Cartesian		*Cylindrical*		*Spherical*
x	=	$r\cos\phi$	=	$r\sin\theta\cos\phi$
y	=	$r\sin\phi$	=	$r\sin\theta\sin\phi$
z	=	z	=	$r\cos\theta$
$\mathbf{i}_x$	=	$\cos\phi\,\mathbf{i}_r - \sin\phi\,\mathbf{i}_\phi$	=	$\sin\theta\cos\phi\,\mathbf{i}_r + \cos\theta\cos\phi\,\mathbf{i}_\theta - \sin\phi\,\mathbf{i}_\phi$
$\mathbf{i}_y$	=	$\sin\phi\,\mathbf{i}_r + \cos\phi\,\mathbf{i}_\phi$	=	$\sin\theta\sin\phi\,\mathbf{i}_r + \cos\theta\sin\phi\,\mathbf{i}_\theta + \cos\phi\,\mathbf{i}_\phi$
$\mathbf{i}_z$	=	$\mathbf{i}_z$	=	$\cos\theta\,\mathbf{i}_r - \sin\theta\,\mathbf{i}_\theta$

Cylindrical		*Cartesian*		*Spherical*
r	=	$\sqrt{x^2+y^2}$	=	$r\sin\theta$
ϕ	=	$\tan^{-1} y/x$	=	ϕ
z	=	z	=	$r\cos\theta$
$\mathbf{i}_r$	=	$\cos\phi\,\mathbf{i}_x + \sin\phi\,\mathbf{i}_y$	=	$\sin\theta\,\mathbf{i}_r + \cos\theta\,\mathbf{i}_\theta$
$\mathbf{i}_\phi$	=	$-\sin\phi\,\mathbf{i}_x + \cos\phi\,\mathbf{i}_y$	=	$\mathbf{i}_\phi$
$\mathbf{i}_z$	=	$\mathbf{i}_z$	=	$\cos\theta\,\mathbf{i}_r - \sin\theta\,\mathbf{i}_\theta$

Spherical		*Cartesian*		*Cylindrical*
r	=	$\sqrt{x^2+y^2+z^2}$	=	$\sqrt{r^2+z^2}$
θ	=	$\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$	=	$\cos^{-1}\frac{z}{\sqrt{r^2+z^2}}$
ϕ	=	$\cot^{-1} x/y$	=	ϕ
$\mathbf{i}_r$	=	$\sin\theta\cos\phi\,\mathbf{i}_x + \sin\theta\sin\phi\,\mathbf{i}_y + \cos\theta\,\mathbf{i}_z$	=	$\sin\theta\,\mathbf{i}_r + \cos\theta\,\mathbf{i}_z$
$\mathbf{i}_\theta$	=	$\cos\theta\cos\phi\,\mathbf{i}_x + \cos\theta\sin\phi\,\mathbf{i}_y - \sin\theta\,\mathbf{i}_z$	=	$\cos\theta\,\mathbf{i}_r - \sin\theta\,\mathbf{i}_z$
$\mathbf{i}_\phi$	=	$-\sin\phi\,\mathbf{i}_x + \cos\phi\,\mathbf{i}_y$	=	$\mathbf{i}_\phi$

Geometric relations between coordinates and unit vectors for Cartesian, cylindrical, and spherical coordinate systems.

Cartesian Coordinates (x, y, z)

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i}_x + \frac{\partial f}{\partial y}\mathbf{i}_y + \frac{\partial f}{\partial z}\mathbf{i}_z$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{i}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{i}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{i}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Cylindrical Coordinates $(\mathrm{r}, \phi, \mathrm{z})$

$$\nabla f = \frac{\partial f}{\partial \mathrm{r}}\mathbf{i}_\mathrm{r} + \frac{1}{\mathrm{r}}\frac{\partial f}{\partial \phi}\mathbf{i}_\phi + \frac{\partial f}{\partial \mathrm{z}}\mathbf{i}_\mathrm{z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}(\mathrm{r}A_\mathrm{r}) + \frac{1}{\mathrm{r}}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_\mathrm{z}}{\partial \mathrm{z}}$$

$$\nabla \times \mathbf{A} = \mathbf{i}_\mathrm{r}\left(\frac{1}{\mathrm{r}}\frac{\partial A_\mathrm{z}}{\partial \phi} - \frac{\partial A_\phi}{\partial \mathrm{z}}\right) + \mathbf{i}_\phi\left(\frac{\partial A_\mathrm{r}}{\partial \mathrm{z}} - \frac{\partial A_\mathrm{z}}{\partial \mathrm{r}}\right) + \mathbf{i}_\mathrm{z}\frac{1}{\mathrm{r}}\left[\frac{\partial(\mathrm{r}A_\phi)}{\partial \mathrm{r}} - \frac{\partial A_\mathrm{r}}{\partial \phi}\right]$$

$$\nabla^2 f = \frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}\frac{\partial f}{\partial \mathrm{r}}\right) + \frac{1}{\mathrm{r}^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial \mathrm{z}^2}$$

Spherical Coordinates (r, θ, ϕ)

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{i}_\phi$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \mathbf{i}_r\frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta A_\phi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right]$$
$$+ \mathbf{i}_\theta\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial(rA_\phi)}{\partial r}\right] + \mathbf{i}_\phi\frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right]$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

MAXWELL'S EQUATIONS

	Integral	*Differential*	*Boundary Conditions*
Faraday's Law	$\oint_L \mathbf{E}' \cdot \mathbf{dl} = -\frac{d}{dt}\int_S \mathbf{B} \cdot \mathbf{dS}$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\mathbf{n} \times (\mathbf{E}_2' - \mathbf{E}_1') = 0.$
Ampere's Law with Maxwell's Displacement Current Correction	$\oint_L \mathbf{H} \cdot \mathbf{dl} = \int_S \mathbf{J}_f \cdot \mathbf{dS} + \frac{d}{dt}\int_S \mathbf{D} \cdot \mathbf{dS}$	$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$	$\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}_f$
Gauss's Law	$\oint_S \mathbf{D} \cdot \mathbf{dS} = \int_V \rho_f dV$	$\nabla \cdot \mathbf{D} = \rho_f$	$\mathbf{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma_f$
	$\oint_S \mathbf{B} \cdot \mathbf{dS} = 0$	$\nabla \cdot \mathbf{B} = 0$	$\mathbf{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0$
Conservation of Charge	$\oint_S \mathbf{J}_f \cdot \mathbf{dS} + \frac{d}{dt}\int_V \rho_f dV = 0$	$\nabla \cdot \mathbf{J}_f + \frac{\partial \rho_f}{\partial t} = 0$	$\mathbf{n} \cdot (\mathbf{J}_2 - \mathbf{J}_1) + \frac{\partial \sigma_f}{\partial t} = 0$

Usual Linear Constitutive Laws

$\mathbf{D} = \varepsilon \mathbf{E}$

$\mathbf{B} = \mu \mathbf{H}$

$\mathbf{J}_f = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \sigma \mathbf{E}'$ [Ohm's law for moving media with velocity $\mathbf{v}$]

PHYSICAL CONSTANTS

Constant	Symbol	Value	units
Speed of light in vacuum	c	$2.9979 \times 10^8 \approx 3 \times 10^8$	m/sec
Elementary electron charge	e	1.602×10^{-19}	coul
Electron rest mass	m_e	9.11×10^{-31}	kg
Electron charge to mass ratio	$\frac{e}{m_e}$	1.76×10^{11}	coul/kg
Proton rest mass	m_p	1.67×10^{-27}	kg
Boltzmann constant	k	1.38×10^{-23}	joule/°K
Gravitation constant	G	6.67×10^{-11}	nt-m^2/(kg)^2
Acceleration of gravity	g	9.807	m/(sec)^2
Permittivity of free space	ε_0	$8.854 \times 10^{-12} \approx \frac{10^{-9}}{36\pi}$	farad/m
Permeability of free space	μ_0	$4\pi \times 10^{-7}$	henry/m
Planck's constant	h	6.6256×10^{-34}	joule-sec
Impedance of free space	$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$	$376.73 \approx 120\pi$	ohms
Avogadro's number	$\mathcal{N}$	6.023×10^{23}	atoms/mole

ELECTROMAGNETIC FIELD THEORY:
a problem solving approach

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to my parents

PREFACE

Electromagnetic field theory is often the least popular course in the electrical engineering curriculum. Heavy reliance on vector and integral calculus can obscure physical phenomena so that the student becomes bogged down in the mathematics and loses sight of the applications. This book instills problem solving confidence by teaching through the use of a large number of worked examples. To keep the subject exciting, many of these problems are based on physical processes, devices, and models.

This text is an introductory treatment on the junior level for a two-semester electrical engineering course starting from the Coulomb–Lorentz force law on a point charge. The theory is extended by the continuous superposition of solutions from previously developed simpler problems leading to the general integral and differential field laws. Often the same problem is solved by different methods so that the advantages and limitations of each approach becomes clear. Sample problems and their solutions are presented for each new concept with great emphasis placed on classical models of such physical phenomena as polarization, conduction, and magnetization. A large variety of related problems that reinforce the text material are included at the end of each chapter for exercise and homework.

It is expected that students have had elementary courses in calculus that allow them to easily differentiate and integrate simple functions. The text tries to keep the mathematical development rigorous but simple by typically describing systems with linear, constant coefficient differential and difference equations.

The text is essentially subdivided into three main subject areas: (1) charges as the source of the electric field coupled to polarizable and conducting media with negligible magnetic field; (2) currents as the source of the magnetic field coupled to magnetizable media with electromagnetic induction generating an electric field; and (3) electrodynamics where the electric and magnetic fields are of equal importance resulting in radiating waves. Wherever possible, electrodynamic solutions are examined in various limits to illustrate the appropriateness of the previously developed quasi-static circuit theory approximations.

Many of my students and graduate teaching assistants have helped in checking the text and exercise solutions and have assisted in preparing some of the field plots.

Markus Zahn

A NOTE TO THE STUDENT

In this text I have tried to make it as simple as possible for an interested student to learn the difficult subject of electromagnetic field theory by presenting many worked examples emphasizing physical processes, devices, and models. The problems at the back of each chapter are grouped by chapter sections and extend the text material. To avoid tedium, most integrals needed for problem solution are supplied as hints. The hints also often suggest the approach needed to obtain a solution easily. Answers to selected problems are listed at the back of this book.

A NOTE TO THE INSTRUCTOR

An Instructor's Manual with solutions to all exercise problems at the end of chapters is available from the author for the cost of reproduction and mailing. Please address requests on University or Company letterhead to:

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Massachusetts Institute of Technology
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CONTENTS

ELECTROMAGNETIC FIELD THEORY:

a problem solving approach

chapter 1

review of vector analysis

Electromagnetic field theory is the study of forces between charged particles resulting in energy conversion or signal transmission and reception. These forces vary in magnitude and direction with time and throughout space so that the theory is a heavy user of vector, differential, and integral calculus. This chapter presents a brief review that highlights the essential mathematical tools needed throughout the text. We isolate the mathematical details here so that in later chapters most of our attention can be devoted to the applications of the mathematics rather than to its development. Additional mathematical material will be presented as needed throughout the text.

1-1 COORDINATE SYSTEMS

A coordinate system is a way of uniquely specifying the location of any position in space with respect to a reference origin. Any point is defined by the intersection of three mutually perpendicular surfaces. The coordinate axes are then defined by the normals to these surfaces at the point. Of course the solution to any problem is always independent of the choice of coordinate system used, but by taking advantage of symmetry, computation can often be simplified by proper choice of coordinate description. In this text we only use the familiar rectangular (Cartesian), circular cylindrical, and spherical coordinate systems.

1-1-1 Rectangular (Cartesian) Coordinates

The most common and often preferred coordinate system is defined by the intersection of three mutually perpendicular planes as shown in Figure 1-1*a*. Lines parallel to the lines of intersection between planes define the coordinate axes (x, y, z), where the x axis lies perpendicular to the plane of constant x, the y axis is perpendicular to the plane of constant y, and the z axis is perpendicular to the plane of constant z. Once an origin is selected with coordinate $(0, 0, 0)$, any other point in the plane is found by specifying its x-directed, y-directed, and z-directed distances from this origin as shown for the coordinate points located in Figure 1-1*b*.

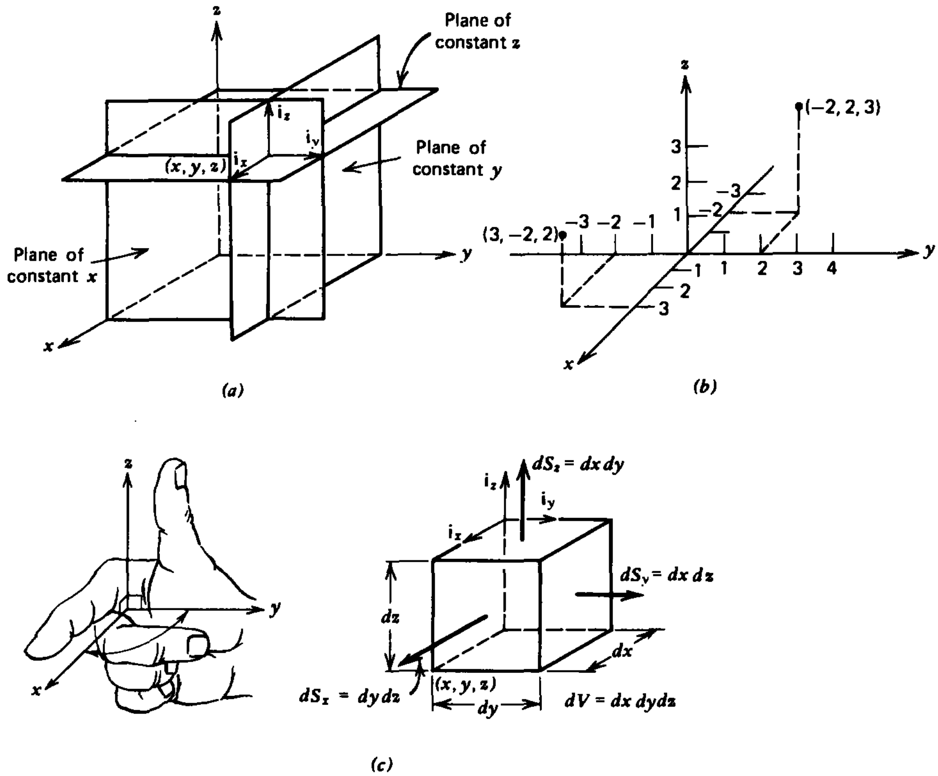


Figure 1-1 Cartesian coordinate system. (*a*) Intersection of three mutually perpendicular planes defines the Cartesian coordinates (x, y, z). (*b*) A point is located in space by specifying its x-, y- and z-directed distances from the origin. (*c*) Differential volume and surface area elements.

By convention, a right-handed coordinate system is always used whereby one curls the fingers of his or her right hand in the direction from x to y so that the forefinger is in the x direction and the middle finger is in the y direction. The thumb then points in the z direction. This convention is necessary to remove directional ambiguities in theorems to be derived later.

Coordinate directions are represented by unit vectors $\mathbf{i}_x$, $\mathbf{i}_y$ and $\mathbf{i}_z$, each of which has a unit length and points in the direction along one of the coordinate axes. Rectangular coordinates are often the simplest to use because the unit vectors always point in the same direction and do not change direction from point to point.

A rectangular differential volume is formed when one moves from a point (x, y, z) by an incremental distance dx, dy, and dz in each of the three coordinate directions as shown in

Figure 1-1c. To distinguish surface elements we subscript the area element of each face with the coordinate perpendicular to the surface.

1-1-2 Circular Cylindrical Coordinates

The cylindrical coordinate system is convenient to use when there is a line of symmetry that is defined as the z axis. As shown in Figure 1-2*a*, any point in space is defined by the intersection of the three perpendicular surfaces of a circular cylinder of radius r, a plane at constant z, and a plane at constant angle ϕ from the x axis.

The unit vectors $\mathbf{i}_r$, $\mathbf{i}_\phi$ and $\mathbf{i}_z$ are perpendicular to each of these surfaces. The direction of $\mathbf{i}_z$ is independent of position, but unlike the rectangular unit vectors the direction of $\mathbf{i}_r$ and $\mathbf{i}_\phi$ change with the angle ϕ as illustrated in Figure 1-2*b*. For instance, when $\phi = 0$ then $\mathbf{i}_r = \mathbf{i}_x$ and $\mathbf{i}_\phi = \mathbf{i}_y$, while if $\phi = \pi/2$, then $\mathbf{i}_r = \mathbf{i}_y$ and $\mathbf{i}_\phi = -\mathbf{i}_x$.

By convention, the triplet (r, ϕ, z) must form a right-handed coordinate system so that curling the fingers of the right hand from $\mathbf{i}_r$ to $\mathbf{i}_\phi$ puts the thumb in the z direction.

A section of differential size cylindrical volume, shown in Figure 1-2*c*, is formed when one moves from a point at coordinate (r, ϕ, z) by an incremental distance dr, r $d\phi$, and dz in each of the three coordinate directions. The differential volume and surface areas now depend on the coordinate r as summarized in Table 1-1.

Table 1-1 Differential lengths, surface area, and volume elements for each geometry. The surface element is subscripted by the coordinate perpendicular to the surface

CARTESIAN	CYLINDRICAL	SPHERICAL
$\mathbf{dl} = dx\,\mathbf{i}_x + dy\,\mathbf{i}_y + dz\,\mathbf{i}_z$	$\mathbf{dl} = dr\,\mathbf{i}_r + r\,d\phi\,\mathbf{i}_\phi + dz\,\mathbf{i}_z$	$\mathbf{dl} = dr\,\mathbf{i}_r + r\,d\theta\,\mathbf{i}_\theta + r\sin\theta\,d\phi\,\mathbf{i}_\phi$
$dS_x = dy\,dz$	$dS_r = r\,d\phi\,dz$	$dS_r = r^2\sin\theta\,d\theta\,d\phi$
$dS_y = dx\,dz$	$dS_\phi = dr\,dz$	$dS_\theta = r\sin\theta\,dr\,d\phi$
$dS_z = dx\,dy$	$dS_z = r\,dr\,d\phi$	$dS_\phi = r\,dr\,d\theta$
$dV = dx\,dy\,dz$	$dV = r\,dr\,d\phi\,dz$	$dV = r^2\sin\theta\,dr\,d\theta\,d\phi$

1-1-3 Spherical Coordinates

A spherical coordinate system is useful when there is a point of symmetry that is taken as the origin. In Figure 1-3*a* we see that the spherical coordinate (r, θ, ϕ) is obtained by the intersection of a sphere with radius r, a plane at constant

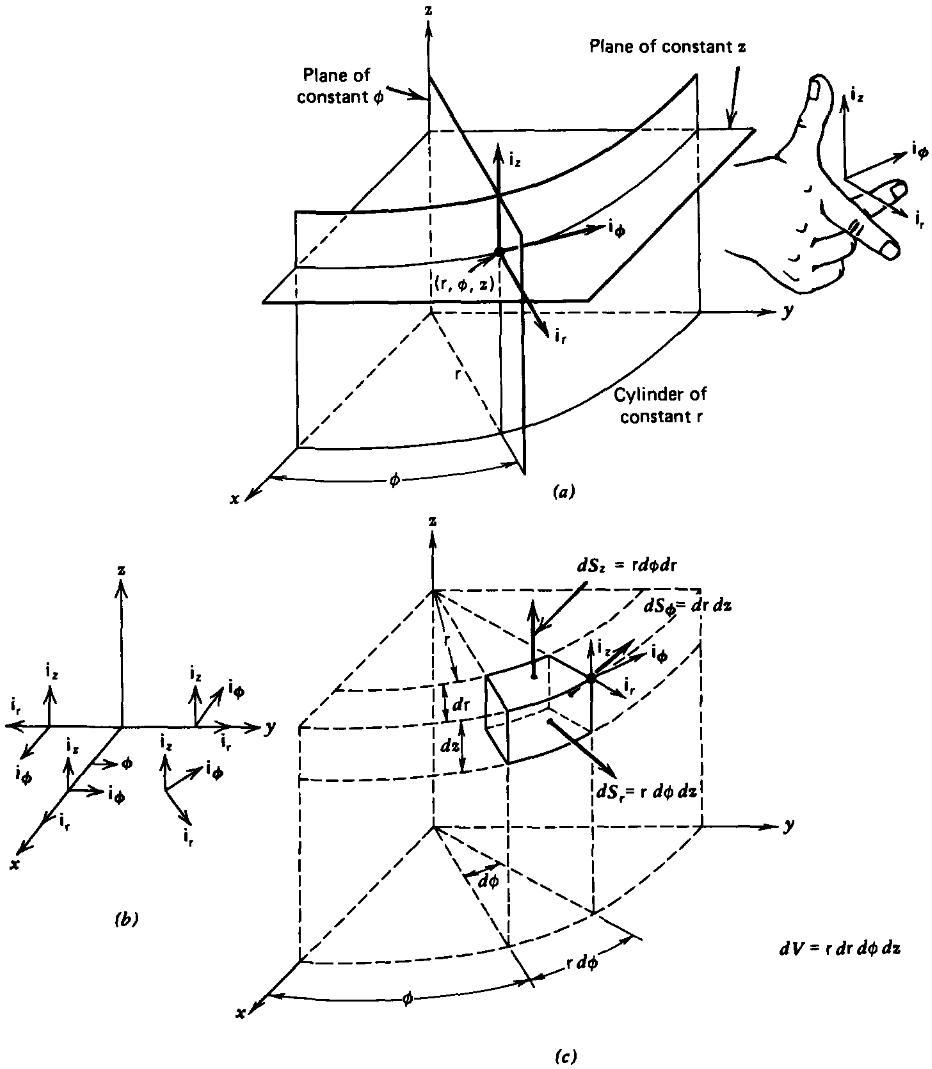
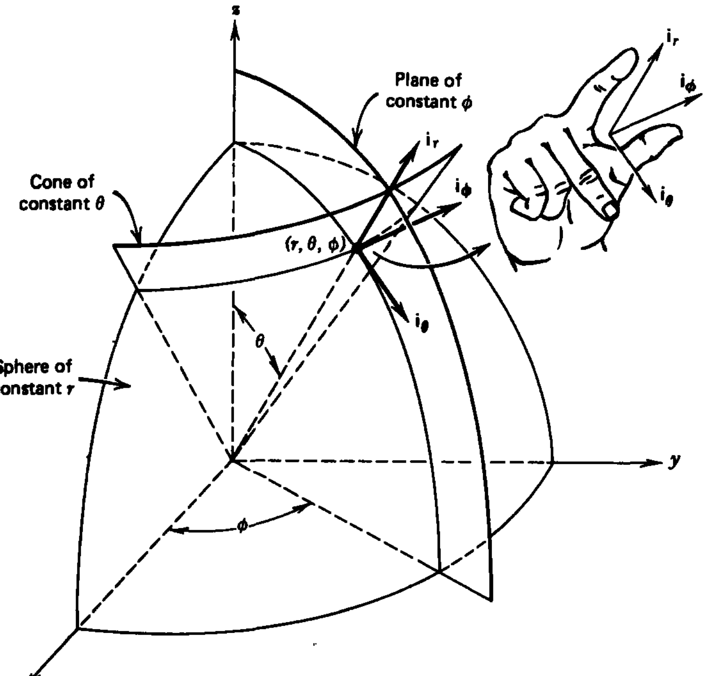


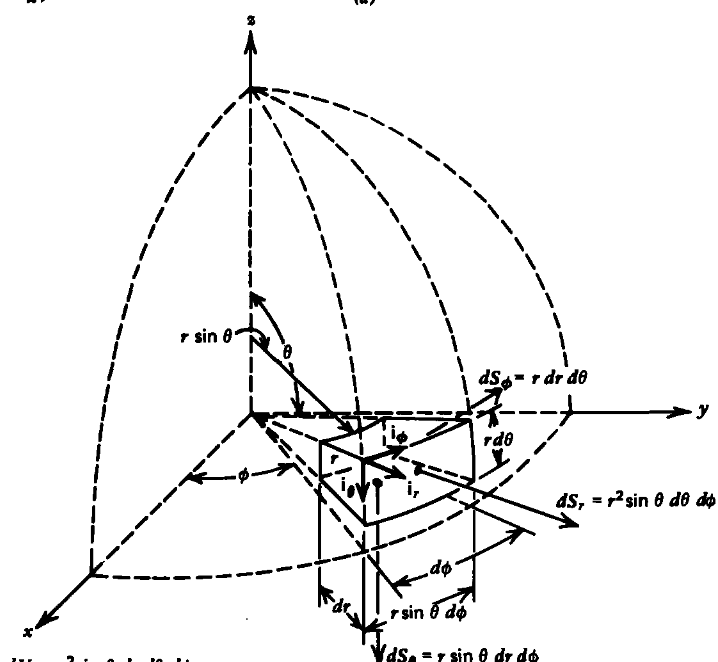
Figure 1-2 Circular cylindrical coordinate system. (*a*) Intersection of planes of constant z and ϕ with a cylinder of constant radius r defines the coordinates (r, ϕ, z). (*b*) The direction of the unit vectors $\mathbf{i}_r$ and $\mathbf{i}_\phi$ vary with the angle ϕ. (*c*) Differential volume and surface area elements.

angle ϕ from the x axis as defined for the cylindrical coordinate system, and a cone at angle θ from the z axis. The unit vectors $\mathbf{i}_r$, $\mathbf{i}_\theta$ and $\mathbf{i}_\phi$ are perpendicular to each of these surfaces and change direction from point to point. The triplet (r, θ, ϕ) must form a right-handed set of coordinates.

The differential-size spherical volume element formed by considering incremental displacements dr, $r\,d\theta$, $r \sin\theta\, d\phi$



(a)



(b)

Figure 1-3 Spherical coordinate system. (*a*) Intersection of plane of constant angle ϕ with cone of constant angle θ and sphere of constant radius r defines the coordinates (r, θ, ϕ). (*b*) Differential volume and surface area elements.

