a^2 - b^2]^{1/2}} \tan^{-1}\left(\frac{[a^2 - b^2]^{1/2} \tan(\frac{1}{2}\phi)}{(a+b)}\right)$$

(c) What is the force per unit length on the cylinder?

(d) A perfectly conducting cylinder of radius a carrying a total current I has its center a distance d above a perfectly conducting plane. What image currents satisfy the boundary conditions?

(e) What is the force per unit length on the cylinder?

29. A current sheet $K_0 \cos ay\mathbf{i}_z$ is placed at $x = 0$. Because there are no volume currents for $x \neq 0$, a scalar magnetic potential can be defined $\mathbf{H} = \nabla\chi$.



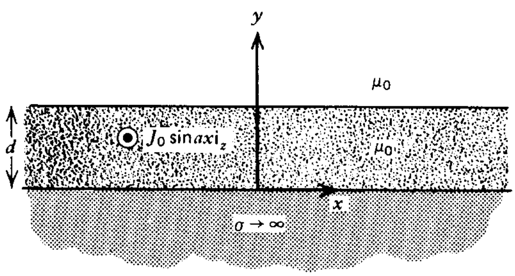
(a) What is the general form of solution for χ? (**Hint:** See Section 4-2-3.)

(b) What boundary conditions must be satisfied?

(c) What is the magnetic field and vector potential everywhere?

(d) What is the equation of the magnetic field lines?

30. A slab of thickness d carries a volume current distribution $J_0 \sin ax\mathbf{i}_z$ and is placed upon a perfectly conducting ground plane.



(a) Find a particular solution for the vector potential. Are all the boundary conditions satisfied?

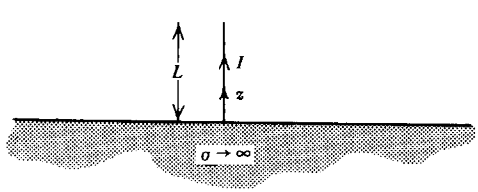
(b) Show that additional solutions to Laplace's equations can be added to the vector potential to satisfy the boundary conditions. What is the magnetic field everywhere?

(c) What is the surface current distribution on the ground plane?

(d) What is the force per unit length on a section of ground plane of width $2\pi/a$? What is the body force per unit length on a section of the current carrying slab of width $2\pi/a$?

(e) What is the magnetic field if the slab carries no current but is permanently magnetized as $M_0 \sin ax\mathbf{i}_y$ Repeat (c) and (d).

31. A line current of length L stands perpendicularly upon a perfectly conducting ground plane.

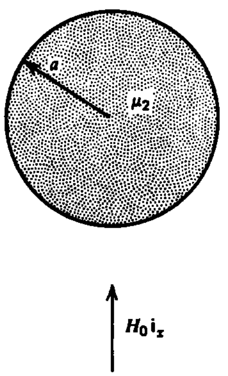


(a) Find a suitable image current that is equivalent to the induced current on the $z=0$ plane. Does the direction of the image current surprise you?

(b) What is the magnetic field everywhere? (**Hint:** See Section 5-4-3*a*.)

(c) What is the surface current distribution on the conducting plane?

32. A cylinder of radius a is placed within a uniform magnetic field $H_0\mathbf{i}_x$. Find the magnetic field for each of the following cases:

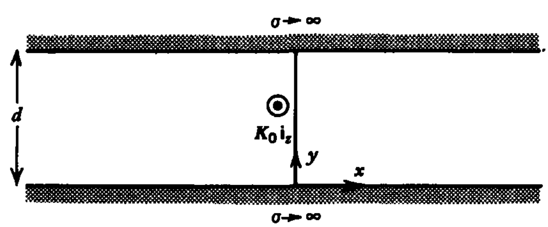


(a) Cylinder has permeability μ_2 and surrounding medium has permeability μ_1.

(b) Perfectly conducting cylinder in free space.

(c) Uniformly magnetized cylinder $M_2\mathbf{i}_x$ in a uniformly magnetized medium $M_1\mathbf{i}_x$.

33. A current sheet $K_0\mathbf{i}_z$ is placed along the y axis at $x=0$ between two parallel perfectly conducting planes a distance d apart.



(a) Write the constant current at $x=0$ as an infinite Fourier series of fundamental period $2d$. (**Hint:** See Section 4-2-5.)

(b) What general form of a scalar potential χ, where $\mathbf{H}=\nabla\chi$, will satisfy the boundary conditions?

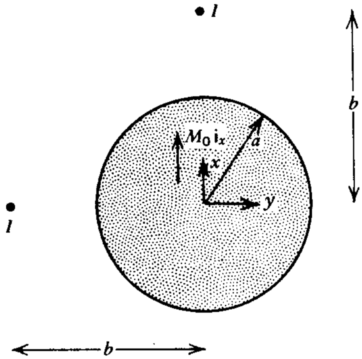
(c) What is the magnetic field everywhere?

(d) What is the surface current distribution and the total current on the conducting planes? **Hint:**

$$\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$$

Section 5.8

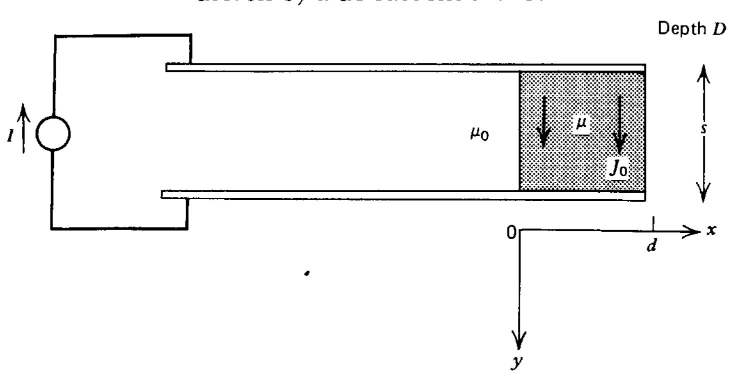
34. An infinitely long cylinder of radius a is permanently magnetized as $M_0\mathbf{i}_x$.



(a) Find the magnetic field everywhere.

(b) An infinitely long line current I is placed either at $y=-b$ or at $x=b$ $(b>a)$. For each of these cases, what is the force per unit length on the line current? (**Hint:** See problem 32c.)

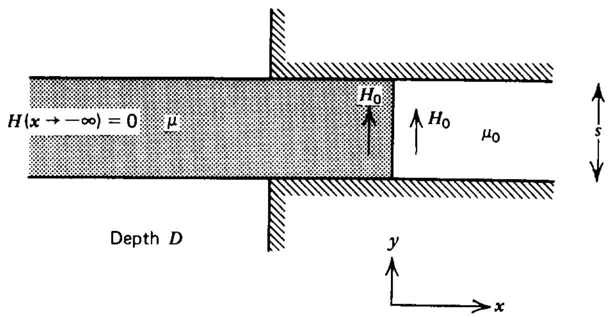
35. Parallel plate electrodes are separated by a rectangular conducting slab that has a permeability μ. The system is driven by a dc current source.



(a) Neglecting fringing field effects assume the magnetic field is $H_z(x)\mathbf{i}_z$. If the current is uniformly distributed throughout the slab, find the magnetic field everywhere.

(b) What is the total force on the slab? Does the force change with different slab permeability? Why not?

36. A permeable slab is partially inserted into the air gap of a magnetic circuit with uniform field H_0. There is a nonuniform fringing field right outside the magnetic circuit near the edges.



(a) What is the total force on the slab in the x direction?

(b) Repeat (a) if the slab is permanently magnetized $\mathbf{M} = M_0\mathbf{i}_y$. (**Hint:** What is $H_x(x = -\infty)$? See Example 5-2a.)

chapter 6

electromagnetic induction

In our development thus far, we have found the electric and magnetic fields to be uncoupled. A net charge generates an electric field while a current is the source of a magnetic field. In 1831 Michael Faraday experimentally discovered that a time varying magnetic flux through a conducting loop also generated a voltage and thus an electric field, proving that electric and magnetic fields are coupled.

6-1 FARADAY'S LAW OF INDUCTION

6-1-1 The Electromotive Force (EMF)

Faraday's original experiments consisted of a conducting loop through which he could impose a dc current via a switch. Another short circuited loop with no source attached was nearby, as shown in Figure 6-1. When a dc current flowed in loop 1, no current flowed in loop 2. However, when the voltage was first applied to loop 1 by closing the switch, a transient current flowed in the opposite direction in loop 2.

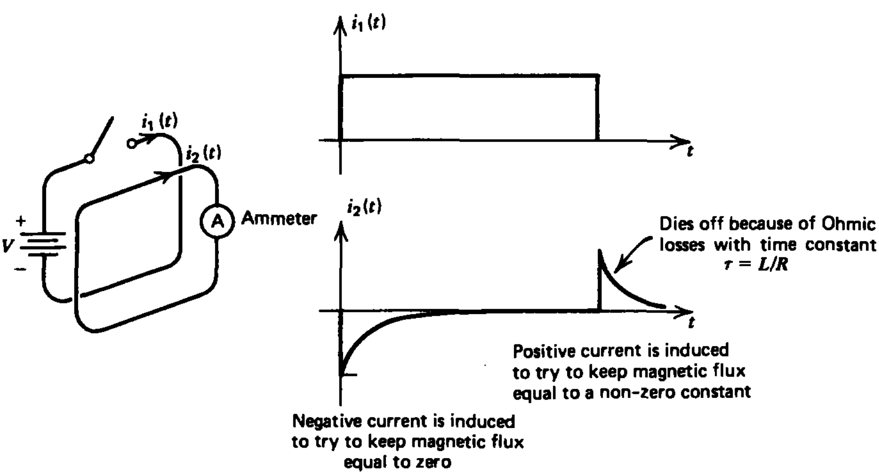


Figure 6-1 Faraday's experiments showed that a time varying magnetic flux through a closed conducting loop induced a current in the direction so as to keep the flux through the loop constant.

When the switch was later opened, another transient current flowed in loop 2, this time in the same direction as the original current in loop 1. Currents are induced in loop 2 whenever a time varying magnetic flux due to loop 1 passes through it.

In general, a time varying magnetic flux can pass through a circuit due to its own or nearby time varying current or by the motion of the circuit through a magnetic field. For any loop, as in Figure 6-2, Faraday's law is

$$\text{EMF} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \tag{1}$$

where EMF is the electromotive force defined as the line integral of the electric field. The minus sign is introduced on the right-hand side of (1) as we take the convention that positive flux flows in the direction perpendicular to the direction of the contour by the right-hand rule.

6-1-2 Lenz's Law

The direction of induced currents is always such as to oppose any changes in the magnetic flux already present. Thus in Faraday's experiment, illustrated in Figure 6-1, when the switch in loop 1 is first closed there is no magnetic flux in loop 2 so that the induced current flows in the opposite direction with its self-magnetic field opposite to the imposed field. The induced current tries to keep a zero flux through

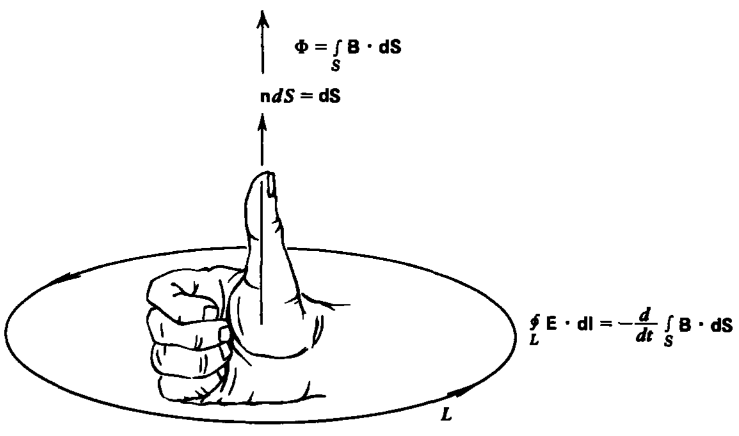


Figure 6-2 Faraday's law states that the line integral of the electric field around a closed loop equals the time rate of change of magnetic flux through the loop. The positive convention for flux is determined by the right-hand rule of curling the fingers on the right hand in the direction of traversal around the loop. The thumb then points in the direction of positive magnetic flux.

loop 2. If the loop is perfectly conducting, the induced current flows as long as current flows in loop 1, with zero net flux through the loop. However, in a real loop, resistive losses cause the current to exponentially decay with an L/R time constant, where L is the self-inductance of the loop and R is its resistance. Thus, in the dc steady state the induced current has decayed to zero so that a constant magnetic flux passes through loop 2 due to the current in loop 1.

When the switch is later opened so that the current in loop 1 goes to zero, the second loop tries to maintain the constant flux already present by inducing a current flow in the same direction as the original current in loop 1. Ohmic losses again make this induced current die off with time.

If a circuit or any part of a circuit is made to move through a magnetic field, currents will be induced in the direction such as to try to keep the magnetic flux through the loop constant. The force on the moving current will always be opposite to the direction of motion.

Lenz's law is clearly demonstrated by the experiments shown in Figure 6-3. When a conducting ax is moved into a magnetic field, eddy currents are induced in the direction where their self-flux is opposite to the applied magnetic field. The Lorentz force is then in the direction opposite to the motion of the ax. This force decreases with time as the currents decay with time due to Ohmic dissipation. If the ax was slotted, effectively creating a very high resistance to the eddy currents, the reaction force becomes very small as the induced current is small.

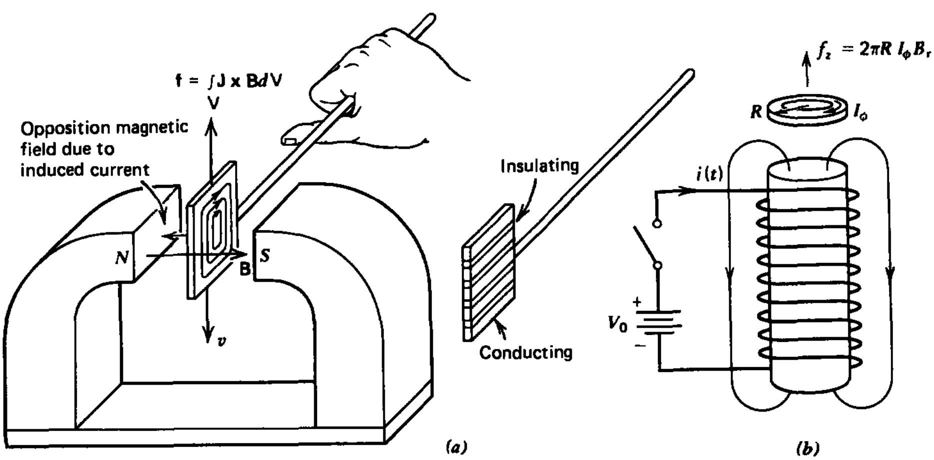


Figure 6-3 Lenz's law. (*a*) Currents induced in a conductor moving into a magnetic field exert a force opposite to the motion. The induced currents can be made small by slotting the ax. (*b*) A conducting ring on top of a coil is flipped off when a current is suddenly applied, as the induced currents try to keep a zero flux through the ring.

When the current is first turned on in the coil in Figure 6-3*b*, the conducting ring that sits on top has zero flux through it. Lenz's law requires that a current be induced opposite to that in the coil. Instantaneously there is no z component of magnetic field through the ring so the flux must return radially. This creates an upwards force:

$$\mathbf{f} = 2\pi R\mathbf{I}\times\mathbf{B} = 2\pi R I_\phi B_r \mathbf{i}_z \tag{2}$$

which flips the ring off the coil. If the ring is cut radially so that no circulating current can flow, the force is zero and the ring does not move.

(a) Short Circuited Loop

To be quantitative, consider the infinitely long time varying line current $I(t)$ in Figure 6-4, a distance r from a rectangular loop of wire with Ohmic conductivity σ, cross-sectional area A, and total length $l = 2(D+d)$. The magnetic flux through the loop due to $I(t)$ is

$$\Phi_m = \int_{z=-D/2}^{D/2}\int_{r}^{r+d} \mu_0 H_\phi(r')\, dr'\, dz$$

$$= \frac{\mu_0 ID}{2\pi}\int_r^{r+d}\frac{dr'}{r'} = \frac{\mu_0 ID}{2\pi}\ln\frac{r+d}{r} \tag{3}$$

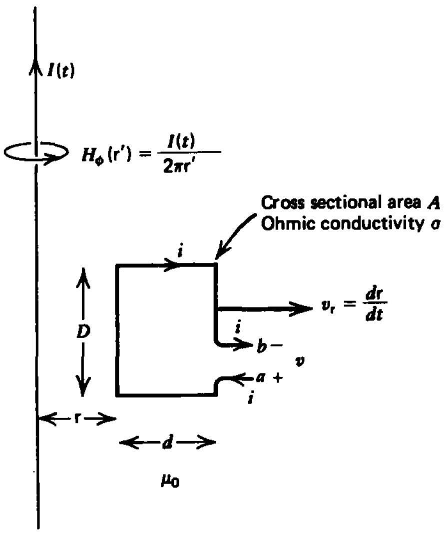


Figure 6-4 A rectangular loop near a time varying line current. When the terminals are short circuited the electromotive force induces a current due to the time varying mutual flux and/or because of the motion of the circuit through the imposed nonuniform magnetic field of the line current. If the loop terminals are open circuited there is no induced current but a voltage develops.

The mutual inductance M is defined as the flux to current ratio where the flux through the loop is due to an external current. Then (3) becomes

$$\Phi_m = M(\mathrm{r})I, \qquad M(\mathrm{r}) = \frac{\mu_0 D}{2\pi}\ln\frac{\mathrm{r}+d}{\mathrm{r}} \tag{4}$$

When the loop is short circuited ($v = 0$), the induced Ohmic current i gives rise to an electric field $[E = J/\sigma = i/(A\sigma)]$ so that Faraday's law applied to a contour within the wire yields an electromotive force just equal to the Ohmic voltage drop:

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = \frac{il}{\sigma A} = iR = -\frac{d\Phi}{dt} \tag{5}$$

where $R = l/(\sigma A)$ is the resistance of the loop. By convention, the current is taken as positive in the direction of the line integral.

The flux in (5) has contributions both from the imposed current as given in (3) and from the induced current proportional to the loop's self-inductance L, which for example is given in Section 5-4-3c for a square loop ($D = d$):

$$\Phi = M(\mathrm{r})I + Li \tag{6}$$

If the loop is also moving radially outward with velocity $v_\mathrm{r} = d\mathrm{r}/dt$, the electromotively induced Ohmic voltage is

$$\begin{aligned} -iR &= \frac{d\Phi}{dt} \\ &= M(\mathrm{r})\frac{dI}{dt} + I\frac{dM(\mathrm{r})}{dt} + L\frac{di}{dt} \\ &= M(\mathrm{r})\frac{dI}{dt} + I\frac{dM}{d\mathrm{r}}\frac{d\mathrm{r}}{dt} + L\frac{di}{dt} \end{aligned} \tag{7}$$

where L is not a function of the loop's radial position.

If the loop is stationary, only the first and third terms on the right-hand side contribute. They are nonzero only if the currents change with time. The second term is due to the motion and it has a contribution even for dc currents.

Turn-on Transient. If the loop is stationary ($d\mathrm{r}/dt = 0$) at $\mathrm{r} = \mathrm{r}_0$, (7) reduces to

$$L\frac{di}{dt} + iR = -M(\mathrm{r}_0)\frac{dI}{dt} \tag{8}$$

If the applied current I is a dc step turned on at $t = 0$, the solution to (8) is

$$i(t) = -\frac{M(\mathrm{r}_0)I}{L}e^{-(R/L)t}, \quad t > 0 \tag{9}$$

where the impulse term on the right-hand side of (8) imposes the initial condition $i(t=0)=-M(\mathrm{r}_0)I/L$. The current is negative, as Lenz's law requires the self-flux to oppose the applied flux.

Turn-off Transient. If after a long time T the current I is instantaneously turned off, the solution is

$$i(t)=\frac{M(\mathrm{r}_0)I}{L}e^{-(R/L)(t-T)},\quad t>T \tag{10}$$

where now the step decrease in current I at $t=T$ reverses the direction of the initial current.

Motion with a dc Current. With a dc current, the first term on the right-hand side in (7) is zero yielding

$$L\frac{di}{dt}+iR=\frac{\mu_0 IDd}{2\pi\mathrm{r}(\mathrm{r}+d)}\frac{d\mathrm{r}}{dt} \tag{11}$$

To continue, we must specify the motion so that we know how r changes with time. Let's consider the simplest case when the loop has no resistance ($R=0$). Then (11) can be directly integrated as

$$Li=-\frac{\mu_0 ID}{2\pi}\ln\frac{1+d/\mathrm{r}}{1+d/\mathrm{r}_0} \tag{12}$$

where we specify that the current is zero when $\mathrm{r}=\mathrm{r}_0$. This solution for a lossless loop only requires that the total flux of (6) remain constant. The current is positive when $\mathrm{r}>\mathrm{r}_0$ as the self-flux must aid the decreasing imposed flux. The current is similarly negative when $\mathrm{r}<\mathrm{r}_0$ as the self-flux must cancel the increasing imposed flux.

The force on the loop for all these cases is only due to the force on the z-directed current legs at r and $\mathrm{r}+d$:

$$\begin{aligned} f_\mathrm{r}&=\frac{\mu_0 DiI}{2\pi}\left(\frac{1}{\mathrm{r}+d}-\frac{1}{\mathrm{r}}\right)\\ &=-\frac{\mu_0 DiId}{2\pi\mathrm{r}(\mathrm{r}+d)}\end{aligned} \tag{13}$$

being attractive if $iI>0$ and repulsive if $iI<0$.

(b) Open Circuited Loop

If the loop is open circuited, no induced current can flow and thus the electric field within the wire is zero ($\mathbf{J}=\sigma\mathbf{E}=0$). The electromotive force then only has a contribution from the gap between terminals equal to the negative of the voltage:

$$\oint_L \mathbf{E}\cdot d\mathbf{l}=\int_b^a \mathbf{E}\cdot d\mathbf{l}=-v=-\frac{d\Phi}{dt}\Rightarrow v=\frac{d\Phi}{dt} \tag{14}$$

Note in Figure 6-4 that our convention is such that the current i is always defined positive flowing out of the positive voltage terminal into the loop. The flux Φ in (14) is now only due to the mutual flux given by (3), as with $i=0$ there is no self-flux. The voltage on the moving open circuited loop is then

$$v = M(\mathrm{r})\frac{dI}{dt} + I\frac{dM}{d\mathrm{r}}\frac{d\mathrm{r}}{dt} \tag{15}$$

(c) Reaction Force

The magnetic force on a short circuited moving loop is always in the direction opposite to its motion. Consider the short circuited loop in Figure 6-5, where one side of the loop moves with velocity v_x. With a uniform magnetic field applied normal to the loop pointing out of the page, an expansion of the loop tends to link more magnetic flux requiring the induced current to flow clockwise so that its self-flux is in the direction given by the right-hand rule, opposite to the applied field. From (1) we have

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = \frac{il}{\sigma A} = iR = -\frac{d\Phi}{dt} = B_0 D\frac{dx}{dt} = B_0 D v_x \tag{16}$$

where $l = 2(D+x)$ also changes with time. The current is then

$$i = \frac{B_0 D v_x}{R} \tag{17}$$

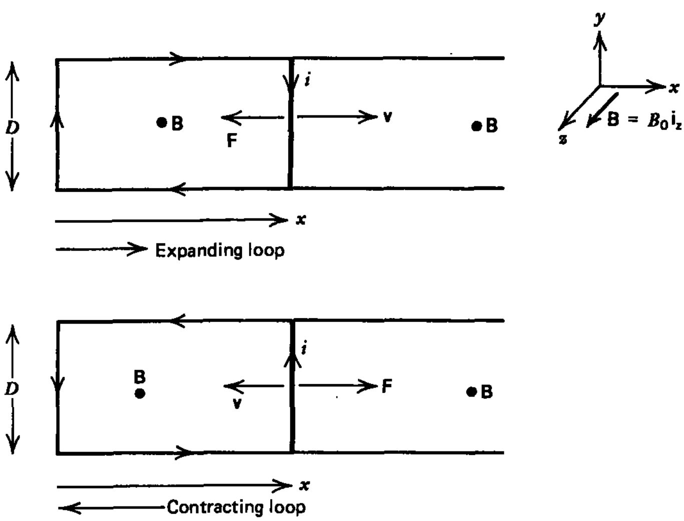


Figure 6-5 If a conductor moves perpendicular to a magnetic field a current is induced in the direction to cause the Lorentz force to be opposite to the motion. The total flux through the closed loop, due to both the imposed field and the self-field generated by the induced current, tries to remain constant.

where we neglected the self-flux generated by i, assuming it to be much smaller than the applied flux due to B_0. Note also that the applied flux is negative, as the right-hand rule applied to the direction of the current defines positive flux into the page, while the applied flux points outwards.

The force on the moving side is then to the left,

$$\mathbf{f} = -iD\mathbf{i}_y \times B_0\mathbf{i}_z = -iDB_0\mathbf{i}_x = -\frac{B_0^2D^2v_x}{R}\mathbf{i}_x \tag{18}$$

opposite to the velocity.

However if the side moves to the left ($v_x < 0$), decreasing the loop's area thereby linking less flux, the current reverses direction as does the force.

6-1-3 Laminations

The induced eddy currents in Ohmic conductors results in Ohmic heating. This is useful in induction furnaces that melt metals, but is undesired in many iron core devices. To reduce this power loss, the cores are often sliced into many thin sheets electrically insulated from each other by thin oxide coatings. The current flow is then confined to lie within a thin sheet and cannot cross over between sheets. The insulating laminations serve the same purpose as the cuts in the slotted ax in Figure 6-3a.

The rectangular conductor in Figure 6-6a has a time varying magnetic field $B(t)$ passing through it. We approximate the current path as following the rectangular shape so that

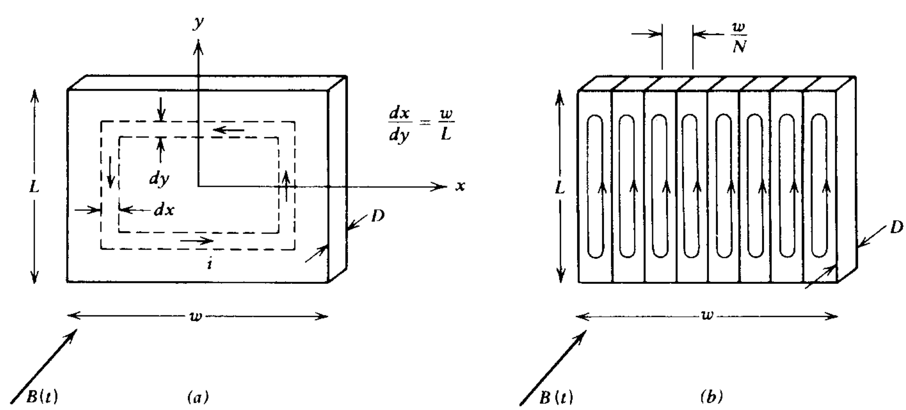


Figure 6-6 (a) A time varying magnetic field through a conductor induces eddy currents that cause Ohmic heating. (b) If the conductor is laminated so that the induced currents are confined to thin strips, the dissipated power decreases.

the flux through the loop of incremental width dx and dy of area $4xy$ is

$$\Phi = -4xyB(t) \tag{19}$$

where we neglect the reaction field of the induced current assuming it to be much smaller than the imposed field. The minus sign arises because, by the right-hand rule illustrated in Figure 6-2, positive flux flows in the direction opposite to $B(t)$. The resistance of the loop is

$$R_x = \frac{4}{\sigma D}\left(\frac{y}{dx}+\frac{x}{dy}\right) = \frac{4}{\sigma D}\frac{L}{w}\frac{x}{dx}\left[1+\left(\frac{w}{L}\right)^2\right] \tag{20}$$

The electromotive force around the loop then just results in an Ohmic current:

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = iR_x = \frac{-d\Phi}{dt} = 4xy\frac{dB}{dt} = \frac{4L}{w}x^2\frac{dB}{dt} \tag{21}$$

with dissipated power

$$dp = i^2R_x = \frac{4Dx^3\sigma L(dB/dt)^2\,dx}{w[1+(w/L)^2]} \tag{22}$$

The total power dissipated over the whole sheet is then found by adding the powers dissipated in each incremental loop:

$$\begin{aligned} P &= \int_0^{w/2} dp \\ &= \frac{4D(dB/dt)^2\sigma L}{w[1+(w/L)^2]}\int_0^{w/2} x^3\,dx \\ &= \frac{LDw^3\sigma(dB/dt)^2}{16[1+(w/L)^2]} \end{aligned} \tag{23}$$

If the sheet is laminated into N smaller ones, as in Figure 6-6b, each section has the same solution as (23) if we replace w by w/N. The total power dissipated is then N times the power dissipated in a single section:

$$P = \frac{LD(w/N)^3\sigma(dB/dt)^2N}{16[1+(w/NL)^2]} = \frac{\sigma LDw^3(dB/dt)^2}{16N^2[1+(w/NL)^2]} \tag{24}$$

As N becomes large so that $w/NL \ll 1$, the dissipated power decreases as $1/N^2$.