Table 1-2 Geometric relations between coordinates and unit vectors for Cartesian, cylindrical, and spherical coordinate systems*

CARTESIAN		CYLINDRICAL		SPHERICAL
x	=	$\mathrm{r}\cos\phi$	=	$r\sin\theta\cos\phi$
y	=	$\mathrm{r}\sin\phi$	=	$r\sin\theta\sin\phi$
z	=	z	=	$r\cos\theta$
$\mathbf{i}_x$	=	$\cos\phi\,\mathbf{i}_\mathrm{r}-\sin\phi\,\mathbf{i}_\phi$	=	$\sin\theta\cos\phi\,\mathbf{i}_r+\cos\theta\cos\phi\,\mathbf{i}_\theta-\sin\phi\,\mathbf{i}_\phi$
$\mathbf{i}_y$	=	$\sin\phi\,\mathbf{i}_\mathrm{r}+\cos\phi\,\mathbf{i}_\phi$	=	$\sin\theta\sin\phi\,\mathbf{i}_r+\cos\theta\sin\phi\,\mathbf{i}_\theta+\cos\phi\,\mathbf{i}_\phi$
$\mathbf{i}_z$	=	$\mathbf{i}_z$	=	$\cos\theta\,\mathbf{i}_r-\sin\theta\,\mathbf{i}_\theta$
CYLINDRICAL		CARTESIAN		SPHERICAL
r	=	$\sqrt{x^2+y^2}$	=	$r\sin\theta$
ϕ	=	$\tan^{-1}\frac{y}{x}$	=	ϕ
z	=	z	=	$r\cos\theta$
$\mathbf{i}_\mathrm{r}$	=	$\cos\phi\,\mathbf{i}_x+\sin\phi\,\mathbf{i}_y$	=	$\sin\theta\,\mathbf{i}_r+\cos\theta\,\mathbf{i}_\theta$
$\mathbf{i}_\phi$	=	$-\sin\phi\,\mathbf{i}_x+\cos\phi\,\mathbf{i}_y$	=	$\mathbf{i}_\phi$
$\mathbf{i}_z$	=	$\mathbf{i}_z$	=	$\cos\theta\,\mathbf{i}_r-\sin\theta\,\mathbf{i}_\theta$
SPHERICAL		CARTESIAN		CYLINDRICAL
r	=	$\sqrt{x^2+y^2+z^2}$	=	$\sqrt{\mathrm{r}^2+z^2}$
θ	=	$\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$	=	$\cos^{-1}\frac{z}{\sqrt{\mathrm{r}^2+z^2}}$
ϕ	=	$\cot^{-1}\frac{x}{y}$	=	ϕ
$\mathbf{i}_r$	=	$\sin\theta\cos\phi\,\mathbf{i}_x+\sin\theta\sin\phi\,\mathbf{i}_y+\cos\theta\,\mathbf{i}_z$	=	$\sin\theta\,\mathbf{i}_\mathrm{r}+\cos\theta\,\mathbf{i}_z$
$\mathbf{i}_\theta$	=	$\cos\theta\cos\phi\,\mathbf{i}_x+\cos\theta\sin\phi\,\mathbf{i}_y-\sin\theta\,\mathbf{i}_z$	=	$\cos\theta\,\mathbf{i}_\mathrm{r}-\sin\theta\,\mathbf{i}_z$
$\mathbf{i}_\phi$	=	$-\sin\phi\,\mathbf{i}_x+\cos\phi\,\mathbf{i}_y$	=	$\mathbf{i}_\phi$

* *Note that throughout this text a lower case roman* r *is used for the cylindrical radial coordinate while an italicized* r *is used for the spherical radial coordinate.*

from the coordinate (r, θ, ϕ) now depends on the angle θ and the radial position r as shown in Figure 1-3b and summarized in Table 1-1. Table 1-2 summarizes the geometric relations between coordinates and unit vectors for the three coordinate systems considered. Using this table, it is possible to convert coordinate positions and unit vectors from one system to another.

1-2 VECTOR ALGEBRA

1-2-1 Scalars and Vectors

A scalar quantity is a number completely determined by its magnitude, such as temperature, mass, and charge, the last being especially important in our future study. Vectors, such as velocity and force, must also have their direction specified and in this text are printed in boldface type. They are completely described by their components along three coordinate directions as shown for rectangular coordinates in Figure 1-4. A vector is represented by a directed line segment in the direction of the vector with its length proportional to its magnitude. The vector

$$\mathbf{A} = A_x \mathbf{i}_x + A_y \mathbf{i}_y + A_z \mathbf{i}_z \tag{1}$$

in Figure 1-4 has magnitude

$$A = |\mathbf{A}| = [A_x^2 + A_y^2 + A_z^2]^{1/2} \tag{2}$$

Note that each of the components in (1) (A_x, A_y, and A_z) are themselves scalars. The direction of each of the components is given by the unit vectors. We could describe a vector in any of the coordinate systems replacing the subscripts (x, y, z) by (r, ϕ, z) or (r, θ, ϕ); however, for conciseness we often use rectangular coordinates for general discussion.

1-2-2 Multiplication of a Vector by a Scalar

If a vector is multiplied by a positive scalar, its direction remains unchanged but its magnitude is multiplied by the

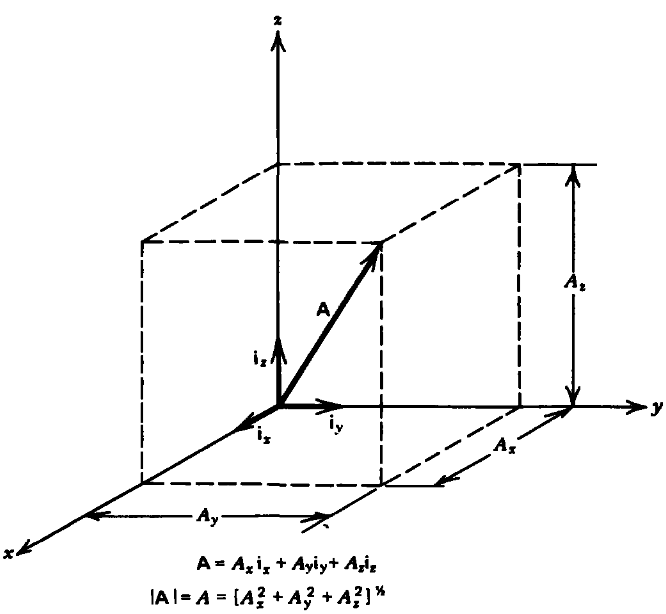


Figure 1-4 A vector is described by its components along the three coordinate directions.

scalar. If the scalar is negative, the direction of the vector is reversed:

$$a\mathbf{A} = aA_x\mathbf{i}_x + aA_y\mathbf{i}_y + aA_z\mathbf{i}_z \tag{3}$$

1-2-3 Addition and Subtraction

The sum of two vectors is obtained by adding their components while their difference is obtained by subtracting their components. If the vector **B**

$$\mathbf{B} = B_x\mathbf{i}_x + B_y\mathbf{i}_y + B_z\mathbf{i}_z \tag{4}$$

is added or subtracted to the vector **A** of (1), the result is a new vector **C**:

$$\mathbf{C} = \mathbf{A} \pm \mathbf{B} = (A_x \pm B_x)\mathbf{i}_x + (A_y \pm B_y)\mathbf{i}_y + (A_z \pm B_z)\mathbf{i}_z \tag{5}$$

Geometrically, the vector sum is obtained from the diagonal of the resulting parallelogram formed from **A** and **B** as shown in Figure 1-5*a*. The difference is found by first

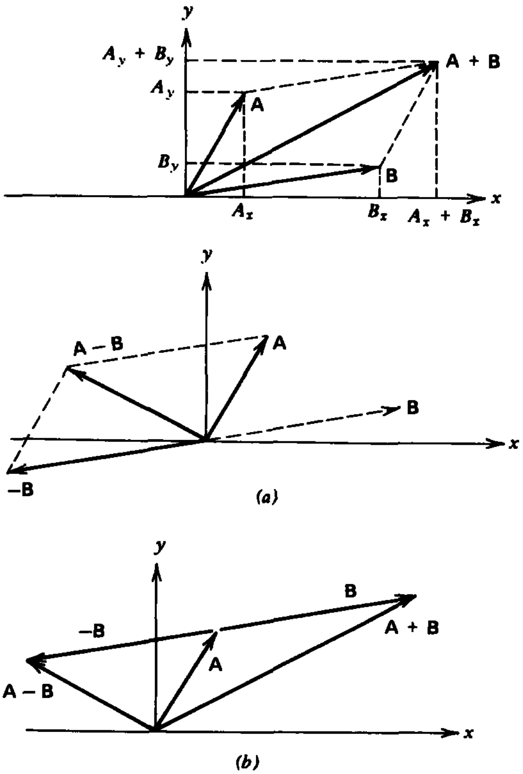


Figure 1-5 The sum and difference of two vectors (*a*) by finding the diagonal of the parallelogram formed by the two vectors, and (*b*) by placing the tail of a vector at the head of the other.

drawing $-\mathbf{B}$ and then finding the diagonal of the parallelogram formed from the sum of $\mathbf{A}$ and $-\mathbf{B}$. The sum of the two vectors is equivalently found by placing the tail of a vector at the head of the other as in Figure 1-5*b*.

Subtraction is the same as addition of the negative of a vector.

EXAMPLE 1-1 VECTOR ADDITION AND SUBTRACTION

Given the vectors

$$\mathbf{A} = 4\mathbf{i}_x + 4\mathbf{i}_y, \quad \mathbf{B} = \mathbf{i}_x + 8\mathbf{i}_y$$

find the vectors $\mathbf{B} \pm \mathbf{A}$ and their magnitudes. For the geometric solution, see Figure 1-6.

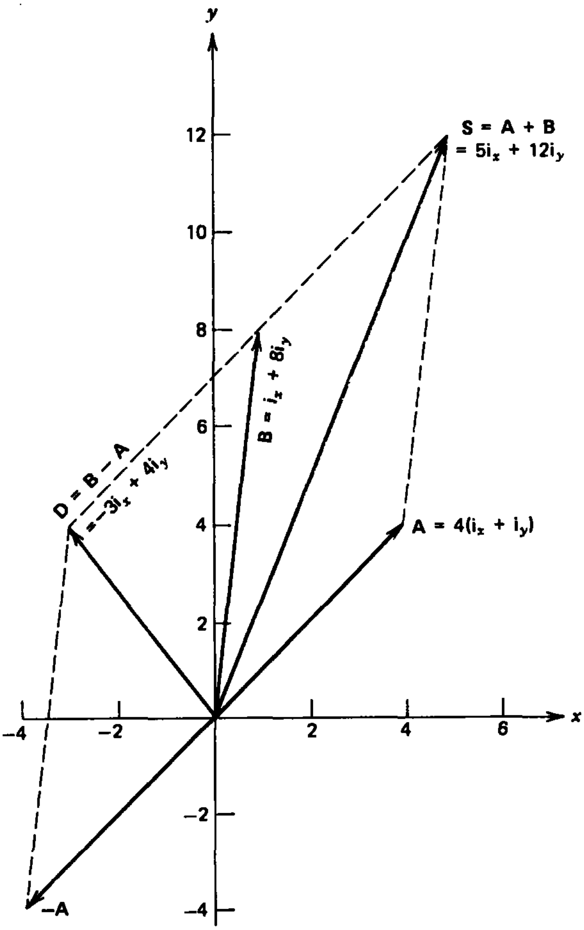


Figure 1-6 The sum and difference of vectors $\mathbf{A}$ and $\mathbf{B}$ given in Example 1-1.

SOLUTION

Sum

$$\mathbf{S} = \mathbf{A}+\mathbf{B} = (4+1)\mathbf{i}_x + (4+8)\mathbf{i}_y = 5\mathbf{i}_x + 12\mathbf{i}_y$$

$$S = [5^2 + 12^2]^{1/2} = 13$$

Difference

$$\mathbf{D} = \mathbf{B}-\mathbf{A} = (1-4)\mathbf{i}_x + (8-4)\mathbf{i}_y = -3\mathbf{i}_x + 4\mathbf{i}_y$$

$$D = [(-3)^2 + 4^2]^{1/2} = 5$$

1-2-4 The Dot (Scalar) Product

The dot product between two vectors results in a scalar and is defined as

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta \tag{6}$$

where θ is the smaller angle between the two vectors. The term $A\cos\theta$ is the component of the vector **A** in the direction of **B** shown in Figure 1-7. One application of the dot product arises in computing the incremental work dW necessary to move an object a differential vector distance **dl** by a force **F**. Only the component of force in the direction of displacement contributes to the work

$$dW = \mathbf{F}\cdot\mathbf{dl} \tag{7}$$

The dot product has maximum value when the two vectors are colinear ($\theta = 0$) so that the dot product of a vector with itself is just the square of its magnitude. The dot product is zero if the vectors are perpendicular ($\theta = \pi/2$). These properties mean that the dot product between different orthogonal unit vectors at the same point is zero, while the dot

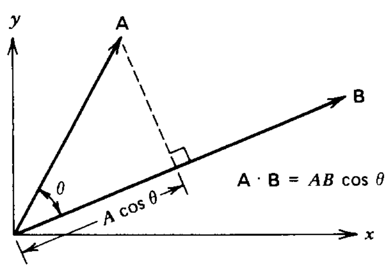


Figure 1-7 The dot product between two vectors.

product between a unit vector and itself is unity

$$\begin{aligned} \mathbf{i}_x \cdot \mathbf{i}_x &= 1, \quad & \mathbf{i}_x \cdot \mathbf{i}_y &= 0 \\ \mathbf{i}_y \cdot \mathbf{i}_y &= 1, \quad & \mathbf{i}_x \cdot \mathbf{i}_z &= 0 \\ \mathbf{i}_z \cdot \mathbf{i}_z &= 1, \quad & \mathbf{i}_y \cdot \mathbf{i}_z &= 0 \end{aligned} \tag{8}$$

Then the dot product can also be written as

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= (A_x\mathbf{i}_x + A_y\mathbf{i}_y + A_z\mathbf{i}_z) \cdot (B_x\mathbf{i}_x + B_y\mathbf{i}_y + B_z\mathbf{i}_z) \\ &= A_xB_x + A_yB_y + A_zB_z \end{aligned} \tag{9}$$

From (6) and (9) we see that the dot product does not depend on the order of the vectors

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \tag{10}$$

By equating (6) to (9) we can find the angle between vectors as

$$\cos \theta = \frac{A_xB_x + A_yB_y + A_zB_z}{AB} \tag{11}$$

Similar relations to (8) also hold in cylindrical and spherical coordinates if we replace (x, y, z) by (r, ϕ, z) or (r, θ, ϕ). Then (9) to (11) are also true with these coordinate substitutions.

EXAMPLE 1-2 DOT PRODUCT

Find the angle between the vectors shown in Figure 1-8,

$$\mathbf{A} = \sqrt{3}\,\mathbf{i}_x + \mathbf{i}_y, \quad \mathbf{B} = 2\mathbf{i}_x$$

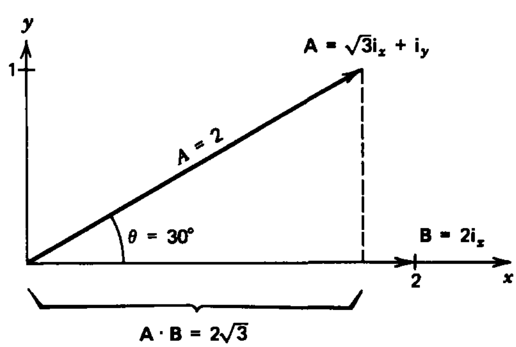


Figure 1-8 The angle between the two vectors **A** and **B** in Example 1-2 can be found using the dot product.

SOLUTION

From (11)

$$\cos\theta = \frac{A_x B_x}{[A_x^2 + A_y^2]^{1/2} B_x} = \frac{\sqrt{3}}{2}$$

$$\theta = \cos^{-1}\frac{\sqrt{3}}{2} = 30°$$

1-2-5 The Cross (Vector) Product

The cross product between two vectors $\mathbf{A}\times\mathbf{B}$ is defined as a vector perpendicular to both **A** and **B**, which is in the direction of the thumb when using the right-hand rule of curling the fingers of the right hand from **A** to **B** as shown in Figure 1-9. The magnitude of the cross product is

$$|\mathbf{A}\times\mathbf{B}| = AB\sin\theta \qquad (12)$$

where θ is the enclosed angle between **A** and **B**. Geometrically, (12) gives the area of the parallelogram formed with **A** and **B** as adjacent sides. Interchanging the order of **A** and **B** reverses the sign of the cross product:

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A} \qquad (13)$$

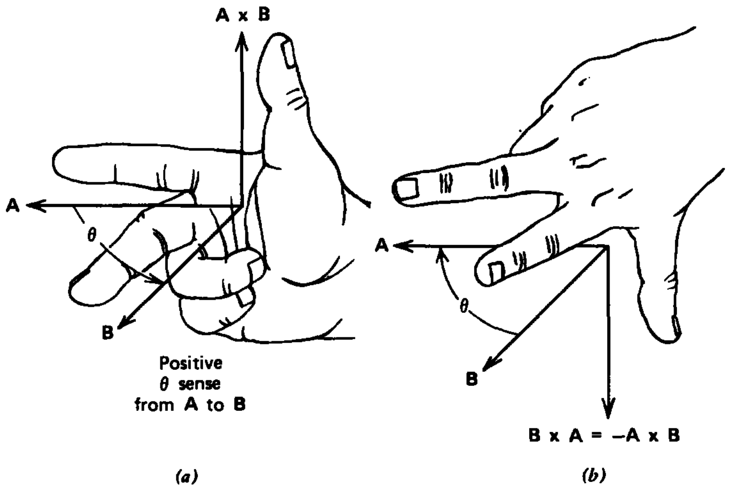


Figure 1-9 (*a*) The cross product between two vectors results in a vector perpendicular to both vectors in the direction given by the right-hand rule. (*b*) Changing the order of vectors in the cross product reverses the direction of the resultant vector.

The cross product is zero for colinear vectors ($\theta = 0$) so that the cross product between a vector and itself is zero and is maximum for perpendicular vectors ($\theta = \pi/2$). For rectangular unit vectors we have

$$\begin{aligned} &\mathbf{i}_x \times \mathbf{i}_x = 0, \quad &\mathbf{i}_x \times \mathbf{i}_y = \mathbf{i}_z, \quad &\mathbf{i}_y \times \mathbf{i}_x = -\mathbf{i}_z \\ &\mathbf{i}_y \times \mathbf{i}_y = 0, \quad &\mathbf{i}_y \times \mathbf{i}_z = \mathbf{i}_x, \quad &\mathbf{i}_z \times \mathbf{i}_y = -\mathbf{i}_x \\ &\mathbf{i}_z \times \mathbf{i}_z = 0, \quad &\mathbf{i}_z \times \mathbf{i}_x = \mathbf{i}_y, \quad &\mathbf{i}_x \times \mathbf{i}_z = -\mathbf{i}_y \end{aligned} \tag{14}$$

These relations allow us to simply define a right-handed coordinate system as one where

$$\mathbf{i}_x \times \mathbf{i}_y = \mathbf{i}_z \tag{15}$$

Similarly, for cylindrical and spherical coordinates, right-handed coordinate systems have

$$\mathbf{i}_r \times \mathbf{i}_\phi = \mathbf{i}_z, \quad \mathbf{i}_r \times \mathbf{i}_\theta = \mathbf{i}_\phi \tag{16}$$

The relations of (14) allow us to write the cross product between **A** and **B** as

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= (A_x\mathbf{i}_x + A_y\mathbf{i}_y + A_z\mathbf{i}_z) \times (B_x\mathbf{i}_x + B_y\mathbf{i}_y + B_z\mathbf{i}_z) \\ &= \mathbf{i}_x(A_yB_z - A_zB_y) + \mathbf{i}_y(A_zB_x - A_xB_z) + \mathbf{i}_z(A_xB_y - A_yB_x) \end{aligned} \tag{17}$$

which can be compactly expressed as the determinantal expansion

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= \det \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\ &= \mathbf{i}_x(A_yB_z - A_zB_y) + \mathbf{i}_y(A_zB_x - A_xB_z) + \mathbf{i}_z(A_xB_y - A_yB_x) \end{aligned} \tag{18}$$

The cyclical and orderly permutation of (x, y, z) allows easy recall of (17) and (18). If we think of xyz as a three-day week where the last day z is followed by the first day x, the days progress as

$$\underline{xyz}\,x\underline{yzx}\,y\underline{zxy}\,z \cdots \tag{19}$$

where the three possible positive permutations are underlined. Such permutations of xyz in the subscripts of (18) have positive coefficients while the odd permutations, where xyz do not follow sequentially

$$xzy, \; yxz, \; zyx \tag{20}$$

have negative coefficients in the cross product.

In (14)–(20) we used Cartesian coordinates, but the results remain unchanged if we sequentially replace (x, y, z) by the cylindrical coordinates (r, ϕ, z) or the spherical coordinates (r, θ, ϕ).

EXAMPLE 1-3 CROSS PRODUCT

Find the unit vector $\mathbf{i}_n$ perpendicular in the right-hand sense to the vectors shown in Figure 1-10.

$$\mathbf{A} = -\mathbf{i}_x + \mathbf{i}_y + \mathbf{i}_z, \quad \mathbf{B} = \mathbf{i}_x - \mathbf{i}_y + \mathbf{i}_z$$

What is the angle between **A** and **B**?

SOLUTION

The cross product $\mathbf{A} \times \mathbf{B}$ is perpendicular to both **A** and **B**

$$\mathbf{A} \times \mathbf{B} = \det \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 2(\mathbf{i}_x + \mathbf{i}_y)$$

The unit vector $\mathbf{i}_n$ is in this direction but it must have a magnitude of unity

$$\mathbf{i}_n = \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|} = \frac{1}{\sqrt{2}} (\mathbf{i}_x + \mathbf{i}_y)$$

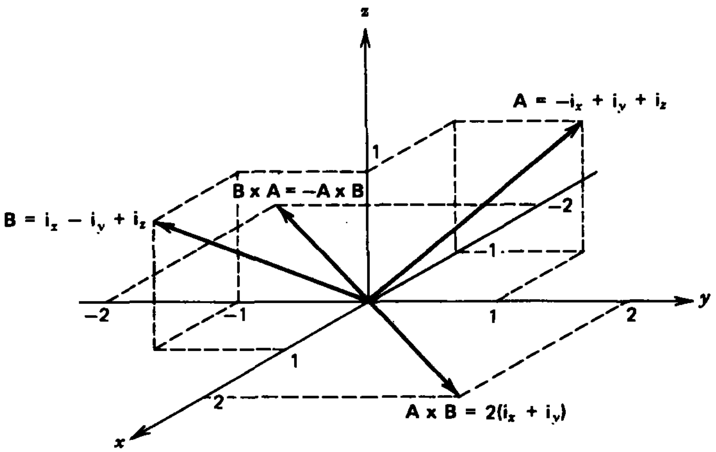


Figure 1-10 The cross product between the two vectors in Example 1-3.

The angle between **A** and **B** is found using (12) as

$$\sin\theta = \frac{|\mathbf{A}\times\mathbf{B}|}{AB} = \frac{2\sqrt{2}}{\sqrt{3}\sqrt{3}}$$
$$= \tfrac{2}{3}\sqrt{2} \Rightarrow \theta = 70.5° \text{ or } 109.5°$$

The ambiguity in solutions can be resolved by using the dot product of (11)

$$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{AB} = \frac{-1}{\sqrt{3}\sqrt{3}} = -\tfrac{1}{3} \Rightarrow \theta = 109.5°$$

1-3 THE GRADIENT AND THE DEL OPERATOR

1-3-1 The Gradient

Often we are concerned with the properties of a scalar field $f(x, y, z)$ around a particular point. The chain rule of differentiation then gives us the incremental change df in f for a small change in position from (x, y, z) to $(x+dx, y+dy, z+dz)$:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \tag{1}$$

If the general differential distance vector **dl** is defined as

$$\mathbf{dl} = dx\,\mathbf{i}_x + dy\,\mathbf{i}_y + dz\,\mathbf{i}_z \tag{2}$$

(1) can be written as the dot product:

$$df = \left(\frac{\partial f}{\partial x}\mathbf{i}_x + \frac{\partial f}{\partial y}\mathbf{i}_y + \frac{\partial f}{\partial z}\mathbf{i}_z\right)\cdot\mathbf{dl}$$
$$= \operatorname{grad} f\cdot\mathbf{dl} \tag{3}$$

where the spatial derivative terms in brackets are defined as the gradient of f:

$$\operatorname{grad} f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i}_x + \frac{\partial f}{\partial y}\mathbf{i}_y + \frac{\partial f}{\partial z}\mathbf{i}_z \tag{4}$$

The symbol ∇ with the gradient term is introduced as a general vector operator, termed the del operator:

$$\nabla = \mathbf{i}_x\frac{\partial}{\partial x} + \mathbf{i}_y\frac{\partial}{\partial y} + \mathbf{i}_z\frac{\partial}{\partial z} \tag{5}$$

By itself the del operator is meaningless, but when it premultiplies a scalar function, the gradient operation is defined. We will soon see that the dot and cross products between the del operator and a vector also define useful operations.

With these definitions, the change in f of (3) can be written as

$$df = \nabla f \cdot \mathbf{dl} = |\nabla f|\, dl \cos\theta \tag{6}$$

where θ is the angle between ∇f and the position vector $\mathbf{dl}$. The direction that maximizes the change in the function f is when $\mathbf{dl}$ is colinear with $\nabla f(\theta = 0)$. The gradient thus has the direction of maximum change in f. Motions in the direction along lines of constant f have $\theta = \pi/2$ and thus by definition $df = 0$.

1-3-2 Curvilinear Coordinates

(a) Cylindrical

The gradient of a scalar function is defined for any coordinate system as that vector function that when dotted with $\mathbf{dl}$ gives df. In cylindrical coordinates the differential change in $f(\mathrm{r}, \phi, z)$ is

$$df = \frac{\partial f}{\partial \mathrm{r}} d\mathrm{r} + \frac{\partial f}{\partial \phi} d\phi + \frac{\partial f}{\partial z} dz \tag{7}$$

The differential distance vector is

$$\mathbf{dl} = d\mathrm{r}\mathbf{i}_\mathrm{r} + \mathrm{r}\, d\phi\, \mathbf{i}_\phi + dz\, \mathbf{i}_z \tag{8}$$

so that the gradient in cylindrical coordinates is

$$df = \nabla f \cdot \mathbf{dl} \Rightarrow \nabla f = \frac{\partial f}{\partial \mathrm{r}} \mathbf{i}_\mathrm{r} + \frac{1}{\mathrm{r}} \frac{\partial f}{\partial \phi} \mathbf{i}_\phi + \frac{\partial f}{\partial z} \mathbf{i}_z \tag{9}$$

(b) Spherical

Similarly in spherical coordinates the distance vector is

$$\mathbf{dl} = dr\, \mathbf{i}_r + r\, d\theta\, \mathbf{i}_\theta + r \sin\theta\, d\phi\, \mathbf{i}_\phi \tag{10}$$

with the differential change of $f(r, \theta, \phi)$ as

$$df = \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \phi} d\phi = \nabla f \cdot \mathbf{dl} \tag{11}$$

Using (10) in (11) gives the gradient in spherical coordinates as

$$\nabla f = \frac{\partial f}{\partial r} \mathbf{i}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \mathbf{i}_\theta + \frac{1}{r \sin\theta} \frac{\partial f}{\partial \phi} \mathbf{i}_\phi \tag{12}$$

EXAMPLE 1-4 GRADIENT

Find the gradient of each of the following functions where a and b are constants:

(a) $f = ax^2y + by^3z$

SOLUTION

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i}_x + \frac{\partial f}{\partial y}\mathbf{i}_y + \frac{\partial f}{\partial z}\mathbf{i}_z$$
$$= 2axy\mathbf{i}_x + (ax^2 + 3by^2z)\mathbf{i}_y + by^3\mathbf{i}_z$$

(b) $f = ar^2 \sin\phi + brz\cos 2\phi$

SOLUTION

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\mathbf{i}_\phi + \frac{\partial f}{\partial z}\mathbf{i}_z$$
$$= (2ar\sin\phi + bz\cos 2\phi)\mathbf{i}_r$$
$$+ (ar\cos\phi - 2bz\sin 2\phi)\mathbf{i}_\phi + br\cos 2\phi\mathbf{i}_z$$

(c) $f = \frac{a}{r} + br\sin\theta\cos\phi$

SOLUTION

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{i}_\phi$$
$$= \left(-\frac{a}{r^2} + b\sin\theta\cos\phi\right)\mathbf{i}_r + b\cos\theta\cos\phi\mathbf{i}_\theta - b\sin\phi\mathbf{i}_\phi$$

1-3-3 The Line Integral

In Section 1-2-4 we motivated the use of the dot product through the definition of incremental work as depending only on the component of force $\mathbf{F}$ in the direction of an object's differential displacement $\mathbf{dl}$. If the object moves along a path, the total work is obtained by adding up the incremental works along each small displacement on the path as in Figure 1-11. If we break the path into N small displacements

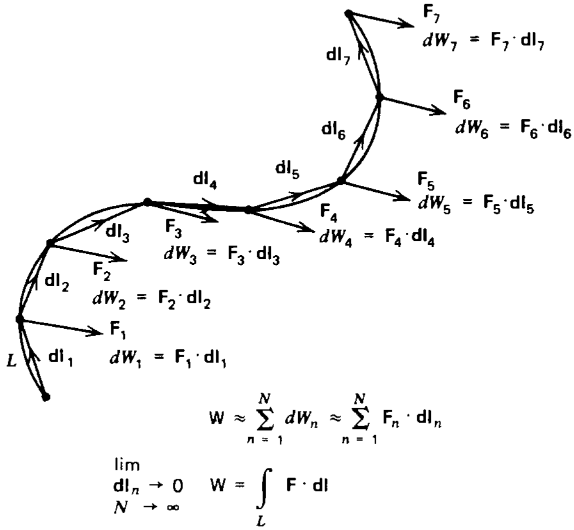


Figure 1-11 The total work in moving a body over a path is approximately equal to the sum of incremental works in moving the body each small incremental distance **dl**. As the differential distances approach zero length, the summation becomes a line integral and the result is exact.

$\mathbf{dl}_1, \mathbf{dl}_2, \ldots, \mathbf{dl}_N$, the work performed is approximately

$$W \approx \mathbf{F}_1 \cdot \mathbf{dl}_1 + \mathbf{F}_2 \cdot \mathbf{dl}_2 + \mathbf{F}_3 \cdot \mathbf{dl}_3 + \cdots + \mathbf{F}_N \cdot \mathbf{dl}_N$$
$$\approx \sum_{n=1}^{N} \mathbf{F}_n \cdot \mathbf{dl}_n \tag{13}$$

The result becomes exact in the limit as N becomes large with each displacement $\mathbf{dl}_n$ becoming infinitesimally small:

$$W = \lim_{\substack{N \to \infty \\ \mathbf{dl}_n \to 0}} \sum_{n=1}^{N} \mathbf{F}_n \cdot \mathbf{dl}_n = \int_L \mathbf{F} \cdot \mathbf{dl} \tag{14}$$

In particular, let us integrate (3) over a path between the two points a and b in Figure 1-12a:

$$\int_a^b df = f|_b - f|_a = \int_a^b \nabla f \cdot \mathbf{dl} \tag{15}$$

Because df is an exact differential, its line integral depends only on the end points and not on the shape of the contour itself. Thus, all of the paths between a and b in Figure 1-12a have the same line integral of ∇f, no matter what the function f may be. If the contour is a closed path so that $a = b$, as in

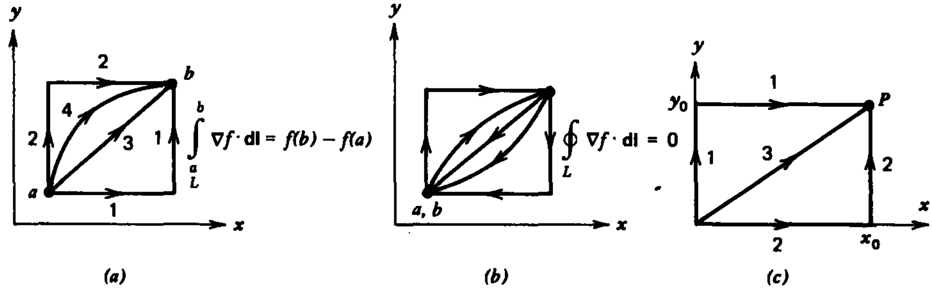


Figure 1-12 The component of the gradient of a function integrated along a line contour depends only on the end points and not on the contour itself. (*a*) Each of the contours have the same starting and ending points at *a* and *b* so that they all have the same line integral of ∇f. (*b*) When all the contours are closed with the same beginning and ending point at *a*, the line integral of ∇f is zero. (*c*) The line integral of the gradient of the function in Example (1-5) from the origin to the point *P* is the same for all paths.

Figure 1-12*b*, then (15) is zero:

$$\oint_L \nabla f \cdot d\mathbf{l} = f|_a - f|_a = 0 \tag{16}$$

where we indicate that the path is closed by the small circle in the integral sign $\oint$. The line integral of the gradient of a function around a closed path is zero.

EXAMPLE 1-5 LINE INTEGRAL

For $f = x^2 y$, verify (15) for the paths shown in Figure 1-12*c* between the origin and the point *P* at (x_0, y_0).

SOLUTION

The total change in *f* between 0 and *P* is

$$\int_0^P df = f|_P - f|_0 = x_0^2 y_0$$

From the line integral along path 1 we find

$$\int_0^P \nabla f \cdot d\mathbf{l} = \int_{\substack{y=0\\x=0}}^{y_0} \underbrace{\cancel{\frac{\partial f}{\partial y}}^{\,0}}_{x^2} dy + \int_{\substack{x=0\\y=y_0}}^{x_0} \underbrace{\frac{\partial f}{\partial x}}_{2xy} dx = x_0^2 y_0$$

Similarly, along path 2 we also obtain

$$\int_0^P \nabla f \cdot \mathbf{dl} = \int_{\substack{x=0\\y=0}}^{x_0} \underbrace{\frac{\partial f}{\partial x}}_{2xy}{}^{0}\, dx + \int_{\substack{y=0\\x=x_0}}^{y_0} \underbrace{\frac{\partial f}{\partial y}}_{x^2} dy = x_0^2 y_0$$

while along path 3 we must relate x and y along the straight line as

$$y = \frac{y_0}{x_0} x \Rightarrow dy = \frac{y_0}{x_0} dx$$

to yield

$$\int_0^P \nabla f \cdot \mathbf{dl} = \int_0^P \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\right) = \int_{x=0}^{x_0} \frac{3 y_0 x^2}{x_0} dx = x_0^2 y_0$$

1-4 FLUX AND DIVERGENCE

If we measure the total mass of fluid entering the volume in Figure 1-13 and find it to be less than the mass leaving, we know that there must be an additional source of fluid within the pipe. If the mass leaving is less than that entering, then

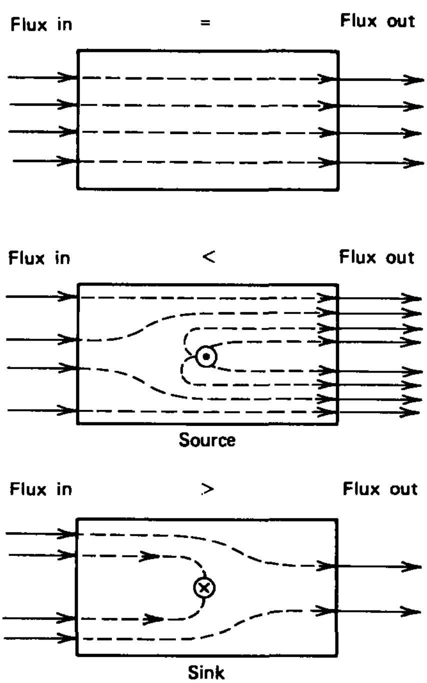


Figure 1-13 The net flux through a closed surface tells us whether there is a source or sink within an enclosed volume.

there is a sink (or drain) within the volume. In the absence of sources or sinks, the mass of fluid leaving equals that entering so the flow lines are continuous. Flow lines originate at a source and terminate at a sink.

1-4-1 Flux

We are illustrating with a fluid analogy what is called the flux Φ of a vector **A** through a closed surface:

$$\Phi = \oint_S \mathbf{A} \cdot \mathbf{dS} \tag{1}$$

The differential surface element **dS** is a vector that has magnitude equal to an incremental area on the surface but points in the direction of the outgoing unit normal **n** to the surface S, as in Figure 1-14. Only the component of **A** perpendicular to the surface contributes to the flux, as the tangential component only results in flow of the vector **A** along the surface and not through it. A positive contribution to the flux occurs if **A** has a component in the direction of **dS** out from the surface. If the normal component of **A** points into the volume, we have a negative contribution to the flux.

If there is no source for **A** within the volume V enclosed by the surface S, all the flux entering the volume equals that leaving and the net flux is zero. A source of **A** within the volume generates more flux leaving than entering so that the flux is positive ($\Phi > 0$) while a sink has more flux entering than leaving so that $\Phi < 0$.

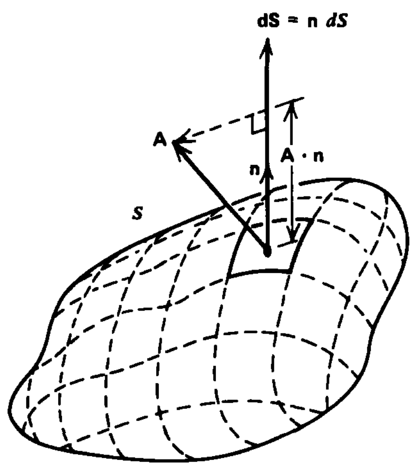


Figure 1-14 The flux of a vector **A** through the closed surface S is given by the surface integral of the component of **A** perpendicular to the surface S. The differential vector surface area element **dS** is in the direction of the unit normal **n**.

Thus we see that the sign and magnitude of the net flux relates the quantity of a field through a surface to the sources or sinks of the vector field within the enclosed volume.

1-4-2 Divergence

We can be more explicit about the relationship between the rate of change of a vector field and its sources by applying (1) to a volume of differential size, which for simplicity we take to be rectangular in Figure 1-15. There are three pairs of plane parallel surfaces perpendicular to the coordinate axes so that (1) gives the flux as

$$\begin{aligned}\Phi = &\int_1 A_x(x)\,dy\,dz - \int_{1'} A_x(x-\Delta x)\,dy\,dz\\ &+\int_2 A_y(y+\Delta y)\,dx\,dz - \int_{2'} A_y(y)\,dx\,dz\\ &+\int_3 A_z(z+\Delta z)\,dx\,dy - \int_{3'} A_z(z)\,dx\,dy \end{aligned} \tag{2}$$

where the primed surfaces are differential distances behind the corresponding unprimed surfaces. The minus signs arise because the outgoing normals on the primed surfaces point in the negative coordinate directions.

Because the surfaces are of differential size, the components of **A** are approximately constant along each surface so that the surface integrals in (2) become pure

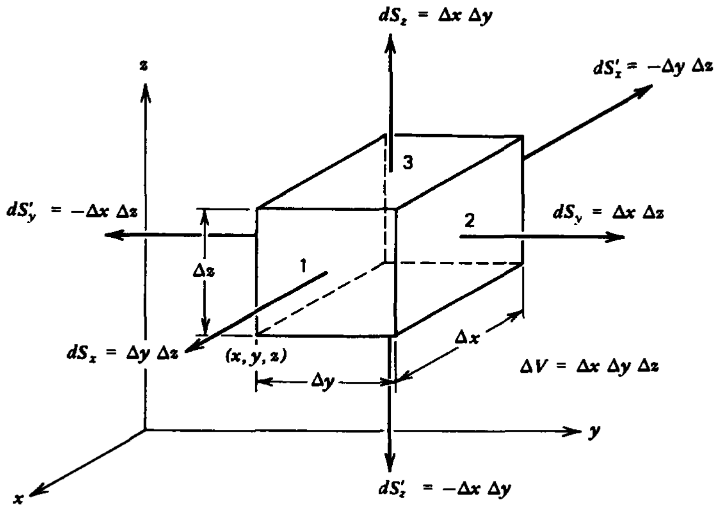


Figure 1-15 Infinitesimal rectangular volume used to define the divergence of a vector.

multiplications of the component of **A** perpendicular to the surface and the surface area. The flux then reduces to the form

$$\Phi \approx \left(\frac{[A_x(x)-A_x(x-\Delta x)]}{\Delta x}+\frac{[A_y(y+\Delta y)-A_y(y)]}{\Delta y}+\frac{[A_z(z+\Delta z)-A_z(z)]}{\Delta z}\right)\Delta x\,\Delta y\,\Delta z \tag{3}$$

We have written (3) in this form so that in the limit as the volume becomes infinitesimally small, each of the bracketed terms defines a partial derivative

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0\\ \Delta z\to 0}} \Phi = \left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)\Delta V \tag{4}$$

where $\Delta V = \Delta x\,\Delta y\,\Delta z$ is the volume enclosed by the surface S.

The coefficient of ΔV in (4) is a scalar and is called the divergence of **A**. It can be recognized as the dot product between the vector del operator of Section 1-3-1 and the vector **A**:

$$\operatorname{div}\mathbf{A} = \nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z} \tag{5}$$

1-4-3 Curvilinear Coordinates

In cylindrical and spherical coordinates, the divergence operation is not simply the dot product between a vector and the del operator because the directions of the unit vectors are a function of the coordinates. Thus, derivatives of the unit vectors have nonzero contributions. It is easiest to use the generalized definition of the divergence independent of the coordinate system, obtained from (1)–(5) as

$$\nabla\cdot\mathbf{A} = \lim_{\Delta V\to 0}\frac{\oint_S \mathbf{A}\cdot d\mathbf{S}}{\Delta V} \tag{6}$$

(a) Cylindrical Coordinates

In cylindrical coordinates we use the small volume shown in Figure 1-16a to evaluate the net flux as

$$\begin{aligned}\Phi = \oint_S \mathbf{A}\cdot d\mathbf{S} &= \int_1 (r+\Delta r)A_{r|_{r+\Delta r}}\,d\phi\,dz - \int_{1'} rA_{r|_r}\,d\phi\,dz\\ &\quad+\int_2 A_{\phi|_{\phi+\Delta\phi}}\,dr\,dz - \int_{2'} A_{\phi|_\phi}\,dr\,dz\\ &\quad+\int_3 rA_{z|_{z+\Delta z}}\,dr\,d\phi - \int_{3'} rA_{z|_z}\,dr\,d\phi\end{aligned} \tag{7}$$

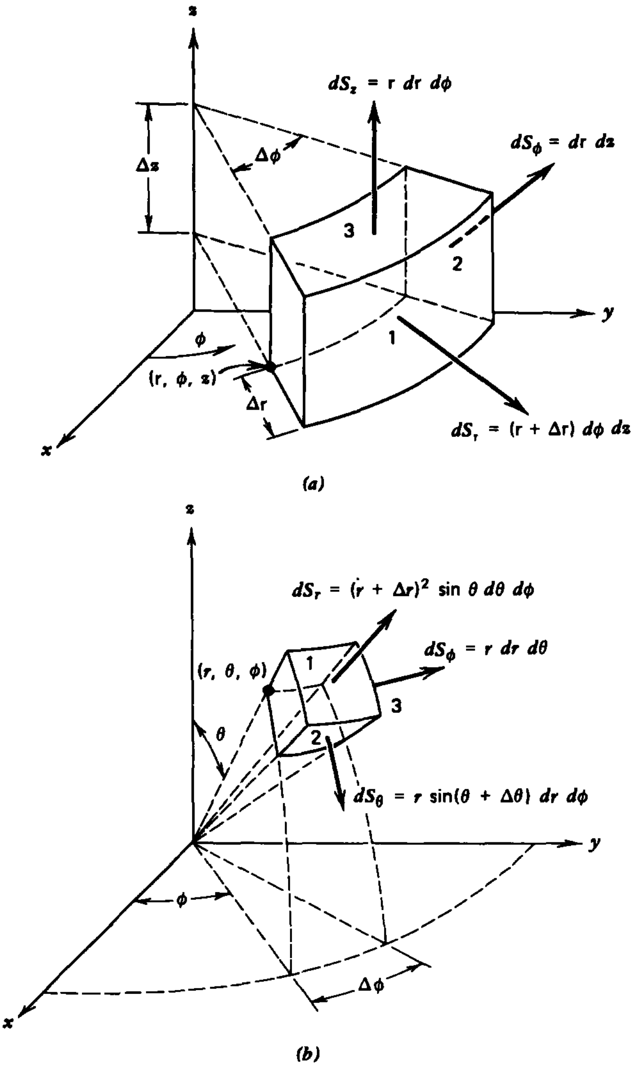


Figure 1-16 Infinitesimal volumes used to define the divergence of a vector in (*a*) cylindrical and (*b*) spherical geometries.

Again, because the volume is small, we can treat it as approximately rectangular with the components of **A** approximately constant along each face. Then factoring out the volume $\Delta V = r\Delta r\, \Delta\phi\, \Delta z$ in (7),

$$\Phi \approx \left(\frac{[(r+\Delta r)A_{r|_{r+\Delta r}} - rA_{r|_r}]}{r\,\Delta r} + \frac{[A_{\phi|_{\phi+\Delta\phi}} - A_{\phi|_\phi}]}{r\,\Delta\phi} + \frac{[A_{z|_{z+\Delta z}} - A_{z|_z}]}{\Delta z}\right) r\Delta r\, \Delta\phi\, \Delta z \qquad (8)$$

lets each of the bracketed terms become a partial derivative as the differential lengths approach zero and (8) becomes an exact relation. The divergence is then

$$\nabla \cdot \mathbf{A} = \lim_{\substack{\Delta r \to 0 \\ \Delta\phi \to 0 \\ \Delta z \to 0}} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta V} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \tag{9}$$

(b) Spherical Coordinates

Similar operations on the spherical volume element $\Delta V = r^2 \sin\theta\, \Delta r\, \Delta\theta\, \Delta\phi$ in Figure 1-16*b* defines the net flux through the surfaces:

$$\begin{aligned}\Phi &= \oint_S \mathbf{A} \cdot d\mathbf{S} \\ &\approx \left(\frac{[(r+\Delta r)^2 A_r|_{r+\Delta r} - r^2 A_r|_r]}{r^2 \Delta r} \right. \\ &\quad + \frac{[A_\theta|_{\theta+\Delta\theta} \sin(\theta + \Delta\theta) - A_\theta|_\theta \sin\theta]}{r \sin\theta\, \Delta\theta} \\ &\quad \left. + \frac{[A_\phi|_{\phi+\Delta\phi} - A_\phi|_\phi]}{r \sin\theta\, \Delta\phi} \right) r^2 \sin\theta\, \Delta r\, \Delta\theta\, \Delta\phi\end{aligned} \tag{10}$$

The divergence in spherical coordinates is then

$$\begin{aligned}\nabla \cdot \mathbf{A} &= \lim_{\substack{\Delta r \to 0 \\ \Delta\theta \to 0 \\ \Delta\phi \to 0}} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta V} \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}\end{aligned} \tag{11}$$

1-4-4 The Divergence Theorem

If we now take many adjoining incremental volumes of any shape, we form a macroscopic volume V with enclosing surface S as shown in Figure 1-17*a*. However, each interior common surface between incremental volumes has the flux leaving one volume (positive flux contribution) just entering the adjacent volume (negative flux contribution) as in Figure 1-17*b*. The net contribution to the flux for the surface integral of (1) is zero for all interior surfaces. Nonzero contributions to the flux are obtained only for those surfaces which bound the outer surface S of V. Although the surface contributions to the flux using (1) cancel for all interior volumes, the flux obtained from (4) in terms of the divergence operation for

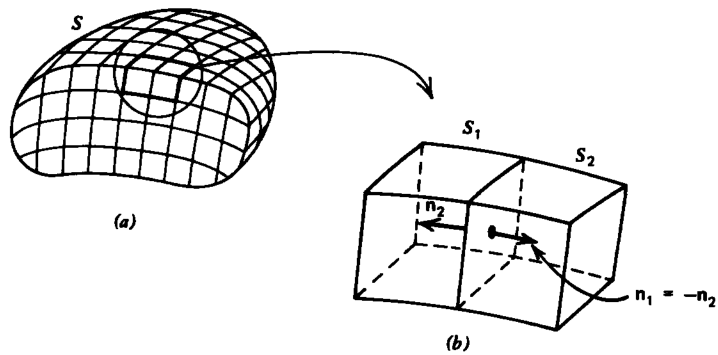


Figure 1-17 Nonzero contributions to the flux of a vector are only obtained across those surfaces that bound the outside of a volume. (a) Within the volume the flux leaving one incremental volume just enters the adjacent volume where (b) the outgoing normals to the common surface separating the volumes are in opposite directions.

each incremental volume add. By adding all contributions from each differential volume, we obtain the divergence theorem:

$$\Phi = \oint_S \mathbf{A} \cdot d\mathbf{S} = \lim_{\substack{N \to \infty \\ \Delta V_n \to 0}} \sum_{n=1}^{\infty} (\nabla \cdot \mathbf{A}) \Delta V_n = \int_V \nabla \cdot \mathbf{A} \, dV \quad (12)$$

where the volume V may be of macroscopic size and is enclosed by the outer surface S. This powerful theorem converts a surface integral into an equivalent volume integral and will be used many times in our development of electromagnetic field theory.

EXAMPLE 1-6 THE DIVERGENCE THEOREM

Verify the divergence theorem for the vector

$$\mathbf{A} = x\mathbf{i}_x + y\mathbf{i}_y + z\mathbf{i}_z = r\mathbf{i}_r$$

by evaluating both sides of (12) for the rectangular volume shown in Figure 1-18.

SOLUTION

The volume integral is easier to evaluate as the divergence of $\mathbf{A}$ is a constant

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 3$$

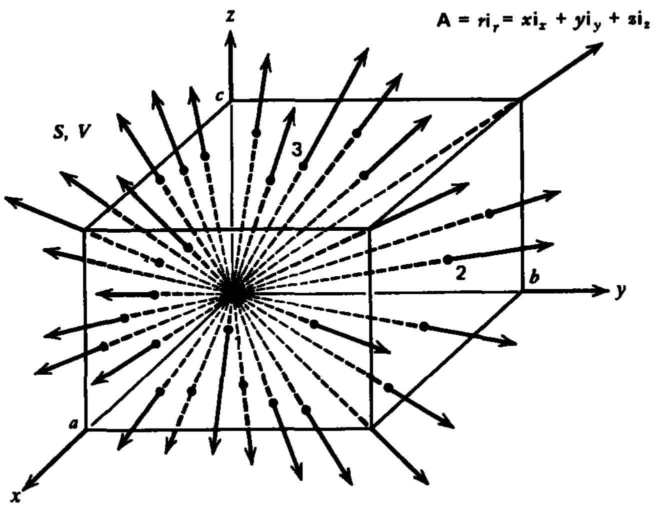


Figure 1-18 The divergence theorem is verified in Example 1-6 for the radial vector through a rectangular volume.

(In spherical coordinates $\nabla \cdot \mathbf{A} = (1/r^2)(\partial/\partial r)(r^3) = 3$) so that the volume integral in (12) is

$$\int_V \nabla \cdot \mathbf{A}\, dV = 3abc$$

The flux passes through the six plane surfaces shown:

$$\Phi = \oint_S \mathbf{A} \cdot d\mathbf{S} = \int_1 \underbrace{A_x(a)}_{a}\, dy\, dz - \int_{1'} \underbrace{A_x(0)}_{0}\, dy\, dz$$

$$+ \int_2 \underbrace{A_y(b)}_{b}\, dx\, dz - \int_{2'} \underbrace{A_y(0)}_{0}\, dx\, dz$$

$$+ \int_3 \underbrace{A_z(c)}_{c}\, dx\, dy - \int_{3'} \underbrace{A_z(0)}_{0}\, dx\, dy = 3abc$$

which verifies the divergence theorem.

1.5 THE CURL AND STOKES' THEOREM

1-5-1 Curl

We have used the example of work a few times previously to motivate particular vector and integral relations. Let us do so once again by considering the line integral of a vector around a closed path called the circulation:

$$C = \oint_L \mathbf{A} \cdot \mathbf{dl} \tag{1}$$

where if C is the work, $\mathbf{A}$ would be the force. We evaluate (1) for the infinitesimal rectangular contour in Figure 1-19*a*:

$$C = \int_{\underset{1}{x}}^{x+\Delta x} A_x(y)\,dx + \int_{\underset{2}{y}}^{y+\Delta y} A_y(x+\Delta x)\,dy + \int_{\underset{3}{x+\Delta x}}^{x} A_x(y+\Delta y)\,dx + \int_{\underset{4}{y+\Delta y}}^{y} A_y(x)\,dy \tag{2}$$

The components of $\mathbf{A}$ are approximately constant over each differential sized contour leg so that (2) is approximated as

$$C \approx \left(\frac{[A_x(y) - A_x(y+\Delta y)]}{\Delta y} + \frac{[A_y(x+\Delta x) - A_y(x)]}{\Delta x}\right)\Delta x\,\Delta y \tag{3}$$

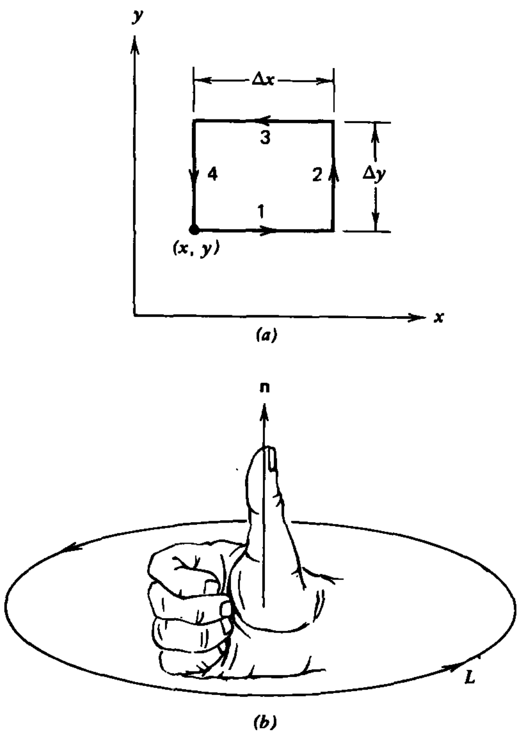


Figure 1-19 (*a*) Infinitesimal rectangular contour used to define the circulation. (*b*) The right-hand rule determines the positive direction perpendicular to a contour.

where terms are factored so that in the limit as Δx and Δy become infinitesimally small, (3) becomes exact and the bracketed terms define partial derivatives:

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta S_z = \Delta x\, \Delta y}} C = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \Delta S_z \tag{4}$$

The contour in Figure 1-19*a* could just have as easily been in the xz or yz planes where (4) would equivalently become

$$C = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) \Delta S_x \quad (yz \text{ plane})$$

$$C = \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) \Delta S_y \quad (xz \text{ plane}) \tag{5}$$

by simple positive permutations of x, y, and z.

The partial derivatives in (4) and (5) are just components of the cross product between the vector del operator of Section 1-3-1 and the vector **A**. This operation is called the curl of **A** and it is also a vector:

$$\text{curl}\, \mathbf{A} = \nabla \times \mathbf{A} = \det \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \mathbf{i}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{i}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$$

$$+ \mathbf{i}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \tag{6}$$

The cyclical permutation of (x, y, z) allows easy recall of (6) as described in Section 1-2-5.

In terms of the curl operation, the circulation for any differential sized contour can be compactly written as

$$C = (\nabla \times \mathbf{A}) \cdot \mathbf{dS} \tag{7}$$

where $\mathbf{dS} = \mathbf{n}\, dS$ is the area element in the direction of the normal vector **n** perpendicular to the plane of the contour in the sense given by the right-hand rule in traversing the contour, illustrated in Figure 1-19*b*. Curling the fingers on the right hand in the direction of traversal around the contour puts the thumb in the direction of the normal **n**.

For a physical interpretation of the curl it is convenient to continue to use a fluid velocity field as a model although the general results and theorems are valid for any vector field. If

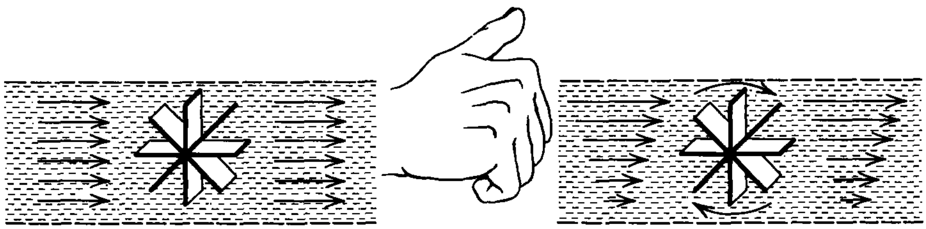


Figure 1-20 A fluid with a velocity field that has a curl tends to turn the paddle wheel. The curl component found is in the same direction as the thumb when the fingers of the right hand are curled in the direction of rotation.

a small paddle wheel is imagined to be placed without disturbance in a fluid flow, the velocity field is said to have circulation, that is, a nonzero curl, if the paddle wheel rotates as illustrated in Figure 1-20. The curl component found is in the direction of the axis of the paddle wheel.