6-1-4 Betatron

The cyclotron, discussed in Section 5-1-4, is not used to accelerate electrons because their small mass allows them to reach relativistic speeds, thereby increasing their mass and decreasing their angular speed. This puts them out of phase with the applied voltage. The betatron in Figure 6-7 uses the transformer principle where the electrons circulating about the evacuated toroid act like a secondary winding. The imposed time varying magnetic flux generates an electric field that accelerates the electrons.

Faraday's law applied to a contour following the charge's trajectory at radius R yields

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = E_\phi 2\pi R = -\frac{d\Phi}{dt} \tag{25}$$

which accelerates the electrons as

$$m\frac{dv_\phi}{dt} = -eE_\phi = \frac{e}{2\pi R}\frac{d\Phi}{dt} \Rightarrow v_\phi = \frac{e}{2\pi m R}\Phi \tag{26}$$

The electrons move in the direction so that their self-magnetic flux is opposite to the applied flux. The resulting Lorentz force is radially inward. A stable orbit of constant radius R is achieved if this force balances the centrifugal force:

$$m\frac{dv_r}{dt} = \frac{mv_\phi^2}{R} - ev_\phi B_z(R) = 0 \tag{27}$$

which from (26) requires the flux and magnetic field to be related as

$$\Phi = 2\pi R^2 B_z(R) \tag{28}$$

This condition cannot be met by a uniform field (as then $\Phi = \pi R^2 B_z$) so in practice the imposed field is made to approximately vary with radial position as

$$B_z(r) = B_0\left(\frac{R}{r}\right) \Rightarrow \Phi = 2\pi\int_{r=0}^{R} B_z(r) r\,dr = 2\pi R^2 B_0 \tag{29}$$

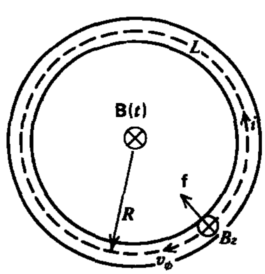


Figure 6-7 The betatron accelerates electrons to high speeds using the electric field generated by a time varying magnetic field.

where R is the equilibrium orbit radius, so that (28) is satisfied.

The magnetic field must remain curl free where there is no current so that the spatial variation in (29) requires a radial magnetic field component:

$$\nabla\times\mathbf{B}=\left(\frac{\partial B_r}{\partial z}-\frac{\partial B_z}{\partial r}\right)\mathbf{i}_\phi=0\Rightarrow B_r=-\frac{B_0R}{r^2}z \qquad (30)$$

Then any z-directed perturbation displacements

$$\frac{d^2z}{dt^2}=\frac{ev_\phi}{m}B_r(R)=-\left(\frac{eB_0}{m}\right)^2z$$

$$\Rightarrow z=A_1\sin\omega_0t+A_2\cos\omega_0t,\qquad \omega_0=\frac{eB_0}{m} \qquad (31)$$

have sinusoidal solutions at the cyclotron frequency $\omega_0=eB_0/m$, known as betatron oscillations.

6-1-5 Faraday's Law and Stokes' Theorem

The integral form of Faraday's law in (1) shows that with magnetic induction the electric field is no longer conservative as its line integral around a closed path is non-zero. We may convert (1) to its equivalent differential form by considering a stationary contour whose shape does not vary with time. Because the area for the surface integral does not change with time, the time derivative on the right-hand side in (1) may be brought inside the integral but becomes a partial derivative because $\mathbf{B}$ is also a function of position:

$$\oint_L\mathbf{E}\cdot d\mathbf{l}=-\int_S\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{S} \qquad (32)$$

Using Stokes' theorem, the left-hand side of (32) can be converted to a surface integral,

$$\oint_L\mathbf{E}\cdot d\mathbf{l}=\int_S\nabla\times\mathbf{E}\cdot d\mathbf{S}=-\int_S\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{S} \qquad (33)$$

which is equivalent to

$$\int_S\left(\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}\right)\cdot d\mathbf{S}=0 \qquad (34)$$

Since this last relation is true for any surface, the integrand itself must be zero, which yields Faraday's law of induction in differential form as

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \qquad (35)$$

6-2 MAGNETIC CIRCUITS

Various alloys of iron having very high values of relative permeability are typically used in relays and machines to constrain the magnetic flux to mostly lie within the permeable material.

6-2-1 Self-Inductance

The simple magnetic circuit in Figure 6-8 has an N turn coil wrapped around a core with very high relative permeability idealized to be infinite. There is a small air gap of length s in the core. In the core, the magnetic flux density $\mathbf{B}$ is proportional to the magnetic field intensity $\mathbf{H}$ by an infinite permeability μ. The $\mathbf{B}$ field must remain finite to keep the flux and coil voltage finite so that the $\mathbf{H}$ field in the core must be zero:

$$\lim_{\mu\to\infty} \mathbf{B} = \mu\mathbf{H} \Rightarrow \begin{cases} \mathbf{H} = 0 \\ \mathbf{B} \text{ finite} \end{cases} \tag{1}$$

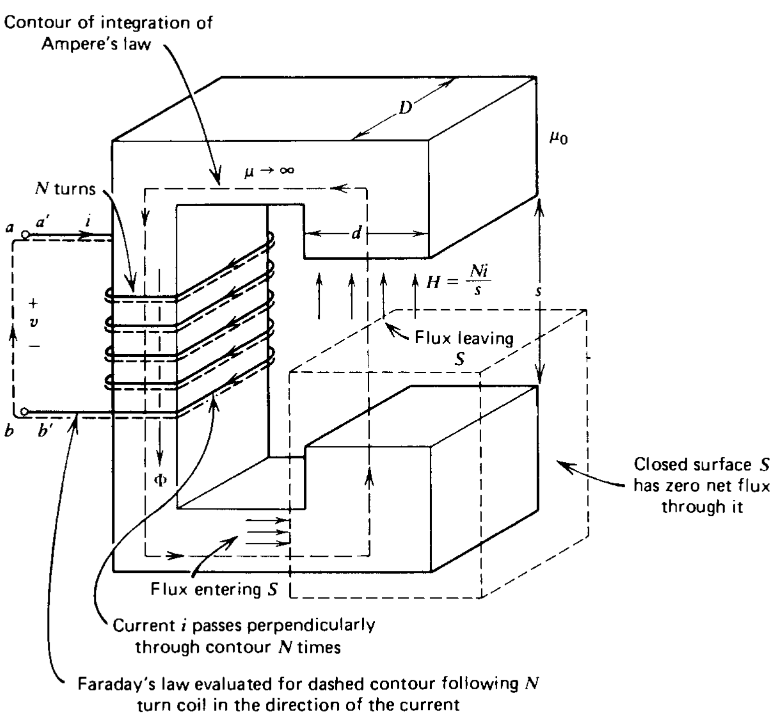


Figure 6-8 The magnetic field is zero within an infinitely permeable magnetic core and is constant in the air gap if we neglect fringing. The flux through the air gap is constant at every cross section of the magnetic circuit and links the N turn coil N times.

The $\mathbf{H}$ field can then only be nonzero in the air gap. This field emanates perpendicularly from the pole faces as no surface currents are present so that the tangential component of $\mathbf{H}$ is continuous and thus zero. If we neglect fringing field effects, assuming the gap s to be much smaller than the width d or depth D, the $\mathbf{H}$ field is uniform throughout the gap. Using Ampere's circuital law with the contour shown, the only nonzero contribution is in the air gap,

$$\oint_L \mathbf{H}\cdot d\mathbf{l} = Hs = I_{\text{total enclosed}} = Ni \tag{2}$$

where we realize that the coil current crosses perpendicularly through our contour N times. The total flux in the air gap is then

$$\Phi = \mu_0 HDd = \frac{\mu_0 NDd}{s} i \tag{3}$$

Because the total flux through any closed surface is zero,

$$\oint_S \mathbf{B}\cdot d\mathbf{S} = 0 \tag{4}$$

all the flux leaving S in Figure 6-8 on the air gap side enters the surface through the iron core, as we neglect leakage flux in the fringing field. The flux at any cross section in the iron core is thus constant, given by (3).

If the coil current i varies with time, the flux in (3) also varies with time so that a voltage is induced across the coil. We use the integral form of Faraday's law for a contour that lies within the winding with Ohmic conductivity σ, cross sectional area A, and total length l. Then the current density and electric field within the wire is

$$J = \frac{i}{A}, \qquad E = \frac{J}{\sigma} = \frac{i}{\sigma A} \tag{5}$$

so that the electromotive force has an Ohmic part as well as a contribution due to the voltage across the terminals:

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = \underbrace{\int_{a'}^{b'} \frac{i}{\sigma A}\, dl}_{\substack{iR \\ \text{in wire}}} + \underbrace{\int_b^a \mathbf{E}\cdot d\mathbf{l}}_{\substack{-v \\ \text{across} \\ \text{terminals}}} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S} \tag{6}$$

The surface S on the right-hand side is quite complicated because of the spiral nature of the contour. If the coil only had one turn, the right-hand side of (6) would just be the time derivative of the flux of (3). For two turns, as in Figure 6-9, the flux links the coil twice, while for N turns the total flux

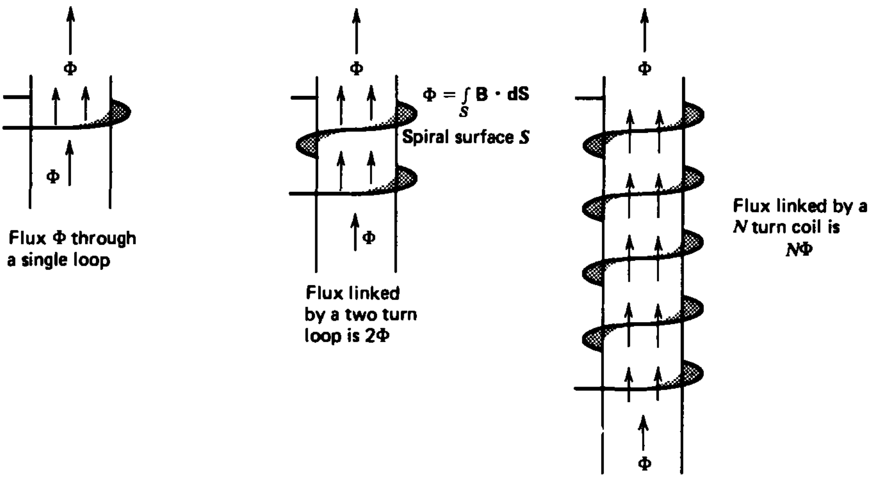


Figure 6-9 The complicated spiral surface for computation of the linked flux by an N turn coil can be considered as N single loops each linking the same flux Φ.

linked by the coil is $N\Phi$. Then (6) reduces to

$$v = iR + L\frac{di}{dt} \tag{7}$$

where the self-inductance is defined as

$$L = \frac{N\Phi}{i} = \frac{N\int_S \mathbf{B}\cdot d\mathbf{S}}{\oint_L \mathbf{H}\cdot d\mathbf{l}} = \frac{\mu_0 N^2 Dd}{s} \text{ henry [kg-m}^2\text{-A}^{-2}\text{-s}^{-2}] \tag{8}$$

For linearly permeable materials, the inductance is always independent of the excitations and only depends on the geometry. Because of the fixed geometry, the inductance is a constant and thus was taken outside the time derivative in (7). In geometries that change with time, the inductance will also be a function of time and must remain under the derivative. The inductance is always proportional to the square of the number of coil turns. This is because the flux Φ in the air gap is itself proportional to N and it links the coil N times.

EXAMPLE 6-1 SELF-INDUCTANCES

Find the self-inductances for the coils shown in Figure 6-10.

(a) Solenoid

An N turn coil is tightly wound upon a cylindrical core of radius a, length l, and permeability μ.

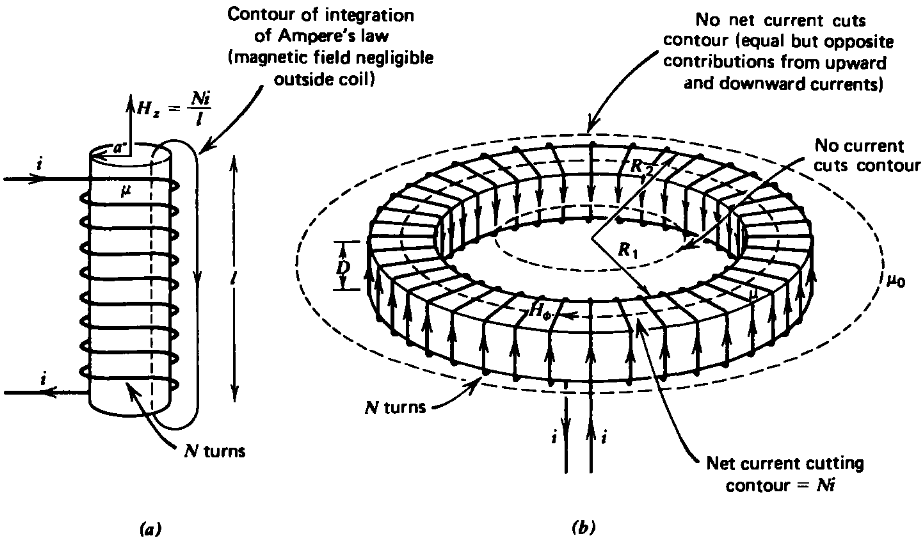


Figure 6-10 Inductances. (*a*) Solenoidal coil; (*b*) toroidal coil.

SOLUTION

A current i flowing in the wire approximates a surface current

$$K_\phi = Ni/l$$

If the length l is much larger than the radius a, we can neglect fringing field effects at the ends and the internal magnetic field is approximately uniform and equal to the surface current,

$$H_z = K_\phi = \frac{Ni}{l}$$

as we assume the exterior magnetic field is negligible. The same result is obtained using Ampere's circuital law for the contour shown in Figure 6-10*a*. The flux links the coil N times:

$$L = \frac{N\Phi}{i} = \frac{N\mu H_z \pi a^2}{i} = \frac{N^2 \mu \pi a^2}{l}$$

(b) Toroid

An N turn coil is tightly wound around a donut-shaped core of permeability μ with a rectangular cross section and inner and outer radii R_1 and R_2.

SOLUTION

Applying Ampere's circuital law to the three contours shown in Figure 6-10*b*, only the contour within the core has a net current passing through it:

$$\oint_L \mathbf{H}\cdot d\mathbf{l} = H_\phi 2\pi r = \begin{cases} 0, & r<R_1 \\ Ni, & R_1<r<R_2 \\ 0, & r>R_2 \end{cases}$$

The inner contour has no current through it while the outer contour enclosing the whole toroid has equal but opposite contributions from upward and downward currents.

The flux through any single loop is

$$\Phi = \mu D \int_{R_1}^{R_2} H_\phi\, dr$$

$$= \frac{\mu D N i}{2\pi}\int_{R_1}^{R_2}\frac{dr}{r}$$

$$= \frac{\mu D N i}{2\pi}\ln\frac{R_2}{R_1}$$

so that the self-inductance is

$$L = \frac{N\Phi}{i} = \frac{\mu D N^2}{2\pi}\ln\frac{R_2}{R_1}$$

6-2-2 Reluctance

Magnetic circuits are analogous to resistive electronic circuits if we define the magnetomotive force (MMF) $\mathcal{F}$ analogous to the voltage (EMF) as

$$\mathcal{F} = Ni \tag{9}$$

The flux then plays the same role as the current in electronic circuits so that we define the magnetic analog to resistance as the reluctance:

$$\mathcal{R} = \frac{\mathcal{F}}{\Phi} = \frac{N^2}{L} = \frac{\text{(length)}}{\text{(permeability)(cross-sectional area)}} \tag{10}$$

which is proportional to the reciprocal of the inductance.

The advantage to this analogy is that the rules of adding reluctances in series and parallel obey the same rules as resistances.

(a) Reluctances in Series

For the iron core of infinite permeability in Figure 6-11*a*, with two finitely permeable gaps the reluctance of each gap is found from (8) and (10) as

$$\mathcal{R}_1 = \frac{s_1}{\mu_1 a_1 D}, \qquad \mathcal{R}_2 = \frac{s_2}{\mu_2 a_2 D} \tag{11}$$

so that the flux is

$$\Phi = \frac{\mathcal{F}}{\mathcal{R}_1 + \mathcal{R}_2} = \frac{Ni}{\mathcal{R}_1 + \mathcal{R}_2} \Rightarrow L = \frac{N\Phi}{i} = \frac{N^2}{\mathcal{R}_1 + \mathcal{R}_2} \tag{12}$$

The iron core with infinite permeability has zero reluctance. If the permeable gaps were also iron with infinite permeability, the reluctances of (11) would also be zero so that the flux

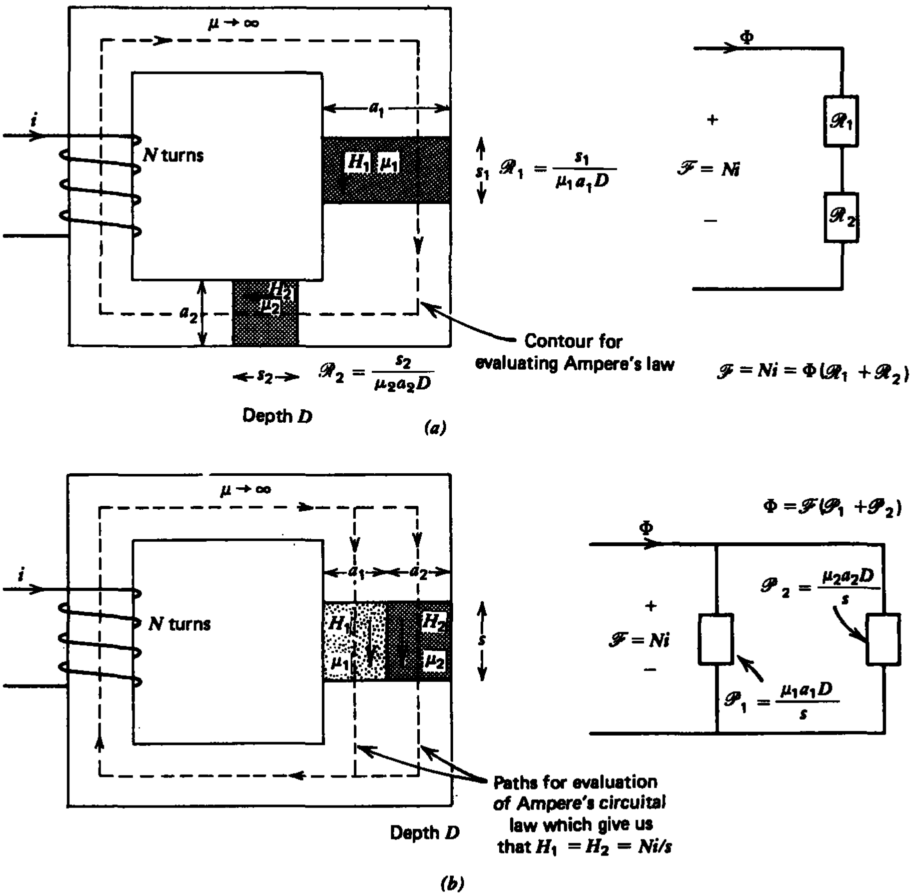


Figure 6-11 Magnetic circuits are most easily analyzed from a circuit approach where (*a*) reluctances in series add and (*b*) permeances in parallel add.

in (12) becomes infinite. This is analogous to applying a voltage across a short circuit resulting in an infinite current. Then the small resistance in the wires determines the large but finite current. Similarly, in magnetic circuits the small reluctance of a closed iron core of high permeability with no gaps limits the large but finite flux determined by the saturation value of magnetization.

The **H** field is nonzero only in the permeable gaps so that Ampere's law yields

$$H_1 s_1 + H_2 s_2 = Ni \tag{13}$$

Since the flux must be continuous at every cross section,

$$\Phi = \mu_1 H_1 a_1 D = \mu_2 H_2 a_2 D \tag{14}$$

we solve for the **H** fields as

$$H_1 = \frac{\mu_2 a_2 Ni}{\mu_1 a_1 s_2 + \mu_2 a_2 s_1}, \qquad H_2 = \frac{\mu_1 a_1 Ni}{\mu_1 a_1 s_2 + \mu_2 a_2 s_1} \tag{15}$$

(b) Reluctances in Parallel

If a gap in the iron core is filled with two permeable materials, as in Figure 6-11*b*, the reluctance of each material is still given by (11). Since each material sees the same magnetomotive force, as shown by applying Ampere's circuital law to contours passing through each material,

$$H_1 s = H_2 s = Ni \Rightarrow H_1 = H_2 = \frac{Ni}{s} \tag{16}$$

the H fields in each material are equal. The flux is then

$$\Phi = (\mu_1 H_1 a_1 + \mu_2 H_2 a_2) D = \frac{Ni(\mathcal{R}_1 + \mathcal{R}_2)}{\mathcal{R}_1 \mathcal{R}_2} = Ni(\mathcal{P}_1 + \mathcal{P}_2) \tag{17}$$

where the permeances $\mathcal{P}_1$ and $\mathcal{P}_2$ are just the reciprocal reluctances analogous to conductance.

6-2-3 Transformer Action

(a) Voltages are not Unique

Consider two small resistors R_1 and R_2 forming a loop enclosing one leg of a closed magnetic circuit with permeability μ, as in Figure 6-12. An N turn coil excited on one leg with a time varying current generates a time varying flux that is approximately

$$\Phi(t) = \mu N A i_1 / l \tag{18}$$

where l is the average length around the core.

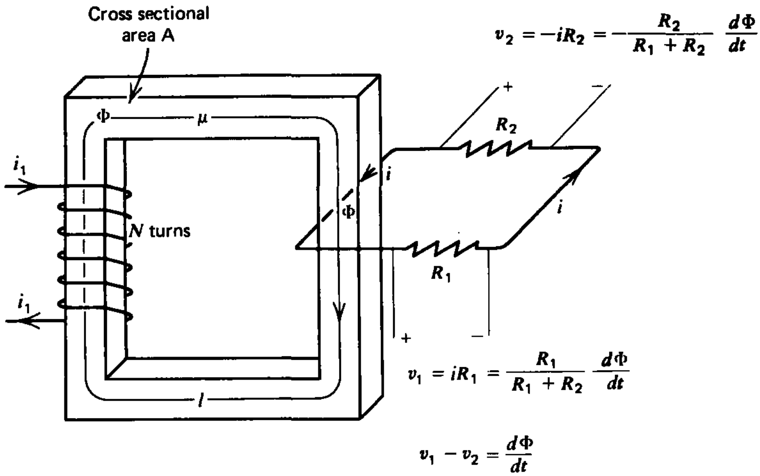


Figure 6-12 Voltages are not unique in the presence of a time varying magnetic field. A resistive loop encircling a magnetic circuit has different measured voltages across the same node pair. The voltage difference is equal to the time rate of magnetic flux through the loop.

Applying Faraday's law to the resistive loop we have

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = i(R_1+R_2) = +\frac{d\Phi(t)}{dt} \Rightarrow i = \frac{1}{R_1+R_2}\frac{d\Phi}{dt} \tag{19}$$

where we neglect the self-flux produced by the induced current i and reverse the sign on the magnetic flux term because Φ penetrates the loop in Figure 6-12 in the direction opposite to the positive convention given by the right-hand rule illustrated in Figure 6-2.

If we now measured the voltage across each resistor, we would find different values and opposite polarities even though our voltmeter was connected to the same nodes:

$$\begin{aligned} v_1 &= iR_1 = +\frac{R_1}{R_1+R_2}\frac{d\Phi}{dt} \\ v_2 &= -iR_2 = \frac{-R_2}{R_1+R_2}\frac{d\Phi}{dt} \end{aligned} \tag{20}$$

This nonuniqueness of the voltage arises because the electric field is no longer curl free. The voltage difference between two points depends on the path of the connecting wires. If any time varying magnetic flux passes through the contour defined by the measurement, an additional contribution results.

(b) Ideal Transformers

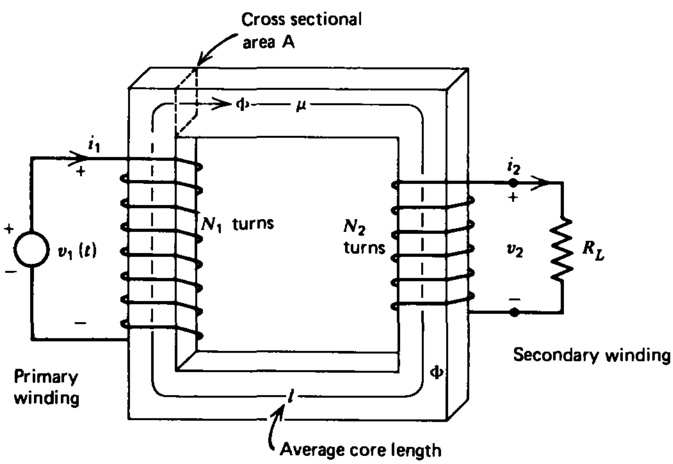
Two coils tightly wound on a highly permeable core, so that all the flux of one coil links the other, forms an ideal transformer, as in Figure 6-13. Because the iron core has an infinite permeability, all the flux is confined within the core. The currents flowing in each coil, i_1 and i_2, are defined so that when they are positive the fluxes generated by each coil are in the opposite direction. The total flux in the core is then

$$\Phi=\frac{N_1i_1-N_2i_2}{\mathcal{R}},\qquad \mathcal{R}=\frac{l}{\mu A} \tag{21}$$

where $\mathcal{R}$ is the reluctance of the core and l is the average length of the core.

The flux linked by each coil is then

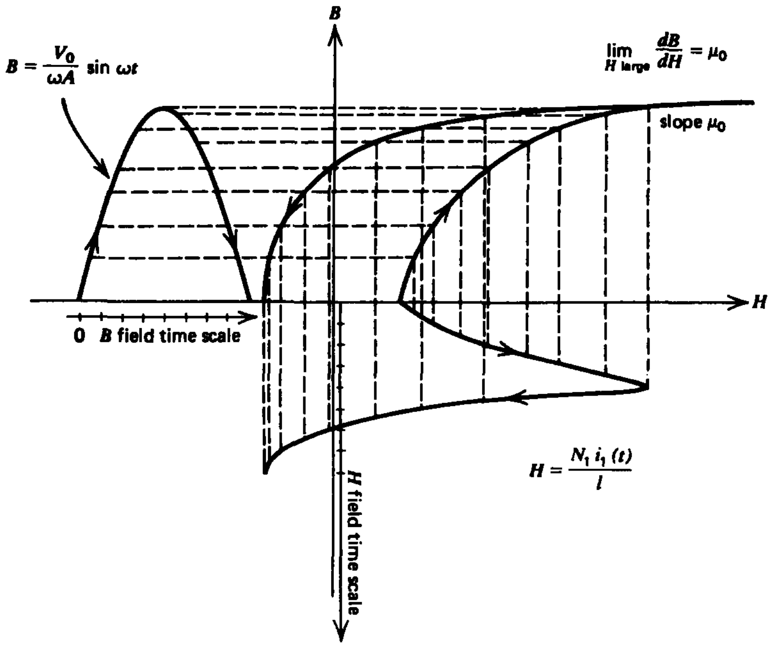
$$\begin{aligned}\lambda_1&=N_1\Phi=\frac{\mu A}{l}(N_1^2i_1-N_1N_2i_2)\\ \lambda_2&=N_2\Phi=\frac{\mu A}{l}(N_1N_2i_1-N_2^2i_2)\end{aligned} \tag{22}$$



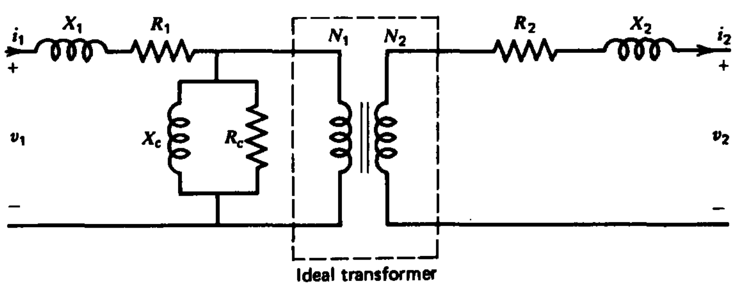
$$\left.\begin{aligned}\frac{v_1}{v_2}&=\frac{N_1}{N_2}\\ \frac{i_1}{i_2}&=\frac{N_2}{N_1}\end{aligned}\right\}\Rightarrow v_1i_1=v_2i_2$$

(*a*)

Figure 6-13 (*a*) An ideal transformer relates primary and secondary voltages by the ratio of turns while the currents are in the inverse ratio so that the input power equals the output power. The **H** field is zero within the infinitely permeable core. (*b*) In a real transformer the nonlinear B-H hysteresis loop causes a nonlinear primary current i_1 with an open circuited secondary ($i_2=0$) even though the imposed sinusoidal voltage $v_1=V_0\cos\omega t$ fixes the flux to be sinusoidal. (*c*) A more complete transformer equivalent circuit.