1-5-2 The Curl for Curvilinear Coordinates

A coordinate independent definition of the curl is obtained using (7) in (1) as

$$(\nabla\times\mathbf{A})_n=\lim_{dS_n\to 0}\frac{\oint_L \mathbf{A}\cdot d\mathbf{l}}{dS_n} \tag{8}$$

where the subscript n indicates the component of the curl perpendicular to the contour. The derivation of the curl operation (8) in cylindrical and spherical coordinates is straightforward but lengthy.

(a) Cylindrical Coordinates

To express each of the components of the curl in cylindrical coordinates, we use the three orthogonal contours in Figure 1-21. We evaluate the line integral around contour a:

$$\begin{aligned}\oint_a \mathbf{A}\cdot d\mathbf{l}&=\int_z^{z-\Delta z}A_z(\phi)\,dz+\int_\phi^{\phi+\Delta\phi}A_\phi(z-\Delta z)\,\mathrm{r}\,d\phi\\&\quad+\int_{z-\Delta z}^{z}A_z(\phi+\Delta\phi)\,dz+\int_{\phi+\Delta\phi}^{\phi}A_\phi(z)\,\mathrm{r}\,d\phi\\&\approx\left(\frac{[A_z(\phi+\Delta\phi)-A_z(\phi)]}{\mathrm{r}\Delta\phi}-\frac{[A_\phi(z)-A_\phi(z-\Delta z)]}{\Delta z}\right)\mathrm{r}\,\Delta\phi\,\Delta z\end{aligned} \tag{9}$$

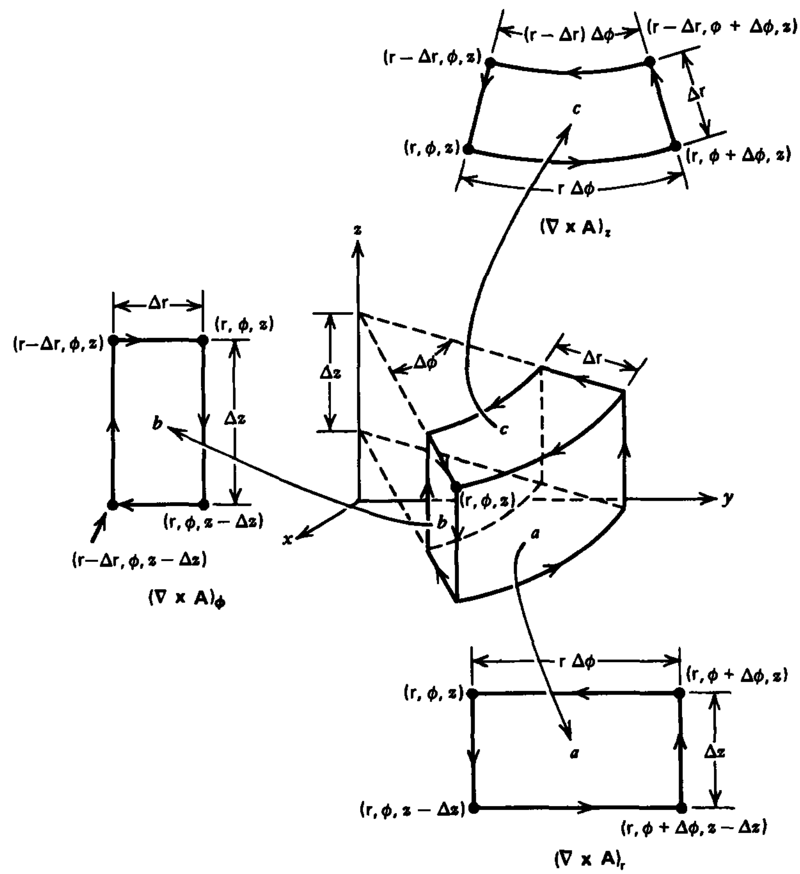


Figure 1-21 Incremental contours along cylindrical surface area elements used to calculate each component of the curl of a vector in cylindrical coordinates.

to find the radial component of the curl as

$$(\nabla\times\mathbf{A})_r=\lim_{\substack{\Delta\phi\to 0\\ \Delta z\to 0}}\frac{\oint_a \mathbf{A}\cdot d\mathbf{l}}{r\,\Delta\phi\,\Delta z}=\frac{1}{r}\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z} \tag{10}$$

We evaluate the line integral around contour b:

$$\begin{aligned}\oint_b \mathbf{A}\cdot d\mathbf{l}&=\int_{r-\Delta r}^{r} A_r(z)\,dr+\int_{z}^{z-\Delta z} A_z(r)\,dz+\int_{r}^{r-\Delta r} A_r(z-\Delta z)\,dr\\&\quad+\int_{z-\Delta z}^{z} A_z(r-\Delta r)\,dz\\&\approx\left(\frac{[A_r(z)-A_r(z-\Delta z)]}{\Delta z}-\frac{[A_z(r)-A_z(r-\Delta r)]}{\Delta r}\right)\Delta r\,\Delta z\end{aligned} \tag{11}$$

to find the ϕ component of the curl,

$$(\nabla\times\mathbf{A})_\phi=\lim_{\substack{\Delta r\to 0\\ \Delta z\to 0}}\frac{\oint_b \mathbf{A}\cdot\mathbf{dl}}{\Delta r\,\Delta z}=\left(\frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r}\right)\tag{12}$$

The z component of the curl is found using contour c:

$$\begin{aligned}\oint_c \mathbf{A}\cdot\mathbf{dl}&=\int_{r-\Delta r}^{r} A_{r|_\phi}\,dr+\int_{\phi}^{\phi+\Delta\phi} rA_{\phi|_r}\,d\phi+\int_{r}^{r-\Delta r} A_{r|_{\phi+\Delta\phi}}\,dr\\ &\quad+\int_{\phi+\Delta\phi}^{\phi}(r-\Delta r)A_{\phi|_{r-\Delta r}}\,d\phi\\ &\approx\left(\frac{[rA_{\phi|_r}-(r-\Delta r)A_{\phi|_{r-\Delta r}}]}{r\Delta r}-\frac{[A_{r|_{\phi+\Delta\phi}}-A_{r|_\phi}]}{r\Delta\phi}\right)r\,\Delta r\,\Delta\phi\end{aligned}\tag{13}$$

to yield

$$(\nabla\times\mathbf{A})_z=\lim_{\substack{\Delta r\to 0\\ \Delta\phi\to 0}}\frac{\oint_c \mathbf{A}\cdot\mathbf{dl}}{r\,\Delta r\,\Delta\phi}=\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\phi)-\frac{\partial A_r}{\partial\phi}\right)\tag{14}$$

The curl of a vector in cylindrical coordinates is thus

$$\begin{aligned}\nabla\times\mathbf{A}&=\left(\frac{1}{r}\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z}\right)\mathbf{i}_r+\left(\frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r}\right)\mathbf{i}_\phi\\ &\quad+\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\phi)-\frac{\partial A_r}{\partial\phi}\right)\mathbf{i}_z\end{aligned}\tag{15}$$

(b) Spherical Coordinates

Similar operations on the three incremental contours for the spherical element in Figure 1-22 give the curl in spherical coordinates. We use contour a for the radial component of the curl:

$$\begin{aligned}\oint_a \mathbf{A}\cdot\mathbf{dl}&=\int_{\phi}^{\phi+\Delta\phi} A_{\phi|_\theta}r\sin\theta\,d\phi+\int_{\theta}^{\theta-\Delta\theta} rA_{\theta|_{\phi+\Delta\phi}}\,d\theta\\ &\quad+\int_{\phi+\Delta\phi}^{\phi} r\sin(\theta-\Delta\theta)A_{\phi|_{\theta-\Delta\theta}}\,d\phi+\int_{\theta-\Delta\theta}^{\theta} rA_{\theta|_\phi}\,d\theta\\ &\approx\left(\frac{[A_{\phi|_\theta}\sin\theta-A_{\phi|_{\theta-\Delta\theta}}\sin(\theta-\Delta\theta)]}{r\sin\theta\,\Delta\theta}\right.\\ &\quad\left.-\frac{[A_{\theta|_{\phi+\Delta\phi}}-A_{\theta|_\phi}]}{r\sin\theta\,\Delta\phi}\right)r^2\sin\theta\,\Delta\theta\,\Delta\phi\end{aligned}\tag{16}$$

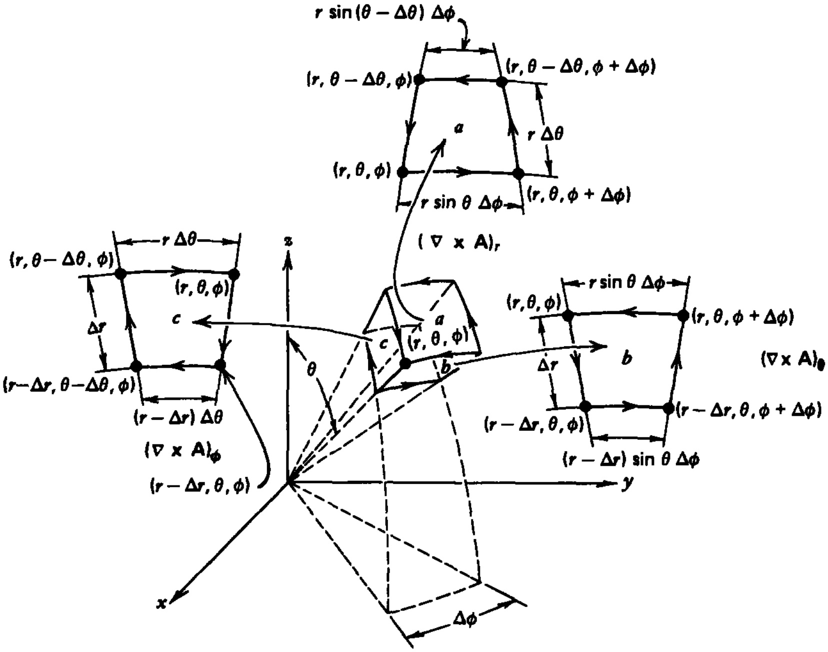


Figure 1-22 Incremental contours along spherical surface area elements used to calculate each component of the curl of a vector in spherical coordinates.

to obtain

$$(\nabla\times\mathbf{A})_r=\lim_{\substack{\Delta\theta\to 0\\ \Delta\phi\to 0}}\frac{\oint_a \mathbf{A}\cdot d\mathbf{l}}{r^2\sin\theta\,\Delta\theta\,\Delta\phi}=\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi\sin\theta)-\frac{\partial A_\theta}{\partial\phi}\right) \tag{17}$$

The θ component is found using contour b:

$$\oint_b \mathbf{A}\cdot d\mathbf{l}=\int_r^{r-\Delta r} A_r|_\phi\,dr+\int_\phi^{\phi+\Delta\phi}(r-\Delta r)A_\phi|_{r-\Delta r}\sin\theta\,d\phi$$

$$+\int_{r-\Delta r}^{r} A_r|_{\phi+\Delta\phi}\,dr+\int_{\phi+\Delta\phi}^{\phi} rA_\phi|_r\sin\theta\,d\phi$$

$$\approx\left(\frac{[A_r|_{\phi+\Delta\phi}-A_r|_\phi]}{r\sin\theta\,\Delta\phi}\right.$$

$$\left.-\frac{[rA_\phi|_r-(r-\Delta r)A_\phi|_{r-\Delta r}]}{r\,\Delta r}\right)r\sin\theta\,\Delta r\,\Delta\phi \tag{18}$$

as

$$(\nabla\times\mathbf{A})_\theta=\lim_{\substack{\Delta r\to 0\\ \Delta\phi\to 0}}\frac{\oint_b \mathbf{A}\cdot\mathbf{dl}}{r\sin\theta\,\Delta r\,\Delta\phi}=\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{\partial}{\partial r}(rA_\phi)\right) \tag{19}$$

The ϕ component of the curl is found using contour c:

$$\begin{aligned}\oint_c \mathbf{A}\cdot\mathbf{dl}&=\int_{\theta-\Delta\theta}^{\theta} rA_{\theta|_r}\,d\theta+\int_r^{r-\Delta r} A_{r|_\theta}\,dr\\&\quad+\int_\theta^{\theta-\Delta\theta}(r-\Delta r)A_{\theta|_{r-\Delta r}}\,d\theta+\int_{r-\Delta r}^{r} A_{r|_{\theta-\Delta\theta}}\,dr\\&\approx\left(\frac{[rA_{\theta|_r}-(r-\Delta r)A_{\theta|_{r-\Delta r}}]}{r\,\Delta r}-\frac{[A_{r|_\theta}-A_{r|_{\theta-\Delta\theta}}]}{r\,\Delta\theta}\right)r\,\Delta r\,\Delta\theta\end{aligned} \tag{20}$$

as

$$(\nabla\times\mathbf{A})_\phi=\lim_{\substack{\Delta r\to 0\\ \Delta\theta\to 0}}\frac{\oint_c \mathbf{A}\cdot\mathbf{dl}}{r\,\Delta r\,\Delta\theta}=\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial A_r}{\partial\theta}\right) \tag{21}$$

The curl of a vector in spherical coordinates is thus given from (17), (19), and (21) as

$$\begin{aligned}\nabla\times\mathbf{A}&=\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi\sin\theta)-\frac{\partial A_\theta}{\partial\phi}\right)\mathbf{i}_r\\&\quad+\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{\partial}{\partial r}(rA_\phi)\right)\mathbf{i}_\theta\\&\quad+\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial A_r}{\partial\theta}\right)\mathbf{i}_\phi\end{aligned} \tag{22}$$

1-5-3 Stokes' Theorem

We now piece together many incremental line contours of the type used in Figures 1-19–1-21 to form a macroscopic surface S like those shown in Figure 1-23. Then each small contour generates a contribution to the circulation

$$dC=(\nabla\times\mathbf{A})\cdot\mathbf{dS} \tag{23}$$

so that the total circulation is obtained by the sum of all the small surface elements

$$C=\int_S(\nabla\times\mathbf{A})\cdot\mathbf{dS} \tag{24}$$

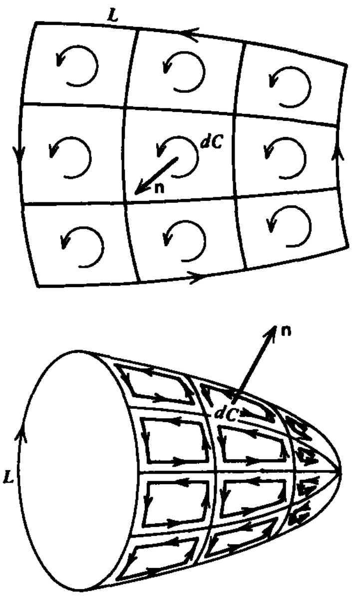


Figure 1-23 Many incremental line contours distributed over any surface, have nonzero contribution to the circulation only along those parts of the surface on the boundary contour L.

Each of the terms of (23) are equivalent to the line integral around each small contour. However, all interior contours share common sides with adjacent contours but which are twice traversed in opposite directions yielding no net line integral contribution, as illustrated in Figure 1-23. Only those contours with a side on the open boundary L have a nonzero contribution. The total result of adding the contributions for all the contours is Stokes' theorem, which converts the line integral over the bounding contour L of the outer edge to a surface integral over any area S bounded by the contour

$$\oint_L \mathbf{A} \cdot \mathbf{dl} = \int_S (\nabla \times \mathbf{A}) \cdot \mathbf{dS} \tag{25}$$

Note that there are an infinite number of surfaces that are bounded by the same contour L. Stokes' theorem of (25) is satisfied for all these surfaces.

EXAMPLE 1-7 STOKES' THEOREM

Verify Stokes' theorem of (25) for the circular bounding contour in the xy plane shown in Figure 1-24 with a vector

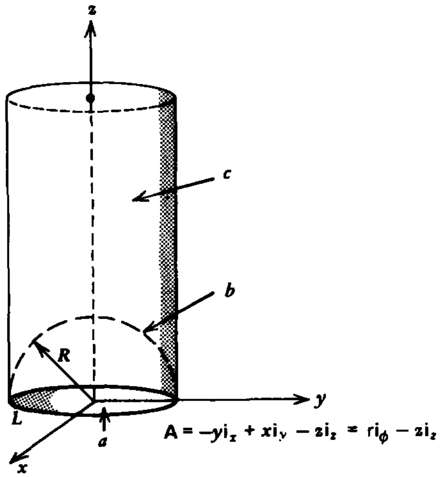


Figure 1-24 Stokes' theorem for the vector given in Example 1-7 can be applied to any surface that is bounded by the same contour L.

field

$$\mathbf{A} = -y\mathbf{i}_x + x\mathbf{i}_y - z\mathbf{i}_z = r\mathbf{i}_\phi - z\mathbf{i}_z$$

Check the result for the (a) flat circular surface in the xy plane, (b) for the hemispherical surface bounded by the contour, and (c) for the cylindrical surface bounded by the contour.

SOLUTION

For the contour shown

$$\mathbf{dl} = R\, d\phi\, \mathbf{i}_\phi$$

so that

$$\mathbf{A} \cdot \mathbf{dl} = R^2\, d\phi$$

where on L, $r = R$. Then the circulation is

$$C = \oint_L \mathbf{A} \cdot \mathbf{dl} = \int_0^{2\pi} R^2\, d\phi = 2\pi R^2$$

The z component of $\mathbf{A}$ had no contribution because $\mathbf{dl}$ was entirely in the xy plane.

The curl of $\mathbf{A}$ is

$$\nabla \times \mathbf{A} = \mathbf{i}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = 2\mathbf{i}_z$$

(a) For the circular area in the plane of the contour, we have that

$$\int_S (\nabla\times\mathbf{A})\cdot\mathbf{dS}=2\int_S dS_z=2\pi R^2$$

which agrees with the line integral result.

(b) For the hemispherical surface

$$\int_S(\nabla\times\mathbf{A})\cdot\mathbf{dS}=\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}2\mathbf{i}_z\cdot\mathbf{i}_rR^2\sin\theta\,d\theta\,d\phi$$

From Table 1-2 we use the dot product relation

$$\mathbf{i}_z\cdot\mathbf{i}_r=\cos\theta$$

which again gives the circulation as

$$C=\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}R^2\sin 2\theta\,d\theta\,d\phi=-2\pi R^2\frac{\cos 2\theta}{2}\bigg|_{\theta=0}^{\pi/2}=2\pi R^2$$

(c) Similarly, for the cylindrical surface, we only obtain nonzero contributions to the surface integral at the upper circular area that is perpendicular to $\nabla\times\mathbf{A}$. The integral is then the same as part (a) as $\nabla\times\mathbf{A}$ is independent of z.

1-5-4 Some Useful Vector Identities

The curl, divergence, and gradient operations have some simple but useful properties that are used throughout the text.

(a) The Curl of the Gradient is Zero [$\nabla\times(\nabla f)=0$]

We integrate the normal component of the vector $\nabla\times(\nabla f)$ over a surface and use Stokes' theorem

$$\int_S\nabla\times(\nabla f)\cdot\mathbf{dS}=\oint_L\nabla f\cdot\mathbf{dl}=0 \tag{26}$$

where the zero result is obtained from Section 1-3-3, that the line integral of the gradient of a function around a closed path is zero. Since the equality is true for any surface, the vector coefficient of $\mathbf{dS}$ in (26) must be zero

$$\nabla\times(\nabla f)=0$$

The identity is also easily proved by direct computation using the determinantal relation in Section 1-5-1 defining the curl operation:

$$\nabla\times(\nabla f)=\det\begin{vmatrix}\mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z}\end{vmatrix}$$

$$=\mathbf{i}_x\left(\frac{\partial^2 f}{\partial y\partial z}-\frac{\partial^2 f}{\partial z\partial y}\right)+\mathbf{i}_y\left(\frac{\partial^2 f}{\partial z\partial x}-\frac{\partial^2 f}{\partial x\partial z}\right)+\mathbf{i}_z\left(\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial^2 f}{\partial y\partial x}\right)=0 \tag{28}$$

Each bracketed term in (28) is zero because the order of differentiation does not matter.

(b) The Divergence of the Curl of a Vector is Zero [$\nabla\cdot(\nabla\times\mathbf{A})=0$]

One might be tempted to apply the divergence theorem to the surface integral in Stokes' theorem of (25). However, the divergence theorem requires a closed surface while Stokes' theorem is true in general for an open surface. Stokes' theorem for a closed surface requires the contour L to shrink to zero giving a zero result for the line integral. The divergence theorem applied to the closed surface with vector $\nabla\times\mathbf{A}$ is then

$$\oint_S \nabla\times\mathbf{A}\cdot d\mathbf{S}=0\Rightarrow\int_V \nabla\cdot(\nabla\times\mathbf{A})\,dV=0\Rightarrow\nabla\cdot(\nabla\times\mathbf{A})=0 \tag{29}$$

which proves the identity because the volume is arbitrary.

More directly we can perform the required differentiations

$$\nabla\cdot(\nabla\times\mathbf{A})$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)$$

$$=\left(\frac{\partial^2 A_z}{\partial x\partial y}-\frac{\partial^2 A_z}{\partial y\partial x}\right)+\left(\frac{\partial^2 A_x}{\partial y\partial z}-\frac{\partial^2 A_x}{\partial z\partial y}\right)+\left(\frac{\partial^2 A_y}{\partial z\partial x}-\frac{\partial^2 A_y}{\partial x\partial z}\right)=0 \tag{30}$$

where again the order of differentiation does not matter.

PROBLEMS

Section 1-1

1. Find the area of a circle in the xy plane centered at the origin using:

(a) rectangular coordinates $x^2+y^2=a^2$ (**Hint:**

$$\int\sqrt{a^2-x^2}\,dx=\tfrac{1}{2}[x\sqrt{a^2-x^2}+a^2\sin^{-1}(x/a)])$$

(b) cylindrical coordinates $r = a$.
Which coordinate system is easier to use?

2. Find the volume of a sphere of radius R centered at the origin using:

(a) rectangular coordinates $x^2+y^2+z^2=R^2$ (**Hint:**
$$\int \sqrt{a^2-x^2}\,dx = \tfrac{1}{2}[x\sqrt{a^2-x^2}+a^2\sin^{-1}(x/a)])$$
(b) cylindrical coordinates $r^2+z^2=R^2$;
(c) spherical coordinates $r = R$.
Which coordinate system is easiest?

Section 1-2
3. Given the three vectors

$$\mathbf{A} = 3\mathbf{i}_x + 2\mathbf{i}_y - \mathbf{i}_z$$
$$\mathbf{B} = 3\mathbf{i}_x - 4\mathbf{i}_y - 5\mathbf{i}_z$$
$$\mathbf{C} = \mathbf{i}_x - \mathbf{i}_y + \mathbf{i}_z$$

find the following:

(a) $\mathbf{A} \pm \mathbf{B}$, $\mathbf{B} \pm \mathbf{C}$, $\mathbf{A} \pm \mathbf{C}$
(b) $\mathbf{A} \cdot \mathbf{B}$, $\mathbf{B} \cdot \mathbf{C}$, $\mathbf{A} \cdot \mathbf{C}$
(c) $\mathbf{A} \times \mathbf{B}$, $\mathbf{B} \times \mathbf{C}$, $\mathbf{A} \times \mathbf{C}$
(d) $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$, $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ [Are they equal?]
(e) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$, $\mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$ [Are they equal?]
(f) What is the angle between **A** and **C** and between **B** and $\mathbf{A} \times \mathbf{C}$?

4. Given the sum and difference between two vectors,

$$\mathbf{A} + \mathbf{B} = -\mathbf{i}_x + 5\mathbf{i}_y - 4\mathbf{i}_z$$
$$\mathbf{A} - \mathbf{B} = 3\mathbf{i}_x - \mathbf{i}_y - 2\mathbf{i}_z$$

find the individual vectors **A** and **B**.
5. (a) Given two vectors **A** and **B**, show that the component of **B** parallel to **A** is

$$\mathbf{B}_\parallel = \frac{\mathbf{B} \cdot \mathbf{A}}{\mathbf{A} \cdot \mathbf{A}}\mathbf{A}$$

(**Hint:** $\mathbf{B}_\parallel = \alpha\mathbf{A}$. What is α?)
(b) If the vectors are

$$\mathbf{A} = \mathbf{i}_x - 2\mathbf{i}_y + \mathbf{i}_z$$
$$\mathbf{B} = 3\mathbf{i}_x + 5\mathbf{i}_y - 5\mathbf{i}_z$$

what are the components of **B** parallel and perpendicular to **A**?

$$\mathbf{B} = \mathbf{B}_\perp + \mathbf{B}_\parallel$$

6. What are the angles between each of the following vectors:

$$\mathbf{A} = 4\mathbf{i}_x - 2\mathbf{i}_y + 2\mathbf{i}_z$$

$$\mathbf{B} = -6\mathbf{i}_x + 3\mathbf{i}_y - 3\mathbf{i}_z$$

$$\mathbf{C} = \mathbf{i}_x + 3\mathbf{i}_y + \mathbf{i}_z$$

7. Given the two vectors

$$\mathbf{A} = 3\mathbf{i}_x + 4\mathbf{i}_y \quad \text{and} \quad \mathbf{B} = 7\mathbf{i}_x - 24\mathbf{i}_y$$

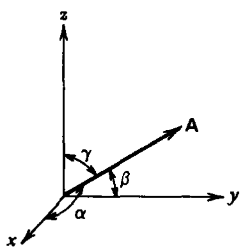
(a) What is their dot product?
(b) What is their cross product?
(c) What is the angle θ between the two vectors?

8. Given the vector

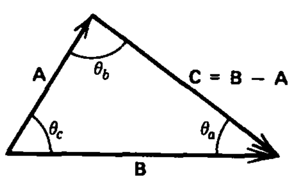
$$\mathbf{A} = A_x\mathbf{i}_x + A_y\mathbf{i}_y + A_z\mathbf{i}_z$$

the directional cosines are defined as the cosines of the angles between $\mathbf{A}$ and each of the Cartesian coordinate axes. Find each of these directional cosines and show that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$



9. A triangle is formed by the three vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C} = \mathbf{B} - \mathbf{A}$.



(a) Find the length of the vector $\mathbf{C}$ in terms of the lengths of $\mathbf{A}$ and $\mathbf{B}$ and the enclosed angle θ_c. The result is known as the law of cosines. (**Hint:** $\mathbf{C}\cdot\mathbf{C} = (\mathbf{B}-\mathbf{A})\cdot(\mathbf{B}-\mathbf{A})$.)

(b) For the same triangle, prove the law of sines:

$$\frac{\sin\theta_a}{A} = \frac{\sin\theta_b}{B} = \frac{\sin\theta_c}{C}$$

(**Hint:** $\mathbf{B}\times\mathbf{A} = (\mathbf{C}+\mathbf{A})\times\mathbf{A}$.)

10. (a) Prove that the dot and cross can be interchanged in the scalar triple product

$$(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}=(\mathbf{B}\times\mathbf{C})\cdot\mathbf{A}=(\mathbf{C}\times\mathbf{A})\cdot\mathbf{B}$$

(b) Show that this product gives the volume of a parallelepiped whose base is defined by the vectors **A** and **B** and whose height is given by **C**.

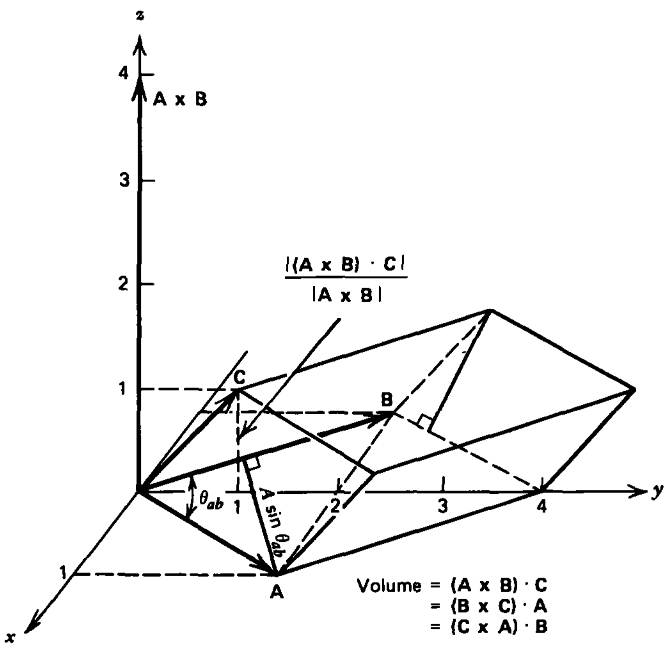
(c) If

$$\mathbf{A}=\mathbf{i}_x+2\mathbf{i}_y,\quad \mathbf{B}=-\mathbf{i}_x+2\mathbf{i}_y,\quad \mathbf{C}=\mathbf{i}_x+\mathbf{i}_z$$

verify the identities of (a) and find the volume of the parallelepiped formed by the vectors.

(d) Prove the vector triple product identity

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

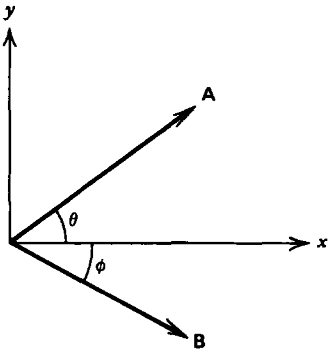


11. (a) Write the vectors **A** and **B** using Cartesian coordinates in terms of their angles θ and ϕ from the x axis.

(b) Using the results of (a) derive the trigonometric expansions

$$\sin(\theta+\phi)=\sin\theta\cos\phi+\sin\phi\cos\theta$$

$$\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi$$



Section 1-3

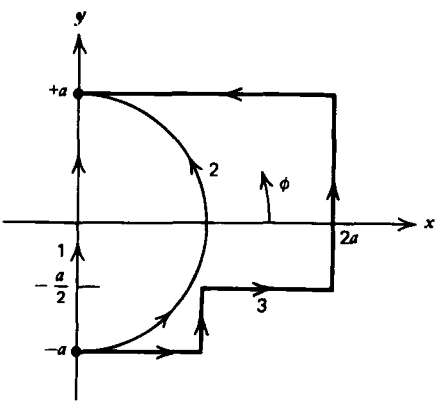
12. Find the gradient of each of the following functions where a and b are constants:

(a) $f = axz + bx^3y$
(b) $f = (a/r)\sin\phi + brz^2\cos 3\phi$
(c) $f = ar\cos\theta + (b/r^2)\sin\phi$

13. Evaluate the line integral of the gradient of the function

$$f = r\sin\phi$$

over each of the contours shown.



Section 1-4

14. Find the divergence of the following vectors:

(a) $\mathbf{A} = x\mathbf{i}_x + y\mathbf{i}_y + z\mathbf{i}_z = r\mathbf{i}_r$
(b) $\mathbf{A} = (xy^2z^3)[\mathbf{i}_x + \mathbf{i}_y + \mathbf{i}_z]$
(c) $\mathbf{A} = r\cos\phi\,\mathbf{i}_r + [(z/r)\sin\phi)]\mathbf{i}_z$
(d) $\mathbf{A} = r^2\sin\theta\cos\phi\,[\mathbf{i}_r + \mathbf{i}_\theta + \mathbf{i}_\phi]$

15. Using the divergence theorem prove the following integral identities:

(a) $\int_V \nabla f\,dV = \oint_S f\,\mathbf{dS}$

(**Hint:** Let $\mathbf{A} = \mathbf{i}f$, where $\mathbf{i}$ is any *constant* unit vector.)

(b) $\int_V \nabla \times \mathbf{F}\, dV = -\oint_S \mathbf{F} \times \mathbf{dS}$

(**Hint:** Let $\mathbf{A} = \mathbf{i} \times \mathbf{F}$.)

(c) Using the results of (a) show that the normal vector integrated over a surface is zero:

$$\oint_S \mathbf{dS} = 0$$

(d) Verify (c) for the case of a sphere of radius R. (**Hint:** $\mathbf{i}_r = \sin\theta\cos\phi\,\mathbf{i}_x + \sin\theta\sin\phi\,\mathbf{i}_y + \cos\theta\,\mathbf{i}_z$.

16. Using the divergence theorem prove Green's theorem

$$\oint_S [f\nabla g - g\nabla f] \cdot \mathbf{dS} = \int_V [f\nabla^2 g - g\nabla^2 f]\, dV$$

(**Hint:** $\nabla \cdot (f\nabla g) = f\nabla^2 g + \nabla f \cdot \nabla g$.)

17. (a) Find the area element $\mathbf{dS}$ (magnitude and direction) on each of the four surfaces of the pyramidal figure shown.

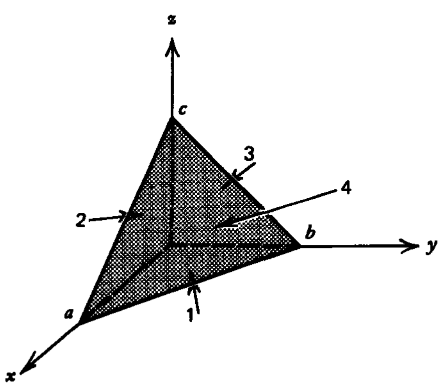
(b) Find the flux of the vector

$$\mathbf{A} = r\mathbf{i}_r = x\mathbf{i}_x + y\mathbf{i}_y + z\mathbf{i}_z$$

through the surface of (a).

(c) Verify the divergence theorem by also evaluating the flux as

$$\Phi = \int_V \nabla \cdot \mathbf{A}\, dV$$



Section 1-5

18. Find the curl of the following vectors:

(a) $\mathbf{A} = x^2y\mathbf{i}_x + y^2z\mathbf{i}_y + xy\mathbf{i}_z$

(b) $\mathbf{A} = r\cos\phi\,\mathbf{i}_z + \dfrac{z\sin\phi}{r}\mathbf{i}_r$

(c) $\mathbf{A} = r^2\sin\theta\cos\phi\,\mathbf{i}_r + \dfrac{\cos\theta\sin\phi}{r^2}\mathbf{i}_\theta$

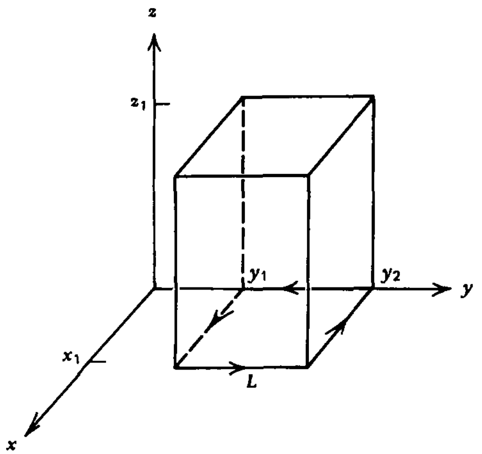
19. Using Stokes' theorem prove that

$$\oint_L f\,\mathbf{dl} = -\int_S \nabla f \times \mathbf{dS}$$

(**Hint:** Let $\mathbf{A} = \mathbf{i}f$, where $\mathbf{i}$ is any *constant* unit vector.)

20. Verify Stokes' theorem for the rectangular bounding contour in the xy plane with a vector field

$$\mathbf{A} = (x+a)(y+b)(z+c)\mathbf{i}_x$$



Check the result for (a) a flat rectangular surface in the xy plane, and (b) for the rectangular cylinder.

21. Show that the order of differentiation for the mixed second derivative

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

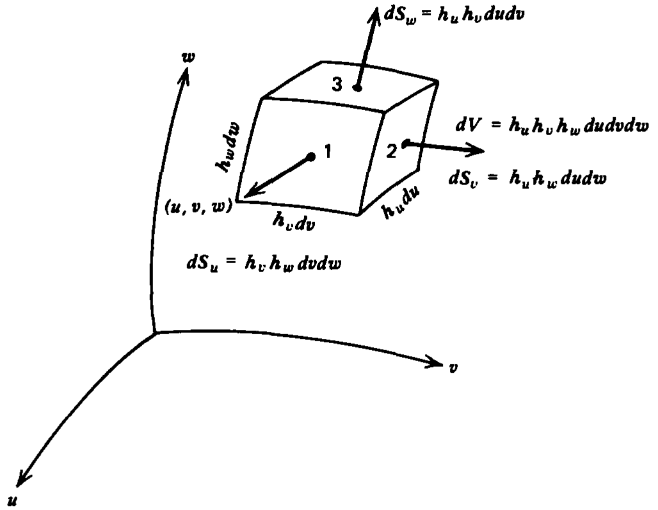
does not matter for the function

$$f = \frac{x^2 \ln y}{y}$$

22. Some of the unit vectors in cylindrical and spherical coordinates change direction in space and thus, unlike Cartesian unit vectors, are not constant vectors. This means that spatial derivatives of these unit vectors are generally nonzero. Find the divergence and curl of all the unit vectors.

23. A general right-handed orthogonal curvilinear coordinate system is described by variables (u, v, w), where

$$\mathbf{i}_u \times \mathbf{i}_v = \mathbf{i}_w$$



Since the incremental coordinate quantities du, dv, and dw do not necessarily have units of length, the differential length elements must be multiplied by coefficients that generally are a function of u, v, and w:

$$dL_u = h_u\, du, \quad dL_v = h_v\, dv, \quad dL_w = h_w\, dw$$

(a) What are the h coefficients for the Cartesian, cylindrical, and spherical coordinate systems?

(b) What is the gradient of any function $f(u, v, w)$?

(c) What is the area of each surface and the volume of a differential size volume element in the (u, v, w) space?

(d) What are the curl and divergence of the vector

$$\mathbf{A} = A_u \mathbf{i}_u + A_v \mathbf{i}_v + A_w \mathbf{i}_w?$$

(e) What is the scalar Laplacian $\nabla^2 f = \nabla \cdot (\nabla f)$?

(f) Check your results of (b)–(e) for the three basic coordinate systems.

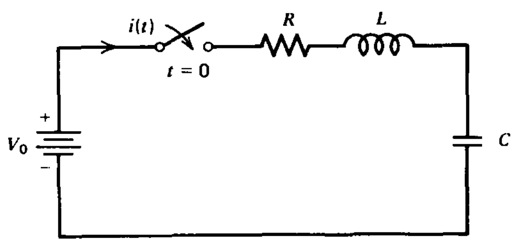
24. Prove the following vector identities:

(a) $\nabla(fg) = f\nabla g + g\nabla f$
(b) $\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$
(c) $\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + (\mathbf{A} \cdot \nabla)f$
(d) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$
(e) $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$

(f) $\nabla \times (f\mathbf{A}) = \nabla f \times \mathbf{A} + f \nabla \times \mathbf{A}$
(g) $(\nabla \times \mathbf{A}) \times \mathbf{A} = (\mathbf{A} \cdot \nabla)\mathbf{A} - \frac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A})$
(h) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

25. Two points have Cartesian coordinates (1, 2, −1) and (2, −3, 1).
(a) What is the distance between these two points?
(b) What is the unit vector along the line joining the two points?
(c) Find a unit vector in the xy plane perpendicular to the unit vector found in (b).

Miscellaneous
26. A series RLC circuit offers a good review in solving linear, constant coefficient ordinary differential equations. A step voltage V_0 is applied to the initially unexcited circuit at $t = 0$.



(a) Write a single differential equation for the current.
(b) Guess an exponential solution of the form

$$i(t) = \hat{I}e^{st}$$

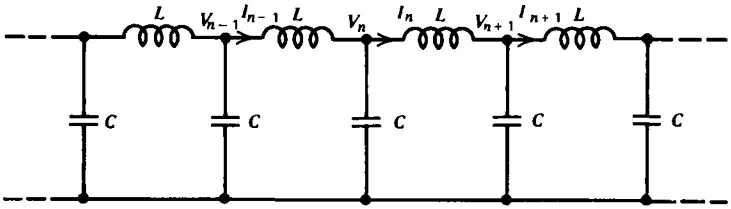
and find the natural frequencies of the circuit.
(c) What are the initial conditions? What are the steady-state voltages across each element?
(d) Write and sketch the solution for $i(t)$ when

$$\left(\frac{R}{2L}\right)^2 < \frac{1}{LC}, \quad \left(\frac{R}{2L}\right)^2 = \frac{1}{LC}, \quad \left(\frac{R}{2L}\right)^2 > \frac{1}{LC}$$

(e) What is the voltage across each element?
(f) After the circuit has reached the steady state, the terminal voltage is instantly short circuited. What is the short circuit current?

27. Many times in this text we consider systems composed of repetitive sequences of a basic building block. Such discrete element systems are described by difference equations. Consider a distributed series inductance-shunt capacitance system excited by a sinusoidal frequency ω so that the voltage and current in the nth loop vary as

$$i_n = \text{Re}\,(I_n e^{j\omega t}); \quad v_n = \text{Re}(V_n e^{j\omega t})$$



(a) By writing Kirchoff's voltage law for the nth loop, show that the current obeys the difference equation

$$I_{n+1}-\left(2-\frac{\omega^2}{\omega_0^2}\right)I_n+I_{n-1}=0$$

What is ω_0^2?

(b) Just as exponential solutions satisfy linear constant coefficient differential equations, power-law solutions satisfy linear constant coefficient difference equations

$$I_n=\hat{I}\lambda^n$$

What values of λ satisfy (a)?

(c) The general solution to (a) is a linear combination of all the possible solutions. The circuit ladder that has N nodes is excited in the zeroth loop by a current source

$$i_0=\text{Re}\,(I_0\,e^{j\omega t})$$

Find the general expression for current i_n and voltage v_n for any loop when the last loop N is either open ($I_N=0$) or short circuited ($V_N=0$). (**Hint:** $a+\sqrt{a^2-1}=1/(a-\sqrt{a^2-1})$)

(d) What are the natural frequencies of the system when the last loop is either open or short circuited? (**Hint:** $(1)^{1/(2N)}=e^{j2\pi r/2N}$, $r=1,2,3,\ldots,2N$.)

chapter 2

the electric field

The ancient Greeks observed that when the fossil resin amber was rubbed, small light-weight objects were attracted. Yet, upon contact with the amber, they were then repelled. No further significant advances in the understanding of this mysterious phenomenon were made until the eighteenth century when more quantitative electrification experiments showed that these effects were due to electric charges, the source of all effects we will study in this text.

2-1 ELECTRIC CHARGE

2-1-1 Charging by Contact

We now know that all matter is held together by the attractive force between equal numbers of negatively charged electrons and positively charged protons. The early researchers in the 1700s discovered the existence of these two species of charges by performing experiments like those in Figures 2-1 to 2-4. When a glass rod is rubbed by a dry cloth, as in Figure 2-1, some of the electrons in the glass are rubbed off onto the cloth. The cloth then becomes negatively charged because it now has more electrons than protons. The glass rod becomes

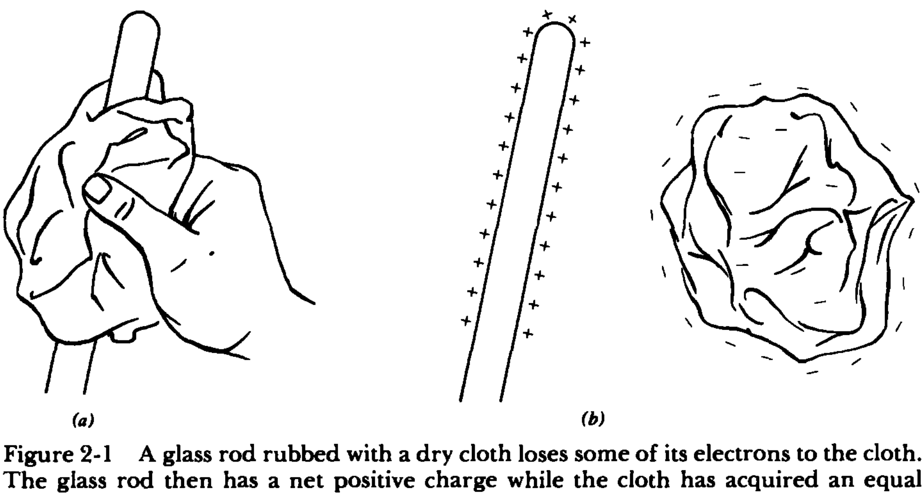


Figure 2-1 A glass rod rubbed with a dry cloth loses some of its electrons to the cloth. The glass rod then has a net positive charge while the cloth has acquired an equal amount of negative charge. The total charge in the system remains zero.

positively charged as it has lost electrons leaving behind a surplus number of protons. If the positively charged glass rod is brought near a metal ball that is free to move as in Figure 2-2*a*, the electrons in the ball near the rod are attracted to the surface leaving uncovered positive charge on the other side of the ball. This is called electrostatic induction. There is then an attractive force of the ball to the rod. Upon contact with the rod, the negative charges are neutralized by some of the positive charges on the rod, the whole combination still retaining a net positive charge as in Figure 2-2*b*. This transfer of charge is called conduction. It is then found that the now positively charged ball is repelled from the similarly charged rod. The metal ball is said to be conducting as charges are easily induced and conducted. It is important that the supporting string not be conducting, that is, insulating, otherwise charge would also distribute itself over the whole structure and not just on the ball.

If two such positively charged balls are brought near each other, they will also repel as in Figure 2-3*a*. Similarly, these balls could be negatively charged if brought into contact with the negatively charged cloth. Then it is also found that two negatively charged balls repel each other. On the other hand, if one ball is charged positively while the other is charged negatively, they will attract. These circumstances are summarized by the simple rules:

Opposite Charges Attract. Like Charges Repel.

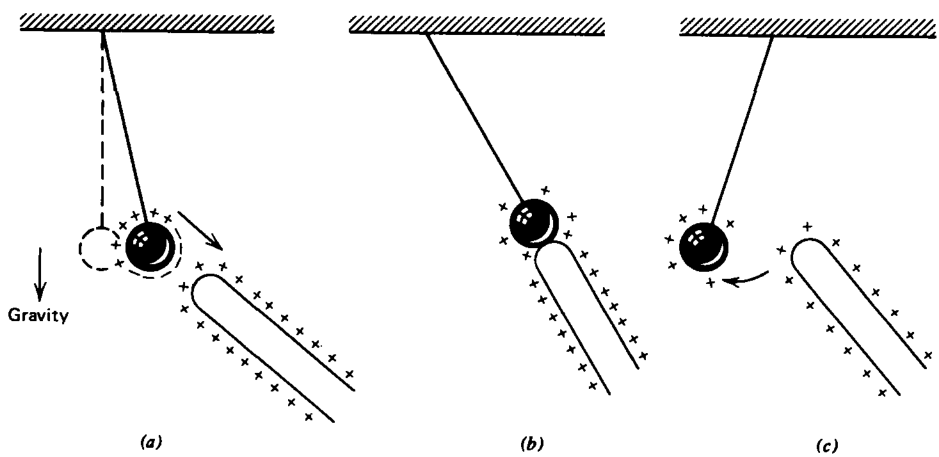


Figure 2-2 (*a*) A charged rod near a neutral ball will induce an opposite charge on the near surface. Since the ball is initially neutral, an equal amount of positive charge remains on the far surface. Because the negative charge is closer to the rod, it feels a stronger attractive force than the repelling force due to the like charges. (*b*) Upon contact with the rod the negative charge is neutralized leaving the ball positively charged. (*c*) The like charges then repel causing the ball to deflect away.

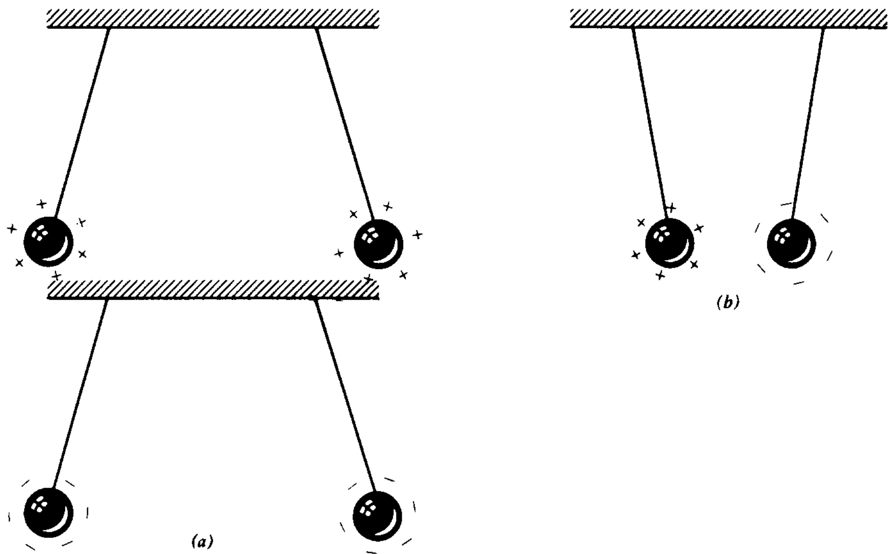


Figure 2-3 (*a*) Like charged bodies repel while (*b*) oppositely charged bodies attract.

In Figure 2-2*a*, the positively charged rod attracts the negative induced charge but repels the uncovered positive charge on the far end of the ball. The net force is attractive because the positive charge on the ball is farther away from the glass rod so that the repulsive force is less than the attractive force.

We often experience nuisance frictional electrification when we walk across a carpet or pull clothes out of a dryer. When we comb our hair with a plastic comb, our hair often becomes charged. When the comb is removed our hair still stands up, as like charged hairs repel one another. Often these effects result in sparks because the presence of large amounts of charge actually pulls electrons from air molecules.

2-1-2 Electrostatic Induction

Even without direct contact net charge can also be placed on a body by electrostatic induction. In Figure 2-4*a* we see two initially neutral suspended balls in contact acquiring opposite charges on each end because of the presence of a charged rod. If the balls are now separated, each half retains its net charge even if the inducing rod is removed. The net charge on the two balls is zero, but we have been able to isolate net positive and negative charges on each ball.

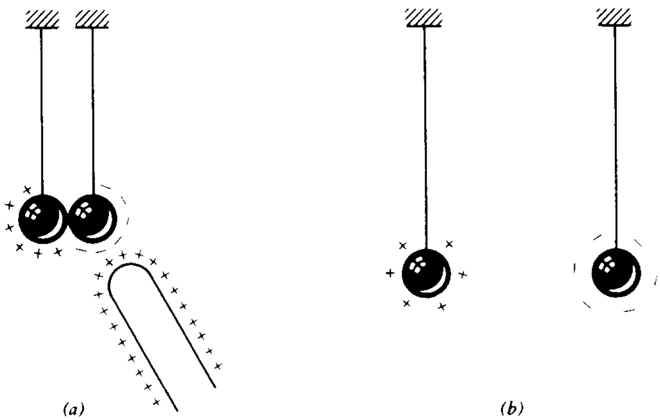


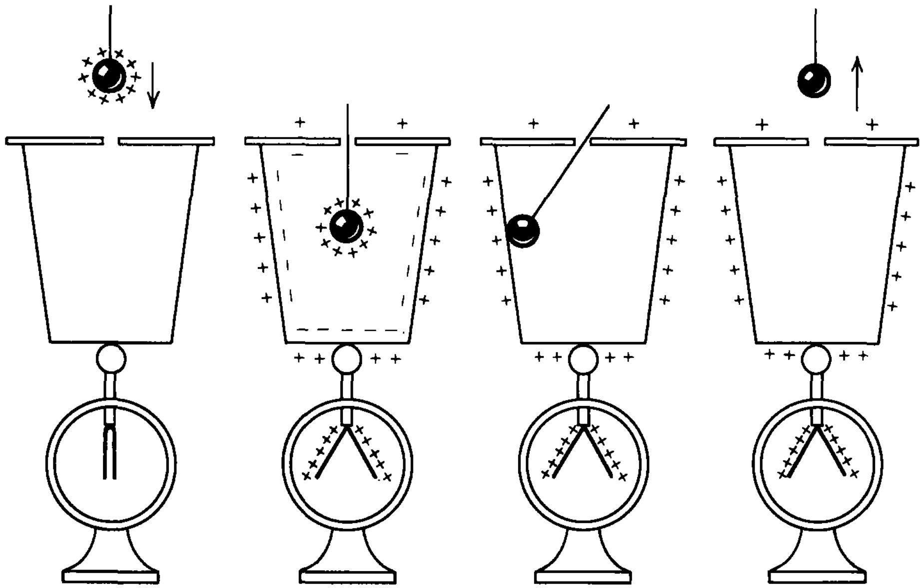
Figure 2-4 A net charge can be placed on a body without contact by electrostatic induction. (*a*) When a charged body is brought near a neutral body, the near side acquires the opposite charge. Being neutral, the far side takes on an equal but opposite charge. (*b*) If the initially neutral body is separated, each half retains its charge.

2-1-3 Faraday's "Ice-Pail" Experiment

These experiments showed that when a charged conductor contacted another conductor, whether charged or not, the total charge on both bodies was shared. The presence of charge was first qualitatively measured by an electroscope that consisted of two attached metal foil leaves. When charged, the mutual repulsion caused the leaves to diverge.

In 1843 Michael Faraday used an electroscope to perform the simple but illuminating "ice-pail" experiment illustrated in Figure 2-5. When a charged body is inside a closed isolated conductor, an equal amount of charge appears on the outside of the conductor as evidenced by the divergence of the electroscope leaves. This is true whether or not the charged body has contacted the inside walls of the surrounding conductor. If it has not, opposite charges are induced on the inside wall leaving unbalanced charge on the outside. If the charged body is removed, the charge on the inside and outside of the conductor drops to zero. However, if the charged body does contact an inside wall, as in Figure 2-5*c*, all the charge on the inside wall and ball is neutralized leaving the outside charged. Removing the initially charged body as in Figure 2-5*d* will find it uncharged, while the ice-pail now holds the original charge.

If the process shown in Figure 2-5 is repeated, the charge on the pail can be built up indefinitely. This is the principle of electrostatic generators where large amounts of charge are stored by continuous deposition of small amounts of charge.



(a) (b) (c) (d)

Figure 2-5 Faraday first demonstrated the principles of charge conservation by attaching an electroscope to an initially uncharged metal ice pail. (*a*) When all charges are far away from the pail, there is no charge on the pail nor on the flexible gold leaves of the electroscope attached to the outside of the can, which thus hang limply. (*b*) As a charged ball comes within the pail, opposite charges are induced on the inner surface. Since the pail and electroscope were originally neutral, unbalanced charge appears on the outside of which some is on the electroscope leaves. The leaves being like charged repel each other and thus diverge. (*c*) Once the charged ball is within a closed conducting body, the charge on the outside of the pail is independent of the position of the charged ball. If the charged ball contacts the inner surface of the pail, the inner charges neutralize each other. The outside charges remain unchanged. (*d*) As the now uncharged ball leaves the pail, the distributed charge on the outside of the pail and electroscope remains unchanged.