(b)



(c)

Figure 6.13.

which can be written as

$$\begin{aligned} \lambda_1 &= L_1 i_1 - M i_2 \\ \lambda_2 &= M i_1 - L_2 i_2 \end{aligned} \tag{23}$$

where L_1 and L_2 are the self-inductances of each coil alone and M is the mutual inductance between coils:

$$L_1 = N_1^2 L_0, \qquad L_2 = N_2^2 L_0, \qquad M = N_1 N_2 L_0, \qquad L_0 = \mu A/l \tag{24}$$

In general, the mutual inductance obeys the equality:

$$M = k(L_1 L_2)^{1/2}, \qquad 0 \le k \le 1 \tag{25}$$

where k is called the coefficient of coupling. For a noninfinite core permeability, k is less than unity because some of the flux of each coil goes into the free space region and does not link the other coil. In an ideal transformer, where the permeability is infinite, there is no leakage flux so that $k=1$.

From (23), the voltage across each coil is

$$\begin{aligned} v_1 &= \frac{d\lambda_1}{dt} = L_1\frac{di_1}{dt} - M\frac{di_2}{dt} \\ v_2 &= \frac{d\lambda_2}{dt} = M\frac{di_1}{dt} - L_2\frac{di_2}{dt} \end{aligned} \tag{26}$$

Because with no leakage, the mutual inductance is related to the self-inductances as

$$M = \frac{N_2}{N_1}L_1 = \frac{N_1}{N_2}L_2 \tag{27}$$

the ratio of coil voltages is the same as the turns ratio:

$$\frac{v_1}{v_2} = \frac{d\lambda_1/dt}{d\lambda_2/dt} = \frac{N_1}{N_2} \tag{28}$$

In the ideal transformer of infinite core permeability, the inductances of (24) are also infinite. To keep the voltages and fluxes in (26) finite, the currents must be in the inverse turns ratio

$$\frac{i_1}{i_2} = \frac{N_2}{N_1} \tag{29}$$

The electrical power delivered by the source to coil 1, called the primary winding, just equals the power delivered to the load across coil 2, called the secondary winding:

$$v_1 i_1 = v_2 i_2 \tag{30}$$

If $N_2 > N_1$, the voltage on winding 2 is greater than the voltage on winding 1 but current i_2 is less than i_1 keeping the powers equal.

If primary winding 1 is excited by a time varying voltage $v_1(t)$ with secondary winding 2 loaded by a resistor R_L so that

$$v_2 = i_2 R_L \tag{31}$$

the effective resistance seen by the primary winding is

$$R_{\text{eff}} = \frac{v_1}{i_1} = \frac{N_1}{N_2}\frac{v_2}{(N_2/N_1)i_2} = \left(\frac{N_1}{N_2}\right)^2 R_L \tag{32}$$

A transformer is used in this way as an impedance transformer where the effective resistance seen at the primary winding is increased by the square of the turns ratio.

(c) Real Transformers

When the secondary is open circuited ($i_2 = 0$), (29) shows that the primary current of an ideal transformer is also zero. In practice, applying a primary sinusoidal voltage $V_0 \cos \omega t$ will result in a small current due to the finite self-inductance of the primary coil. Even though this self-inductance is large if the core permeability μ is large, we must consider its effect because there is no opposing flux as a result of the open circuited secondary coil. Furthermore, the nonlinear hysteresis curve of the iron as discussed in Section 5-5-3*c* will result in a nonsinusoidal current even though the voltage is sinusoidal. In the magnetic circuit of Figure 6.13*a* with $i_2 = 0$, the magnetic field is

$$\mathbf{H} = \frac{N_1 i_1}{l} \tag{33}$$

while the imposed sinusoidal voltage also fixes the magnetic flux to be sinusoidal

$$v_1 = \frac{d\Phi}{dt} = V_0 \cos \omega t \Rightarrow \Phi = BA = \frac{V_0}{\omega} \sin \omega t \tag{34}$$

Thus the upper half of the nonlinear B-H magnetization characteristic in Figure 6-13*b* has the same shape as the flux-current characteristic with proportionality factors related to the geometry. Note that in saturation the B-H curve approaches a straight line with slope μ_0. For a half-cycle of flux given by (34), the nonlinear open circuit magnetizing current is found graphically as a function of time in Figure 6-13*b*. The current is symmetric over the negative half of the flux cycle. Fourier analysis shows that this nonlinear current is composed of all the odd harmonics of the driving frequency dominated by the third and fifth harmonics. This causes problems in power systems and requires extra transformer windings to trap the higher harmonic currents, thus preventing their transmission.

A more realistic transformer equivalent circuit is shown in Figure 6-13*c* where the leakage reactances X_1 and X_2 represent the fact that all the flux produced by one coil does not link the other. Some small amount of flux is in the free space region surrounding the windings. The nonlinear inductive reactance X_c represents the nonlinear magnetization characteristic illustrated in Figure 6-13*b*, while R_c represents the power dissipated in traversing the hysteresis loop over a cycle. This dissipated power per cycle equals the area enclosed by the hysteresis loop. The winding resistances are R_1 and R_2.

6-3 FARADAY'S LAW FOR MOVING MEDIA

6-3-1 The Electric Field Transformation

If a point charge q travels with a velocity $\mathbf{v}$ through a region with electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$, it experiences the combined Coulomb–Lorentz force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1}$$

Now consider another observer who is travelling at the same velocity $\mathbf{v}$ as the charge carrier so that their relative velocity is zero. This moving observer will then say that there is no Lorentz force, only a Coulombic force

$$\mathbf{F}' = q\mathbf{E}' \tag{2}$$

where we indicate quantities measured by the moving observer with a prime. A fundamental postulate of mechanics is that all physical laws are the same in every inertial coordinate system (systems that travel at constant relative velocity). This requires that the force measured by two inertial observers be the same so that $\mathbf{F}' = \mathbf{F}$:

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B} \tag{3}$$

The electric field measured by the two observers in relative motion will be different. This result is correct for material velocities much less than the speed of light and is called a Galilean field transformation. The complete relativistically correct transformation slightly modifies (3) and is called a Lorentzian transformation but will not be considered here.

In using Faraday's law of Section 6-1-1, the question remains as to which electric field should be used if the contour L and surface S are moving. One uses the electric field that is measured by an observer moving at the same velocity as the convecting contour. The time derivative of the flux term cannot be brought inside the integral if the surface S is itself a function of time.

6-3-2 Ohm's Law for Moving Conductors

The electric field transformation of (3) is especially important in modifying Ohm's law for moving conductors. For nonrelativistic velocities, an observer moving along at the same velocity as an Ohmic conductor measures the usual Ohm's law in his reference frame,

$$\mathbf{J}'_f=\sigma\mathbf{E}' \tag{4}$$

where we assume the conduction process is unaffected by the motion. Then in Galilean relativity for systems with no free charge, the current density in all inertial frames is the same so that (3) in (4) gives us the generalized Ohm's law as

$$\mathbf{J}'_f=\mathbf{J}_f=\sigma(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \tag{5}$$

where $\mathbf{v}$ is the velocity of the conductor.

The effects of material motion are illustrated by the parallel plate geometry shown in Figure 6-14. A current source is applied at the left-hand side that distributes itself uniformly as a surface current $K_x=\pm I/D$ on the planes. The electrodes are connected by a conducting slab that moves to the right with constant velocity U. The voltage across the current source can be computed using Faraday's law with the contour shown. Let us have the contour continually expanding with the 2–3 leg moving with the conductor. Applying Faraday's law we have

$$\oint_L \mathbf{E}'\cdot d\mathbf{l}=\int_1^2 \cancelto{0}{\mathbf{E}'}\cdot d\mathbf{l}+\underbrace{\int_2^3 \mathbf{E}'\cdot d\mathbf{l}}_{iR}+\int_3^4 \cancelto{0}{\mathbf{E}'}\cdot d\mathbf{l}+\underbrace{\int_4^1 \mathbf{E}\cdot d\mathbf{l}}_{-v}$$

$$=-\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S} \tag{6}$$

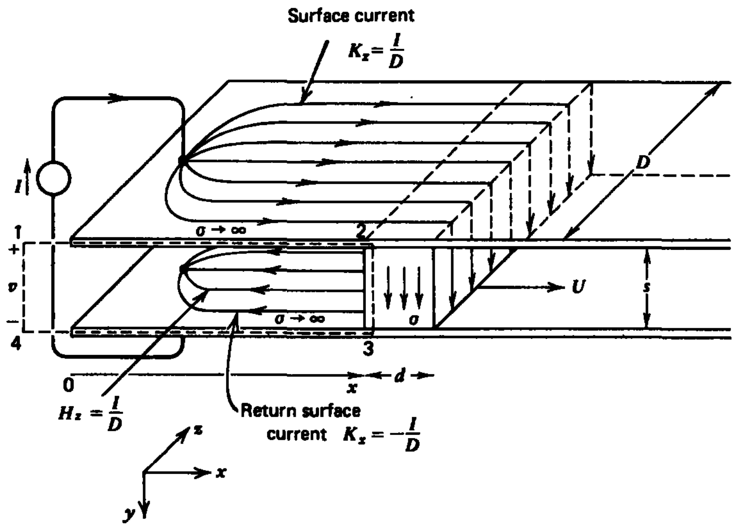


Figure 6-14 A moving, current-carrying Ohmic conductor generates a speed voltage as well as the usual resistive voltage drop.

where the electric field used along each leg is that measured by an observer in the frame of reference of the contour. Along the 1–2 and 3–4 legs, the electric field is zero within the stationary perfect conductors. The second integral within the moving Ohmic conductor uses the electric field $\mathbf{E}'$, as measured by a moving observer because the contour is also expanding at the same velocity, and from (4) and (5) is related to the terminal current as

$$\mathbf{E}' = \frac{\mathbf{J}'}{\sigma} = \frac{I}{\sigma Dd}\mathbf{i}_y \tag{7}$$

In (6), the last line integral across the terminals defines the voltage.

$$\frac{Is}{\sigma Dd} - v = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S} = -\frac{d}{dt}(\mu_0 H_z xs) \tag{8}$$

The first term is just the resistive voltage drop across the conductor, present even if there is no motion. The term on the right-hand side in (8) only has a contribution due to the linearly increasing area ($dx/dt = U$) in the free space region with constant magnetic field,

$$H_z = I/D \tag{9}$$

The terminal voltage is then

$$v = I\left(R + \frac{\mu_0 Us}{D}\right), \qquad R = \frac{s}{\sigma Dd} \tag{10}$$

We see that the speed voltage contribution arose from the flux term in Faraday's law. We can obtain the same solution using a contour that is stationary and does not expand with the conductor. We pick the contour to just lie within the conductor at the time of interest. Because the contour does not expand with time so that both the magnetic field and the contour area does not change with time, the right-hand side of (6) is zero. The only difference now is that along the 2–3 leg we use the electric field as measured by a stationary observer,

$$\mathbf{E} = \mathbf{E}' - \mathbf{v}\times\mathbf{B} \tag{11}$$

so that (6) becomes

$$IR + \frac{\mu_0 UIs}{D} - v = 0 \tag{12}$$

which agrees with (10) but with the speed voltage term now arising from the electric field side of Faraday's law.

This speed voltage contribution is the principle of electric generators converting mechanical work to electric power when moving a current-carrying conductor through a magnetic field. The resistance term accounts for the electric power dissipated. Note in (10) that the speed voltage contribution just adds with the conductor's resistance so that the effective terminal resistance is $v/I = R + (\mu_o Us/D)$. If the slab moves in the opposite direction such that U is negative, the terminal resistance can also become negative for sufficiently large U ($U < -RD/\mu_0 s$). Such systems are unstable where the natural modes grow rather than decay with time with any small perturbation, as illustrated in Section 6-3-3*b*.

6-3-3 Faraday's Disk (Homopolar Generator)*

(a) Imposed Magnetic Field

A disk of conductivity σ rotating at angular velocity ω transverse to a uniform magnetic field $B_0\mathbf{i}_z$, illustrates the basic principles of electromechanical energy conversion. In Figure 6-15*a* we assume that the magnetic field is generated by an N turn coil wound on the surrounding magnetic circuit,

$$B_0 = \frac{\mu_0 N i_f}{s} \tag{13}$$

The disk and shaft have a permeability of free space μ_0, so that the applied field is not disturbed by the assembly. The shaft and outside surface at $\mathrm{r} = R_0$ are highly conducting and make electrical connection to the terminals via sliding contacts.

We evaluate Faraday's law using the contour shown in Figure 6-15*a* where the 1–2 leg within the disk is stationary so the appropriate electric field to be used is given by (11):

$$E_\mathrm{r} = \frac{J_\mathrm{r}}{\sigma} - \omega \mathrm{r} B_0 = \frac{i_r}{2\pi\sigma d\mathrm{r}} - \omega \mathrm{r} B_0 \tag{14}$$

where the electric field and current density are radial and i_r is the total rotor terminal current. For the stationary contour with a constant magnetic field, there is no time varying flux through the contour:

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = \int_1^2 E_\mathrm{r}\, d\mathrm{r} + \int_3^4 \underbrace{\mathbf{E} \cdot d\mathbf{l}}_{-v_r} = 0 \tag{15}$$

* *Some of the treatment in this section is similar to that developed in: H. H. Woodson and J. R. Melcher,* Electromechanical Dynamics, *Part I, Wiley, N.Y., 1968, Ch. 6.*

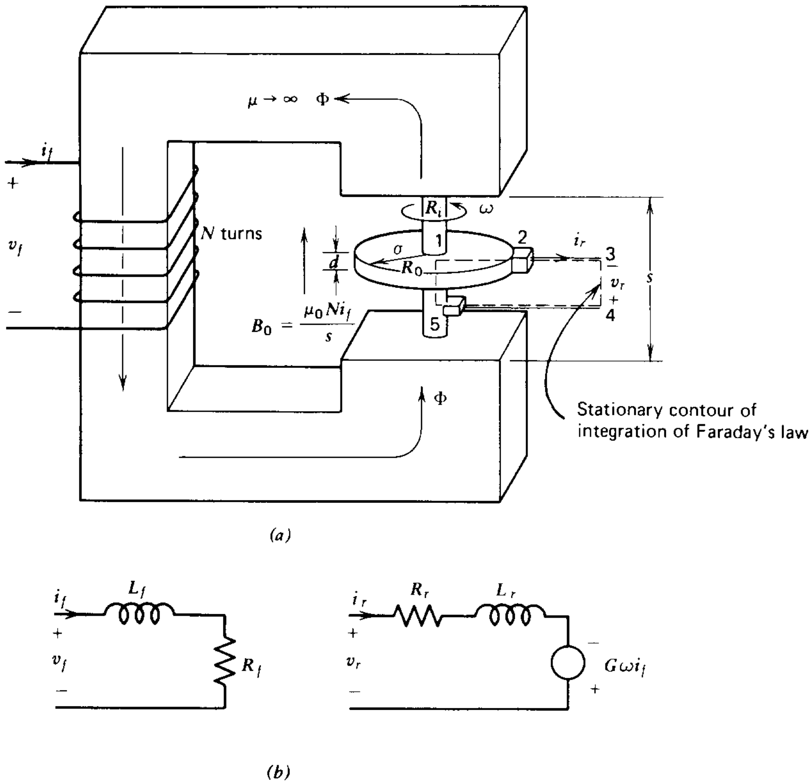


Figure 6-15 (*a*) A conducting disk rotating in an axial magnetic field is called a homopolar generator. (*b*) In addition to Ohmic and inductive voltages there is a speed voltage contribution proportional to the speed of the disk and the magnetic field.

Using (14) in (15) yields the terminal voltage as

$$\begin{aligned} v_r &= \int_{R_i}^{R_0} \left(\frac{i_r}{2\pi r \sigma d} - \omega r B_0 \right) dr \\ &= \frac{i_r}{2\pi\sigma d} \ln \frac{R_0}{R_i} - \frac{\omega B_0}{2} (R_0^2 - R_i^2) \\ &= i_r R_r - G\omega i_f \end{aligned} \tag{16}$$

where R_r is the internal rotor resistance of the disk and G is called the speed coefficient:

$$R_r = \frac{\ln (R_0/R_i)}{2\pi\sigma d}, \qquad G = \frac{\mu_0 N}{2s} (R_0^2 - R_i^2) \tag{17}$$

We neglected the self-magnetic field due to the rotor current, assuming it to be much smaller than the applied field B_0, but it is represented in the equivalent rotor circuit in Figure 6-15*b* as the self-inductance L_r in series with a resistor and a speed voltage source linearly dependent on the field current. The stationary field coil is represented by its self-inductance and resistance.

For a copper disk ($\sigma \approx 6 \times 10^7$ siemen/m) of thickness 1 mm rotating at 3600 rpm ($\omega = 120\pi$ radian/sec) with outer and inner radii $R_0 = 10$ cm and $R_i = 1$ cm in a magnetic field of $B_0 = 1$ tesla, the open circuit voltage is

$$v_{oc} = -\frac{\omega B_0}{2}(R_0^2 - R_i^2) \approx -1.9 \text{ V} \tag{18}$$

while the short circuit current is

$$i_{sc} = \frac{v_{oc}}{\ln(R_0/R_i)} 2\pi\sigma d \approx 3 \times 10^5 \text{ amp} \tag{19}$$

Homopolar generators are typically high current, low voltage devices. The electromagnetic torque on the disk due to the Lorentz force is

$$\begin{aligned}
\mathbf{T} &= \int_{\phi=0}^{2\pi} \int_{z=0}^{d} \int_{r=R_i}^{R_0} r\mathbf{i}_r \times (\mathbf{J} \times \mathbf{B}) r \, dr \, d\phi \, dz \\
&= -i_r B_0 \mathbf{i}_z \int_{R_i}^{R_0} r \, dr \\
&= -\frac{i_r B_0}{2}(R_0^2 - R_i^2)\mathbf{i}_z \\
&= -G i_f i_r \mathbf{i}_z
\end{aligned} \tag{20}$$

The negative sign indicates that the Lorentz force acts on the disk in the direction opposite to the motion. An external torque equal in magnitude but opposite in direction to (20) is necessary to turn the shaft.

This device can also be operated as a motor if a rotor current into the disk ($i_r < 0$) is imposed. Then the electrical torque causes the disk to turn.

(b) Self-Excited Generator

For generator operation it is necessary to turn the shaft and supply a field current to generate the magnetic field. However, if the field coil is connected to the rotor terminals, as in Figure 6-16*a*, the generator can supply its own field current. The equivalent circuit for self-excited operation is shown in Figure 6-16*b* where the series connection has $i_r = i_f$.

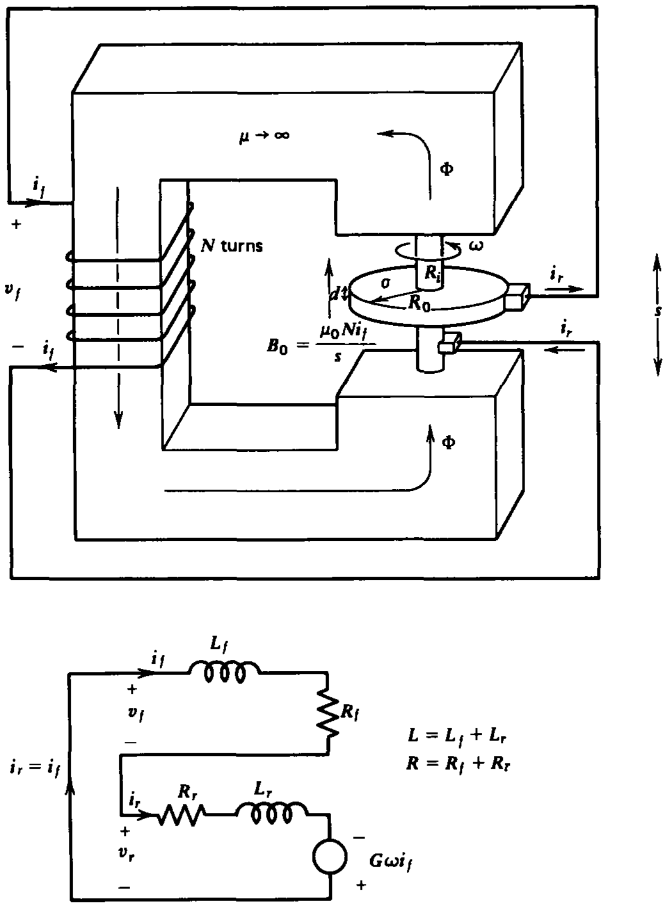


Figure 6-16 A homopolar generator can be self-excited where the generated rotor current is fed back to the field winding to generate its own magnetic field.

Kirchoff's voltage law around the loop is

$$L\frac{di}{dt}+i(R-G\omega)=0, \qquad R=R_r+R_f, \qquad L=L_r+L_f \tag{21}$$

where R and L are the series resistance and inductance of the coil and disk. The solution to (21) is

$$i=I_0\,e^{-[(R-G\omega)/L]t} \tag{22}$$

where I_0 is the initial current at $t=0$. If the exponential factor is positive

$$G\omega>R \tag{23}$$

the current grows with time no matter how small I_0 is. In practice, I_0 is generated by random fluctuations (noise) due to residual magnetism in the iron core. The exponential growth is limited by magnetic core saturation so that the current reaches a steady-state value. If the disk is rotating in the opposite direction ($\omega < 0$), the condition of (23) cannot be satisfied. It is then necessary for the field coil connection to be reversed so that $i_r = -i_f$. Such a dynamo model has been used as a model of the origin of the earth's magnetic field.

(c) Self-Excited ac Operation

Two such coupled generators can spontaneously generate two phase ac power if two independent field windings are connected, as in Figure 6-17. The field windings are connected so that if the flux through the two windings on one machine add, they subtract on the other machine. This accounts for the sign difference in the speed voltages in the equivalent circuits,

$$\begin{aligned} L\frac{di_1}{dt}+(R-G\omega)i_1+G\omega i_2=0 \\ L\frac{di_2}{dt}+(R-G\omega)i_2-G\omega i_1=0 \end{aligned} \tag{24}$$

where L and R are the total series inductance and resistance. The disks are each turned at the same angular speed ω.

Since (24) are linear with constant coefficients, solutions are of the form

$$i_1=I_1 e^{st}, \qquad i_2=I_2 e^{st} \tag{25}$$

which when substituted back into (24) yields

$$\begin{aligned} (Ls+R-G\omega)I_1+G\omega I_2=0 \\ -G\omega I_1+(Ls+R-G\omega)I_2=0 \end{aligned} \tag{26}$$

For nontrivial solutions, the determinant of the coefficients of I_1 and I_2 must be zero,

$$(Ls+R-G\omega)^2=-(G\omega)^2 \tag{27}$$

which when solved for s yields the complex conjugate natural frequencies,

$$\begin{aligned} s=-\frac{(R-G\omega)}{L}\pm j\frac{G\omega}{L} \\ I_1/I_2=\pm j \end{aligned} \tag{28}$$

where the currents are 90° out of phase. If the real part of s is positive, the system is self-excited so that any perturbation

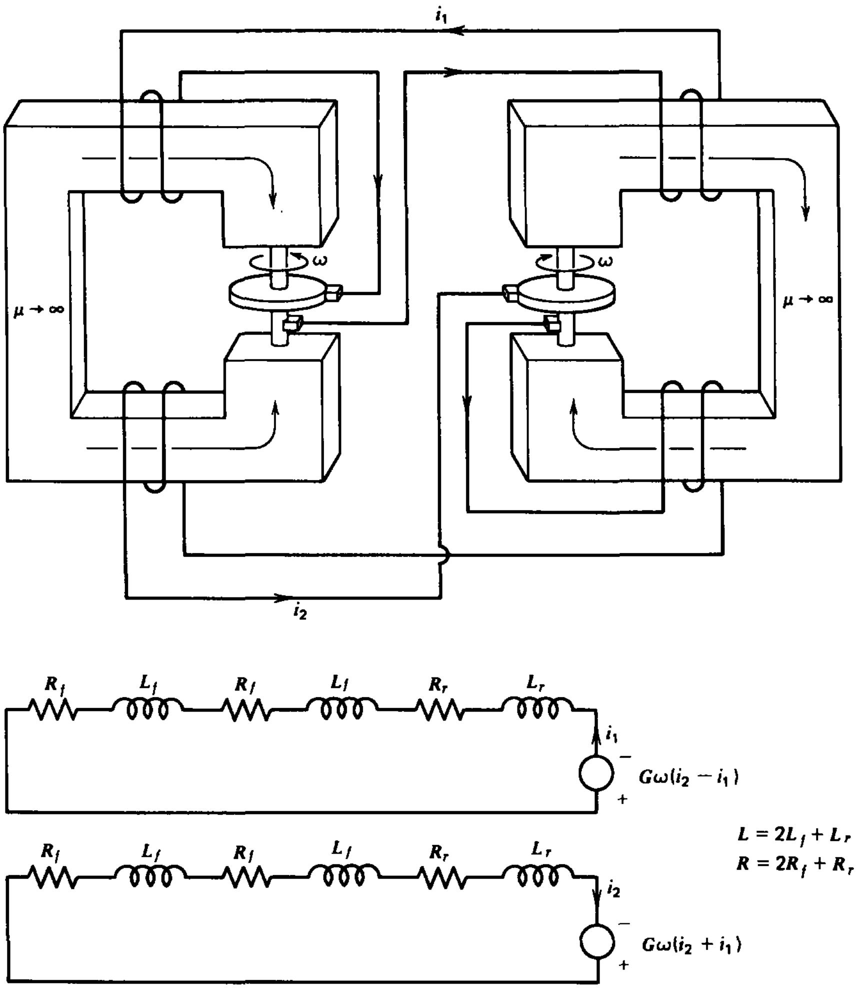


Figure 6-17 Cross-connecting two homopolar generators can result in self-excited two-phase alternating currents. Two independent field windings are required where on one machine the fluxes add while on the other they subtract.

grows at an exponential rate:

$$G\omega > R \tag{29}$$

The imaginary part of s yields the oscillation frequency

$$\omega_0 = \text{Im}\,(s) = G\omega/L \tag{30}$$

Again, core saturation limits the exponential growth so that two-phase power results. Such a model may help explain the periodic reversals in the earth's magnetic field every few hundred thousand years.