This large accumulation of charge gives rise to a large force on any other nearby charge, which is why electrostatic generators have been used to accelerate charged particles to very high speeds in atomic studies.

2-2 THE COULOMB FORCE LAW BETWEEN STATIONARY CHARGES

2-2-1 Coulomb's Law

It remained for Charles Coulomb in 1785 to express these experimental observations in a quantitative form. He used a very sensitive torsional balance to measure the force between two stationary charged balls as a function of their distance apart. He discovered that the force between two small charges q_1 and q_2 (idealized as point charges of zero size) is proportional to their magnitudes and inversely proportional to the square of the distance r_{12} between them, as illustrated in Figure 2-6. The force acts along the line joining the charges in the same or opposite direction of the unit vector $\mathbf{i}_{12}$ and is attractive if the charges are of opposite sign and repulsive if like charged. The force $\mathbf{F}_2$ on charge q_2 due to charge q_1 is equal in magnitude but opposite in direction to the force $\mathbf{F}_1$ on q_1, the net force on the pair of charges being zero.

$$\mathbf{F}_2 = -\mathbf{F}_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}^2}\mathbf{i}_{12} \text{ nt } [\text{kg}-\text{m}-\text{s}^{-2}] \tag{1}$$

2-2-2 Units

The value of the proportionality constant $1/4\pi\varepsilon_0$ depends on the system of units used. Throughout this book we use SI units (Système International d'Unités) for which the base units are taken from the rationalized MKSA system of units where distances are measured in meters (m), mass in kilograms (kg), time in seconds (s), and electric current in amperes (A). The unit of charge is a coulomb where 1 coulomb = 1 ampere-second. The adjective "rationalized" is used because the factor of 4π is arbitrarily introduced into the proportionality factor in Coulomb's law of (1). It is done this way so as to cancel a 4π that will arise from other more often used laws we will introduce shortly. Other derived units are formed by combining base units.

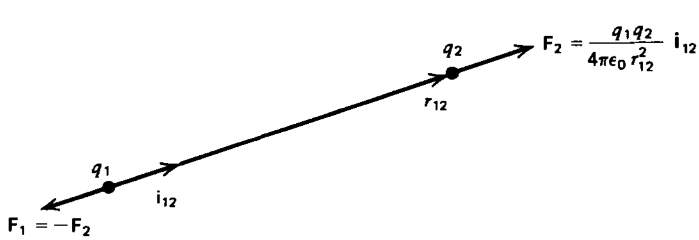


Figure 2-6 The Coulomb force between two point charges is proportional to the magnitude of the charges and inversely proportional to the square of the distance between them. The force on each charge is equal in magnitude but opposite in direction. The force vectors are drawn as if q_1 and q_2 are of the same sign so that the charges repel. If q_1 and q_2 are of opposite sign, both force vectors would point in the opposite directions, as opposite charges attract.

The parameter ε_0 is called the permittivity of free space and has a value

$$\varepsilon_0 = (4\pi \times 10^{-7} c^2)^{-1}$$
$$\approx \frac{10^{-9}}{36\pi} \approx 8.8542 \times 10^{-12} \text{ farad/m } [\text{A}^2 - \text{s}^4 - \text{kg}^{-1} - \text{m}^{-3}] \quad (2)$$

where c is the speed of light in vacuum ($c \approx 3 \times 10^8$ m/sec).

This relationship between the speed of light and a physical constant was an important result of the early electromagnetic theory in the late nineteenth century, and showed that light is an electromagnetic wave; see the discussion in Chapter 7.

To obtain a feel of how large the force in (1) is, we compare it with the gravitational force that is also an inverse square law with distance. The smallest unit of charge known is that of an electron with charge e and mass m_e

$$e \approx 1.60 \times 10^{-19} \text{ Coul}, \quad m_e \approx 9.11 \times 10^{-31} \text{ kg}$$

Then, the ratio of electric to gravitational force magnitudes for two electrons is independent of their separation:

$$\frac{\mathbf{F}_e}{\mathbf{F}_g} = -\frac{e^2/(4\pi\varepsilon_0 r^2)}{Gm_e^2/r^2} = -\frac{e^2}{m_e^2}\frac{1}{4\pi\varepsilon_0 G} \approx -4.16 \times 10^{42} \quad (3)$$

where $G = 6.67 \times 10^{-11}$ [m^3-s^{-2}-kg^{-1}] is the gravitational constant. This ratio is so huge that it exemplifies why electrical forces often dominate physical phenomena. The minus sign is used in (3) because the gravitational force between two masses is always attractive while for two like charges the electrical force is repulsive.

2-2-3 The Electric Field

If the charge q_1 exists alone, it feels no force. If we now bring charge q_2 within the vicinity of q_1, then q_2 feels a force that varies in magnitude and direction as it is moved about in space and is thus a way of mapping out the vector force field due to q_1. A charge other than q_2 would feel a different force from q_2 proportional to its own magnitude and sign. It becomes convenient to work with the quantity of force per unit charge that is called the electric field, because this quantity is independent of the particular value of charge used in mapping the force field. Considering q_2 as the test charge, the electric field due to q_1 at the position of q_2 is defined as

$$\mathbf{E}_2 = \lim_{q_2 \to 0} \frac{\mathbf{F}_2}{q_2} = \frac{q_1}{4\pi\varepsilon_0 r_{12}^2} \mathbf{i}_{12} \text{ volts/m } [\text{kg-m-s}^{-3} - \text{A}^{-1}] \quad (4)$$

In the definition of (4) the charge q_1 must remain stationary. This requires that the test charge q_2 be negligibly small so that its force on q_1 does not cause q_1 to move. In the presence of nearby materials, the test charge q_2 could also induce or cause redistribution of the charges in the material. To avoid these effects in our definition of the electric field, we make the test charge infinitely small so its effects on nearby materials and charges are also negligibly small. Then (4) will also be a valid definition of the electric field when we consider the effects of materials. To correctly map the electric field, the test charge must not alter the charge distribution from what it is in the absence of the test charge.

2-2-4 Superposition

If our system only consists of two charges, Coulomb's law (1) completely describes their interaction and the definition of an electric field is unnecessary. The electric field concept is only useful when there are large numbers of charge present as each charge exerts a force on all the others. Since the forces on a particular charge are linear, we can use superposition, whereby if a charge q_1 alone sets up an electric field $\mathbf{E}_1$, and another charge q_2 alone gives rise to an electric field $\mathbf{E}_2$, then the resultant electric field with both charges present is the vector sum $\mathbf{E}_1+\mathbf{E}_2$. This means that if a test charge q_p is placed at point P in Figure 2-7, in the vicinity of N charges it will feel a force

$$\mathbf{F}_p = q_p\mathbf{E}_P \tag{5}$$

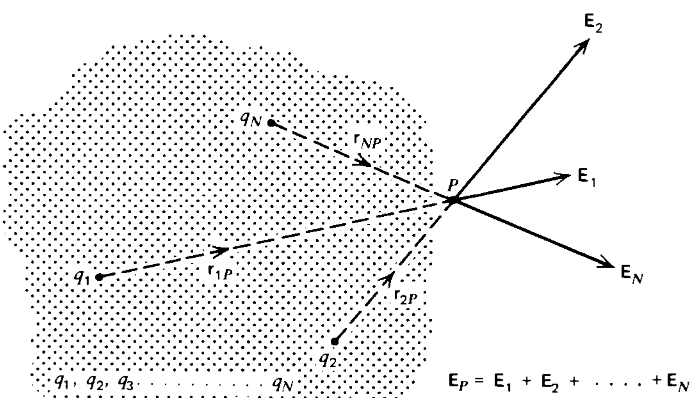


Figure 2-7 The electric field due to a collection of point charges is equal to the vector sum of electric fields from each charge alone.

where $\mathbf{E}_P$ is the vector sum of the electric fields due to all the N-point charges,

$$\mathbf{E}_P=\frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_{1P}^2}\mathbf{i}_{1P}+\frac{q_2}{r_{2P}^2}\mathbf{i}_{2P}+\frac{q_3}{r_{3P}^2}\mathbf{i}_{3P}+\cdots+\frac{q_N}{r_{NP}^2}\mathbf{i}_{NP}\right)$$

$$=\frac{1}{4\pi\varepsilon_0}\sum_{n=1}^{N}\frac{q_n}{r_{nP}^2}\mathbf{i}_{nP} \qquad (6)$$

Note that $\mathbf{E}_P$ has no contribution due to q_p since a charge cannot exert a force upon itself.

EXAMPLE 2-1 TWO-POINT CHARGES

Two-point charges are a distance a apart along the z axis as shown in Figure 2-8. Find the electric field at any point in the $z=0$ plane when the charges are:

(a) both equal to q
(b) of opposite polarity but equal magnitude $\pm q$. This configuration is called an electric dipole.

SOLUTION

(a) In the $z=0$ plane, each point charge alone gives rise to field components in the $\mathbf{i}_r$ and $\mathbf{i}_z$ directions. When both charges are equal, the superposition of field components due to both charges cancel in the z direction but add radially:

$$E_r(z=0)=\frac{q}{4\pi\varepsilon_0}\frac{2r}{[r^2+(a/2)^2]^{3/2}}$$

As a check, note that far away from the point charges ($r\gg a$) the field approaches that of a point charge of value $2q$:

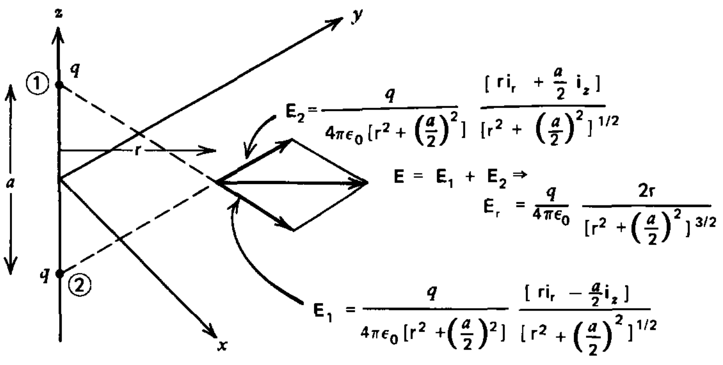
$$\lim_{r\gg a} E_r(z=0)=\frac{2q}{4\pi\varepsilon_0 r^2}$$

(b) When the charges have opposite polarity, the total electric field due to both charges now cancel in the radial direction but add in the z direction:

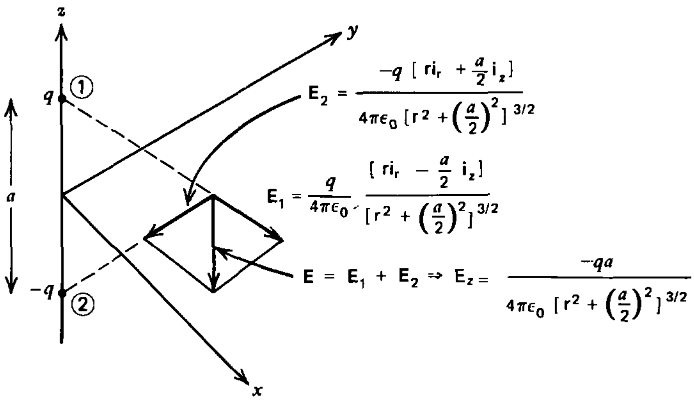
$$E_z(z=0)=\frac{-q}{4\pi\varepsilon_0}\frac{a}{[r^2+(a/2)^2]^{3/2}}$$

Far away from the point charges the electric field dies off as the inverse cube of distance:

$$\lim_{r\gg a} E_z(z=0)=\frac{-qa}{4\pi\varepsilon_0 r^3}$$



(*a*)



(*b*)

Figure 2-8 Two equal magnitude point charges are a distance a apart along the z axis. (*a*) When the charges are of the same polarity, the electric field due to each is radially directed away. In the $z=0$ symmetry plane, the net field component is radial. (*b*) When the charges are of opposite polarity, the electric field due to the negative charge is directed radially inwards. In the $z=0$ symmetry plane, the net field is now $-z$ directed.

The faster rate of decay of a dipole field is because the net charge is zero so that the fields due to each charge tend to cancel each other out.

2-3 CHARGE DISTRIBUTIONS

The method of superposition used in Section 2.2.4 will be used throughout the text in relating fields to their sources. We first find the field due to a single-point source. Because the field equations are linear, the net field due to many point sources is just the superposition of the fields from each source alone. Thus, knowing the electric field for a single-point charge at an arbitrary position immediately gives us the total field for any distribution of point charges.

In typical situations, one coulomb of total charge may be present requiring 6.25×10^{18} elementary charges ($e\approx1.60\times10^{-19}$ coul). When dealing with such a large number of particles, the discrete nature of the charges is often not important and we can consider them as a continuum. We can then describe the charge distribution by its density. The same model is used in the classical treatment of matter. When we talk about mass we do not go to the molecular scale and count the number of molecules, but describe the material by its mass density that is the product of the local average number of molecules in a unit volume and the mass per molecule.

2-3-1 Line, Surface, and Volume Charge Distributions

We similarly speak of charge densities. Charges can distribute themselves on a line with line charge density λ (coul/m), on a surface with surface charge density σ (coul/m^2) or throughout a volume with volume charge density ρ (coul/m^3).

Consider a distribution of free charge dq of differential size within a macroscopic distribution of line, surface, or volume charge as shown in Figure 2-9. Then, the total charge q within each distribution is obtained by summing up all the differential elements. This requires an integration over the line, surface, or volume occupied by the charge.

$$dq=\begin{cases}\lambda\, dl\\ \sigma\, dS\\ \rho\, dV\end{cases}\Rightarrow q=\begin{cases}\int_L \lambda\, dl & \text{(line charge)}\\ \int_S \sigma\, dS & \text{(surface charge)}\\ \int_V \rho\, dV & \text{(volume charge)}\end{cases} \qquad (1)$$

EXAMPLE 2-2 CHARGE DISTRIBUTIONS

Find the total charge within each of the following distributions illustrated in Figure 2-10.

(a) Line charge λ_0 uniformly distributed in a circular hoop of radius a.

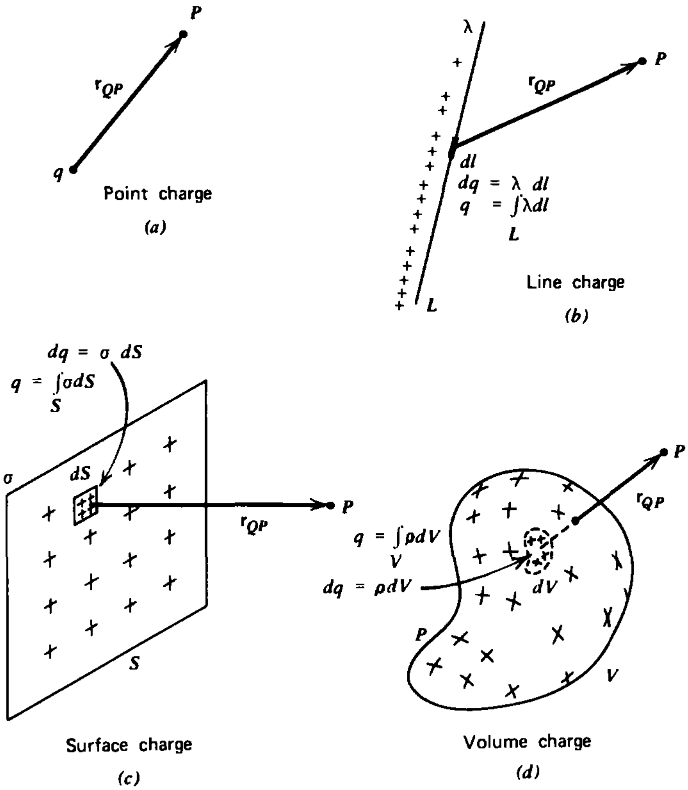


Figure 2-9 Charge distributions. (*a*) Point charge; (*b*) Line charge; (*c*) Surface charge; (*d*) Volume charge.

SOLUTION

$$q = \int_L \lambda \, dl = \int_0^{2\pi} \lambda_0 a \, d\phi = 2\pi a \lambda_0$$

(b) Surface charge σ_0 uniformly distributed on a circular disk of radius a.

SOLUTION

$$q = \int_S \sigma \, dS = \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \sigma_0 r \, dr \, d\phi = \pi a^2 \sigma_0$$

(c) Volume charge ρ_0 uniformly distributed throughout a sphere of radius R.

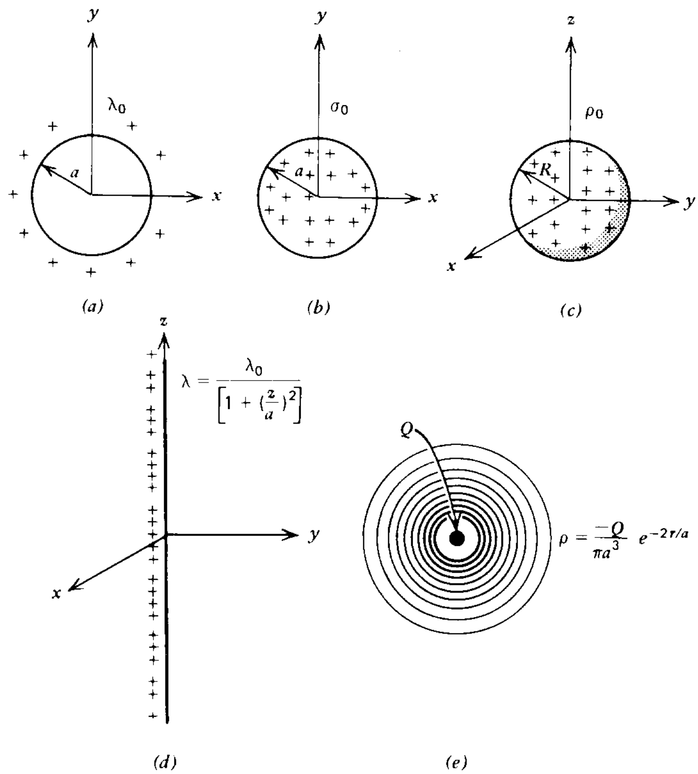


Figure 2-10 Charge distributions of Example 2-2. (*a*) Uniformly distributed line charge on a circular hoop. (*b*) Uniformly distributed surface charge on a circular disk. (*c*) Uniformly distributed volume charge throughout a sphere. (*d*) Nonuniform line charge distribution. (*e*) Smooth radially dependent volume charge distribution throughout all space, as a simple model of the electron cloud around the positively charged nucleus of the hydrogen atom.

SOLUTION

$$q = \int_V \rho\, dV = \int_{r=0}^{R} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \rho_0 r^2 \sin\theta\, dr\, d\theta\, d\phi = \tfrac{4}{3}\pi R^3 \rho_0$$

(d) A line charge of infinite extent in the z direction with charge density distribution

$$\lambda = \frac{\lambda_0}{[1+(z/a)^2]}$$

SOLUTION

$$q = \int_L \lambda\, dl = \int_{-\infty}^{+\infty} \frac{\lambda_0\, dz}{[1+(z/a)^2]} = \lambda_0 a \tan^{-1}\frac{z}{a}\bigg|_{-\infty}^{+\infty} = \lambda_0 \pi a$$

(e) The electron cloud around the positively charged nucleus Q in the hydrogen atom is simply modeled as the spherically symmetric distribution

$$\rho(r) = -\frac{Q}{\pi a^3} e^{-2r/a}$$

where a is called the Bohr radius.

SOLUTION

The total charge in the cloud is

$$\begin{aligned} q &= \int_V \rho\, dV \\ &= -\int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{Q}{\pi a^3} e^{-2r/a} r^2 \sin\theta\, dr\, d\theta\, d\phi \\ &= -\int_{r=0}^{\infty} \frac{4Q}{a^3} e^{-2r/a} r^2\, dr \\ &= -\frac{4Q}{a^3}\left(-\frac{a}{2}\right) e^{-2r/a}\left[r^2 - \frac{a^2}{2}\left(-\frac{2r}{a} - 1\right)\right]\Bigg|_{r=0}^{\infty} \\ &= -Q \end{aligned}$$

2-3-2 The Electric Field Due to a Charge Distribution

Each differential charge element dq as a source at point Q contributes to the electric field at a point P as

$$d\mathbf{E} = \frac{dq}{4\pi\varepsilon_0 r_{QP}^2} \mathbf{i}_{QP} \tag{2}$$

where r_{QP} is the distance between Q and P with $\mathbf{i}_{QP}$ the unit vector directed from Q to P. To find the total electric field, it is necessary to sum up the contributions from each charge element. This is equivalent to integrating (2) over the entire charge distribution, remembering that both the distance r_{QP} and direction $\mathbf{i}_{QP}$ vary for each differential element throughout the distribution

$$\mathbf{E} = \int_{\text{all } q} \frac{dq}{4\pi\varepsilon_0 r_{QP}^2} \mathbf{i}_{QP} \tag{3}$$

where (3) is a line integral for line charges ($dq = \lambda\, dl$), a surface integral for surface charges ($dq = \sigma\, dS$), a volume integral for a volume charge distribution ($dq=\rho\, dV$), or in general, a combination of all three.

If the total charge distribution is known, the electric field is obtained by performing the integration of (3). Some general rules and hints in using (3) are:

1. It is necessary to distinguish between the coordinates of the field points and the charge source points. Always integrate over the coordinates of the charges.
2. Equation (3) is a vector equation and so generally has three components requiring three integrations. Symmetry arguments can often be used to show that particular field components are zero.
3. The distance r_{QP} is always positive. In taking square roots, always make sure that the positive square root is taken.
4. The solution to a particular problem can often be obtained by integrating the contributions from simpler differential size structures.

2-3-3 Field Due to an Infinitely Long Line Charge

An infinitely long uniformly distributed line charge λ_0 along the z axis is shown in Figure 2-11. Consider the two symmetrically located charge elements dq_1 and dq_2 a distance z above and below the point P, a radial distance r away. Each charge element alone contributes radial and z components to the electric field. However, just as we found in Example 2-1a, the two charge elements together cause equal magnitude but oppositely directed z field components that thus cancel leaving only additive radial components:

$$dE_r=\frac{\lambda_0\, dz}{4\pi\varepsilon_0(z^2+r^2)}\cos\theta=\frac{\lambda_0 r\, dz}{4\pi\varepsilon_0(z^2+r^2)^{3/2}} \tag{4}$$

To find the total electric field we integrate over the length of the line charge:

$$\begin{aligned} E_r&=\frac{\lambda_0 r}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty}\frac{dz}{(z^2+r^2)^{3/2}}\\ &=\frac{\lambda_0 r}{4\pi\varepsilon_0}\frac{z}{r^2(z^2+r^2)^{1/2}}\bigg|_{z=-\infty}^{+\infty}\\ &=\frac{\lambda_0}{2\pi\varepsilon_0 r} \end{aligned} \tag{5}$$

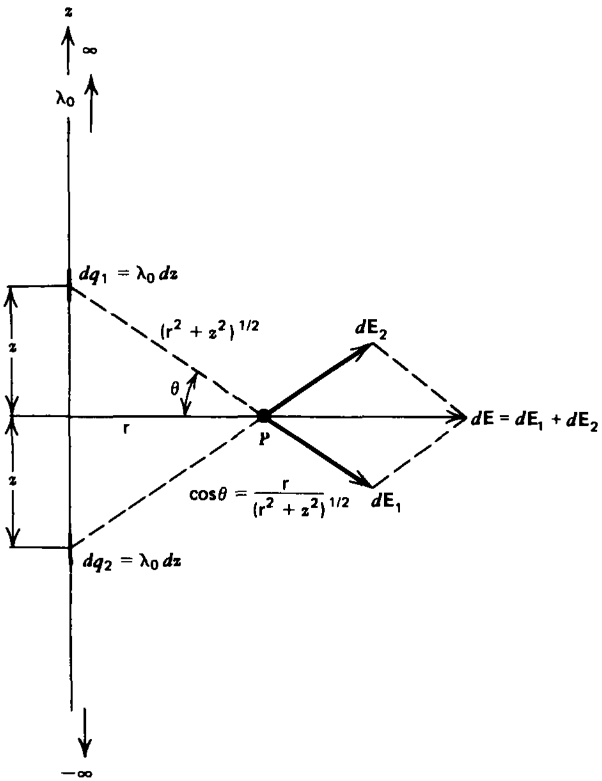


Figure 2-11 An infinitely long uniform distribution of line charge only has a radially directed electric field because the z components of the electric field are canceled out by symmetrically located incremental charge elements as also shown in Figure 2-8*a*.

2-3-4 Field Due to Infinite Sheets of Surface Charge

(a) Single Sheet

A surface charge sheet of infinite extent in the $y=0$ plane has a uniform surface charge density σ_0 as in Figure 2-12*a*. We break the sheet into many incremental line charges of thickness dx with $d\lambda = \sigma_0\,dx$. We could equivalently break the surface into incremental horizontal line charges of thickness dz. Each incremental line charge alone has a radial field component as given by (5) that in Cartesian coordinates results in x and y components. Consider the line charge $d\lambda_1$, a distance x to the left of P, and the symmetrically placed line charge $d\lambda_2$ the same distance x to the right of P. The x components of the resultant fields cancel while the y

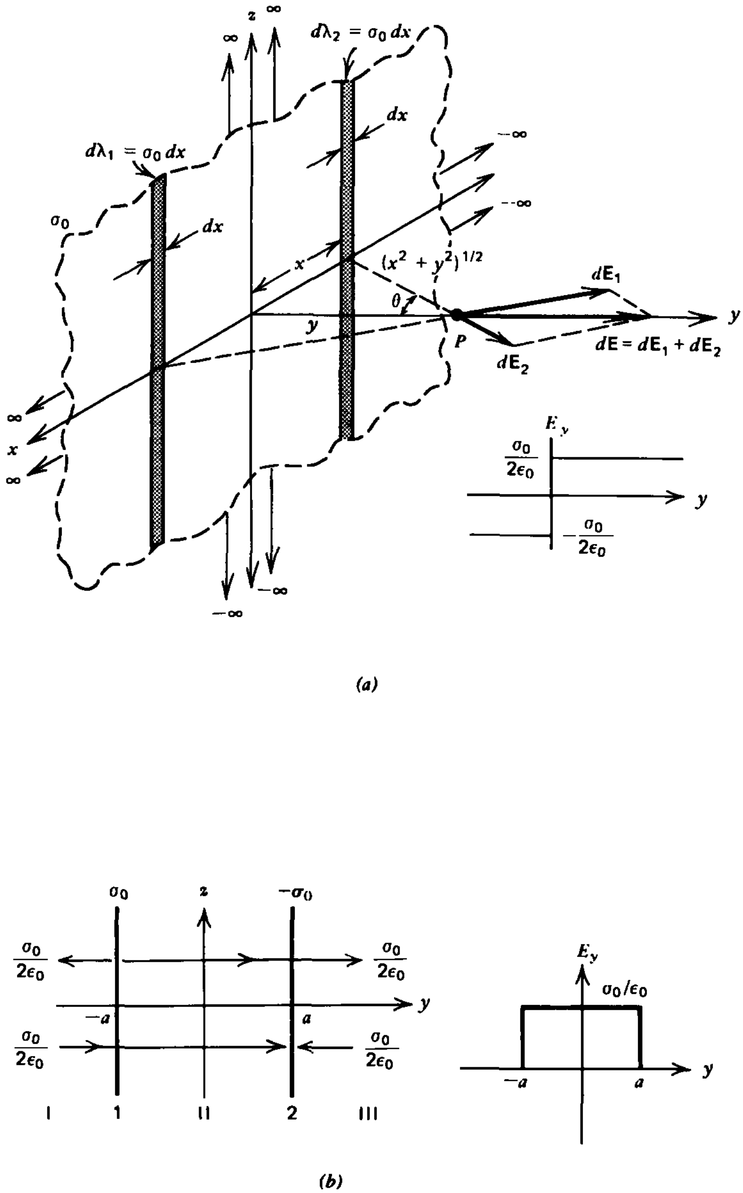


Figure 2-12 (*a*) The electric field from a uniformly surface charged sheet of infinite extent is found by summing the contributions from each incremental line charge element. Symmetrically placed line charge elements have x field components that cancel, but y field components that add. (*b*) Two parallel but oppositely charged sheets of surface charge have fields that add in the region between the sheets but cancel outside. (*c*) The electric field from a volume charge distribution is obtained by summing the contributions from each incremental surface charge element.