(d) Periodic Motor Speed Reversals

If the field winding of a motor is excited by a dc current, as in Figure 6-18, with the rotor terminals connected to a generator whose field and rotor terminals are in series, the circuit equation is

$$\frac{di}{dt}+\frac{(R-G_g\omega_g)}{L}i=\frac{G_m\omega_m}{L}I_f \tag{31}$$

where L and R are the total series inductances and resistances. The angular speed of the generator ω_g is externally

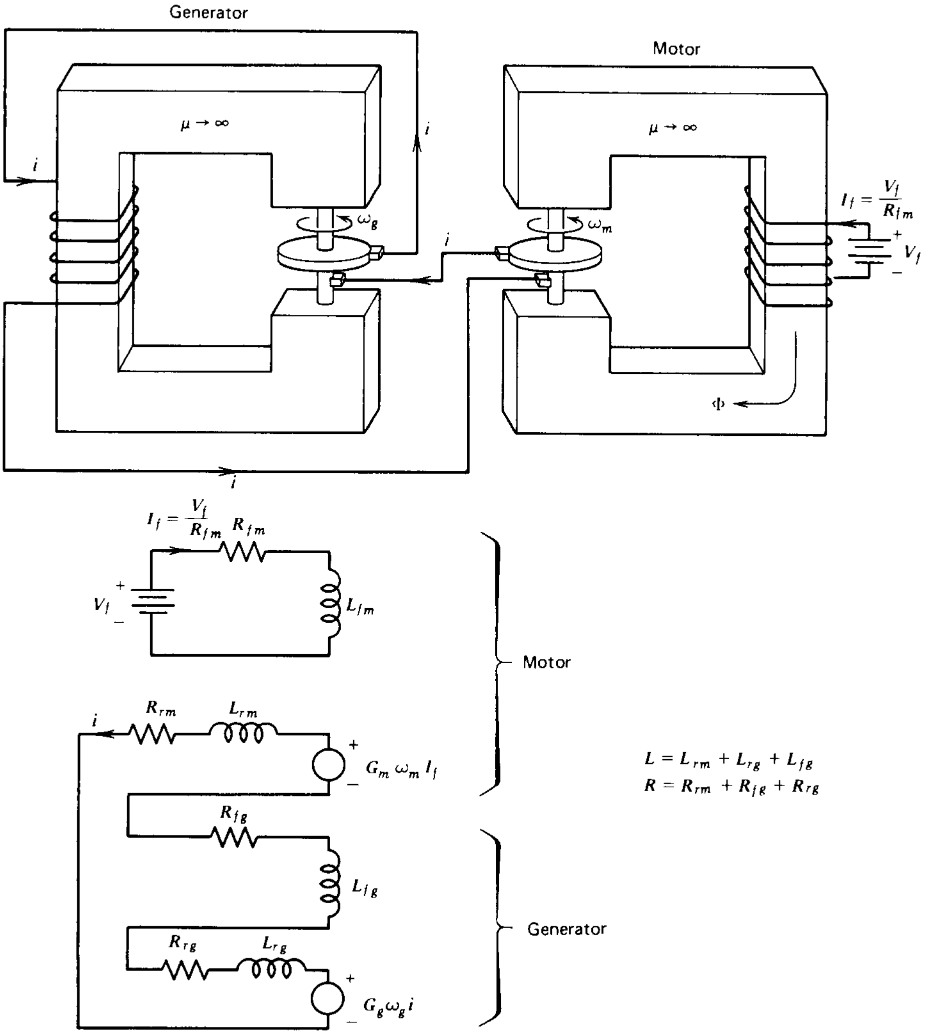


Figure 6-18 Cross connecting a homopolar generator and motor can result in spontaneous periodic speed reversals of the motor's shaft.

constrained to be a constant. The angular acceleration of the motor's shaft is equal to the torque of (20),

$$J\frac{d\omega_m}{dt} = -G_m I_f i \tag{32}$$

where J is the moment of inertia of the shaft and $I_f = V_f/R_{fm}$ is the constant motor field current.

Solutions of these coupled, linear constant coefficient differential equations are of the form

$$\begin{aligned} i &= I\,e^{st} \\ \omega &= W\,e^{st} \end{aligned} \tag{33}$$

which when substituted back into (31) and (32) yield

$$\begin{aligned} I\left(s+\frac{R-G_g\omega_g}{L}\right)-W\left(\frac{G_m I_f}{L}\right)&=0 \\ I\left(\frac{G_m I_f}{J}\right)+Ws&=0 \end{aligned} \tag{34}$$

Again, for nontrivial solutions the determinant of coefficients of I and W must be zero,

$$s\left(s+\frac{R-G_g\omega_g}{L}\right)+\frac{(G_m I_f)^2}{JL}=0 \tag{35}$$

which when solved for s yields

$$s=-\frac{(R-G_g\omega_g)}{2L}\pm\left[\left(\frac{R-G_g\omega_g}{2L}\right)^2-\frac{(G_m I_f)^2}{JL}\right]^{1/2} \tag{36}$$

For self-excitation the real part of s must be positive,

$$G_g\omega_g > R \tag{37}$$

while oscillations will occur if s has an imaginary part,

$$\frac{(G_m I_f)^2}{JL} > \left(\frac{R-G_g\omega_g}{2L}\right)^2 \tag{38}$$

Now, both the current and shaft's angular velocity spontaneously oscillate with time.

6-3-4 Basic Motors and Generators

(a) ac Machines

Alternating voltages are generated from a dc magnetic field by rotating a coil, as in Figure 6-19. An output voltage is measured via slip rings through carbon brushes. If the loop of area A is vertical at $t=0$ linking zero flux, the imposed flux

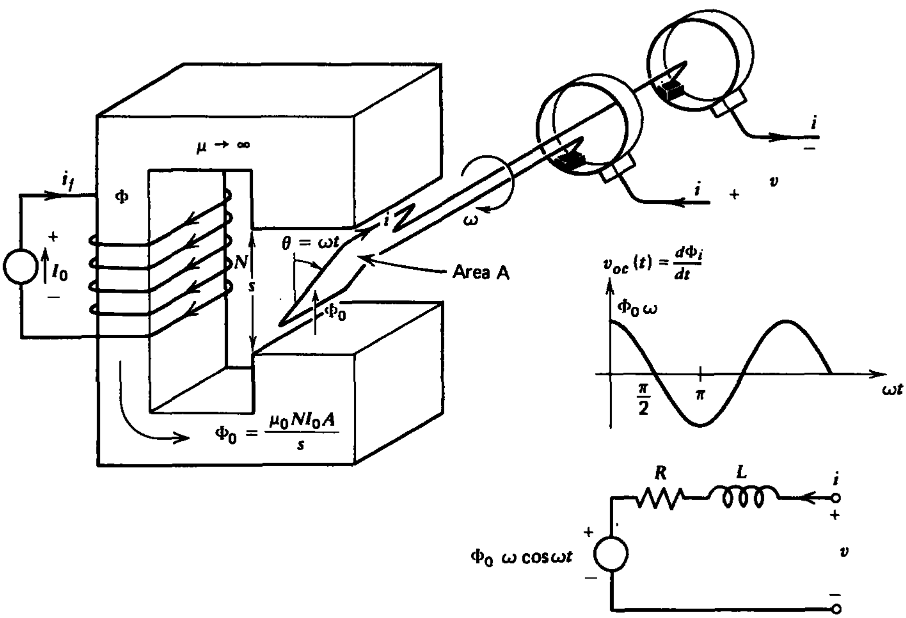


Figure 6-19 A coil rotated within a constant magnetic field generates a sinusoidal voltage.

through the loop at any time, varies sinusoidally with time due to the rotation as

$$\Phi_i = \Phi_0 \sin \omega t \tag{39}$$

Faraday's law applied to a stationary contour instantaneously passing through the wire then gives the terminal voltage as

$$v = iR + \frac{d\Phi}{dt} = iR + L\frac{di}{dt} + \Phi_0\omega \cos \omega t \tag{40}$$

where R and L are the resistance and inductance of the wire. The total flux is equal to the imposed flux of (39) as well as self-flux (accounted for by L) generated by the current i. The equivalent circuit is then similar to that of the homopolar generator, but the speed voltage term is now sinusoidal in time.

(b) dc Machines

DC machines have a similar configuration except that the slip ring is split into two sections, as in Figure 6-20*a*. Then whenever the output voltage tends to change sign, the terminals are also reversed yielding the waveform shown, which is of one polarity with periodic variations from zero to a peak value.

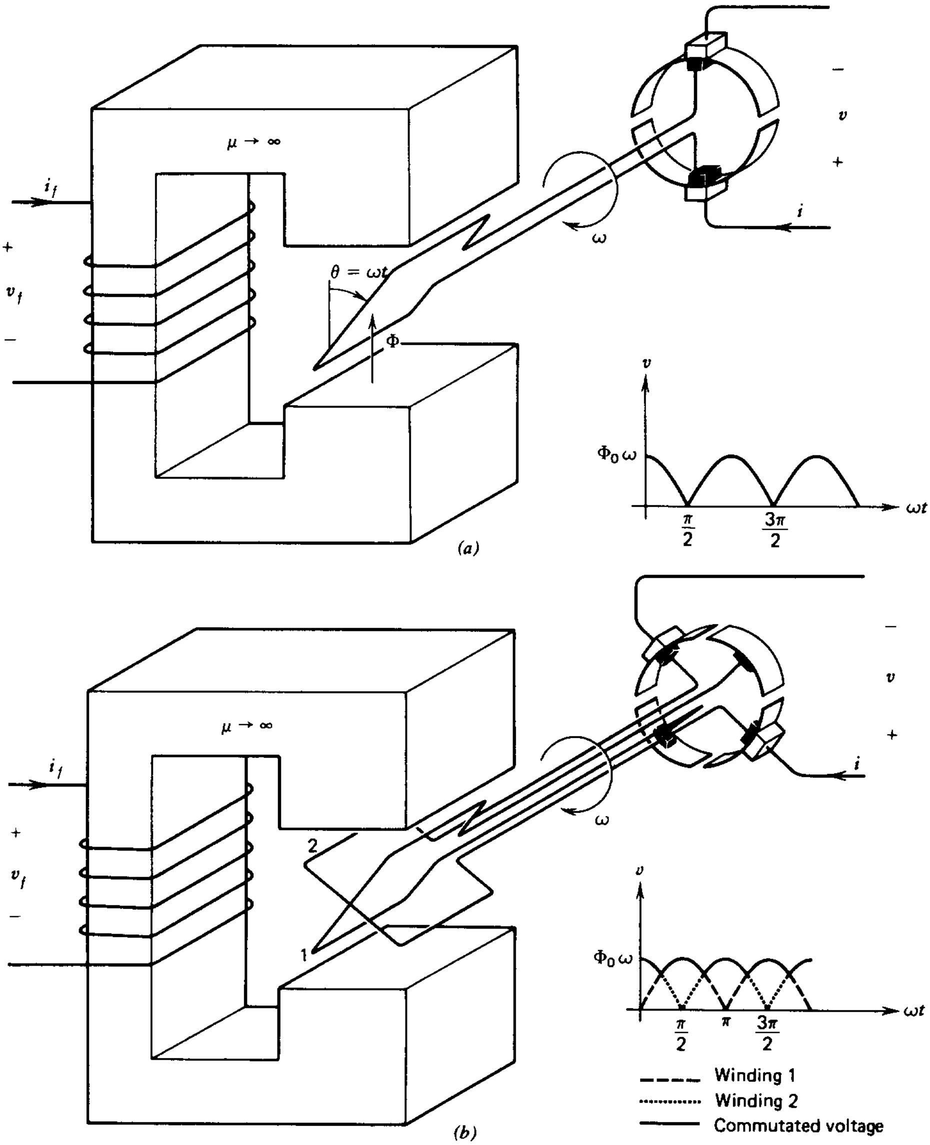


Figure 6-20 (*a*) If the slip rings are split so that when the voltage tends to change sign the terminals are also reversed, the resulting voltage is of one polarity. (*b*) The voltage waveform can be smoothed out by placing a second coil at right angles to the first and using a four-section commutator.

The voltage waveform can be smoothed out by using a four-section commutator and placing a second coil perpendicular to the first, as in Figure 6-20*b*. This second coil now generates its peak voltage when the first coil generates zero voltage. With more commutator sections and more coils, the dc voltage can be made as smooth as desired.

6-3-5 MHD Machines

Magnetohydrodynamic machines are based on the same principles as rotating machines, replacing the rigid rotor by a conducting fluid. For the linear machine in Figure 6-21, a fluid with Ohmic conductivity σ flowing with velocity v_y moves perpendicularly to an applied magnetic field $B_0\mathbf{i}_z$. The terminal voltage V is related to the electric field and current as

$$\mathbf{E}=\mathbf{i}_x\frac{V}{s},\qquad \mathbf{J}=\sigma(\mathbf{E}+\mathbf{v}\times\mathbf{B})=\sigma\left(\frac{V}{s}+v_yB_0\right)\mathbf{i}_x=\frac{i}{Dd}\mathbf{i}_x \tag{41}$$

which can be rewritten as

$$V=iR-v_yB_0s \tag{42}$$

which has a similar equivalent circuit as for the homopolar generator.

The force on the channel is then

$$\mathbf{f}=\int_V \mathbf{J}\times\mathbf{B}\,dV$$
$$=-iB_0s\mathbf{i}_y \tag{43}$$

again opposite to the fluid motion.

6-3-6 Paradoxes

Faraday's law is prone to misuse, which has led to numerous paradoxes. The confusion arises because the same

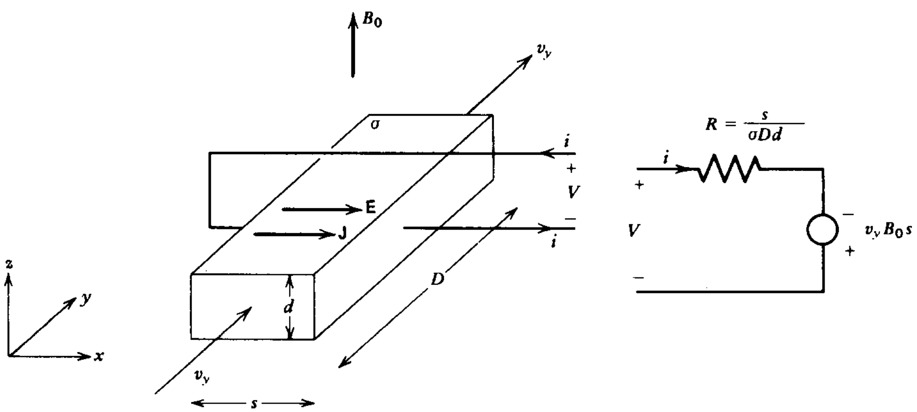


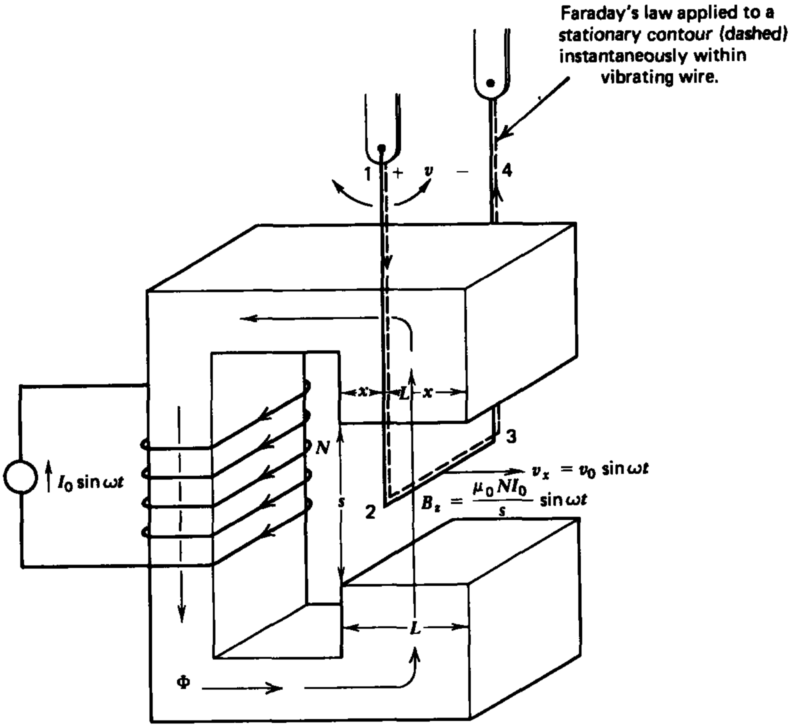
Figure 6-21 An MHD (magnetohydrodynamic) machine replaces a rotating conductor by a moving fluid.

contribution can arise from either the electromotive force side of the law, as a speed voltage when a conductor moves orthogonal to a magnetic field, or as a time rate of change of flux through the contour. This flux term itself has two contributions due to a time varying magnetic field or due to a contour that changes its shape, size, or orientation. With all these potential contributions it is often easy to miss a term or to double count.

(a) A Commutatorless dc Machine*

Many persons have tried to make a commutatorless dc machine but to no avail. One novel unsuccessful attempt is illustrated in Figure 6-22, where a highly conducting wire is vibrated within the gap of a magnetic circuit with sinusoidal velocity:

$$v_x = v_0 \sin \omega t \tag{44}$$



Fɪɢ 6-22 It is impossible to design a commutatorless dc machine. Although the speed voltage alone can have a dc average, it will be canceled by the transformer electromotive force due to the time rate of change of magnetic flux through the loop. The total terminal voltage will always have a zero time average.

* *H. Sohon, Electrical Essays for Recreation.* Electrical Engineering, *May (1946), p. 294.*

The sinusoidal current imposes the air gap flux density at the same frequency ω:

$$B_z = B_0 \sin \omega t, \qquad B_0 = \mu_0 N I_0/s \tag{45}$$

Applying Faraday's law to a stationary contour instantaneously within the open circuited wire yields

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = \int_1^2 \overset{0}{\cancel{\mathbf{E}'}}\cdot d\mathbf{l} + \int_2^3 \underbrace{\mathbf{E}\cdot d\mathbf{l}}_{\mathbf{E}=-\mathbf{v}\times\mathbf{B}} + \int_3^4 \overset{0}{\cancel{\mathbf{E}'}}\cdot d\mathbf{l} + \int_4^1 \underbrace{\mathbf{E}\cdot d\mathbf{l}}_{-v}$$

$$= -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S} \tag{46}$$

where the electric field within the highly conducting wire as measured by an observer moving with the wire is zero. The electric field on the 2–3 leg within the air gap is given by (11), where $\mathbf{E}'=0$, while the 4–1 leg defines the terminal voltage. If we erroneously argue that the flux term on the right-hand side is zero because the magnetic field $\mathbf{B}$ is perpendicular to $\mathbf{dS}$, the terminal voltage is

$$v = v_x B_z l = v_0 B_0 l \sin^2 \omega t \tag{47}$$

which has a dc time-average value. Unfortunately, this result is not complete because we forgot to include the flux that turns the corner in the magnetic core and passes perpendicularly through our contour. Only the flux to the right of the wire passes through our contour, which is the fraction $(L-x)/L$ of the total flux. Then the correct evaluation of (46) is

$$-v + v_x B_z l = +\frac{d}{dt}[(L-x)B_z l] \tag{48}$$

where x is treated as a constant because the contour is stationary. The change in sign on the right-hand side arises because the flux passes through the contour in the direction opposite to its normal defined by the right-hand rule. The voltage is then

$$v = v_x B_z l - (L-x) l \frac{dB_z}{dt} \tag{49}$$

where the wire position is obtained by integrating (44),

$$x = \int v_x\, dt = -\frac{v_0}{\omega}(\cos \omega t - 1) + x_0 \tag{50}$$

and x_0 is the wire's position at $t=0$. Then (49) becomes

$$\begin{aligned} v &= l\frac{d}{dt}(xB_z) - Ll\frac{dB_z}{dt} \\ &= B_0 l v_0\left[\left(\frac{x_0\omega}{v_0}+1\right)\cos\omega t - \cos 2\omega t\right] - LlB_0\omega\cos\omega t \end{aligned} \tag{51}$$

which has a zero time average.

(b) Changes in Magnetic Flux Due to Switching

Changing the configuration of a circuit using a switch does not result in an electromotive force unless the magnetic flux itself changes.

In Figure 6-23*a*, the magnetic field through the loop is externally imposed and is independent of the switch position. Moving the switch does not induce an EMF because the magnetic flux through any surface remains unchanged.

In Figure 6-23*b*, a dc current source is connected to a circuit through a switch *S*. If the switch is instantaneously moved from contact 1 to contact 2, the magnetic field due to the source current *I* changes. The flux through any fixed area has thus changed resulting in an EMF.

(c) Time Varying Number of Turns on a Coil*

If the number of turns on a coil is changing with time, as in Figure 6-24, the voltage is equal to the time rate of change of flux through the coil. Is the voltage then

$$v \stackrel{?}{=} N\frac{d\Phi}{dt} \tag{52}$$

or

$$v \stackrel{?}{=} \frac{d}{dt}(N\Phi) = N\frac{d\Phi}{dt} + \Phi\frac{dN}{dt} \tag{53}$$

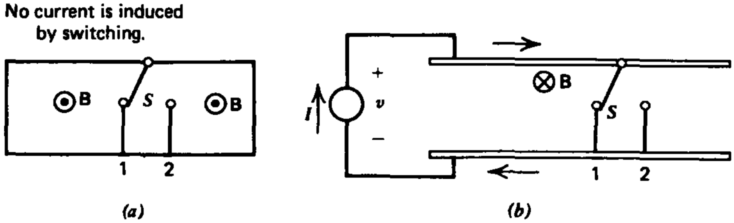


Figure 6-23 (*a*) Changes in a circuit through the use of a switch does not by itself generate an EMF. (*b*) However, an EMF can be generated if the switch changes the magnetic field.

* *L. V. Bewley*. Flux Linkages and Electromagnetic Induction. *Macmillan, New York, 1952.*

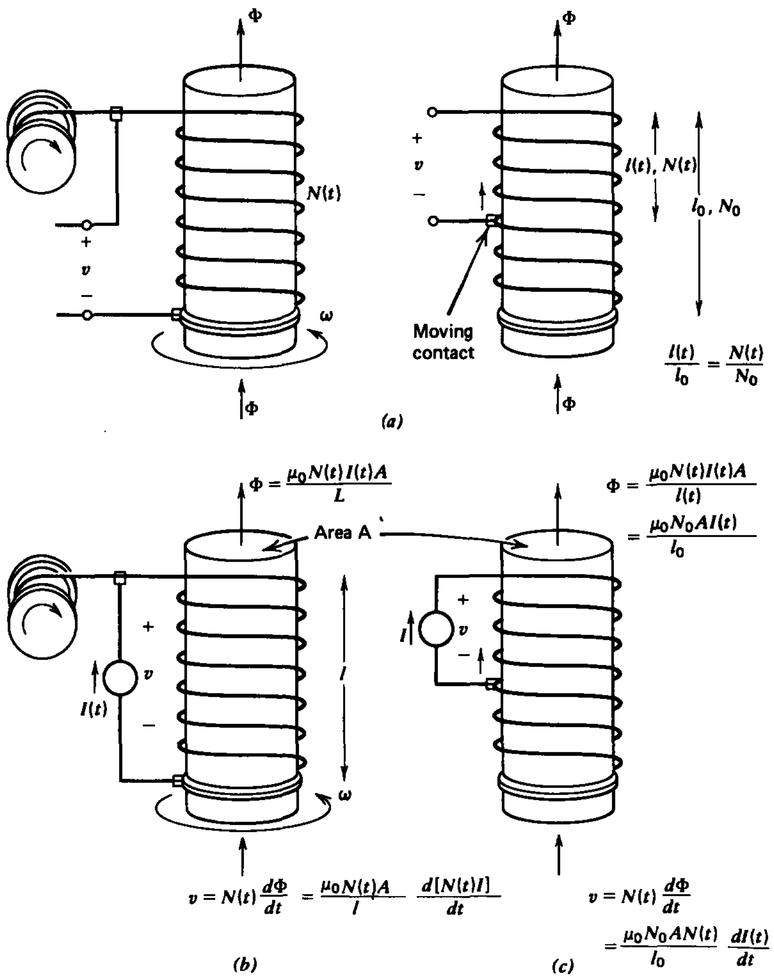


Figure 6-24 (*a*) If the number of turns on a coil is changing with time, the induced voltage is $v = N(t)\, d\Phi/dt$. A constant flux does not generate any voltage. (*b*) If the flux itself is proportional to the number of turns, a dc current can generate a voltage. (*c*) With the tap changing coil, the number of turns per unit length remains constant so that a dc current generates no voltage because the flux does not change with time.

For the first case a dc flux generates no voltage while the second does.

We use Faraday's law with a stationary contour instantaneously within the wire. Because the contour is stationary, its area of NA is not changing with time and so can be taken outside the time derivative in the flux term of Faraday's law so that the voltage is given by (52) and (53) is wrong. Note that there is no speed voltage contribution in the electromotive force because the velocity of the wire is in the same direction as the contour ($\mathbf{v}\times\mathbf{B}\cdot d\mathbf{l}=0$).

If the flux Φ itself depends on the number of turns, as in Figure 6-24*b*, there may be a contribution to the voltage even if the exciting current is dc. This is true for the turns being wound onto the cylinder in Figure 6-24*b*. For the tap changing configuration in Figure 6-24*c*, with uniformly wound turns, the ratio of turns to effective length is constant so that a dc current will still not generate a voltage.

6-4 MAGNETIC DIFFUSION INTO AN OHMIC CONDUCTOR*

If the current distribution is known, the magnetic field can be directly found from the Biot–Savart or Ampere's laws. However, when the magnetic field varies with time, the generated electric field within an Ohmic conductor induces further currents that also contribute to the magnetic field.

6-4-1 Resistor–Inductor Model

A thin conducting shell of radius R_i, thickness Δ, and depth l is placed within a larger conducting cylinder, as shown in Figure 6-25. A step current I_0 is applied at $t=0$ to the larger cylinder, generating a surface current $\mathbf{K}=(I_0/l)\mathbf{i}_\phi$. If the length l is much greater than the outer radius R_0, the magnetic field is zero outside the cylinder and uniform inside for $R_i<r<R_0$. Then from the boundary condition on the discontinuity of tangential $\mathbf{H}$ given in Section 5-6-1, we have

$$\mathbf{H}_0=\frac{I_0}{l}\mathbf{i}_z, \qquad R_i<r<R_0 \tag{1}$$

The magnetic field is different inside the conducting shell because of the induced current, which from Lenz's law, flows in the opposite direction to the applied current. Because the shell is assumed to be very thin ($\Delta \ll R_i$), this induced current can be considered a surface current related to the volume current and electric field in the conductor as

$$K_\phi = J_\phi \Delta = (\sigma\Delta)E_\phi \tag{2}$$

The product $(\sigma\Delta)$ is called the surface conductivity. Then the magnetic fields on either side of the thin shell are also related by the boundary condition of Section 5-6-1:

$$H_i - H_0 = K_\phi = (\sigma\Delta)E_\phi \tag{3}$$

* *Much of the treatment of this section is similar to that of H. H. Woodson and J. R. Melcher,* Electromechanical Dynamics, *Part II, Wiley, N.Y., 1968, Ch. 7.*

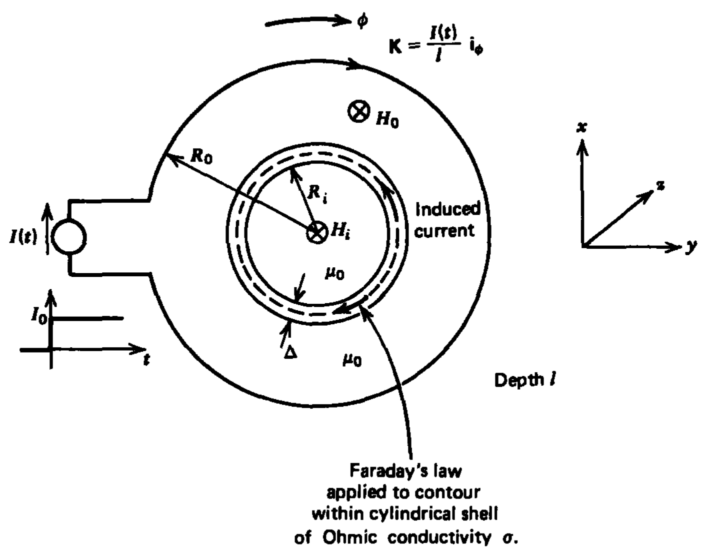


Figure 6-25 A step change in magnetic field causes the induced current within an Ohmic conductor to flow in the direction where its self-flux opposes the externally imposed flux. Ohmic dissipation causes the induced current to exponentially decay with time with a L/R time constant.