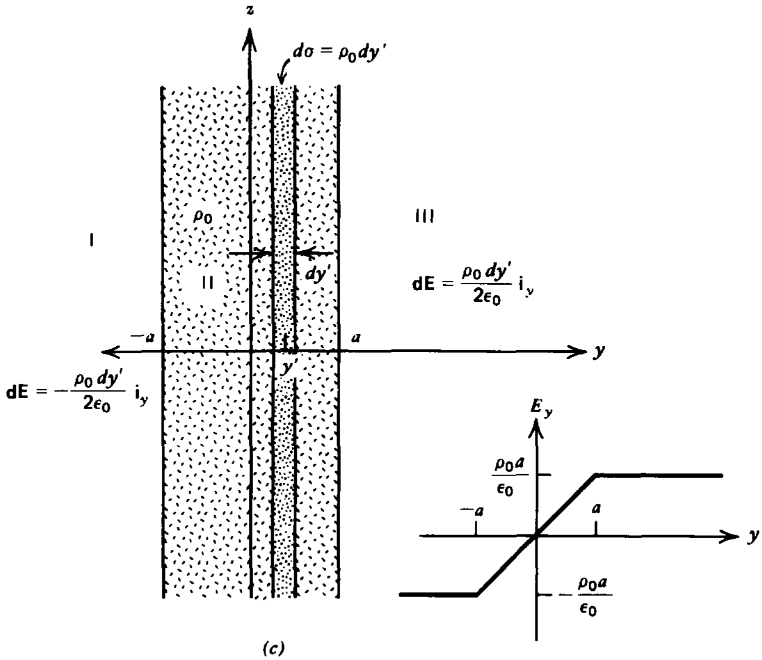


Fig. 2-12(c)

components add:

$$dE_y = \frac{\sigma_0\,dx}{2\pi\varepsilon_0(x^2+y^2)^{1/2}}\cos\theta = \frac{\sigma_0 y\,dx}{2\pi\varepsilon_0(x^2+y^2)} \tag{6}$$

The total field is then obtained by integration over all line charge elements:

$$\begin{aligned} E_y &= \frac{\sigma_0 y}{2\pi\varepsilon_0}\int_{-\infty}^{+\infty}\frac{dx}{x^2+y^2} \\ &= \frac{\sigma_0 y}{2\pi\varepsilon_0}\frac{1}{y}\tan^{-1}\frac{x}{y}\bigg|_{x=-\infty}^{+\infty} \\ &= \begin{cases}\sigma_0/2\varepsilon_0, & y>0\\ -\sigma_0/2\varepsilon_0, & y<0\end{cases} \end{aligned} \tag{7}$$

where we realized that the inverse tangent term takes the sign of the ratio x/y so that the field reverses direction on each side of the sheet. The field strength does not decrease with distance from the infinite sheet.

(b) Parallel Sheets of Opposite Sign

A capacitor is formed by two oppositely charged sheets of surface charge a distance $2a$ apart as shown in Figure 2-12*b*.

The fields due to each charged sheet alone are obtained from (7) as

$$\mathbf{E}_1=\begin{cases}\dfrac{\sigma_0}{2\varepsilon_0}\mathbf{i}_y, & y>-a\\[2ex] -\dfrac{\sigma_0}{2\varepsilon_0}\mathbf{i}_y, & y<-a\end{cases}\qquad \mathbf{E}_2=\begin{cases}-\dfrac{\sigma_0}{2\varepsilon_0}\mathbf{i}_y, & y>a\\[2ex] \dfrac{\sigma_0}{2\varepsilon_0}\mathbf{i}_y, & y<a\end{cases}\tag{8}$$

Thus, outside the sheets in regions I and III the fields cancel while they add in the enclosed region II. The nonzero field is confined to the region between the charged sheets and is independent of the spacing:

$$\mathbf{E}=\mathbf{E}_1+\mathbf{E}_2=\begin{cases}(\sigma_0/\varepsilon_0)\mathbf{i}_y, & |y|<a\\ 0 & |y|>a\end{cases}\tag{9}$$

(c) Uniformly Charged Volume

A uniformly charged volume with charge density ρ_0 of infinite extent in the x and z directions and of width $2a$ is centered about the y axis, as shown in Figure 2-12c. We break the volume distribution into incremental sheets of surface charge of width dy' with differential surface charge density $d\sigma=\rho_0\,dy'$. It is necessary to distinguish the position y' of the differential sheet of surface charge from the field point y. The total electric field is the sum of all the fields due to each differentially charged sheet. The problem breaks up into three regions. In region I, where $y\le -a$, each surface charge element causes a field in the negative y direction:

$$E_y=\int_{-a}^{a}-\frac{\rho_0}{2\varepsilon_0}\,dy'=-\frac{\rho_0 a}{\varepsilon_0},\quad y\le -a\tag{10}$$

Similarly, in region III, where $y\ge a$, each charged sheet gives rise to a field in the positive y direction:

$$E_y=\int_{-a}^{a}\frac{\rho_0\,dy'}{2\varepsilon_0}=\frac{\rho_0 a}{\varepsilon_0},\quad y\ge a\tag{11}$$

For any position y in region II, where $-a\le y\le a$, the charge to the right of y gives rise to a negatively directed field while the charge to the left of y causes a positively directed field:

$$E_y=\int_{-a}^{y}\frac{\rho_0\,dy'}{2\varepsilon_0}+\int_{y}^{a}(-)\frac{\rho_0}{2\varepsilon_0}\,dy'=\frac{\rho_0 y}{\varepsilon_0},\quad -a\le y\le a\tag{12}$$

The field is thus constant outside of the volume of charge and in opposite directions on either side being the same as for a surface charged sheet with the same total charge per unit area, $\sigma_0 = \rho_0 2a$. At the boundaries $y = \pm a$, the field is continuous, changing linearly with position between the boundaries:

$$E_y = \begin{cases} -\dfrac{\rho_0 a}{\varepsilon_0}, & y \le -a \\ \dfrac{\rho_0 y}{\varepsilon_0}, & -a \le y \le a \\ \dfrac{\rho_0 a}{\varepsilon_0}, & y \ge a \end{cases} \tag{13}$$

2-3-5 Superposition of Hoops of Line Charge

(a) Single Hoop

Using superposition, we can similarly build up solutions starting from a circular hoop of radius a with uniform line charge density λ_0 centered about the origin in the $z = 0$ plane as shown in Figure 2-13a. Along the z axis, the distance to the hoop perimeter $(a^2 + z^2)^{1/2}$ is the same for all incremental point charge elements $dq = \lambda_0 a\, d\phi$. Each charge element alone contributes z- and r-directed electric field components. However, along the z axis symmetrically placed elements 180° apart have z components that add but radial components that cancel. The z-directed electric field along the z axis is then

$$E_z = \int_0^{2\pi} \frac{\lambda_0 a\, d\phi \cos\theta}{4\pi\varepsilon_0 (z^2 + a^2)} = \frac{\lambda_0 a z}{2\varepsilon_0 (a^2 + z^2)^{3/2}} \tag{14}$$

The electric field is in the $-z$ direction along the z axis below the hoop.

The total charge on the hoop is $q = 2\pi a \lambda_0$ so that (14) can also be written as

$$E_z = \frac{qz}{4\pi\varepsilon_0 (a^2 + z^2)^{3/2}} \tag{15}$$

When we get far away from the hoop ($|z| \gg a$), the field approaches that of a point charge:

$$\lim_{|z| \gg a} E_z = \pm \frac{q}{4\pi\varepsilon_0 z^2} \begin{cases} z > 0 \\ z < 0 \end{cases} \tag{16}$$

(b) Disk of Surface Charge

The solution for a circular disk of uniformly distributed surface charge σ_0 is obtained by breaking the disk into incremental hoops of radius r with line charge $d\lambda = \sigma_0\, dr$ as in

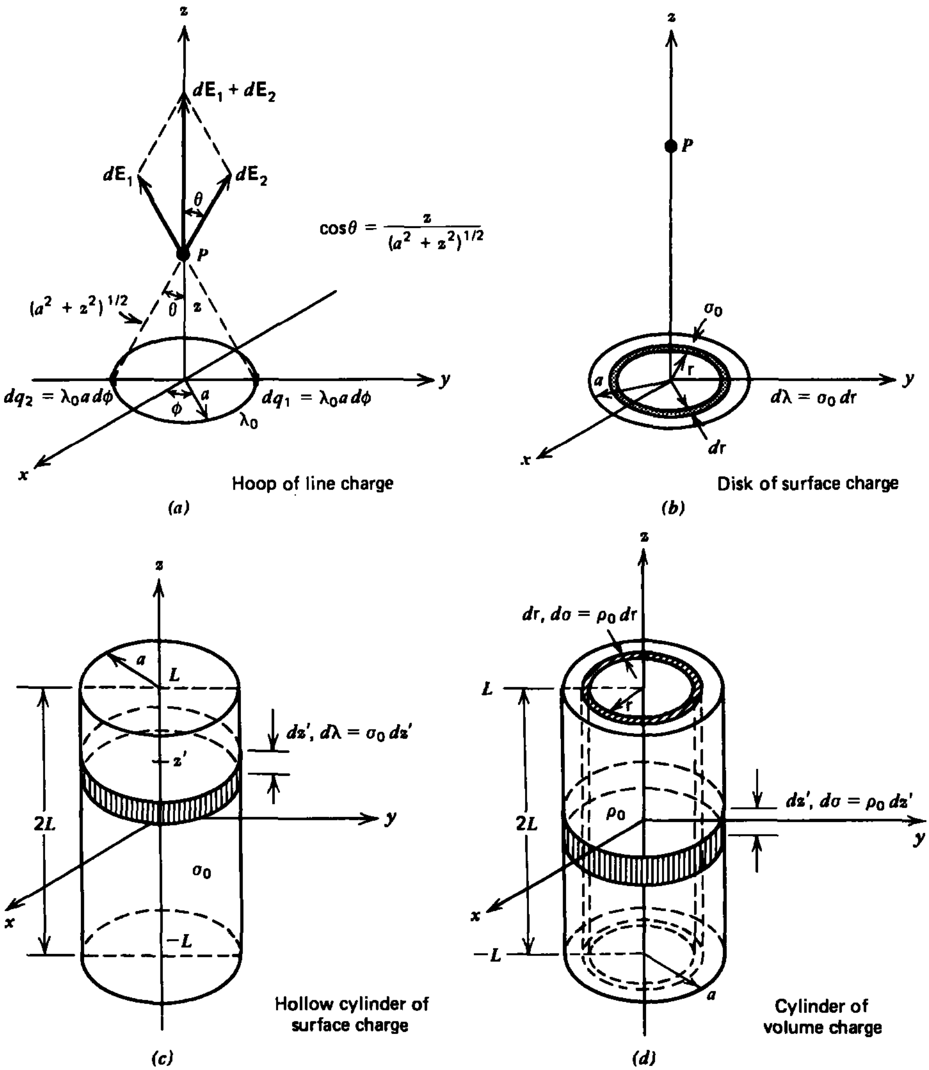


Figure 2-13 (*a*) The electric field along the symmetry z axis of a uniformly distributed hoop of line charge is z directed. (*b*) The axial field from a circular disk of surface charge is obtained by radially summing the contributions of incremental hoops of line charge. (*c*) The axial field from a hollow cylinder of surface charge is obtained by axially summing the contributions of incremental hoops of line charge. (*d*) The axial field from a cylinder of volume charge is found by summing the contributions of axial incremental disks or of radial hollow cylinders of surface charge.

Figure 2-13*b*. Then the incremental z-directed electric field along the z axis due to a hoop of radius r is found from (14) as

$$dE_z = \frac{\sigma_0 \mathrm{r} z\, d\mathrm{r}}{2\varepsilon_0(\mathrm{r}^2 + z^2)^{3/2}} \tag{17}$$

where we replace a with r, the radius of the incremental hoop. The total electric field is then

$$E_z=\frac{\sigma_0 z}{2\varepsilon_0}\int_0^a \frac{\mathrm{r}\,d\mathrm{r}}{(\mathrm{r}^2+z^2)^{3/2}}$$

$$=-\frac{\sigma_0 z}{2\varepsilon_0(\mathrm{r}^2+z^2)^{1/2}}\bigg|_0^a$$

$$=-\frac{\sigma_0}{2\varepsilon_0}\left(\frac{z}{(a^2+z^2)^{1/2}}-\frac{z}{|z|}\right)$$

$$=\pm\frac{\sigma_0}{2\varepsilon_0}-\frac{\sigma_0 z}{2\varepsilon_0(a^2+z^2)^{1/2}}\begin{cases}z>0\\z<0\end{cases} \tag{18}$$

where care was taken at the lower limit (r = 0), as the magnitude of the square root must always be used.

As the radius of the disk gets very large, this result approaches that of the uniform field due to an infinite sheet of surface charge:

$$\lim_{a\to\infty} E_z=\pm\frac{\sigma_0}{2\varepsilon_0}\begin{cases}z>0\\z<0\end{cases} \tag{19}$$

(c) Hollow Cylinder of Surface Charge

A hollow cylinder of length $2L$ and radius a has its axis along the z direction and is centered about the $z=0$ plane as in Figure 2-13c. Its outer surface at $\mathrm{r}=a$ has a uniform distribution of surface charge σ_0. It is necessary to distinguish between the coordinate of the field point z and the source point at z' $(-L\leq z'\leq L)$. The hollow cylinder is broken up into incremental hoops of line charge $d\lambda=\sigma_0\,dz'$. Then, the axial distance from the field point at z to any incremental hoop of line charge is $(z-z')$. The contribution to the axial electric field at z due to the incremental hoop at z' is found from (14) as

$$dE_z=\frac{\sigma_0 a(z-z')\,dz'}{2\varepsilon_0[a^2+(z-z')^2]^{3/2}} \tag{20}$$

which when integrated over the length of the cylinder yields

$$E_z=\frac{\sigma_0 a}{2\varepsilon_0}\int_{-L}^{+L}\frac{(z-z')\,dz'}{[a^2+(z-z')^2]^{3/2}}$$

$$=\frac{\sigma_0 a}{2\varepsilon_0}\frac{1}{[a^2+(z-z')^2]^{1/2}}\bigg|_{z'=-L}^{+L}$$

$$=\frac{\sigma_0 a}{2\varepsilon_0}\left(\frac{1}{[a^2+(z-L)^2]^{1/2}}-\frac{1}{[a^2+(z+L)^2]^{1/2}}\right) \tag{21}$$

(d) Cylinder of Volume Charge

If this same cylinder is uniformly charged throughout the volume with charge density ρ_0, we break the volume into differential-size hollow cylinders of thickness dr with incremental surface charge $d\sigma = \rho_0\, dr$ as in Figure 2-13*d*. Then, the z-directed electric field along the z axis is obtained by integration of (21) replacing a by r:

$$\begin{aligned} E_z &= \frac{\rho_0}{2\varepsilon_0}\int_0^a r\left(\frac{1}{[r^2+(z-L)^2]^{1/2}} - \frac{1}{[r^2+(z+L)^2]^{1/2}}\right) dr \\ &= \frac{\rho_0}{2\varepsilon_0}\{[r^2+(z-L)^2]^{1/2} - [r^2+(z+L)^2]^{1/2}\}\Big|_0^a \\ &= \frac{\rho_0}{2\varepsilon_0}\{[a^2+(z-L)^2]^{1/2} - |z-L| - [a^2+(z+L)^2]^{1/2} \\ &\quad + |z+L|\} \end{aligned} \tag{22}$$

where at the lower $r = 0$ limit we always take the positive square root.

This problem could have equally well been solved by breaking the volume charge distribution into many differential-sized surface charged disks at position z' $(-L \le z' \le L)$, thickness dz', and effective surface charge density $d\sigma = \rho_0\, dz'$. The field is then obtained by integrating (18).

2-4 GAUSS'S LAW

We could continue to build up solutions for given charge distributions using the coulomb superposition integral of Section 2.3.2. However, for geometries with spatial symmetry, there is often a simpler way using some vector properties of the inverse square law dependence of the electric field.

2-4-1 Properties of the Vector Distance Between Two Points, $\mathbf{r}_{QP}$

(a) $\mathbf{r}_{QP}$

In Cartesian coordinates the vector distance $\mathbf{r}_{QP}$ between a source point at Q and a field point at P directed from Q to P as illustrated in Figure 2-14 is

$$\mathbf{r}_{QP} = (x - x_Q)\mathbf{i}_x + (y - y_Q)\mathbf{i}_y + (z - z_Q)\mathbf{i}_z \tag{1}$$

with magnitude

$$r_{QP} = [(x - x_Q)^2 + (y - y_Q)^2 + (z - z_Q)^2]^{1/2} \tag{2}$$

The unit vector in the direction of $\mathbf{r}_{QP}$ is

$$\mathbf{i}_{QP} = \frac{\mathbf{r}_{QP}}{r_{QP}} \tag{3}$$

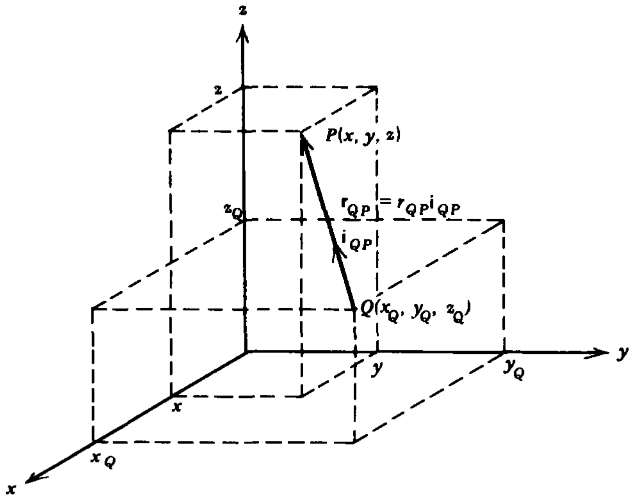


Figure 2-14 The vector distance $\mathbf{r}_{QP}$ between two points Q and P.

(b) Gradient of the Reciprocal Distance, $\nabla(1/r_{QP})$

Taking the gradient of the reciprocal of (2) yields

$$\begin{aligned}\nabla\left(\frac{1}{r_{QP}}\right) &= \mathbf{i}_x\frac{\partial}{\partial x}\left(\frac{1}{r_{QP}}\right)+\mathbf{i}_y\frac{\partial}{\partial y}\left(\frac{1}{r_{QP}}\right)+\mathbf{i}_z\frac{\partial}{\partial z}\left(\frac{1}{r_{QP}}\right)\\ &= -\frac{1}{r_{QP}^3}[(x-x_Q)\mathbf{i}_x+(y-y_Q)\mathbf{i}_y+(z-z_Q)\mathbf{i}_z]\\ &= -\mathbf{i}_{QP}/r_{QP}^2 \qquad (4)\end{aligned}$$

which is the negative of the spatially dependent term that we integrate to find the electric field in Section 2.3.2.

(c) Laplacian of the Reciprocal Distance

Another useful identity is obtained by taking the divergence of the gradient of the reciprocal distance. This operation is called the Laplacian of the reciprocal distance. Taking the divergence of (4) yields

$$\begin{aligned}\nabla^2\left(\frac{1}{r_{QP}}\right) &= \nabla\cdot\left[\nabla\left(\frac{1}{r_{QP}}\right)\right]\\ &= \nabla\cdot\left(\frac{-\mathbf{i}_{QP}}{r_{QP}^2}\right)\\ &= -\frac{\partial}{\partial x}\left(\frac{x-x_Q}{r_{QP}^3}\right)-\frac{\partial}{\partial y}\left(\frac{y-y_Q}{r_{QP}^3}\right)-\frac{\partial}{\partial z}\left(\frac{z-z_Q}{r_{QP}^3}\right)\\ &= -\frac{3}{r_{QP}^3}+\frac{3}{r_{QP}^5}[(x-x_Q)^2+(y-y_Q)^2+(z-z_Q)^2] \qquad (5)\end{aligned}$$

Using (2) we see that (5) reduces to

$$\nabla^2\left(\frac{1}{r_{QP}}\right)=\begin{cases}0, & r_{QP}\neq 0\\ \text{undefined} & r_{QP}=0\end{cases} \tag{6}$$

Thus, the Laplacian of the inverse distance is zero for all nonzero distances but is undefined when the field point is coincident with the source point.

2-4-2 Gauss's Law In Integral Form

(a) Point Charge Inside or Outside a Closed Volume

Now consider the two cases illustrated in Figure 2-15 where an arbitrarily shaped closed volume V either surrounds a point charge q or is near a point charge q outside the surface S. For either case the electric field emanates radially from the point charge with the spatial inverse square law. We wish to calculate the flux of electric field through the surface S surrounding the volume V:

$$\begin{aligned}\Phi&=\oint_S \mathbf{E}\cdot d\mathbf{S}\\ &=\oint_S \frac{q}{4\pi\varepsilon_0 r_{QP}^2}\mathbf{i}_{QP}\cdot d\mathbf{S}\\ &=\oint_S \frac{-q}{4\pi\varepsilon_0}\nabla\left(\frac{1}{r_{QP}}\right)\cdot d\mathbf{S}\end{aligned} \tag{7}$$

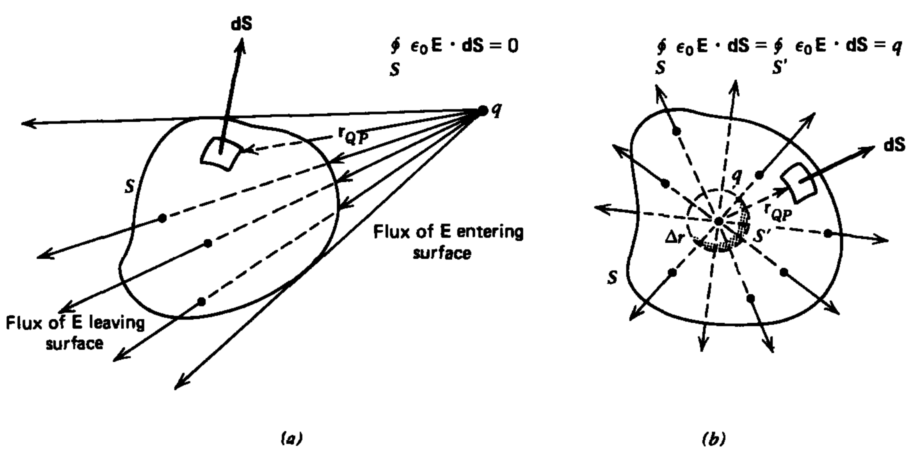


Figure 2-15 (*a*) The net flux of electric field through a closed surface S due to an outside point charge is zero because as much flux enters the near side of the surface as leaves on the far side. (*b*) All the flux of electric field emanating from an enclosed point charge passes through the surface.

where we used (4). We can now use the divergence theorem to convert the surface integral to a volume integral:

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = \frac{-q}{4\pi\varepsilon_0}\int_V \nabla\cdot\left[\nabla\left(\frac{1}{r_{QP}}\right)\right]dV \tag{8}$$

When the point charge q is outside the surface every point in the volume has a nonzero value of r_{QP}. Then, using (6) with $r_{QP}\neq 0$, we see that the net flux of $\mathbf{E}$ through the surface is zero.

This result can be understood by examining Figure 2-15*a*. The electric field emanating from q on that part of the surface S nearest q has its normal component oppositely directed to $\mathbf{dS}$ giving a negative contribution to the flux. However, on the opposite side of S the electric field exits with its normal component in the same direction as $\mathbf{dS}$ giving a positive contribution to the flux. We have shown that these flux contributions are equal in magnitude but opposite in sign so that the net flux is zero.

As illustrated in Figure 2-15*b*, assuming q to be positive, we see that when S surrounds the charge the electric field points outwards with normal component in the direction of $\mathbf{dS}$ everywhere on S so that the flux must be positive. If q were negative, $\mathbf{E}$ and $\mathbf{dS}$ would be oppositely directed everywhere so that the flux is also negative. For either polarity with nonzero q, the flux cannot be zero. To evaluate the value of this flux we realize that (8) is zero everywhere except where $r_{QP}=0$ so that the surface S in (8) can be shrunk down to a small spherical surface S' of infinitesimal radius Δr surrounding the point charge; the rest of the volume has $r_{QP}\neq 0$ so that $\nabla\cdot\nabla(1/r_{QP})=0$. On this incremental surface we know the electric field is purely radial in the same direction as $\mathbf{d}S'$ with the field due to a point charge:

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = \oint_{S'} \mathbf{E}\cdot d\mathbf{S}' = \frac{q}{4\pi\varepsilon_0(\Delta r)^2}4\pi(\Delta r)^2 = \frac{q}{\varepsilon_0} \tag{9}$$

If we had many point charges within the surface S, each charge q_i gives rise to a flux q_i/ε_0 so that Gauss's law states that the net flux of $\varepsilon_0\mathbf{E}$ through a closed surface is equal to the net charge enclosed by the surface:

$$\oint_S \varepsilon_0\mathbf{E}\cdot d\mathbf{S} = \sum_{\substack{\text{all } q_i \\ \text{inside } S}} q_i. \tag{10}$$

Any charges outside S do not contribute to the flux.

(b) Charge Distributions

For continuous charge distributions, the right-hand side of (10) includes the sum of all enclosed incremental charge elements so that the total charge enclosed may be a line, surface, and/or volume integral in addition to the sum of point charges:

$$\oint_S \varepsilon_0 \mathbf{E}\cdot d\mathbf{S} = \sum_{\substack{\text{all } q_i \\ \text{inside } S}} q_i + \int_{\substack{\text{all } q \\ \text{inside } S}} dq$$

$$= \left(\sum q_i + \int_L \lambda\, dl + \int_S \sigma\, dS + \int_V \rho\, dV\right)\Bigg|_{\substack{\text{all charge} \\ \text{inside } S}} \quad (11)$$

Charges outside the volume give no contribution to the total flux through the enclosing surface.

Gauss's law of (11) can be used to great advantage in simplifying computations for those charges distributed with spatial symmetry. The trick is to find a surface S that has sections tangent to the electric field so that the dot product is zero, or has surfaces perpendicular to the electric field and upon which the field is constant so that the dot product and integration become pure multiplications. If the appropriate surface is found, the surface integral becomes very simple to evaluate.

Coulomb's superposition integral derived in Section 2.3.2 is often used with symmetric charge distributions to determine if any field components are zero. Knowing the direction of the electric field often suggests the appropriate Gaussian surface upon which to integrate (11). This integration is usually much simpler than using Coulomb's law for each charge element.

2-4-3 Spherical Symmetry

(a) Surface Charge

A sphere of radius R has a uniform distribution of surface charge σ_0 as in Figure 2-16a. Measure the angle θ from the line joining any point P at radial distance r to the sphere center. Then, the distance from P to any surface charge element on the sphere is independent of the angle ϕ. Each differential surface charge element at angle θ contributes field components in the radial and θ directions, but symmetrically located charge elements at $-\phi$ have equal field magnitude components that add radially but cancel in the θ direction.