Applying Faraday's law to a contour within the conducting shell yields

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S} \Rightarrow E_\phi 2\pi R_i = -\mu_0 \pi R_i^2 \frac{dH_i}{dt} \tag{4}$$

where only the magnetic flux due to H_i passes through the contour. Then using (1)–(3) in (4) yields a single equation in H_i:

$$\frac{dH_i}{dt} + \frac{H_i}{\tau} = \frac{I(t)}{l\tau}, \qquad \tau = \frac{\mu_0 R_i \sigma \Delta}{2} \tag{5}$$

where we recognize the time constant τ as just being the ratio of the shell's self-inductance to resistance:

$$L = \frac{\Phi}{K_\phi l} = \frac{\mu_0 \pi R_i^2}{l}, \qquad R = \frac{2\pi R_i}{\sigma l \Delta}, \qquad \tau = \frac{L}{R} = \frac{\mu_0 R_i \sigma \Delta}{2} \tag{6}$$

The solution to (5) for a step current with zero initial magnetic field is

$$H_i = \frac{I_0}{l}(1 - e^{-t/\tau}) \tag{7}$$

Initially, the magnetic field is excluded from inside the conducting shell by the induced current. However, Ohmic dissipation causes the induced current to decay with time so that the magnetic field may penetrate through the shell with characteristic time constant τ.

6-4-2 The Magnetic Diffusion Equation

The transient solution for a thin conducting shell could be solved using the integral laws because the geometry constrained the induced current to flow azimuthally with no radial variations. If the current density is not directly known, it becomes necessary to self-consistently solve for the current density with the electric and magnetic fields:

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}\quad\text{(Faraday's law)}\tag{8}$$

$$\nabla\times\mathbf{H}=\mathbf{J}_f\quad\text{(Ampere's law)}\tag{9}$$

$$\nabla\cdot\mathbf{B}=0\quad\text{(Gauss's law)}\tag{10}$$

For linear magnetic materials with constant permeability μ and constant Ohmic conductivity σ moving with velocity $\mathbf{U}$, the constitutive laws are

$$\mathbf{B}=\mu\mathbf{H},\qquad \mathbf{J}_f=\sigma(\mathbf{E}+\mathbf{U}\times\mu\mathbf{H})\tag{11}$$

We can reduce (8)–(11) to a single equation in the magnetic field by taking the curl of (9), using (8) and (11) as

$$\begin{aligned}\nabla\times(\nabla\times\mathbf{H})&=\nabla\times\mathbf{J}_f\\&=\sigma[\nabla\times\mathbf{E}+\mu\nabla\times(\mathbf{U}\times\mathbf{H})]\\&=\mu\sigma\left(-\frac{\partial\mathbf{H}}{\partial t}+\nabla\times(\mathbf{U}\times\mathbf{H})\right)\end{aligned}\tag{12}$$

The double cross product of $\mathbf{H}$ can be simplified using the vector identity

$$\begin{aligned}\nabla\times(\nabla\times\mathbf{H})&=\nabla(\overset{0}{\nabla\cdot\mathbf{H}})-\nabla^2\mathbf{H}\\\Rightarrow\frac{1}{\mu\sigma}\nabla^2\mathbf{H}&=\frac{\partial\mathbf{H}}{\partial t}-\nabla\times(\mathbf{U}\times\mathbf{H})\end{aligned}\tag{13}$$

where $\mathbf{H}$ has no divergence from (10). Remember that the Laplacian operator on the left-hand side of (13) also differentiates the directionally dependent unit vectors in cylindrical ($\mathbf{i}_r$ and $\mathbf{i}_\phi$) and spherical ($\mathbf{i}_r$, $\mathbf{i}_\theta$, and $\mathbf{i}_\phi$) coordinates.

6-4-3 Transient Solution with No Motion (U = 0)

A step current is turned on at $t=0$, in the parallel plate geometry shown in Figure 6-26. By the right-hand rule and with the neglect of fringing, the magnetic field is in the z direction and only depends on the x coordinate, $B_z(x,t)$, so that (13) reduces to

$$\frac{\partial^2 H_z}{\partial x^2} - \sigma\mu \frac{\partial H_z}{\partial t} = 0 \tag{14}$$

which is similar in form to the diffusion equation of a distributed resistive-capacitive cable developed in Section 3-6-4.

In the dc steady state, the second term is zero so that the solution in each region is of the form

$$\frac{\partial^2 H_z}{\partial x^2} = 0 \Rightarrow H_z = ax + b \tag{15}$$

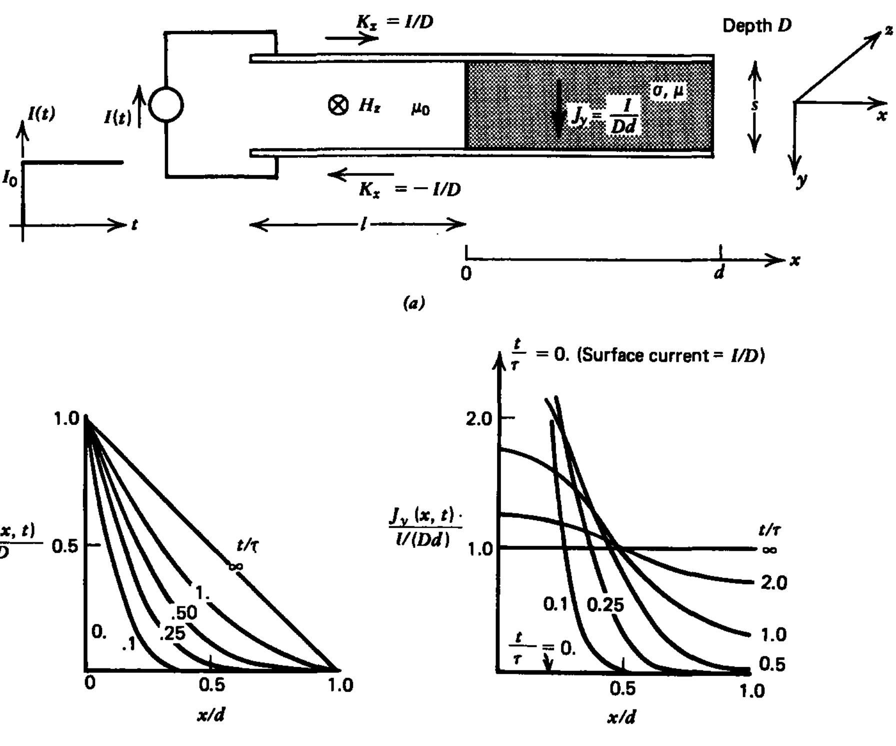


Figure 6-26 (a) A current source is instantaneously turned on at $t=0$. The resulting magnetic field within the Ohmic conductor remains continuous and is thus zero at $t=0$ requiring a surface current at $x=0$. (b) For later times the magnetic field and current diffuse into the conductor with longest time constant $\tau = \sigma\mu d^2/\pi^2$ towards a steady state of uniform current with a linear magnetic field.

where a and b are found from the boundary conditions. The current on the electrodes immediately spreads out to a uniform surface distribution $\pm(I/D)\mathbf{i}_x$ traveling from the upper to lower electrode uniformly through the Ohmic conductor. Then, the magnetic field is uniform in the free space region, decreasing linearly to zero within the Ohmic conductor being continuous across the interface at $x=0$:

$$\lim_{t\to\infty} H_z(x)=\begin{cases}\dfrac{I}{D}, & -l\le x\le 0\\[2mm] \dfrac{I}{Dd}(d-x), & 0\le x\le d\end{cases} \tag{16}$$

In the free space region where $\sigma=0$, the magnetic field remains constant for all time. Within the conducting slab, there is an initial charging transient as the magnetic field builds up to the linear steady-state distribution in (16). Because (14) is a linear equation, for the total solution of the magnetic field as a function of time and space, we use superposition and guess a solution that is the sum of the steady-state solution in (16) and a transient solution which dies off with time:

$$H_z(x,t)=\frac{I}{Dd}(d-x)+\hat{H}(x)\,e^{-\alpha t} \tag{17}$$

We follow the same procedures as for the lossy cable in Section 3-6-4. At this point we do not know the function $\hat{H}(x)$ or the parameter α. Substituting the assumed solution of (17) back into (14) yields the ordinary differential equation

$$\frac{d^2\hat{H}(x)}{dx^2}+\sigma\mu\alpha\hat{H}(x)=0 \tag{18}$$

which has the trigonometric solutions

$$\hat{H}(x)=A_1\sin\sqrt{\sigma\mu\alpha}\,x+A_2\cos\sqrt{\sigma\mu\alpha}\,x \tag{19}$$

Since the time-independent part in (17) already meets the boundary conditions of

$$\begin{aligned} H_z(x=0)&=I/D\\ H_z(x=d)&=0\end{aligned} \tag{20}$$

the transient part of the solution must be zero at the ends

$$\begin{aligned}\hat{H}(x=0)&=0\Rightarrow A_2=0\\ \hat{H}(x=d)&=0\Rightarrow A_1\sin\sqrt{\sigma\mu\alpha}\,d=0\end{aligned} \tag{21}$$

which yields the allowed values of α as

$$\sqrt{\sigma\mu\alpha}\,d=n\pi\Rightarrow\alpha_n=\frac{1}{\mu\sigma}\left(\frac{n\pi}{d}\right)^2,\quad n=1,2,3,\ldots \tag{22}$$

Since there are an infinite number of allowed values of α, the most general solution is the superposition of all allowed solutions:

$$H_z(x,t)=\frac{I}{Dd}(d-x)+\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{d}e^{-\alpha_n t} \qquad (23)$$

This relation satisfies the boundary conditions but not the initial conditions at $t=0$ when the current is first turned on. Before the current takes its step at $t=0$, the magnetic field is zero in the slab. Right after the current is turned on, the magnetic field must remain zero. Faraday's law would otherwise make the electric field and thus the current density infinite within the slab, which is nonphysical. Thus we impose the initial condition

$$H_z(x,t=0)=0=\frac{I}{Dd}(d-x)+\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{d} \qquad (24)$$

which will allow us to solve for the amplitudes A_n by multiplying (24) through by $\sin(m\pi x/d)$ and then integrating over x from 0 to d:

$$0=\frac{I}{Dd}\int_0^d(d-x)\sin\frac{m\pi x}{d}dx+\sum_{n=1}^{\infty}A_n\int_0^d\sin\frac{n\pi x}{d}\sin\frac{m\pi x}{d}dx \qquad (25)$$

The first term on the right-hand side is easily integrable* while the product of sine terms integrates to zero unless $m=n$, yielding

$$A_m=-\frac{2I}{m\pi D} \qquad (26)$$

The total solution is thus

$$H_z(x,t)=\frac{I}{D}\left(1-\frac{x}{d}-2\sum_{n=1}^{\infty}\frac{\sin(n\pi x/d)}{n\pi}e^{-n^2t/\tau}\right) \qquad (27)$$

where we define the fundamental continuum magnetic diffusion time constant τ as

$$\tau=\frac{1}{\alpha_1}=\frac{\mu\sigma d^2}{\pi^2} \qquad (28)$$

analogous to the lumped parameter time constant of (5) and (6).

* $\int_0^d(d-x)\sin\frac{m\pi x}{d}dx=\frac{d^2}{m\pi}$

The magnetic field approaches the steady state in times long compared to τ. For a perfect conductor ($\sigma \to \infty$), this time is infinite and the magnetic field is forever excluded from the slab. The current then flows only along the $x=0$ surface. However, even for copper ($\sigma \approx 6\times 10^7$ siemens/m) 10-cm thick, the time constant is $\tau \approx 80$ msec, which is fast for many applications. The current then diffuses into the conductor where the current density is easily obtained from Ampere's law as

$$\mathbf{J}_f = \nabla\times\mathbf{H} = -\frac{\partial H_z}{\partial x}\mathbf{i}_y$$

$$= \frac{I}{Dd}\left(1+2\sum_{n=1}^{\infty}\cos\frac{n\pi x}{d}e^{-n^2 t/\tau}\right)\mathbf{i}_y \tag{29}$$

The diffusion of the magnetic field and current density are plotted in Figure 6-26*b* for various times

The force on the conducting slab is due to the Lorentz force tending to expand the loop and a magnetization force due to the difference of permeability of the slab and the surrounding free space as derived in Section 5-8-1:

$$\mathbf{F} = \mu_0(\mathbf{M}\cdot\nabla)\mathbf{H} + \mu_0\mathbf{J}_f\times\mathbf{H}$$

$$= (\mu-\mu_0)(\mathbf{H}\cdot\nabla)\mathbf{H} + \mu_0\mathbf{J}_f\times\mathbf{H} \tag{30}$$

For our case with $\mathbf{H} = H_z(x)\mathbf{i}_z$, the magnetization force density has no contribution so that (30) reduces to

$$\begin{aligned}\mathbf{F} &= \mu_0\mathbf{J}_f\times\mathbf{H}\\ &= \mu_0(\nabla\times\mathbf{H})\times\mathbf{H}\\ &= \mu_0(\mathbf{H}\cdot\nabla)\mathbf{H} - \nabla(\tfrac{1}{2}\mu_0\mathbf{H}\cdot\mathbf{H})\\ &= -\frac{d}{dx}(\tfrac{1}{2}\mu_0 H_z^2)\mathbf{i}_x\end{aligned} \tag{31}$$

Integrating (31) over the slab volume with the magnetic field independent of y and z,

$$\begin{aligned}f_x &= -\int_0^d sD\frac{d}{dx}(\tfrac{1}{2}\mu_0 H_z^2)\,dx\\ &= -\tfrac{1}{2}\mu_0 H_z^2 sD\Big|_0^d\\ &= \frac{1}{2}\frac{\mu_0 I^2 s}{D}\end{aligned} \tag{32}$$

gives us a constant force with time that is independent of the permeability. Note that our approach of expressing the current density in terms of the magnetic field in (31) was easier than multiplying the infinite series of (27) and (29), as the result then only depended on the magnetic field at the boundaries that are known from the boundary conditions of (20). The resulting integration in (32) was easy because the force density in (31) was expressed as a pure derivative of x.

6-4-4 The Sinusoidal Steady State (Skin Depth)

We now place an infinitely thick conducting slab a distance d above a sinusoidally varying current sheet $K_0 \cos \omega t \mathbf{i}_y$, which lies on top of a perfect conductor, as in Figure 6-27a. The

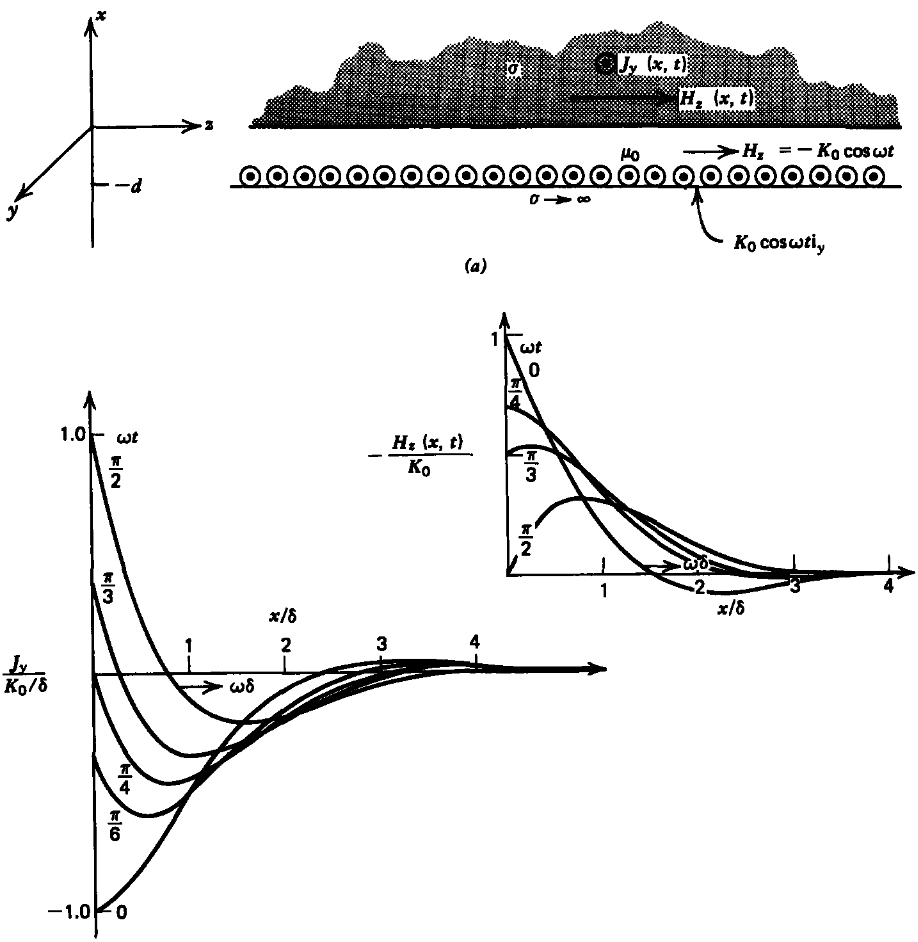


Figure 6-27 (a) A stationary conductor lies above a sinusoidal surface current placed upon a perfect conductor so that $\mathbf{H} = 0$ for $x < -d$. (b) The magnetic field and current density propagates and decays into the conductor with the same characteristic length given by the skin depth $\delta = \sqrt{2/(\omega\mu\sigma)}$. The phase speed of the wave is $\omega\delta$.

magnetic field within the conductor is then also sinusoidally varying with time:

$$H_z(x, t) = \text{Re}\,[\hat{H}_z(x)\, e^{j\omega t}] \tag{33}$$

Substituting (33) into (14) yields

$$\frac{d^2\hat{H}_z}{dx^2} - j\omega\mu\sigma\hat{H}_z = 0 \tag{34}$$

with solution

$$\hat{H}_z(x) = A_1\, e^{(1+j)x/\delta} + A_2\, e^{-(1+j)x/\delta} \tag{35}$$

where the skin depth δ is defined as

$$\delta = \sqrt{2/(\omega\mu\sigma)} \tag{36}$$

Since the magnetic field must remain finite far from the current sheet, A_1 must be zero. The magnetic field is also continuous across the $x = 0$ boundary because there is no surface current, so that the solution is

$$\begin{aligned} H_z(x, t) &= \text{Re}\,[-K_0\, e^{-(1+j)x/\delta}\, e^{j\omega t}] \\ &= -K_0 \cos(\omega t - x/\delta)\, e^{-x/\delta}, \quad x \geq 0 \end{aligned} \tag{37}$$

where the magnetic field in the gap is uniform, determined by the discontinuity in tangential **H** at $x = -d$ to be $H_z = -K_y$ for $-d < x \leq 0$ since within the perfect conductor $(x < -d)\mathbf{H} = 0$. The magnetic field diffuses into the conductor as a strongly damped propagating wave with characteristic penetration depth δ. The skin depth δ is also equal to the propagating wavelength, as drawn in Figure 6-27*b*. The current density within the conductor

$$\begin{aligned} \mathbf{J}_f &= \nabla \times \mathbf{H} = -\frac{\partial H_z}{\partial x}\mathbf{i}_y \\ &= +\frac{K_0\, e^{-x/\delta}}{\delta}\left[\sin\left(\omega t - \frac{x}{\delta}\right) - \cos\left(\omega t - \frac{x}{\delta}\right)\right]\mathbf{i}_y \end{aligned} \tag{38}$$

is also drawn in Figure 6-27*b* at various times in the cycle, being confined near the interface to a depth on the order of δ. For a perfect conductor, $\delta \rightarrow 0$, and the volume current becomes a surface current.

Seawater has a conductivity of ≈ 4 siemens/m so that at a frequency of $f = 1$ MHz $(\omega = 2\pi f)$ the skin depth is $\delta \approx 0.25$ m. This is why radio communications to submarines are difficult. The conductivity of copper is $\sigma \approx 6 \times 10^7$ siemens/m so that at 60 Hz the skin depth is $\delta \approx 8$ mm. Power cables with larger radii have most of the current confined near the surface so that the center core carries very little current. This reduces the cross-sectional area through which the current flows, raising the cable resistance leading to larger power dissipation.

Again, the magnetization force density has no contribution to the force density since H_z only depends on x:

$$\begin{aligned}\mathbf{F} &= \mu_0(\mathbf{M}\cdot\nabla)\mathbf{H}+\mu_0\mathbf{J}_f\times\mathbf{H}\\ &= \mu_0(\nabla\times\mathbf{H})\times\mathbf{H}\\ &= -\nabla(\tfrac{1}{2}\mu_0\mathbf{H}\cdot\mathbf{H})\end{aligned} \quad (39)$$

The total force per unit area on the slab obtained by integrating (39) over x depends only on the magnetic field at $x=0$:

$$\begin{aligned}f_x &= -\int_0^\infty \frac{d}{dx}\left(\frac{\mu_0}{2}H_z^2\right)dx\\ &= -\tfrac{1}{2}\mu_0 H_z^2\big|_0^\infty\\ &= \tfrac{1}{2}\mu_0 K_0^2\cos^2\omega t\end{aligned} \quad (40)$$

because again $\mathbf{H}$ is independent of y and z and the x component of the force density of (39) was written as a pure derivative with respect to x. Note that this approach was easier than integrating the cross product of (38) with (37).

This force can be used to levitate the conductor. Note that the region for $x>\delta$ is dead weight, as it contributes very little to the magnetic force.

6-4-5 Effects of Convection

A distributed dc surface current $-K_0\mathbf{i}_y$ at $x=0$ flows along parallel electrodes and returns via a conducting fluid moving to the right with constant velocity $v_0\mathbf{i}_x$, as shown in Figure 6-28a. The flow is not impeded by the current source at $x=0$. With the neglect of fringing, the magnetic field is purely z directed and only depends on the x coordinate, so that (13) in the dc steady state, with $\mathbf{U}=v_0\mathbf{i}_x$ being a constant, becomes*

$$\frac{d^2H_z}{dx^2}-\mu\sigma v_0\frac{dH_z}{dx}=0 \quad (41)$$

Solutions of the form

$$H_z(x)=A\,e^{px} \quad (42)$$

* $\nabla\times(\mathbf{U}\times\mathbf{H})=\mathbf{U}(\nabla\cdot\mathbf{H})^{0}-\mathbf{H}(\nabla\cdot\mathbf{U})^{0}+(\mathbf{H}\cdot\nabla)\mathbf{U}^{0}-(\mathbf{U}\cdot\nabla)\mathbf{H}=-v_0\dfrac{d\mathbf{H}}{dx}$

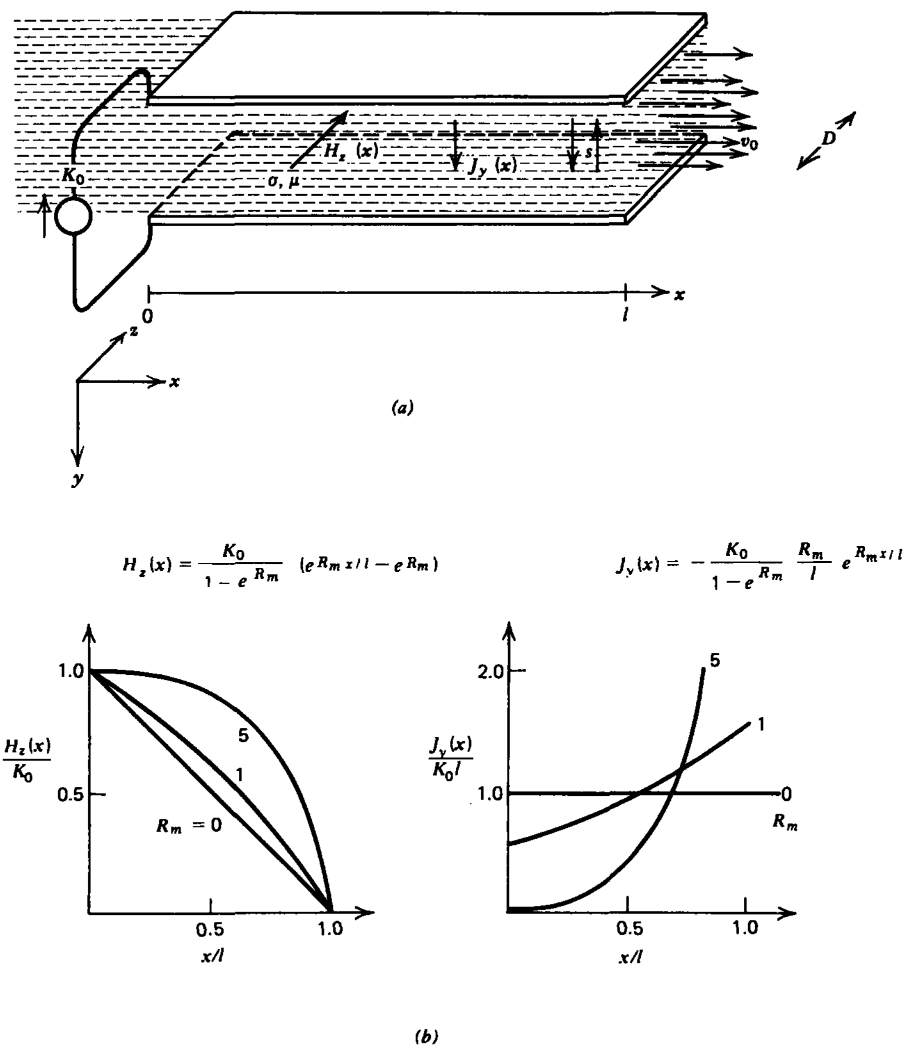


Figure 6-28 (*a*) A conducting material moving through a magnetic field tends to pull the magnetic field and current density with it. (*b*) The magnetic field and current density are greatly disturbed by the flow when the magnetic Reynolds number is large, $R_m = \sigma\mu Ul \gg 1$.

when substituted back into (41) yield two allowed values of p,

$$p^2 - \mu\sigma v_0 p = 0 \Rightarrow p = 0, \qquad p = \mu\sigma v_0 \tag{43}$$

Since (41) is linear, the most general solution is just the sum of the two allowed solutions,

$$H_z(x) = A_1 e^{R_m x/l} + A_2 \tag{44}$$

where the magnetic Reynold's number is defined as

$$R_m = \sigma\mu v_0 l = \frac{\sigma\mu l^2}{l/v_0} \tag{45}$$

and represents the ratio of a representative magnetic diffusion time given by (28) to a fluid transport time (l/v_0). The boundary conditions are

$$H_z(x=0) = K_0, \qquad H_z(x=l) = 0 \tag{46}$$

so that the solution is

$$H_z(x) = \frac{K_0}{1-e^{R_m}}(e^{R_m x/l} - e^{R_m}) \tag{47}$$

The associated current distribution is then

$$\begin{aligned} \mathbf{J}_f = \nabla \times \mathbf{H} &= -\frac{\partial H_z}{\partial x}\mathbf{i}_y \\ &= -\frac{K_0}{1-e^{R_m}}\frac{R_m}{l}e^{R_m x/l}\mathbf{i}_y \end{aligned} \tag{48}$$

The field and current distributions plotted in Figure 6-28*b* for various R_m show that the magnetic field and current are pulled along in the direction of flow. For small R_m the magnetic field is hardly disturbed from the zero flow solution of a linear field and constant current distribution. For very large $R_m \gg 1$, the magnetic field approaches a uniform distribution while the current density approaches a surface current at $x = l$.

The force on the moving fluid is independent of the flow velocity:

$$\begin{aligned} \mathbf{f} &= \int_0^l \mathbf{J} \times \mu_0 \mathbf{H} sD\, dx \\ &= -\frac{K_0^2}{(1-e^{R_m})^2}\mu_0 \frac{R_m}{l} sD \int_0^l e^{R_m x/l}(e^{R_m x/l} - e^{R_m})\, dx\, \mathbf{i}_x \\ &= -\frac{K_0^2 \mu_0 sD}{(1-e^{R_m})^2} e^{R_m x/l}\left(\frac{e^{R_m x/l}}{2} - e^{R_m}\right)\Big|_0^l \mathbf{i}_x \\ &= \tfrac{1}{2}\mu_0 K_0^2 sD \mathbf{i}_x \end{aligned} \tag{49}$$

6-4-6 A Linear Induction Machine

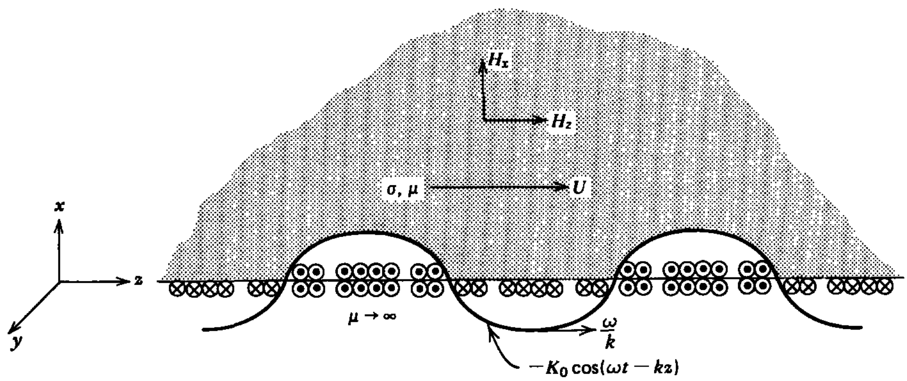
The induced currents in a conductor due to a time varying magnetic field give rise to a force that can cause the conductor to move. This describes a motor. The inverse effect is when we cause a conductor to move through a time varying magnetic field generating a current, which describes a generator.

The linear induction machine shown in Figure 6-29*a* assumes a conductor moves to the right at constant velocity $U\mathbf{i}_z$. Directly below the conductor with no gap is a surface current placed on top of an infinitely permeable medium

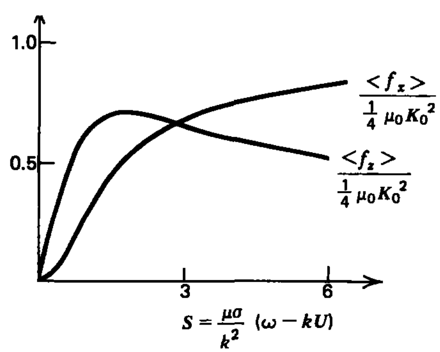
$$\mathbf{K}(t) = -K_0 \cos(\omega t - kz)\mathbf{i}_y = \text{Re}\,[-K_0\, e^{j(\omega t - kz)}\mathbf{i}_y] \tag{50}$$

which is a traveling wave moving to the right at speed ω/k. For $x > 0$, the magnetic field will then have x and z components of the form

$$\begin{aligned} H_z(x, z, t) &= \text{Re}\,[\hat{H}_z(x)\, e^{j(\omega t - kz)}] \\ H_x(x, z, t) &= \text{Re}\,[\hat{H}_x(x)\, e^{j(\omega t - kz)}] \end{aligned} \tag{51}$$



(*a*)



(*b*)

Figure 6-29 (*a*) A traveling wave of surface current induces currents in a conductor that is moving at a velocity U different from the wave speed ω/k. (*b*) The resulting forces can levitate and propel the conductor as a function of the slip S, which measures the difference in speeds of the conductor and traveling wave.

where (10) ($\nabla \cdot \mathbf{B} = 0$) requires these components to be related as

$$\frac{d\hat{H}_x}{dx} - jk\hat{H}_z = 0 \tag{52}$$

The z component of the magnetic diffusion equation of (13) is

$$\frac{d^2\hat{H}_z}{dx^2} - k^2\hat{H}_z = j\mu\sigma(\omega - kU)\hat{H}_z \tag{53}$$

which can also be written as

$$\frac{d^2\hat{H}_z}{dx^2} - \gamma^2\hat{H}_z = 0 \tag{54}$$

where

$$\gamma^2 = k^2(1 + jS), \qquad S = \frac{\mu\sigma}{k^2}(\omega - kU) \tag{55}$$

and S is known as the slip. Solutions of (54) are again exponential but complex because γ is complex:

$$\hat{H}_z = A_1\, e^{\gamma x} + A_2\, e^{-\gamma x} \tag{56}$$

Because $\hat{H}_z$ must remain finite far from the current sheet, $A_1 = 0$, so that using (52) the magnetic field is of the form

$$\hat{\mathbf{H}} = K_0\, e^{-\gamma x}\left(\mathbf{i}_z - \frac{jk}{\gamma}\mathbf{i}_x\right) \tag{57}$$

where we use the fact that the tangential component of $\mathbf{H}$ is discontinuous in the surface current, with $\mathbf{H} = 0$ for $x<0$.

The current density in the conductor is

$$\begin{aligned}\mathbf{J}_f = \nabla \times \mathbf{H} = \mathbf{i}_y\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) \Rightarrow \hat{J}_y &= -jk\hat{H}_x - \frac{d\hat{H}_z}{dx} \\ &= K_0\, e^{-\gamma x}\frac{(\gamma^2 - k^2)}{\gamma} \\ &= \frac{K_0 k^2 jS\, e^{-\gamma x}}{\gamma}\end{aligned} \tag{58}$$

If the conductor and current wave travel at the same speed ($\omega/k = U$), no current is induced as the slip is zero. Currents are only induced if the conductor and wave travel at different velocities. This is the principle of all induction machines.

The force per unit area on the conductor then has x and z components:

$$\mathbf{f}=\int_0^\infty \mathbf{J}\times\mu_0\mathbf{H}\,dx$$
$$=\int_0^\infty \mu_0 J_y(H_z\mathbf{i}_x-H_x\mathbf{i}_z)\,dx \tag{59}$$

These integrations are straightforward but lengthy because first the instantaneous field and current density must be found from (51) by taking the real parts. More important is the time-average force per unit area over a period of excitation:

$$<\mathbf{f}>=\frac{\omega}{2\pi}\int_0^{2\pi/\omega}\mathbf{f}\,dt \tag{60}$$

Since the real part of a complex quantity is equal to half the sum of the quantity and its complex conjugate,

$$\begin{aligned}A&=\text{Re}\,[\hat{A}\,e^{j\omega t}]=\tfrac{1}{2}(\hat{A}\,e^{j\omega t}+\hat{A}^*\,e^{-j\omega t})\\ B&=\text{Re}\,[\hat{B}\,e^{j\omega t}]=\tfrac{1}{2}(\hat{B}\,e^{j\omega t}+\hat{B}^*\,e^{-j\omega t})\end{aligned} \tag{61}$$

the time-average product of two quantities is

$$\frac{\omega}{2\pi}\int_0^{2\pi/\omega}AB\,dt=\frac{1}{4}\frac{\omega}{2\pi}\int_0^{2\pi/\omega}(\hat{A}\hat{B}\,e^{2j\omega t}+\hat{A}^*\hat{B}+\hat{A}\hat{B}^*$$
$$+\hat{A}^*\hat{B}^*\,e^{-2j\omega t})\,dt$$
$$=\tfrac{1}{4}(\hat{A}^*\hat{B}+\hat{A}\hat{B}^*)$$
$$=\tfrac{1}{2}\,\text{Re}\,(\hat{A}\hat{B}^*) \tag{62}$$

which is a formula often used for the time-average power in circuits where A and B are the voltage and current.

Then using (62) in (59), the x component of the time-average force per unit area is

$$<f_x>=\tfrac{1}{2}\,\text{Re}\left(\int_0^\infty \mu_0\hat{J}_y\hat{H}_z^*\,dx\right)$$
$$=\frac{\mu_0}{2}K_0^2k^2S\,\text{Re}\left(\frac{j}{\gamma}\int_0^\infty e^{-(\gamma+\gamma^*)x}\,dx\right)$$
$$=\frac{\mu_0}{2}K_0^2k^2S\,\text{Re}\left(\frac{j}{\gamma(\gamma+\gamma^*)}\right)$$
$$=\frac{1}{4}\frac{\mu_0K_0^2S^2}{[1+S^2+(1+S^2)^{1/2}]}=\tfrac{1}{4}\mu_0K_0^2\left(\frac{\sqrt{1+S^2}-1}{\sqrt{1+S^2}}\right) \tag{63}$$

where the last equalities were evaluated in terms of the slip S from (55).

We similarly compute the time-average shear force per unit area as

$$\begin{aligned} <f_z> &= -\tfrac{1}{2}\,\mathrm{Re}\left(\int_0^\infty \mu_0 J_y H_x^*\,dx\right) \\ &= \frac{\mu_0}{2}\frac{K_0^2 k^3 S}{\gamma\gamma^*}\,\mathrm{Re}\left(\int_0^\infty e^{-(\gamma+\gamma^*)x}\,dx\right) \\ &= \frac{\mu_0}{2}\frac{k^3 K_0^2 S}{\gamma\gamma^*}\,\mathrm{Re}\left(\frac{1}{(\gamma+\gamma^*)}\right) \\ &= \frac{\mu_0 K_0^2 S}{4\sqrt{1+S^2}\,\mathrm{Re}\,(\sqrt{1+jS})} \end{aligned} \tag{64}$$

When the wave speed exceeds the conductor's speed ($\omega/k > U$), the force is positive as $S>0$ so that the wave pulls the conductor along. When $S<0$, the slow wave tends to pull the conductor back as $<f_z> <0$. The forces of (63) and (64), plotted in Figure 6-29*b*, can be used to simultaneously lift and propel a conducting material. There is no force when the wave and conductor travel at the same speed ($\omega/k = U$) as the slip is zero ($S=0$). For large S, the levitating force $<f_x>$ approaches the constant value $\frac{1}{4}\mu_0 K_0^2$ while the shear force approaches zero. There is an optimum value of S that maximizes $<f_z>$. For smaller S, less current is induced while for larger S the phase difference between the imposed and induced currents tend to decrease the time-average force.

6-4-7 Superconductors

In the limit of infinite Ohmic conductivity ($\sigma \to \infty$), the diffusion time constant of (28) becomes infinite while the skin depth of (36) becomes zero. The magnetic field cannot penetrate a perfect conductor and currents are completely confined to the surface.

However, in this limit the Ohmic conduction law is no longer valid and we should use the superconducting constitutive law developed in Section 3-2-2*d* for a single charge carrier:

$$\frac{\partial \mathbf{J}}{\partial t} = \omega_p^2 \varepsilon \mathbf{E} \tag{65}$$

Then for a stationary medium, following the same procedure as in (12) and (13) with the constitutive law of (65), (8)–(11) reduce to

$$\nabla^2 \frac{\partial \mathbf{H}}{\partial t} - \omega_p^2 \varepsilon\mu \frac{\partial \mathbf{H}}{\partial t} = 0 \Rightarrow \nabla^2(\mathbf{H}-\mathbf{H}_0) - \omega_p^2 \varepsilon\mu(\mathbf{H}-\mathbf{H}_0) = 0 \tag{66}$$

where $\mathbf{H}_0$ is the instantaneous magnetic field at $t=0$. If the superconducting material has no initial magnetic field when an excitation is first turned on, then $\mathbf{H}_0=0$.

If the conducting slab in Figure 6-27*a* becomes superconducting, (66) becomes

$$\frac{d^2H_z}{dx^2}-\frac{\omega_p^2}{c^2}H_z=0, \qquad c=\frac{1}{\sqrt{\varepsilon\mu}} \tag{67}$$

where c is the speed of light in the medium.

The solution to (67) is

$$\begin{aligned} H_z &= A_1\,e^{\omega_p x/c}+A_2\,e^{-\omega_p x/c} \\ &= -K_0\cos\omega t\,e^{-\omega_p x/c} \end{aligned} \tag{68}$$

where we use the boundary condition of continuity of tangential $\mathbf{H}$ at $x=0$.

The current density is then

$$\begin{aligned} J_y &= -\frac{\partial H_z}{\partial x} \\ &= -\frac{K_0\omega_p}{c}\cos\omega t\,e^{-\omega_p x/c} \end{aligned} \tag{69}$$

For any frequency ω, including dc ($\omega=0$), the field and current decay with characteristic length:

$$l_c=c/\omega_p \tag{70}$$

Since the plasma frequency ω_p is typically on the order of 10^{15} radian/sec, this characteristic length is very small, $l_c\approx 3\times10^8/10^{15}\approx3\times10^{-7}$ m. Except for this thin sheath, the magnetic field is excluded from the superconductor while the volume current is confined to this region near the interface.

There is one experimental exception to the governing equation in (66), known as the Meissner effect. If an ordinary conductor is placed within a dc magnetic field $\mathbf{H}_0$ and then cooled through the transition temperature for superconductivity, the magnetic flux is pushed out except for a thin sheath of width given by (70). This is contrary to (66), which allows the time-independent solution $\mathbf{H}=\mathbf{H}_0$, where the magnetic field remains trapped within the superconductor. Although the reason is not well understood, superconductors behave as if $\mathbf{H}_0=0$ no matter what the initial value of magnetic field.

6-5 ENERGY STORED IN THE MAGNETIC FIELD

6-5-1 A Single Current Loop

The differential amount of work necessary to overcome the electric and magnetic forces on a charge q moving an incremental distance **ds** at velocity **v** is

$$dW_q = -q(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot \mathbf{ds} \tag{1}$$

(a) Electrical Work

If the charge moves solely under the action of the electrical and magnetic forces with no other forces of mechanical origin, the incremental displacement in a small time dt is related to its velocity as

$$\mathbf{ds} = \mathbf{v}\, dt \tag{2}$$

Then the magnetic field cannot contribute to any work on the charge because the magnetic force is perpendicular to the charge's displacement:

$$dW_q = -q\mathbf{v}\cdot\mathbf{E}\, dt \tag{3}$$

and the work required is entirely due to the electric field. Within a charge neutral wire, the electric field is not due to Coulombic forces but rather arises from Faraday's law. The moving charge constitutes an incremental current element,

$$q\mathbf{v} = i\mathbf{dl} \Rightarrow dW_q = -i\mathbf{E}\cdot\mathbf{dl}\, dt \tag{4}$$

so that the total work necessary to move all the charges in the closed wire is just the sum of the work done on each current element,

$$\begin{aligned} dW &= \oint_L dW_q = -i\, dt \oint_L \mathbf{E}\cdot\mathbf{dl} \\ &= i\, dt \frac{d}{dt}\int_S \mathbf{B}\cdot\mathbf{dS} \\ &= i\, dt \frac{d\Phi}{dt} \\ &= i\, d\Phi \end{aligned} \tag{5}$$

which through Faraday's law is proportional to the change of flux through the current loop. This flux may be due to other currents and magnets (mutual flux) as well as the self-flux due to the current i. Note that the third relation in (5) is just equivalent to the circuit definition of electrical power delivered to the loop:

$$p = \frac{dW}{dt} = i\frac{d\Phi}{dt} = vi \tag{6}$$

All of this energy supplied to accelerate the charges in the wire is stored as no energy is dissipated in the lossless loop and no mechanical work is performed if the loop is held stationary.

(b) Mechanical Work

The magnetic field contributed no work in accelerating the charges. This is not true when the current-carrying wire is itself moved a small vector displacement **ds** requiring us to perform mechanical work,

$$dW = -(i\,\mathbf{dl}\times\mathbf{B})\cdot\mathbf{ds} = i(\mathbf{B}\times\mathbf{dl})\cdot\mathbf{ds}$$
$$= i\mathbf{B}\cdot(\mathbf{dl}\times\mathbf{ds}) \tag{7}$$

where we were able to interchange the dot and the cross using the scalar triple product identity proved in Problem 1-10*a*. We define S_1 as the area originally bounding the loop and S_2 as the bounding area after the loop has moved the distance **ds**, as shown in Figure 6-30. The incremental area $\mathbf{dS}_3$ is then the strip joining the two positions of the loop defined by the bracketed quantity in (7):

$$\mathbf{dS}_3 = \mathbf{dl}\times\mathbf{ds} \tag{8}$$

The flux through each of the contours is

$$\Phi_1 = \int_{S_1} \mathbf{B}\cdot\mathbf{dS}, \qquad \Phi_2 = \int_{S_2} \mathbf{B}\cdot\mathbf{dS} \tag{9}$$

where their difference is just the flux that passes outward through $\mathbf{dS}_3$:

$$d\Phi = \Phi_1 - \Phi_2 = \mathbf{B}\cdot\mathbf{dS}_3 \tag{10}$$

The incremental mechanical work of (7) necessary to move the loop is then identical to (5):

$$dW = i\mathbf{B}\cdot\mathbf{dS}_3 = i\,d\Phi \tag{11}$$

Here there was no change of electrical energy input, with the increase of stored energy due entirely to mechanical work in moving the current loop.

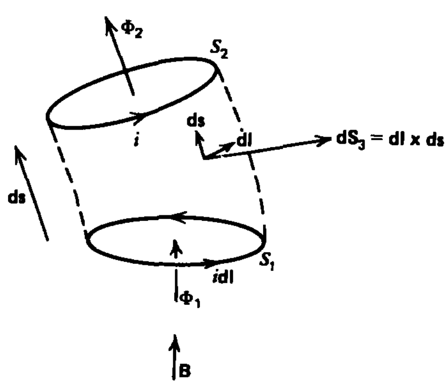


Figure 6-30 The mechanical work necessary to move a current-carrying loop is stored as potential energy in the magnetic field.

6-5-2 Energy and Inductance

If the loop is isolated and is within a linear permeable material, the flux is due entirely to the current, related through the self-inductance of the loop as

$$\Phi = Li \tag{12}$$

so that (5) or (11) can be integrated to find the total energy in a loop with final values of current I and flux Φ:

$$\begin{aligned} W &= \int_0^{\Phi} i\, d\Phi \\ &= \int_0^{\Phi} \frac{\Phi}{L}\, d\Phi \\ &= \frac{1}{2}\frac{\Phi^2}{L} = \frac{1}{2}LI^2 = \frac{1}{2}I\Phi \end{aligned} \tag{13}$$

6-5-3 Current Distributions

The results of (13) are only true for a single current loop. For many interacting current loops or for current distributions, it is convenient to write the flux in terms of the vector potential using Stokes' theorem:

$$\Phi = \int_S \mathbf{B} \cdot \mathbf{dS} = \int_S (\nabla \times \mathbf{A}) \cdot \mathbf{dS} = \oint_L \mathbf{A} \cdot \mathbf{dl} \tag{14}$$

Then each incremental-sized current element carrying a current I with flux $d\Phi$ has stored energy given by (13):

$$dW = \tfrac{1}{2} I\, d\Phi = \tfrac{1}{2}\mathbf{I} \cdot \mathbf{A}\, dl \tag{15}$$

For N current elements, (15) generalizes to

$$\begin{aligned} W &= \tfrac{1}{2}(\mathbf{I}_1 \cdot \mathbf{A}_1\, dl_1 + \mathbf{I}_2 \cdot \mathbf{A}_2\, dl_2 + \cdots + \mathbf{I}_N \cdot \mathbf{A}_N\, dl_N) \\ &= \tfrac{1}{2} \sum_{n=1}^{N} \mathbf{I}_n \cdot \mathbf{A}_n\, dl_n \end{aligned} \tag{16}$$

If the current is distributed over a line, surface, or volume, the summation is replaced by integration:

$$W = \begin{cases} \tfrac{1}{2} \int_L \mathbf{I}_f \cdot \mathbf{A}\, dl & \text{(line current)} \\ \tfrac{1}{2} \int_S \mathbf{K}_f \cdot \mathbf{A}\, dS & \text{(surface current)} \\ \tfrac{1}{2} \int_V \mathbf{J}_f \cdot \mathbf{A}\, dV & \text{(volume current)} \end{cases} \tag{17}$$

Remember that in (16) and (17) the currents and vector potentials are all evaluated at their final values as opposed to (11), where the current must be expressed as a function of flux.

6-5-4 Magnetic Energy Density

This stored energy can be thought of as being stored in the magnetic field. Assuming that we have a free volume distribution of current $\mathbf{J}_f$, we use (17) with Ampere's law to express $\mathbf{J}_f$ in terms of $\mathbf{H}$,

$$W=\frac{1}{2}\int_V \mathbf{J}_f\cdot\mathbf{A}\,dV=\frac{1}{2}\int_V(\nabla\times\mathbf{H})\cdot\mathbf{A}\,dV \tag{18}$$

where the volume V is just the volume occupied by the current. Larger volumes (including all space) can be used in (18), for the region outside the current has $\mathbf{J}_f=0$ so that no additional contributions arise.

Using the vector identity

$$\begin{aligned}\nabla\cdot(\mathbf{A}\times\mathbf{H})&=\mathbf{H}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{H})\\&=\mathbf{H}\cdot\mathbf{B}-\mathbf{A}\cdot(\nabla\times\mathbf{H})\end{aligned} \tag{19}$$

we rewrite (18) as

$$W=\frac{1}{2}\int_V[\mathbf{H}\cdot\mathbf{B}-\nabla\cdot(\mathbf{A}\times\mathbf{H})]\,dV \tag{20}$$

The second term on the right-hand side can be converted to a surface integral using the divergence theorem:

$$\int_V\nabla\cdot(\mathbf{A}\times\mathbf{H})\,dV=\oint_S(\mathbf{A}\times\mathbf{H})\cdot\mathbf{dS} \tag{21}$$

It now becomes convenient to let the volume extend over all space so that the surface is at infinity. If the current distribution does not extend to infinity the vector potential dies off at least as $1/r$ and the magnetic field as $1/r^2$. Then, even though the area increases as r^2, the surface integral in (21) decreases at least as $1/r$ and thus is zero when S is at infinity. Then (20) becomes simply

$$W=\frac{1}{2}\int_V\mathbf{H}\cdot\mathbf{B}\,dV=\frac{1}{2}\int_V\mu H^2\,dV=\frac{1}{2}\int_V\frac{B^2}{\mu}\,dV \tag{22}$$

where the volume V now extends over all space. The magnetic energy density is thus

$$w=\frac{1}{2}\mathbf{H}\cdot\mathbf{B}=\frac{1}{2}\mu H^2=\frac{1}{2}\frac{B^2}{\mu} \tag{23}$$

These results are only true for linear materials where μ does not depend on the magnetic field, although it can depend on position.

For a single coil, the total energy in (22) must be identical to (13), which gives us an alternate method to calculating the self-inductance from the magnetic field.

6-5-5 The Coaxial Cable

(a) External Inductance

A typical cable geometry consists of two perfectly conducting cylindrical shells of radii a and b and length l, as shown in Figure 6-31. An imposed current I flows axially as a surface current in opposite directions on each cylinder. We neglect fringing field effects near the ends so that the magnetic field is the same as if the cylinder were infinitely long. Using Ampere's law we find that

$$H_\phi = \frac{I}{2\pi r}, \quad a < r < b \tag{24}$$

The total magnetic flux between the two conductors is

$$\begin{aligned}\Phi &= \int_a^b \mu_0 H_\phi l\, dr \\ &= \frac{\mu_0 I l}{2\pi} \ln \frac{b}{a}\end{aligned} \tag{25}$$

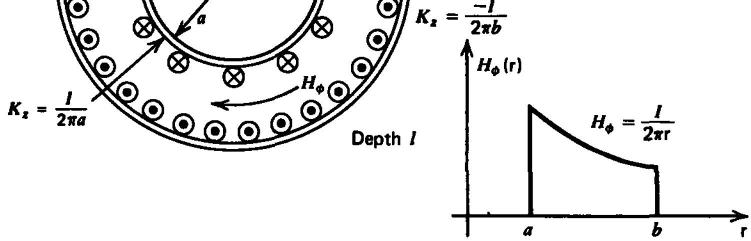


Figure 6-31 The magnetic field between two current-carrying cylindrical shells forming a coaxial cable is confined to the region between cylinders.

giving the self-inductance as

$$L = \frac{\Phi}{I} = \frac{\mu_0 l}{2\pi} \ln \frac{b}{a} \tag{26}$$

The same result can just as easily be found by computing the energy stored in the magnetic field

$$W = \tfrac{1}{2} L I^2 = \tfrac{1}{2}\mu_0 \int_a^b H_\phi^2 2\pi r l \, dr$$

$$= \frac{\mu_0 l I^2}{4\pi} \ln \frac{b}{a} \Rightarrow L = \frac{2W}{I^2} = \frac{\mu_0 l \ln (b/a)}{2\pi} \tag{27}$$

(b) Internal Inductance

If the inner cylinder is now solid, as in Figure 6-32, the current at low enough frequencies where the skin depth is much larger than the radius, is uniformly distributed with density

$$J_z = \frac{I}{\pi a^2} \tag{28}$$

so that a linearly increasing magnetic field is present within the inner cylinder while the outside magnetic field is

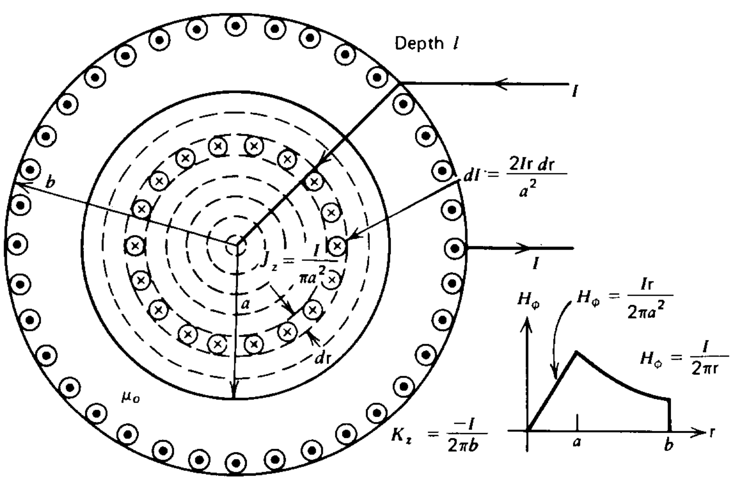


Figure 6-32 At low frequencies the current in a coaxial cable is uniformly distributed over the solid center conductor so that the internal magnetic field increases linearly with radius. The external magnetic field remains unchanged. The inner cylinder can be thought of as many incremental cylindrical shells of thickness dr carrying a fraction of the total current. Each shell links its own self-flux as well as the mutual flux of the other shells of smaller radius. The additional flux within the current-carrying conductor results in the internal inductance of the cable.

unchanged from (24):

$$H_\phi = \begin{cases} \dfrac{I\mathrm{r}}{2\pi a^2}, & 0<\mathrm{r}<a \\ \dfrac{I}{2\pi \mathrm{r}}, & a<\mathrm{r}<b \end{cases} \tag{29}$$

The self-inductance cannot be found using the flux per unit current definition for a current loop since the current is not restricted to a thin filament. The inner cylinder can be thought of as many incremental cylindrical shells, as in Figure 6-32, each linking its own self-flux as well as the mutual flux of the other shells of smaller radius. Note that each shell is at a different voltage due to the differences in enclosed flux, although the terminal wires that are in a region where the magnetic field is negligible have a well-defined unique voltage difference.

The easiest way to compute the self-inductance as seen by the terminal wires is to use the energy definition of (22):

$$\begin{aligned} W &= \tfrac{1}{2}\mu_0 \int_0^b H_\phi^2 2\pi l\mathrm{r}\, d\mathrm{r} \\ &= \pi l \mu_0 \left[\int_0^a \left(\frac{I\mathrm{r}}{2\pi a^2}\right)^2 \mathrm{r}\, d\mathrm{r} + \int_a^b \left(\frac{I}{2\pi \mathrm{r}}\right)^2 \mathrm{r}\, d\mathrm{r}\right] \\ &= \frac{\mu_0 l I^2}{4\pi}\left(\frac{1}{4} + \ln\frac{b}{a}\right) \end{aligned} \tag{30}$$

which gives the self-inductance as

$$L = \frac{2W}{I^2} = \frac{\mu_0 l}{2\pi}\left(\frac{1}{4} + \ln\frac{b}{a}\right) \tag{31}$$

The additional contribution of $\mu_0 l/8\pi$ is called the internal inductance and is due to the flux within the current-carrying conductor.

6-5-6 Self-Inductance, Capacitance, and Resistance

We can often save ourselves further calculations for the external self-inductance if we already know the capacitance or resistance for the same two-dimensional geometry composed of highly conducting electrodes with no internal inductance contribution. For the arbitrary geometry shown in Figure 6-33 of depth d, the capacitance, resistance, and inductance are defined as the ratios of line and surface integrals:

$$C = \frac{\varepsilon d \oint_S \mathbf{E} \cdot \mathbf{n}_s \, ds}{\int_L \mathbf{E} \cdot \mathbf{dl}}$$

$$R = \frac{\int_L \mathbf{E} \cdot \mathbf{dl}}{\sigma d \oint_S \mathbf{E} \cdot \mathbf{n}_s \, ds} \tag{32}$$

$$L = \frac{\mu d \int_L \mathbf{H} \cdot \mathbf{n}_l \, dl}{\oint_S \mathbf{H} \cdot \mathbf{ds}}$$

Because the homogeneous region between electrodes is charge and current free, both the electric and magnetic fields can be derived from a scalar potential that satisfies Laplace's equation. However, the electric field must be incident normally onto the electrodes while the magnetic field is incident tangentially so that **E** and **H** are perpendicular everywhere, each being along the potential lines of the other. This is accounted for in (32) and Figure 6-33 by having $\mathbf{n}_s \, ds$ perpendicular to **ds** and $\mathbf{n}_l \, dl$ perpendicular to **dl**. Then since C, R, and L are independent of the field strengths, we can take **E** and **H** to both have unit magnitude so that in the products of LC and L/R the line and surface integrals cancel:

$$\begin{aligned} LC &= \varepsilon\mu d^2 = d^2/c^2, \qquad c = 1/\sqrt{\varepsilon\mu} \\ L/R &= \mu\sigma d^2, \qquad RC = \varepsilon/\sigma \end{aligned} \tag{33}$$

These products are then independent of the electrode geometry and depend only on the material parameters and the depth of the electrodes.

We recognize the L/R ratio to be proportional to the magnetic diffusion time of Section 6-4-3 while RC is just the charge relaxation time of Section 3-6-1. In Chapter 8 we see that the $\sqrt{LC}$ product is just equal to the time it takes an electromagnetic wave to propagate a distance d at the speed of light c in the medium.

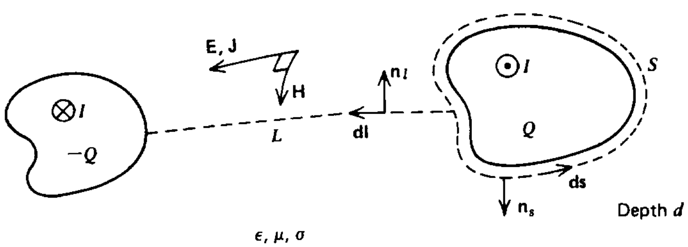


Figure 6-33 The electric and magnetic fields in the two-dimensional homogeneous charge and current-free region between hollow electrodes can be derived from a scalar potential that obeys Laplace's equation. The electric field lines are along the magnetic potential lines and vice versa so **E** and **H** are perpendicular. The inductance-capacitance product is then a constant.

6-6 THE ENERGY METHOD FOR FORCES

6-6-1 The Principle of Virtual Work

In Section 6-5-1 we calculated the energy stored in a current-carrying loop by two methods. First we calculated the electric energy input to a loop with no mechanical work done. We then obtained the same answer by computing the mechanical work necessary to move a current-carrying loop in an external field with no further electrical inputs. In the most general case, an input of electrical energy can result in stored energy dW and mechanical work by the action of a force f_x causing a small displacement dx:

$$i\,d\Phi = dW + f_x\,dx \tag{1}$$

If we knew the total energy stored in the magnetic field as a function of flux and position, the force is simply found as

$$f_x = -\frac{\partial W}{\partial x}\bigg|_{\Phi} \tag{2}$$

We can easily compute the stored energy by realizing that no matter by what process or order the system is assembled, if the final position x and flux Φ are the same, the energy is the same. Since the energy stored is independent of the order that we apply mechanical and electrical inputs, we choose to mechanically assemble a system first to its final position x with no electrical excitations so that $\Phi = 0$. This takes no work as with zero flux there is no force of electrical origin. Once the system is mechanically assembled so that its position remains constant, we apply the electrical excitation to bring the system to its final flux value. The electrical energy required is

$$W = \int_{x\ \text{const}} i\,d\Phi \tag{3}$$

For linear materials, the flux and current are linearly related through the inductance that can now be a function of x because the inductance depends on the geometry:

$$i = \Phi/L(x) \tag{4}$$

Using (4) in (3) allows us to take the inductance outside the integral because x is held constant so that the inductance is also constant:

$$\begin{aligned} W &= \frac{1}{L(x)}\int_0^{\Phi} \Phi\,d\Phi \\ &= \frac{\Phi^2}{2L(x)} = \tfrac{1}{2}L(x)i^2 \end{aligned} \tag{5}$$

The stored energy is the same as found in Section 6-5-2 even when mechanical work is included and the inductance varies with position.

To find the force on the moveable member, we use (2) with the energy expression in (5), which depends only on flux and position:

$$\begin{aligned} f_x &= -\frac{\partial W}{\partial x}\bigg|_{\Phi} \\ &= -\frac{\Phi^2}{2}\frac{d[1/L(x)]}{dx} \\ &= \frac{1}{2}\frac{\Phi^2}{L^2(x)}\frac{dL(x)}{dx} \\ &= \tfrac{1}{2}i^2\frac{dL(x)}{dx} \end{aligned} \tag{6}$$

6-6-2 Circuit Viewpoint

This result can also be obtained using a circuit description with the linear flux-current relation of (4):

$$\begin{aligned} v &= \frac{d\Phi}{dt} \\ &= L(x)\frac{di}{dt} + i\frac{dL(x)}{dt} \\ &= L(x)\frac{di}{dt} + i\frac{dL(x)}{dx}\frac{dx}{dt} \end{aligned} \tag{7}$$

The last term, proportional to the speed of the moveable member, just adds to the usual inductive voltage term. If the geometry is fixed and does not change with time, there is no electromechanical coupling term.

The power delivered to the system is

$$p = vi = i\frac{d}{dt}[L(x)i] \tag{8}$$

which can be expanded as

$$p = \frac{d}{dt}(\tfrac{1}{2}L(x)i^2) + \tfrac{1}{2}i^2\frac{dL(x)}{dx}\frac{dx}{dt} \tag{9}$$

This is in the form

$$p = \frac{dW}{dt} + f_x\frac{dx}{dt}, \quad \begin{cases} W = \tfrac{1}{2}L(x)i^2 \\ f_x = \tfrac{1}{2}i^2\dfrac{dL(x)}{dx} \end{cases} \tag{10}$$

which states that the power delivered to the inductor is equal to the sum of the time rate of energy stored and mechanical power performed on the inductor. This agrees with the energy method approach. If the inductance does not change with time because the geometry is fixed, all the input power is stored as potential energy W.

Example 6-2 MAGNETIC FIELDS AND FORCES

(a) Relay

Find the force on the moveable slug in the magnetic circuit shown in Figure 6-34.

SOLUTION

It is necessary to find the inductance of the system as a function of the slug's position so that we can use (6). Because of the infinitely permeable core and slug, the **H** field is non-zero only in the air gap of length x. We use Ampere's law to obtain

$$H = NI/x$$

The flux through the gap

$$\Phi = \mu_0 NIA/x$$

is equal to the flux through each turn of the coil yielding the inductance as

$$L(x) = \frac{N\Phi}{I} = \frac{\mu_0 N^2 A}{x}$$

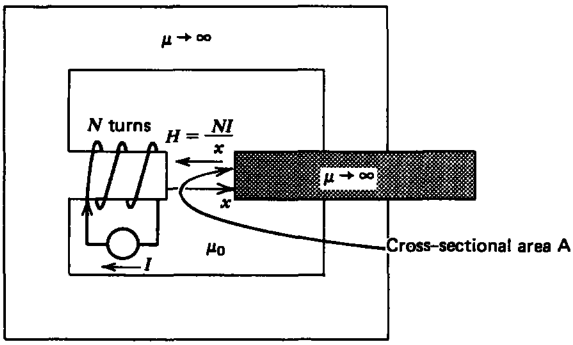


Figure 6-34 The magnetic field exerts a force on the moveable member in the relay pulling it into the magnetic circuit.

The force is then

$$f_x = \tfrac{1}{2} I^2 \frac{dL(x)}{dx}$$

$$= -\frac{\mu_o N^2 A I^2}{2x^2}$$

The minus sign means that the force is opposite to the direction of increasing x, so that the moveable piece is attracted to the coil.

(b) One Turn Loop

Find the force on the moveable upper plate in the one turn loop shown in Figure 6-35.

SOLUTION

The current distributes itself uniformly as a surface current $K = I/D$ on the moveable plate. If we neglect nonuniform field effects near the corners, the **H** field being tangent to the conductors just equals K:

$$H_z = I/D$$

The total flux linked by the current source is then

$$\Phi = \mu_0 H_z x l$$

$$= \frac{\mu_0 x l}{D} I$$

which gives the inductance as

$$L(x) = \frac{\Phi}{I} = \frac{\mu_0 x l}{D}$$

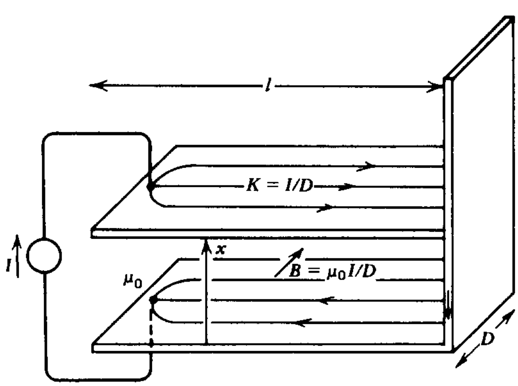


Figure 6-35 The magnetic force on a current-carrying loop tends to expand the loop.

The force is then constant

$$f_x = \frac{1}{2}I^2 \frac{dL(x)}{dx}$$
$$= \frac{1}{2}\frac{\mu_0 l I^2}{D}$$

6-6-3 Magnetization Force

A material with permeability μ is partially inserted into the magnetic circuit shown in Figure 6-36. With no free current in the moveable material, the x-directed force density from Section 5-8-1 is

$$\begin{aligned} F_x &= \mu_0(\mathbf{M}\cdot\nabla)H_x \\ &= (\mu-\mu_0)(\mathbf{H}\cdot\nabla)H_x \\ &= (\mu-\mu_0)\left(H_x\frac{\partial H_x}{\partial x}+H_y\frac{\partial H_x}{\partial y}\right) \end{aligned} \tag{11}$$

where we neglect variations with z. This force arises in the fringing field because within the gap the magnetic field is essentially uniform:

$$H_y = NI/s \tag{12}$$

Because the magnetic field in the permeable block is curl free,

$$\nabla\times\mathbf{H}=0\Rightarrow\frac{\partial H_x}{\partial y}=\frac{\partial H_y}{\partial x} \tag{13}$$

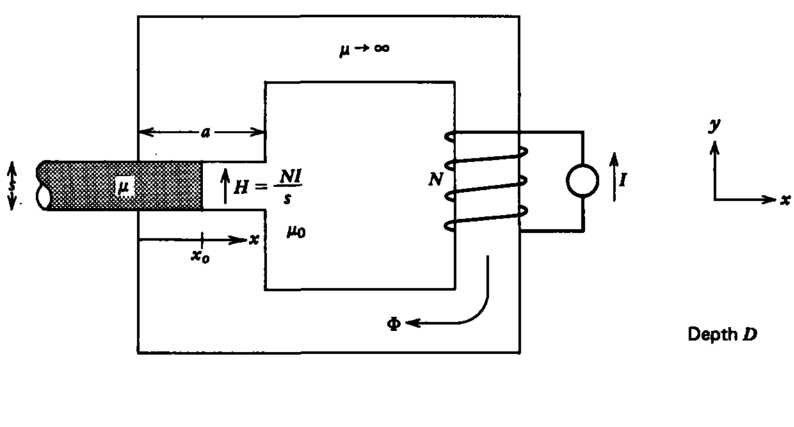


Figure 6-36 · A permeable material tends to be pulled into regions of higher magnetic field.

(11) can be rewritten as

$$F_x = \frac{(\mu-\mu_0)}{2}\frac{\partial}{\partial x}(H_x^2+H_y^2) \tag{14}$$

The total force is then

$$\begin{aligned} f_x &= sD\int_{-\infty}^{x_0} F_x\,dx \\ &= \frac{(\mu-\mu_0)}{2} sD(H_x^2+H_y^2)\Big|_{-\infty}^{x_0} \\ &= \frac{(\mu-\mu_0)}{2}\frac{N^2I^2D}{s} \end{aligned} \tag{15}$$

where the fields at $x=-\infty$ are zero and the field at $x=x_0$ is given by (12). High permeability material is attracted to regions of stronger magnetic field. It is this force that causes iron materials to be attracted towards a magnet. Diamagnetic materials ($\mu<\mu_0$) will be repelled.

This same result can more easily be obtained using (6) where the flux through the gap is

$$\Phi = HD[\mu x+\mu_0(a-x)] = \frac{NID}{s}[(\mu-\mu_0)x+a\mu_0] \tag{16}$$

so that the inductance is

$$L = \frac{N\Phi}{I} = \frac{N^2D}{s}[(\mu-\mu_0)x+a\mu_0] \tag{17}$$

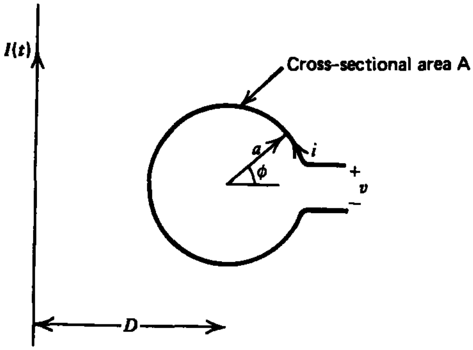
Then the force obtained using (6) agrees with (15)

$$\begin{aligned} f_x &= \tfrac{1}{2}I^2\frac{dL(x)}{dx} \\ &= \frac{(\mu-\mu_0)}{2s}N^2I^2D \end{aligned} \tag{18}$$

PROBLEMS

Section 6-1

1. A circular loop of radius a with Ohmic conductivity σ and cross-sectional area A has its center a small distance D away from an infinitely long time varying current.



(a) Find the mutual inductance M and resistance R of the loop. **Hint:**

$$\int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left[\frac{\sqrt{a^2-b^2}\tan(x/2)}{a+b}\right],$$

$$\int \frac{r\,dr}{\sqrt{D^2-r^2}} = -\sqrt{D^2-r^2}$$

(b) This loop is stationary and has a self-inductance L. What is the time dependence of the induced short circuit current when the line current is instantaneously stepped on to a dc level I at $t=0$?

(c) Repeat (b) when the line current has been on a long time and is suddenly turned off at $t=T$.

(d) If the loop has no resistance and is moving with radial velocity $v_r = dr/dt$, what is the short circuit current and open circuit voltage for a dc line current?

(e) What is the force on the loop when it carries a current i? **Hint:**

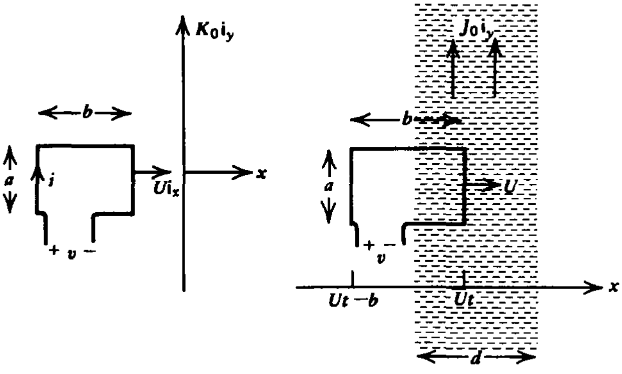
$$\int \frac{\cos\phi\, d\phi}{D+a\cos\phi} = -\frac{1}{a}\sin^{-1}[\cos\phi] + \frac{D}{a\sqrt{D^2-a^2}}\sin^{-1}\left(\frac{a+D\cos\phi}{D+a\cos\phi}\right)$$

2. A rectangular loop at the far left travels with constant velocity $U\mathbf{i}_x$ towards and through a dc surface current sheet $K_0\mathbf{i}_y$ at $x=0$. The right-hand edge of the loop first reaches the current sheet at $t=0$.

(a) What is the loop's open circuit voltage as a function of time?

(b) What is the short circuit current if the loop has self-inductance L and resistance R?

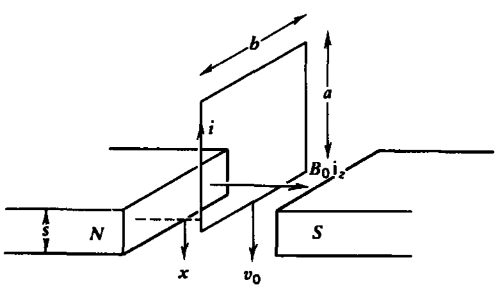
(c) Find the open circuit voltage if the surface current is replaced by a fluid with uniformly distributed volume current. The current is undisturbed as the loop passes through.



(a) (c)

Specifically consider the case when $d > b$ and then sketch the results when $d = b$ and $d < b$. The right edge of the current loop reaches the volume current at $t = 0$.

3. A short circuited rectangular loop of mass m and self-inductance L is dropped with initial velocity $v_0\mathbf{i}_x$ between the pole faces of a magnet that has a concentrated uniform magnetic field $B_0\mathbf{i}_z$. Neglect gravity.



(a) What is the imposed flux through the loop as a function of the loop's position x $(0 < x < s)$ within the magnet?

(b) If the wire has conductivity σ and cross-sectional area A, what equation relates the induced current i in the loop and the loop's velocity?

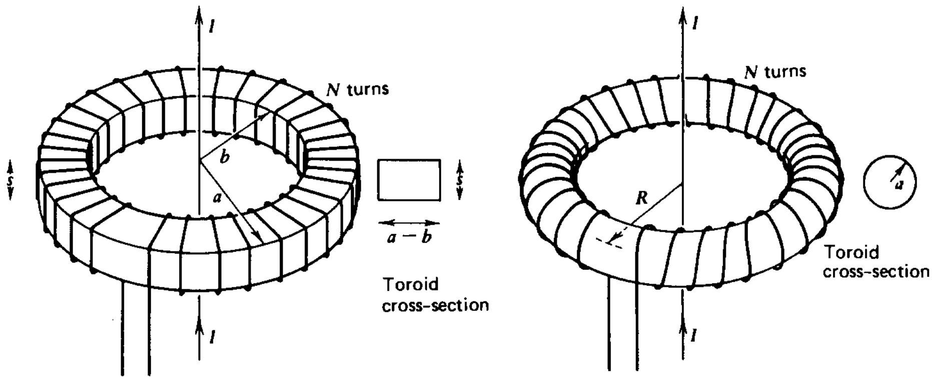
(c) What is the force on the loop in terms of i? Obtain a single equation for the loop's velocity. (**Hint:** Let $\omega_0^2 = B_0^2b^2/mL$, $\alpha = R/L$.)

(d) How does the loop's velocity and induced current vary with time?

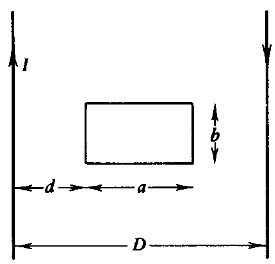
(e) If $\sigma \to \infty$, what minimum initial velocity is necessary for the loop to pass through the magnetic field?

4. Find the mutual inductance between the following currents:

(a) Toroidal coil of rectangular or circular cross section



(a)



(b)

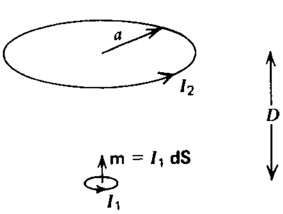
coaxially centered about an infinitely long line current. **Hint:**

$$\int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}} \tan^{-1}\left\{\frac{\sqrt{a^2-b^2}\tan(x/2)}{a+b}\right\},$$

$$\int \frac{r\,dr}{\sqrt{R^2-r^2}} = -\sqrt{R^2-r^2}$$

(b) A very long rectangular current loop, considered as two infinitely long parallel line currents, a distance D apart, carrying the same current I in opposite directions near a small rectangular loop of width a, which is a distance d away from the left line current. Consider the cases $d+a<D$, $d<D<d+a$, and $d>D$.

5. A circular loop of radius a is a distance D above a point magnetic dipole of area dS carrying a current I_1.



(a) What is the vector potential due to the dipole at all points on the circular loop? (**Hint:** See Section 5-5-1.)

(b) How much flux of the dipole passes through the circular loop?

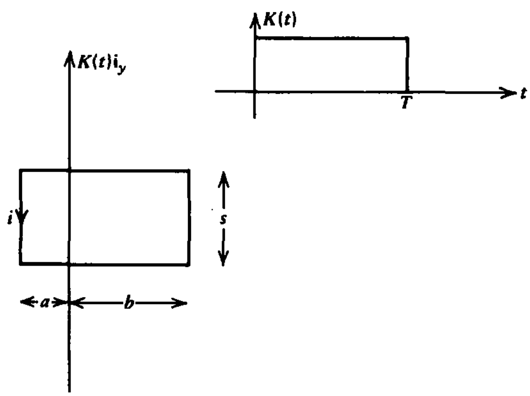
(c) What is the mutual inductance between the dipole and the loop?

(d) If the loop carries a current I_2, what is the magnetic field due to I_2 at the position of the point dipole? (**Hint:** See Section 5-2-4*a*.)

(e) How much flux due to I_2 passes through the magnetic dipole?

(f) What is the mutual inductance? Does your result agree with (c)?

6. A small rectangular loop with self-inductance L, Ohmic conductivity σ, and cross-sectional area A straddles a current sheet.



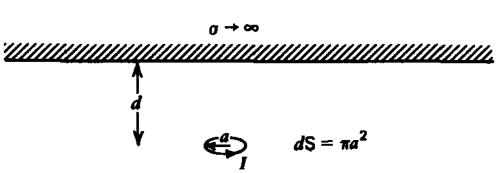
(a) The current sheet is instantaneously turned on to a dc level $K_0\mathbf{i}_y$ at $t=0$. What is the induced loop current?

(b) After a long time T the sheet current is instantaneously set to zero. What is the induced loop current?

(c) What is the induced loop current if the current sheet varies sinusoidally with time as $K_0 \cos \omega t \mathbf{i}_y$.

7. A point magnetic dipole with area dS lies a distance d below a perfectly conducting plane of infinite extent. The dipole current I is instantaneously turned on at $t=0$.

(a) Using the method of images, find the magnetic field everywhere along the conducting plane. (**Hint:** $\mathbf{i}_r \cdot \mathbf{i}_r = \sin\theta$,



$\mathbf{i}_\theta \cdot \mathbf{i}_r = \cos\theta$.)

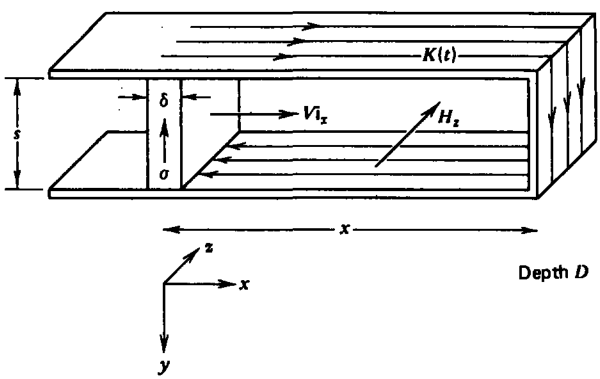
(b) What is the surface current distribution?

(c) What is the force on the plane? **Hint:**

$$\int \frac{r^3\,dr}{(r^2+d^2)^5} = -\frac{(r^2+d^2/4)}{6(r^2+d^2)^4}$$

(d) If the plane has a mass M in the gravity field g, what current I is necessary to just lift the conductor? Evaluate for $M = 10^{-3}$ kg, $d = 10^{-2}$ m, and $a = 10^{-3}$ m.

8. A thin block with Ohmic conductivity σ and thickness δ moves with constant velocity $V\mathbf{i}_x$ between short circuited perfectly conducting parallel plates. An initial surface current K_0 is imposed at $t = 0$ when $x = x_0$, but the source is then removed.



(a) The surface current on the plates $K(t)$ will vary with time. What is the magnetic field in terms of $K(t)$? Neglect fringing effects.

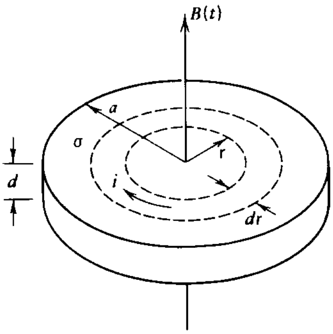
(b) Because the moving block is so thin, the current is uniformly distributed over the thickness δ. Using Faraday's law, find $K(t)$ as a function of time.

(c) What value of velocity will just keep the magnetic field constant with time until the moving block reaches the end?

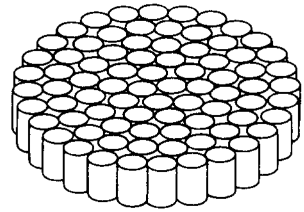
(d) What happens to the magnetic field for larger and smaller velocities?

9. A thin circular disk of radius a, thickness d, and conductivity σ is placed in a uniform time varying magnetic field $B(t)$.

(a) Neglecting the magnetic field of the eddy currents, what is the current induced in a thin circular filament at radius r of thickness dr.



(a)



(d)

(b) What power is dissipated in this incremental current loop?

(c) How much power is dissipated in the whole disk?

(d) If the disk is instead cut up into N smaller circular disks with negligible wastage, what is the approximate radius of each smaller disk?

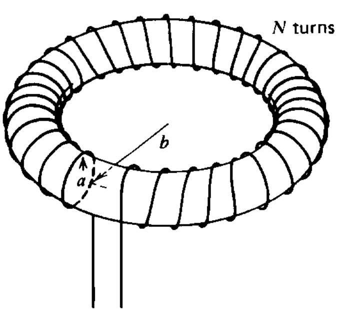
(e) If these N smaller disks are laminated together to form a thin disk of closely packed cylindrical wires, what is the power dissipated?

Section 6-2

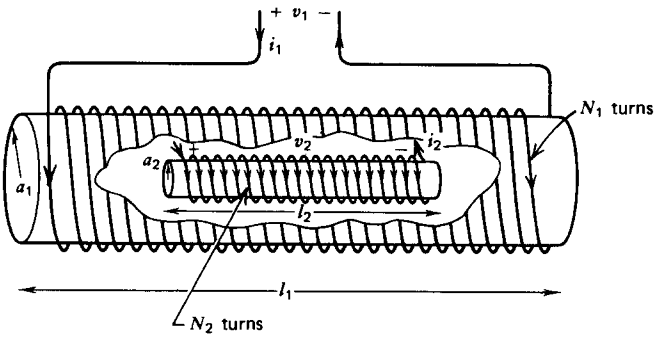
10. Find the self-inductance of an N turn toroidal coil of circular cross-sectional radius a and mean radius b. **Hint:**

$$\int \frac{d\theta}{b + \mathrm{r}\cos\theta} = \frac{2}{\sqrt{b^2 - \mathrm{r}^2}} \tan^{-1} \frac{\sqrt{b^2 - \mathrm{r}^2}\tan(\theta/2)}{b + \mathrm{r}}$$

$$\int \frac{\mathrm{r}\, d\mathrm{r}}{\sqrt{b^2 - \mathrm{r}^2}} = -\sqrt{b^2 - \mathrm{r}^2}$$



11. A large solenoidal coil of long length l_1, radius a_1, and number of turns N_1 coaxially surrounds a smaller coil of long length l_2, radius a_2, and turns N_2.



(a) Neglecting fringing field effects find the self-inductances and mutual inductances of each coil. (**Hint:** Assume the magnetic field is essentially uniform within the cylinders.)

(b) What is the voltage across each coil in terms of i_1 and i_2?

(c) If the coils are connected in series so that $i_1 = i_2$ with the fluxes of each coil in the same direction, what is the total self-inductance?

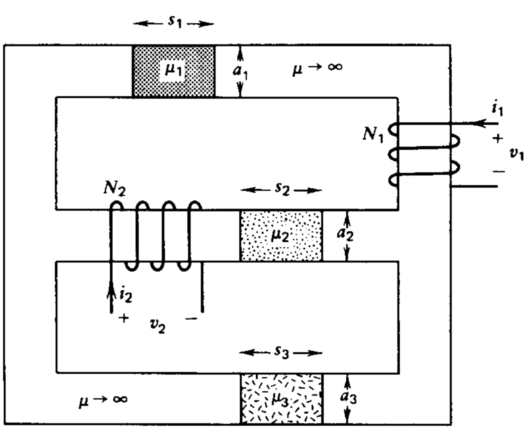
(d) Repeat (c) if the series connection is reversed so that $i_1 = -i_2$ and the fluxes due to each coil are in opposite directions.

(e) What is the total self-inductance if the coils are connected in parallel so that $v_1 = v_2$ or $v_1 = -v_2$?

12. The iron core shown with infinite permeability has three gaps filled with different permeable materials.

(a) What is the equivalent magnetic circuit?

(b) Find the magnetic flux everywhere in terms of the gap reluctances.

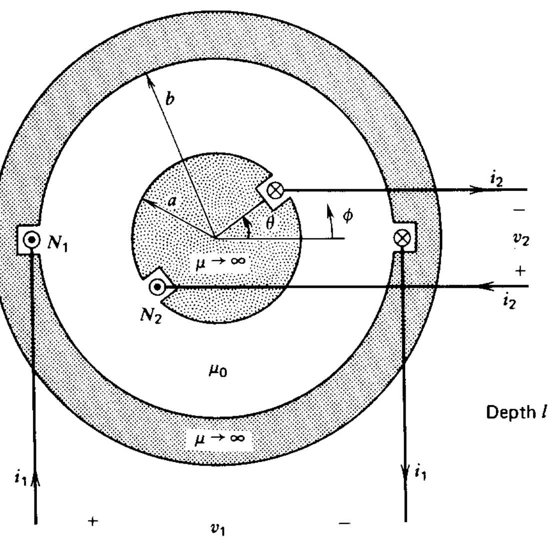


Depth D

(c) What is the total magnetic flux through each winding?

(d) What is the self-inductance and mutual inductance of each winding?

13. A cylindrical shell of infinite permeability, length l and inner radius b coaxially surrounds a solid cylinder also with infinite permeability and length l but with smaller radius a so that there is a small gap $g = b - a$. An N_1 turn coil carrying a current I_1 is placed within two slots on the inner surface of the outer cylinder.



(a) What is the magnetic field everywhere? Neglect all radial variations in the narrow air gap. (**Hint:** Separately consider $0 < \phi < \pi$ and $\pi < \phi < 2\pi$.)

(b) What is the self-inductance of the coil?

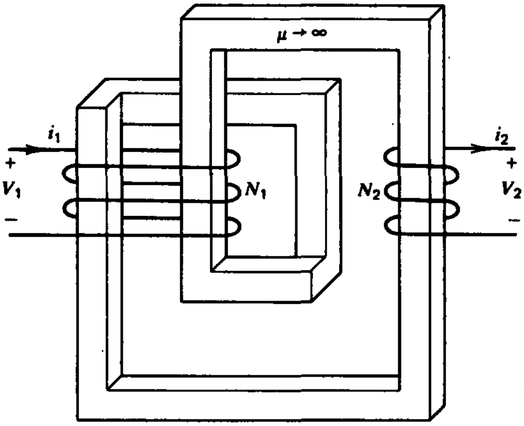
(c) A second coil with N_2 turns carrying a current I_2 is placed in slots on the inner cylinder that is free to rotate. When the rotor is at angle θ, what is the total magnetic field due to currents I_1 and I_2? (**Hint:** Separately consider $0 < \phi < \theta$, $\theta < \phi < \pi$, $\pi < \phi < \pi + \theta$, and $\pi + \theta < \phi < 2\pi$.)

(d) What is the self-inductance and mutual inductance of coil 2 as a function of θ?

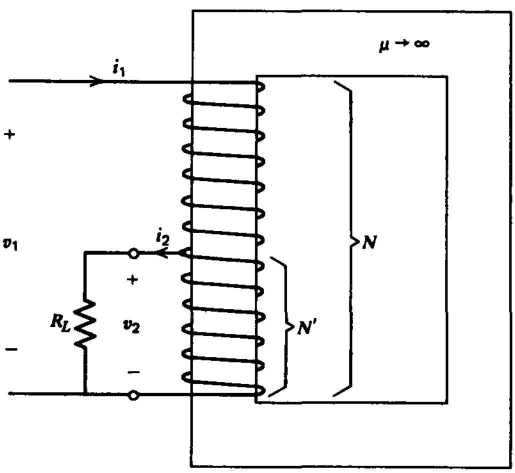
(e) What is the torque on the rotor coil?

14. (a) What is the ratio of terminal voltages and currents for the odd twisted ideal transformer shown?

(b) A resistor R_L is placed across the secondary winding (v_2, i_2). What is the impedance as seen by the primary winding?



15. An N turn coil is wound onto an infinitely permeable magnetic core. An autotransformer is formed by connecting a tap of N' turns.

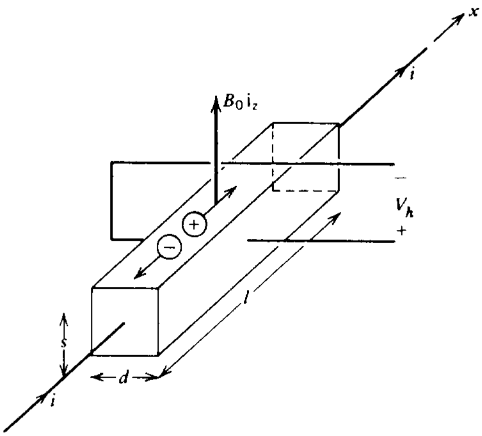


(a) What are the terminal voltage (v_2/v_1) and current (i_2/i_1) ratios?

(b) A load resistor R_L is connected across the terminals of the tap. What is the impedance as seen by the input terminals?

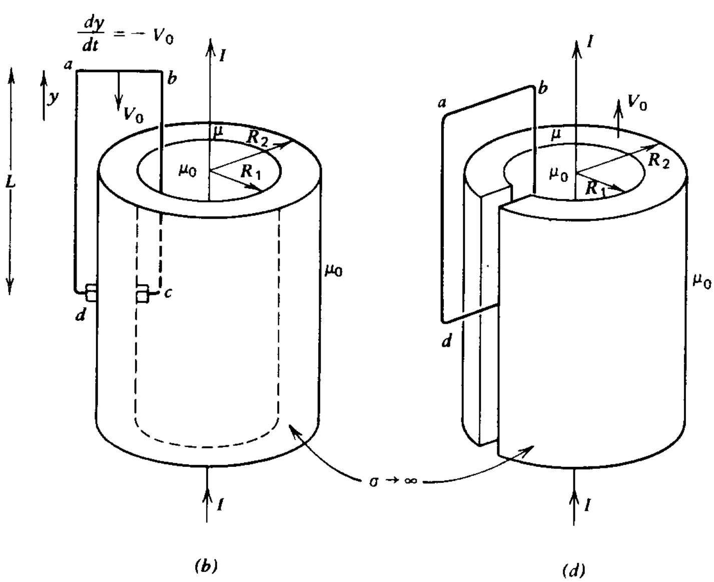
Section 6-3

16. A conducting material with current density $J_x\mathbf{i}_x$ has two species of charge carriers with respective mobilities μ_+ and μ_- and number densities n_+ and n_-. A magnetic field $B_0\mathbf{i}_z$ is imposed perpendicular to the current flow.



(a) What is the open circuit Hall voltage? (**Hint:** The transverse current of each carrier must be zero.)
(b) What is the short circuit Hall current?

17. A highly conducting hollow iron cylinder with permeability μ and inner and outer radii R_1 and R_2 is concentric to an infinitely long dc line current (adapted from L. V. Bewley, *Flux Linkages and Electromagnetic Induction.* Macmillan, New York, 1952, pp. 71–77).



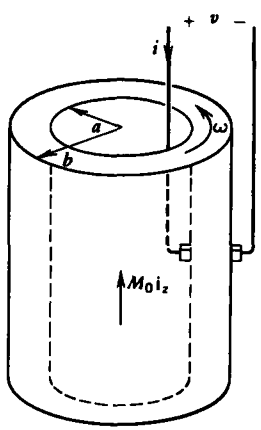
(a) What is the magnetic flux density everywhere? Find the electromotive force (EMF) of the loop for each of the following cases.

(b) A highly conducting circuit *abcd* is moving downward with constant velocity V_0 making contact with the surfaces of the cylinders via sliding brushes. The circuit is completed from c to d via the iron cylinder.

(c) Now the circuit remains stationary and the iron cylinder moves upwards at velocity V_0.

(d) Now a thin axial slot is cut in the cylinder so that it can slip by the complete circuit *abcd*, which remains stationary as the cylinder moves upwards at speed V_0. The brushes are removed and a highly conducting wire completes the c–d path.

18. A very long permanently magnetized cylinder $M_0\mathbf{i}_z$ rotates on a shaft at constant angular speed ω. The inner and outer surfaces at $r = a$ and $r = b$ are perfectly conducting, so that brushes can make electrical contact.



(a) If the cylinder is assumed very long compared to its radius, what are the approximate values of **B** and **H** in the magnet?

(b) What is the open circuit voltage?

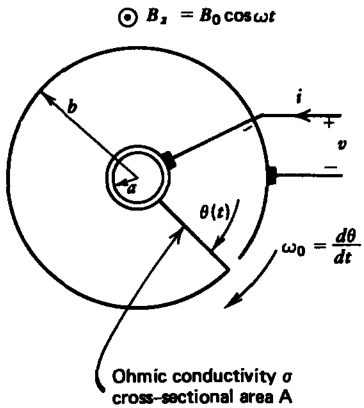
(c) If the magnet has an Ohmic conductivity σ, what is the equivalent circuit of this generator?

(d) What torque is required to turn the magnet if the terminals are short circuited?

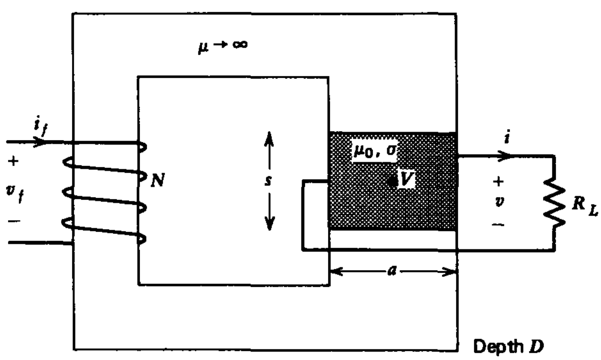
19. A single spoke wheel has a perfectly conducting cut rim. The spoke has Ohmic conductivity σ and cross-sectional area A. The wheel rotates at constant angular speed ω_0 in a sinusoidally varying magnetic field $B_z = B_0 \cos \omega t$.

(a) What is the open circuit voltage and short circuit current?

(b) What is the equivalent circuit?



20. An MHD machine is placed within a magnetic circuit.



(a) A constant dc current $i_f = I_0$ is applied to the N turn coil. How much power is delivered to the load resistor R_L?

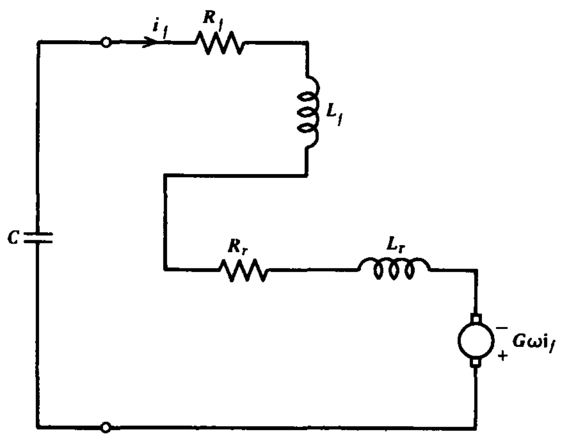
(b) The MHD machine and load resistor R_L are now connected in series with the N turn coil that has a resistance R_f. No current is applied. For what minimum flow speed can the MHD machine be self-excited?

21. The field winding of a homopolar generator is connected in series with the rotor terminals through a capacitor C. The rotor is turned at constant speed ω.

(a) For what minimum value of rotor speed is the system self-excited?

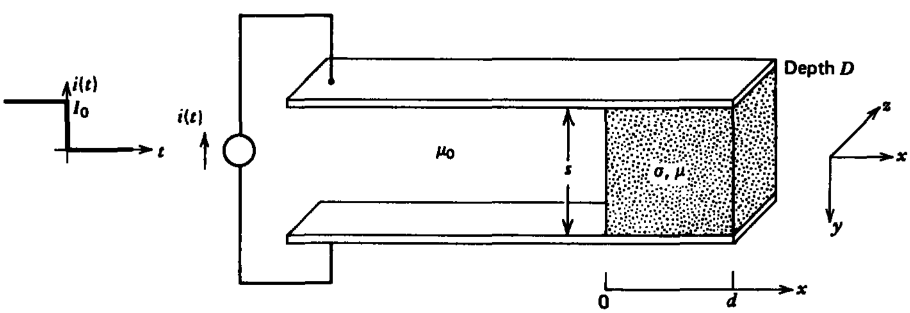
(b) For the self-excited condition of (a) what range of values of C will result in dc self-excitation or in ac self-excitation?

(c) What is the frequency for ac self-excitation?



Section 6-4

22. An Ohmic block separates two perfectly conducting parallel plates. A dc current that has been applied for a long time is instantaneously turned off at $t=0$.



(a) What are the initial and final magnetic field distributions? What are the boundary conditions?

(b) What are the transient magnetic field and current distributions?

(c) What is the force on the block as a function of time?

23. A block of Ohmic material is placed within a magnetic circuit. A step current I_0 is applied at $t=0$.

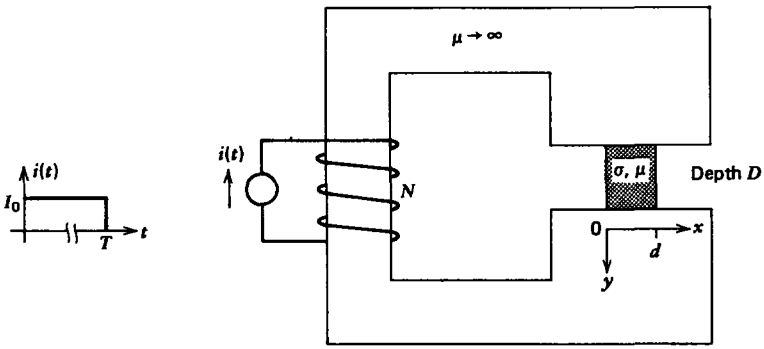
(a) What is the dc steady-state solution for the magnetic field distribution?

(b) What are the boundary and initial conditions for the magnetic field in the conducting block?

(c) What are the transient field and current distributions?

(d) What is the time dependence of the force on the conductor?

(e) The current has been on a long time so that the system is in the dc steady state found in (a) when at $t=T$ the current

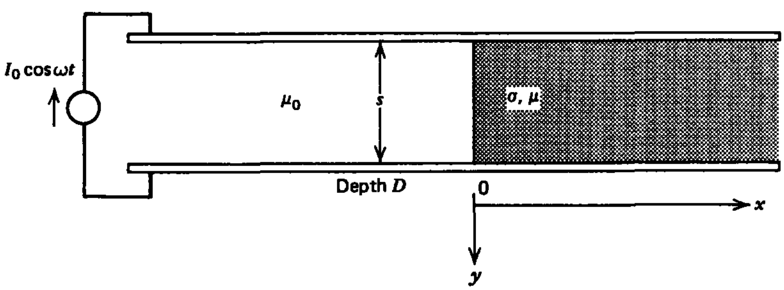


is instantaneously turned off. What are the transient field and current distributions in the conductor?

(f) If the applied coil current varies sinusoidally with time as $i(t) = I_0 \cos \omega t$, what are the sinusoidal steady-state field and current distributions? (**Hint:** Leave your answer in terms of complex amplitudes.)

(g) What is the force on the block?

24. A semi-infinite conducting block is placed between parallel perfect conductors. A sinusoidal current source is applied.



(a) What are the magnetic field and current distributions within the conducting block?

(b) What is the total force on the block?

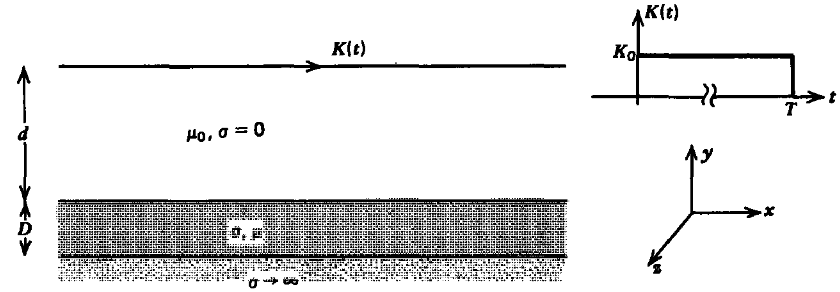
(c) Repeat (a) and (b) if the block has length d.

25. A current sheet that is turned on at $t = 0$ lies a distance d above a conductor of thickness D and conductivity σ. The conductor lies on top of a perfectly conducting plane.

(a) What are the initial and steady-state solutions? What are the boundary conditions?

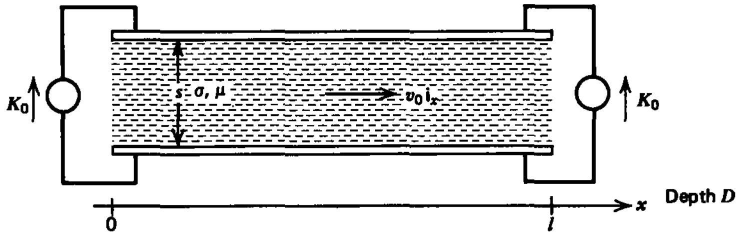
(b) What are the transient magnetic field and current distributions?

(c) After a long time T, so that the system has reached the dc steady state, the surface current is set to zero. What are the subsequent field and current distributions?



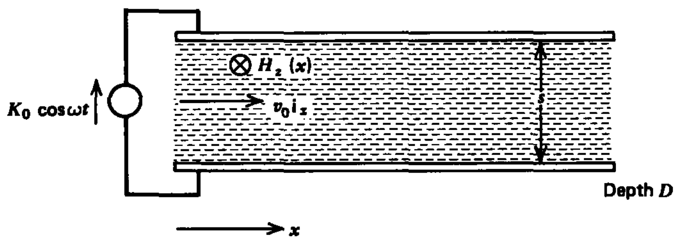
(d) What are the field and current distributions if the current sheet varies as $K_0 \cos \omega t$?

26. Distributed dc currents at $x = 0$ and $x = l$ flow through a conducting fluid moving with constant velocity $v_0\mathbf{i}_x$.



(a) What are the magnetic field and current distributions?
(b) What is the force on the fluid?

27. A sinusoidal surface current at $x = 0$ flows along parallel electrodes and returns through a conducting fluid moving to the right with constant velocity $v_0\mathbf{i}_x$. The flow is not impeded by the current source. The system extends to $x = \infty$.



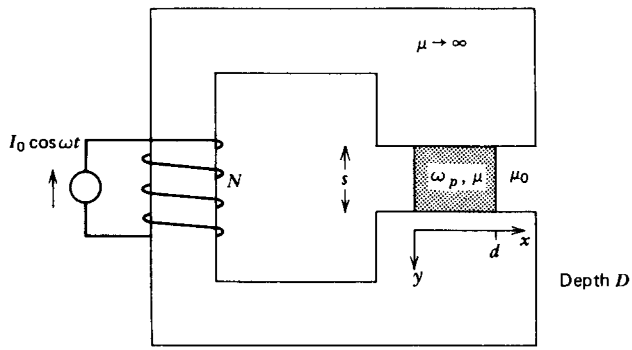
(a) What are the magnetic field and current density distributions?
(b) What is the time-average force on the fluid?

28. The surface current for the linear induction machine treated in Section 6-4-6 is now put a distance s below the conductor.

(a) What are the magnetic field and current distributions in each region of space? (**Hint:** Check your answer with Section 6-4-6 when $s = 0$.)

(b) Repeat (a) if s is set to zero but the conductor has a finite thickness d.

29. A superconducting block with plasma frequency ω_p is placed within a magnetic circuit with exciting current $I_0 \cos \omega t$.

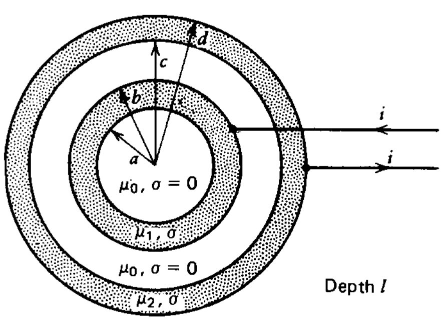


(a) What are the magnetic field and current distributions in the superconductor?

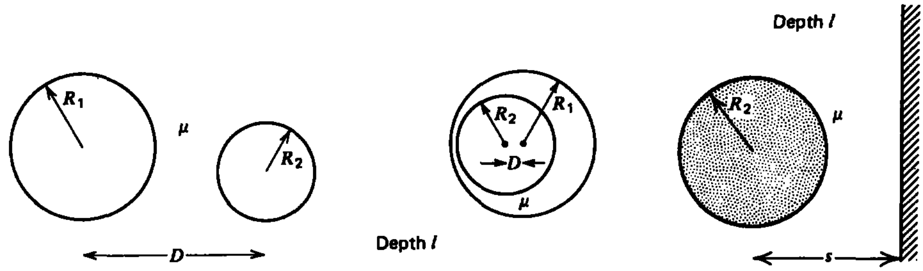
(b) What is the force on the block?

Section 6.5

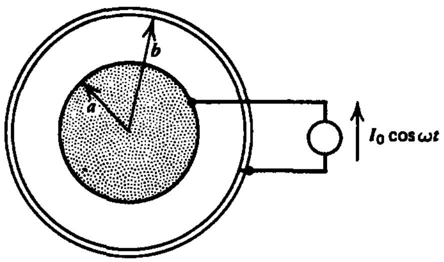
30. Find the magnetic energy stored and the self-inductance for the geometry below where the current in each shell is uniformly distributed.



31. Find the external self-inductance of the two wire lines shown. (**Hint:** See Section 2-6-4c.)



32. A coaxial cable with solid inner conductor is excited by a sinusoidally varying current $I_0 \cos \omega t$ at high enough frequency so that the skin depth is small compared to the radius a. The current is now nonuniformly distributed over the inner conductor.



(a) Assuming that $\mathbf{H} = H_\phi(\mathrm{r})\mathbf{i}_\phi$, what is the governing equation for $H_\phi(\mathrm{r})$ within the inner cylinder. (**Hint:** $\nabla^2 \mathbf{H} = \nabla(\nabla \cdot \mathbf{H})^{\nearrow 0} - \nabla \times (\nabla \times \mathbf{H})$.)

(b) Solve (a) for solutions of the form

$$H_\phi(\mathrm{r}) = \mathrm{Re}\,[\hat{H}_\phi(\mathrm{r})\, e^{j\omega t}]$$

Hint: Bessel's equation is

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - p^2) y = 0$$

with solutions

$$y = A_1 J_p(x) + A_2 Y_p(x)$$

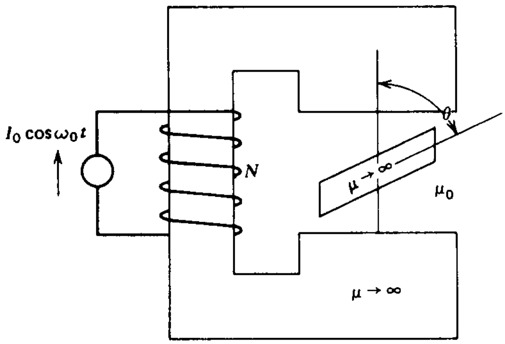
where Y_p is singular at $x = 0$.

(c) What is the current distribution? **Hint:**

$$\frac{d}{dx}[J_1(x)] + \frac{1}{x} J_1(x) = J_0(x)$$

Section 6-6

33. A reluctance motor is made by placing a high permeability material, which is free to rotate, in the air gap of a magnetic circuit excited by a sinusoidal current $I_0 \cos \omega_0 t$.



The inductance of the circuit varies as

$$L(\theta) = L_0 + L_1 \cos 2\theta$$

where the maximum inductance $L_0 + L_1$ occurs when $\theta = 0$ or $\theta = \pi$ and the minimum inductance $L_0 - L_1$ occurs when $\theta = \pm\pi/2$.

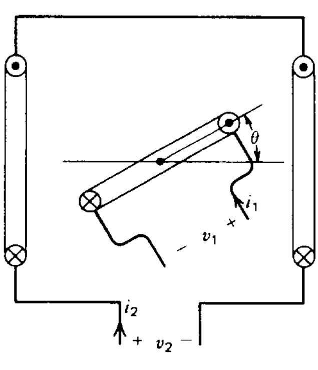
(a) What is the torque on the slab as a function of the angle θ?

(b) The rotor is rotating at constant speed ω, where $\theta = \omega t + \delta$ and δ is the angle of the rotor at $t = 0$. At what value of ω does the torque have a nonzero time average. The reluctance motor is then a synchronous machine. **Hint:**

$$\begin{aligned}\cos^2 \omega_0 t \sin 2\theta &= \tfrac{1}{2}[\sin 2\theta + \cos 2\omega_0 t \sin 2\theta] \\ &= \tfrac{1}{2}\{\sin 2\theta + \tfrac{1}{2}[\sin 2(\omega_0 t + \theta) + \sin 2(\theta - \omega_0 t)]\}\end{aligned}$$

(c) What is the maximum torque that can be delivered by the shaft and at what angle δ does it occur?

34. A system of two coupled coils have the following flux-current relations:



(c)

$$\Phi_1 = L_1(\theta)i_1 + M(\theta)i_2$$

$$\Phi_2 = M(\theta)i_1 + L_2(\theta)i_2$$

(a) What is the power p delivered to the coils?
(b) Write this power in the form

$$p = \frac{dW}{dt} + T\frac{d\theta}{dt}$$

What are W and T?

(c) A small coil is free to rotate in the uniform magnetic field produced by another coil. The flux-current relation is

$$\Phi_1 = L_1 i_1 + M_0 i_2 \sin\theta$$

$$\Phi_2 = M_0 i_1 \sin\theta + L_2 i_2$$

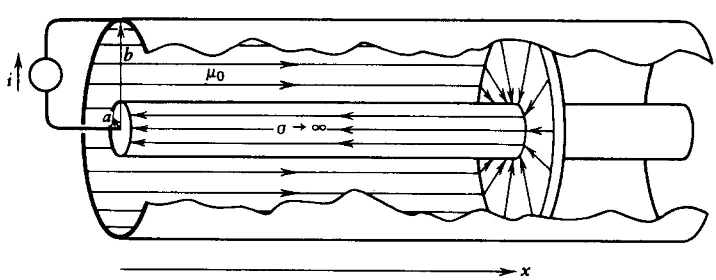
The coils are excited by dc currents I_1 and I_2. What is the torque on the small coil?

(d) If the small coil has conductivity σ, cross-sectional area A, total length l, and is short circuited, what differential equation must the current i_1 obey if θ is a function of time? A dc current I_2 is imposed in coil 2.

(e) The small coil has moment of inertia J. Consider only small motions around $\theta = 0$ so that $\cos\theta \approx 1$. With the torque and current equations linearized, try exponential solutions of the form e^{st} and solve for the natural frequencies.

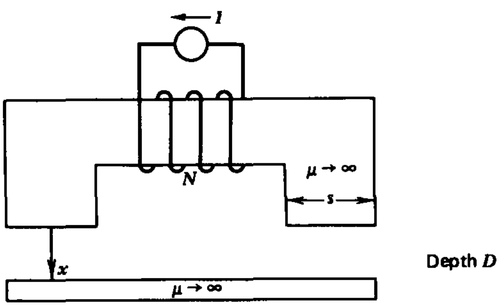
(f) The coil is released from rest at $\theta = \theta_0$. What is $\theta(t)$ and $i_1(t)$? Under what conditions are the solutions oscillatory? Damped?

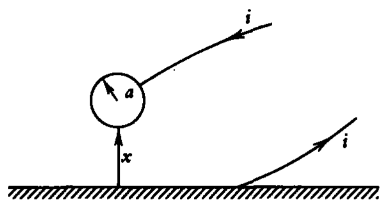
35. A coaxial cable has its short circuited end free to move.



(a) What is the inductance of the cable as a function of x?
(b) What is the force on the end?

36. For the following geometries, find:
(a) The inductance;
(b) The force on the moveable member.





37. A coaxial cylinder is dipped into a magnetizable fluid with permeability μ and mass density ρ_m. How high h does the fluid rise within the cylinder?