Realizing from the symmetry that the electric field is purely radial and only depends on r and not on θ or ϕ, we draw Gaussian spheres of radius r as in Figure 2-16b both inside ($r<R$) and outside ($r>R$) the charged sphere. The Gaussian sphere inside encloses no charge while the outside sphere

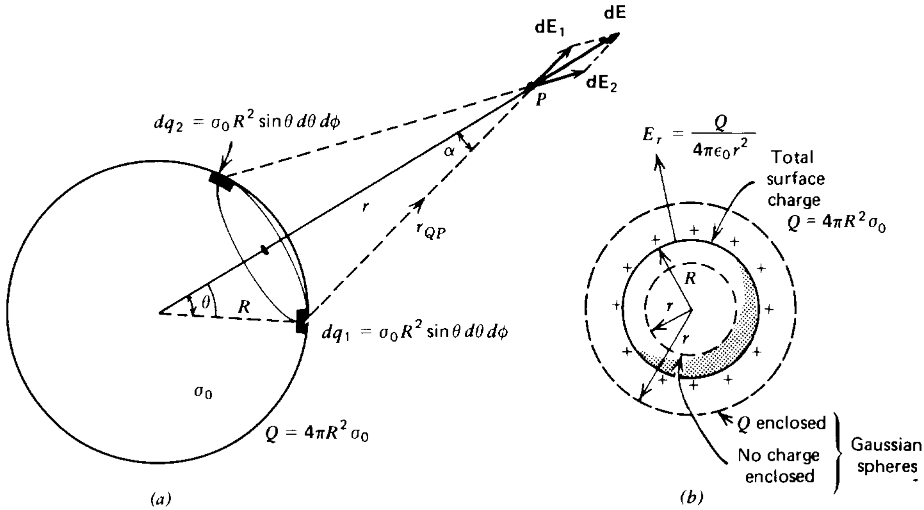


Figure 2-16 A sphere of radius R with uniformly distributed surface charge σ_0. (*a*) Symmetrically located charge elements show that the electric field is purely radial. (*b*) Gauss's law, applied to concentric spherical surfaces inside ($r<R$) and outside ($r>R$) the charged sphere, easily shows that the electric field within the sphere is zero and outside is the same as if all the charge $Q=4\pi R^2\sigma_0$ were concentrated as a point charge at the origin.

encloses all the charge $Q=\sigma_0 4\pi R^2$:

$$\oint_S \varepsilon_0 \mathbf{E}\cdot \mathbf{dS}=\varepsilon_0 E_r 4\pi r^2=\begin{cases}\sigma_0 4\pi R^2=Q, & r>R\\ 0, & r<R\end{cases}\tag{12}$$

so that the electric field is

$$E_r=\begin{cases}\dfrac{\sigma_0 R^2}{\varepsilon_0 r^2}=\dfrac{Q}{4\pi\varepsilon_0 r^2}, & r>R\\ 0, & r<R\end{cases}\tag{13}$$

The integration in (12) amounts to just a multiplication of $\varepsilon_0 E_r$ and the surface area of the Gaussian sphere because on the sphere the electric field is constant and in the same direction as the normal $\mathbf{i}_r$. The electric field outside the sphere is the same as if all the surface charge were concentrated as a point charge at the origin.

The zero field solution for $r<R$ is what really proved Coulomb's law. After all, Coulomb's small spheres were not really point charges and his measurements did have small sources of errors. Perhaps the electric force only varied inversely with distance by some power close to two, $r^{-2+\delta}$, where δ is very small. However, only the inverse square law

gives a zero electric field within a uniformly surface charged sphere. This zero field result is true for any closed conducting body of arbitrary shape charged on its surface with no enclosed charge. Extremely precise measurements were made inside such conducting surface charged bodies and the electric field was always found to be zero. Such a closed conducting body is used for shielding so that a zero field environment can be isolated and is often called a Faraday cage, after Faraday's measurements of actually climbing into a closed hollow conducting body charged on its surface to verify the zero field results.

To appreciate the ease of solution using Gauss's law, let us redo the problem using the superposition integral of Section 2.3.2. From Figure 2-16*a* the incremental radial component of electric field due to a differential charge element is

$$dE_r = \frac{\sigma_0 R^2 \sin\theta\, d\theta\, d\phi}{4\pi\varepsilon_0 r_{QP}^2}\cos\alpha \tag{14}$$

From the law of cosines the angles and distances are related as

$$\begin{aligned} r_{QP}^2 &= r^2 + R^2 - 2rR\cos\theta \\ R^2 &= r^2 + r_{QP}^2 - 2rr_{QP}\cos\alpha \end{aligned} \tag{15}$$

so that α is related to θ as

$$\cos\alpha = \frac{r - R\cos\theta}{[r^2 + R^2 - 2rR\cos\theta]^{1/2}} \tag{16}$$

Then the superposition integral of Section 2.3.2 requires us to integrate (14) as

$$E_r = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{\sigma_0 R^2 \sin\theta (r - R\cos\theta)\, d\theta\, d\phi}{4\pi\varepsilon_0 [r^2 + R^2 - 2rR\cos\theta]^{3/2}} \tag{17}$$

After performing the easy integration over ϕ that yields the factor of 2π, we introduce the change of variable:

$$\begin{aligned} u &= r^2 + R^2 - 2rR\cos\theta \\ du &= 2rR\sin\theta\, d\theta \end{aligned} \tag{18}$$

which allows us to rewrite the electric field integral as

$$\begin{aligned} E_r &= \int_{u=(r-R)^2}^{(r+R)^2} \frac{\sigma_0 R[u + r^2 - R^2]\, du}{8\varepsilon_0 r^2 u^{3/2}} \\ &= \frac{\sigma_0 R}{4\varepsilon_0 r^2}\left(u^{1/2} - \frac{(r^2 - R^2)}{u^{1/2}}\right)\Bigg|_{(r-R)^2}^{(r+R)^2} \\ &= \frac{\sigma_0 R}{4\varepsilon_0 r^2}\left[(r+R) - |r-R| - (r^2 - R^2)\left(\frac{1}{r+R} - \frac{1}{|r-R|}\right)\right] \end{aligned} \tag{19}$$

where we must be very careful to take the positive square root in evaluating the lower limit of the integral for $r<R$. Evaluating (19) for r greater and less than R gives us (13), but with a lot more effort.

(b) Volume Charge Distribution

If the sphere is uniformly charged throughout with density ρ_0, then the Gaussian surface in Figure 2-17a for $r>R$ still encloses the total charge $Q=\frac{4}{3}\pi R^3\rho_0$. However, now the smaller Gaussian surface with $r<R$ encloses a fraction of the total charge:

$$\oint_S \varepsilon_0 \mathbf{E}\cdot d\mathbf{S}=\varepsilon_0 E_r 4\pi r^2=\begin{cases}\rho_0\frac{4}{3}\pi r^3=Q(r/R)^3, & r<R\\ \rho_0\frac{4}{3}\pi R^3=Q, & r>R\end{cases} \tag{20}$$

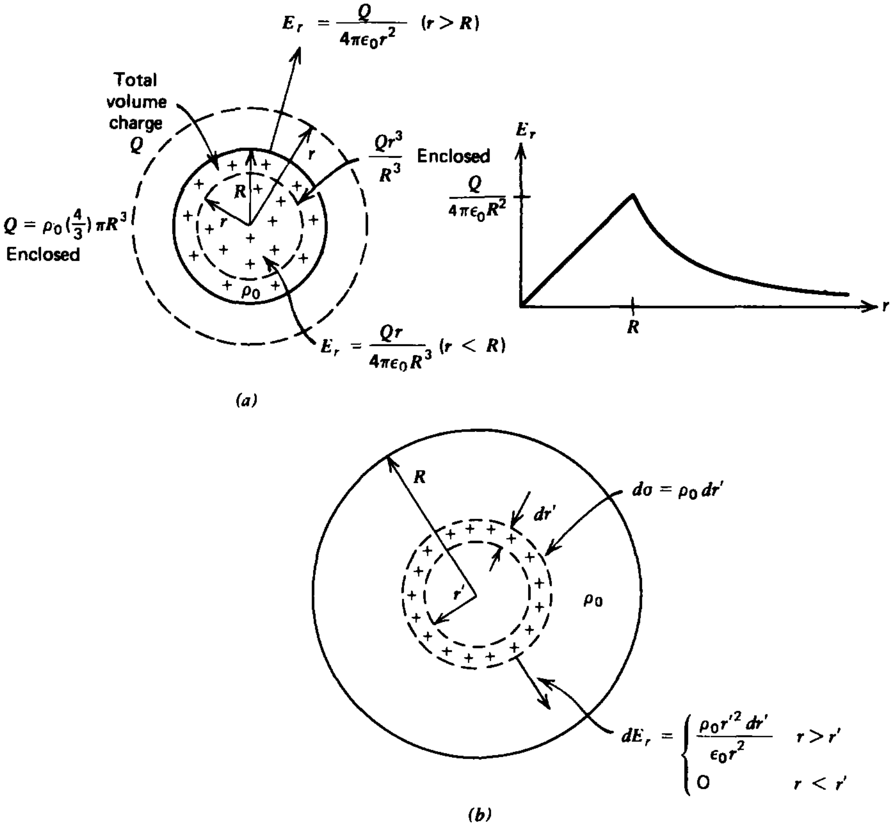


Figure 2-17 (a) Gaussian spheres for a uniformly charged sphere show that the electric field outside the sphere is again the same as if all the charge $Q=\frac{4}{3}\pi R^3\rho_0$ were concentrated as a point charge at $r=0$. (b) The solution is also obtained by summing the contributions from incremental spherical shells of surface charge.

so that the electric field is

$$E_r = \begin{cases} \dfrac{\rho_0 r}{3\varepsilon_0} = \dfrac{Qr}{4\pi\varepsilon_0 R^3}, & r<R \\ \dfrac{\rho_0 R^3}{3\varepsilon_0 r^2} = \dfrac{Q}{4\pi\varepsilon_0 r^2}, & r>R \end{cases} \tag{21}$$

This result could also have been obtained using the results of (13) by breaking the spherical volume into incremental shells of radius r', thickness dr', carrying differential surface charge $d\sigma = \rho_0\, dr'$ as in Figure 2-17b. Then the contribution to the field is zero inside each shell but nonzero outside:

$$dE_r = \begin{cases} 0, & r<r' \\ \dfrac{\rho_0 r'^2\, dr'}{\varepsilon_0 r^2}, & r>r' \end{cases} \tag{22}$$

The total field outside the sphere is due to all the differential shells, while the field inside is due only to the enclosed shells:

$$E_r = \begin{cases} \displaystyle\int_0^r \frac{r'^2 \rho_0\, dr'}{\varepsilon_0 r^2} = \frac{\rho_0 r}{3\varepsilon_0} = \frac{Qr}{4\pi\varepsilon_0 R^3}, & r<R \\ \displaystyle\int_0^R \frac{r'^2 \rho_0\, dr'}{\varepsilon_0 r^2} = \frac{\rho_0 R^3}{3\varepsilon_0 r^2} = \frac{Q}{4\pi\varepsilon_0 r^2}, & r>R \end{cases} \tag{23}$$

which agrees with (21).

2-4-4 Cylindrical Symmetry

(a) Hollow Cylinder of Surface Charge

An infinitely long cylinder of radius a has a uniform distribution of surface charge σ_0, as shown in Figure 2-18a. The angle ϕ is measured from the line joining the field point P to the center of the cylinder. Each incremental line charge element $d\lambda = \sigma_0 a\, d\phi$ contributes to the electric field at P as given by the solution for an infinitely long line charge in Section 2.3.3. However, the symmetrically located element at $-\phi$ gives rise to equal magnitude field components that add radially as measured from the cylinder center but cancel in the ϕ direction.

Because of the symmetry, the electric field is purely radial so that we use Gauss's law with a concentric cylinder of radius r and height L, as in Figure 2-18b where L is arbitrary. There is no contribution to Gauss's law from the upper and lower surfaces because the electric field is purely tangential. Along the cylindrical wall at radius r, the electric field is constant and

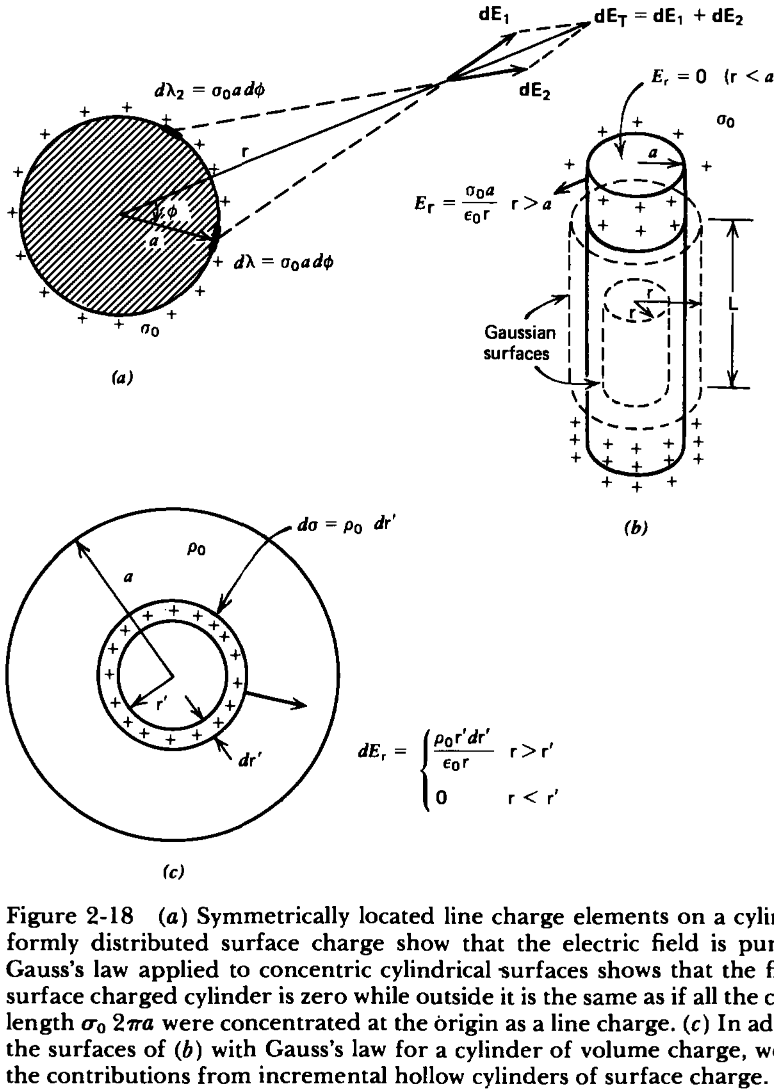


Figure 2-18 (*a*) Symmetrically located line charge elements on a cylinder with uniformly distributed surface charge show that the electric field is purely radial. (*b*) Gauss's law applied to concentric cylindrical surfaces shows that the field inside the surface charged cylinder is zero while outside it is the same as if all the charge per unit length $\sigma_0\, 2\pi a$ were concentrated at the origin as a line charge. (*c*) In addition to using the surfaces of (*b*) with Gauss's law for a cylinder of volume charge, we can also sum the contributions from incremental hollow cylinders of surface charge.

purely normal so that Gauss's law simply yields

$$\oint_S \varepsilon_0 \mathbf{E} \cdot \mathbf{dS} = \varepsilon_0 E_r 2\pi r L = \begin{cases} \sigma_0 2\pi a L, & r > a \\ 0 & r < a \end{cases} \tag{24}$$

where for $r < a$ no charge is enclosed, while for $r > a$ all the charge within a height L is enclosed. The electric field outside the cylinder is then the same as if all the charge per unit length $\lambda = \sigma_0 2\pi a$ were concentrated along the axis of the cylinder:

$$E_r = \begin{cases} \dfrac{\sigma_0 a}{\varepsilon_0 r} = \dfrac{\lambda}{2\pi\varepsilon_0 r} & r > a \\ 0, & r < a \end{cases} \tag{25}$$

Note in (24) that the arbitrary height L canceled out.

(b) Cylinder of Volume Charge

If the cylinder is uniformly charged with density ρ_0, both Gaussian surfaces in Figure 2-18b enclose charge

$$\oint_S \varepsilon_0 \mathbf{E} \cdot \mathbf{dS} = \varepsilon_0 E_r 2\pi r L = \begin{cases} \rho_0 \pi a^2 L, & r > a \\ \rho_0 \pi r^2 L, & r < a \end{cases} \tag{26}$$

so that the electric field is

$$E_r = \begin{cases} \dfrac{\rho_0 a^2}{2\varepsilon_0 r} = \dfrac{\lambda}{2\pi\varepsilon_0 r}, & r > a \\ \dfrac{\rho_0 r}{2\varepsilon_0} = \dfrac{\lambda r}{2\pi\varepsilon_0 a^2} & r < a \end{cases} \tag{27}$$

where $\lambda = \rho_0 \pi a^2$ is the total charge per unit length on the cylinder.

Of course, this result could also have been obtained by integrating (25) for all differential cylindrical shells of radius r' with thickness dr' carrying incremental surface charge $d\sigma = \rho_0\, dr'$, as in Figure 2-18c.

$$E_r = \begin{cases} \displaystyle\int_0^a \frac{\rho_0 r'}{\varepsilon_0 r}\, dr' = \frac{\rho_0 a^2}{2\varepsilon_0 r} = \frac{\lambda}{2\pi\varepsilon_0 r}, & r > a \\ \displaystyle\int_0^r \frac{\rho_0 r'}{\varepsilon_0 r}\, dr' = \frac{\rho_0 r}{2\varepsilon_0} = \frac{\lambda r}{2\pi\varepsilon_0 a^2}, & r < a \end{cases} \tag{28}$$

2-4-5 Gauss's Law and the Divergence Theorem

If a volume distribution of charge ρ is completely surrounded by a closed Gaussian surface S, Gauss's law of (11) is

$$\oint_S \varepsilon_0 \mathbf{E} \cdot \mathbf{dS} = \int_V \rho\, dV \tag{29}$$

The left-hand side of (29) can be changed to a volume integral using the divergence theorem:

$$\oint_S \varepsilon_0 \mathbf{E} \cdot \mathbf{dS} = \int_V \nabla \cdot (\varepsilon_0 \mathbf{E})\, dV = \int_V \rho\, dV \tag{30}$$

Since (30) must hold for any volume, the volume integrands in (30) must be equal, yielding the point form of Gauss's law:

$$\nabla \cdot (\varepsilon_0 \mathbf{E}) = \rho \tag{31}$$

Since the permittivity of free space ε_0 is a constant, it can freely move outside the divergence operator.

2-4-6 Electric Field Discontinuity Across a Sheet of Surface Charge

In Section 2.3.4*a* we found that the electric field changes direction discontinuously on either side of a straight sheet of surface charge. We can be more general by applying the surface integral form of Gauss's law in (30) to the differential-sized pill-box surface shown in Figure 2-19 surrounding a small area dS of surface charge:

$$\oint_S \varepsilon_0 \mathbf{E} \cdot \mathbf{dS} = \int_S \sigma\, dS \Rightarrow \varepsilon_0 (E_{2n} - E_{1n})\, dS = \sigma\, dS \tag{32}$$

where E_{2n} and E_{1n} are the perpendicular components of electric field on each side of the interface. Only the upper and lower surfaces of the pill-box contribute in (32) because the surface charge is assumed to have zero thickness so that the short cylindrical surface has zero area. We thus see that the surface charge density is proportional to the discontinuity in the normal component of electric field across the sheet:

$$\varepsilon_0 (E_{2n} - E_{1n}) = \sigma \Rightarrow \mathbf{n} \cdot \varepsilon_0 (\mathbf{E}_2 - \mathbf{E}_1) = \sigma \tag{33}$$

where $\mathbf{n}$ is perpendicular to the interface directed from region 1 to region 2.

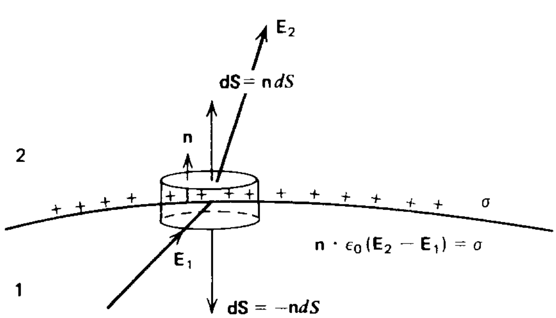


Figure 2-19 Gauss's law applied to a differential sized pill-box surface enclosing some surface charge shows that the normal component of $\varepsilon_0 \mathbf{E}$ is discontinuous in the surface charge density.

2-5 THE ELECTRIC POTENTIAL

If we have two charges of opposite sign, work must be done to separate them in opposition to the attractive coulomb force. This work can be regained if the charges are allowed to come together. Similarly, if the charges have the same sign, work must be done to push them together; this work can be regained if the charges are allowed to separate. A charge gains energy when moved in a direction opposite to a force. This is called potential energy because the amount of energy depends on the position of the charge in a force field.

2-5-1 Work Required to Move a Point Charge

The work W required to move a test charge q_t along any path from the radial distance r_a to the distance r_b with a force that just overcomes the coulombic force from a point charge q, as shown in Figure 2-20, is

$$W = -\int_{r_a}^{r_b} \mathbf{F}\cdot d\mathbf{l}$$

$$= -\frac{qq_t}{4\pi\varepsilon_0}\int_{r_a}^{r_b} \frac{\mathbf{i}_r\cdot d\mathbf{l}}{r^2} \qquad (1)$$

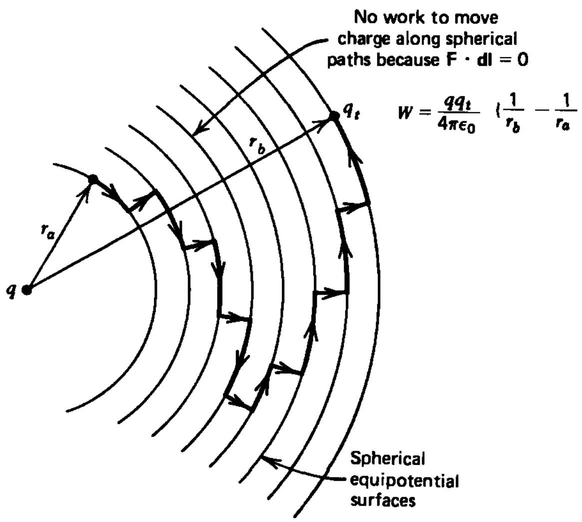


Figure 2-20 It takes no work to move a test charge q_t along the spherical surfaces perpendicular to the electric field due to a point charge q. Such surfaces are called equipotential surfaces.

The minus sign in front of the integral is necessary because the quantity W represents the work we must exert on the test charge in opposition to the coulombic force between charges. The dot product in (1) tells us that it takes no work to move the test charge perpendicular to the electric field, which in this case is along spheres of constant radius. Such surfaces are called equipotential surfaces. Nonzero work is necessary to move q to a different radius for which $\mathbf{dl} = dr\,\mathbf{i}_r$. Then, the work of (1) depends only on the starting and ending positions (r_a and r_b) of the path and not on the shape of the path itself:

$$\begin{aligned} W &= -\frac{qq_t}{4\pi\varepsilon_0}\int_{r_a}^{r_b}\frac{dr}{r^2} \\ &= \frac{qq_t}{4\pi\varepsilon_0}\left(\frac{1}{r_b}-\frac{1}{r_a}\right) \end{aligned} \tag{2}$$

We can convince ourselves that the sign is correct by examining the case when r_b is bigger than r_a and the charges q and q_t are of opposite sign and so attract each other. To separate the charges further requires us to do work on q_t so that W is positive in (2). If q and q_t are the same sign, the repulsive coulomb force would tend to separate the charges further and perform work on q_t. For force equilibrium, we would have to exert a force opposite to the direction of motion so that W is negative.

If the path is closed so that we begin and end at the same point with $r_a = r_b$, the net work required for the motion is zero. If the charges are of the opposite sign, it requires positive work to separate them, but on the return, equal but opposite work is performed on us as the charges attract each other.

If there was a distribution of charges with net field $\mathbf{E}$, the work in moving the test charge against the total field $\mathbf{E}$ is just the sum of the works necessary to move the test charge against the field from each charge alone. Over a closed path this work remains zero:

$$W = \oint_L -q_t\mathbf{E}\cdot\mathbf{dl} = 0 \Rightarrow \oint_L \mathbf{E}\cdot\mathbf{dl} = 0 \tag{3}$$

which requires that the line integral of the electric field around the closed path also be zero.

2-5-2 The Electric Field and Stokes' Theorem

Using Stokes' theorem of Section 1.5.3, we can convert the line integral of the electric field to a surface integral of the curl of the electric field:

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} \tag{4}$$

From Section 1.3.3, we remember that the gradient of a scalar function also has the property that its line integral around a closed path is zero. This means that the electric field can be determined from the gradient of a scalar function V called the potential having units of volts [kg-m^2-s^{-3}-A^{-1}]:

$$\mathbf{E} = -\nabla V \tag{5}$$

The minus sign is introduced by convention so that the electric field points in the direction of decreasing potential. From the properties of the gradient discussed in Section 1.3.1 we see that the electric field is always perpendicular to surfaces of constant potential.

By applying the right-hand side of (4) to an area of differential size or by simply taking the curl of (5) and using the vector identity of Section 1.5.4*a* that the curl of the gradient is zero, we reach the conclusion that the electric field has zero curl:

$$\nabla \times \mathbf{E} = 0 \tag{6}$$

2-5-3 The Potential and the Electric Field

The potential difference between the two points at r_a and r_b is the work per unit charge necessary to move from r_a to r_b:

$$\begin{aligned} V(r_b) - V(r_a) &= \frac{W}{q_t} \\ &= -\int_{r_a}^{r_b} \mathbf{E} \cdot d\mathbf{l} = +\int_{r_b}^{r_a} \mathbf{E} \cdot d\mathbf{l} \end{aligned} \tag{7}$$

Note that (3), (6), and (7) are the fields version of Kirchoff's circuit voltage law that the algebraic sum of voltage drops around a closed loop is zero.

The advantage to introducing the potential is that it is a scalar from which the electric field can be easily calculated. The electric field must be specified by its three components, while if the single potential function V is known, taking its negative gradient immediately yields the three field components. This is often a simpler task than solving for each field component separately. Note in (5) that adding a constant to the potential does not change the electric field, so the potential is only uniquely defined to within a constant. It is necessary to specify a reference zero potential that is often taken at infinity. In actual practice zero potential is often assigned to the earth's surface so that common usage calls the reference point "ground."

The potential due to a single point charge q is

$$V(r_b)-V(r_a)=-\int_{r_a}^{r_b}\frac{q\,dr}{4\pi\varepsilon_0 r^2}=\frac{q}{4\pi\varepsilon_0 r}\bigg|_{r_a}^{r_b}$$

$$=\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_b}-\frac{1}{r_a}\right) \tag{8}$$

If we pick our reference zero potential at $r_a=\infty$, $V(r_a)=0$ so that $r_b=r$ is just the radial distance from the point charge. The scalar potential V is then interpreted as the work per unit charge necessary to bring a charge from infinity to some distance r from the point charge q:

$$V(r)=\frac{q}{4\pi\varepsilon_0 r} \tag{9}$$

The net potential from many point charges is obtained by the sum of the potentials from each charge alone. If there is a continuous distribution of charge, the summation becomes an integration over all the differential charge elements dq:

$$V=\int_{\text{all } q}\frac{dq}{4\pi\varepsilon_0 r_{QP}} \tag{10}$$

where the integration is a line integral for line charges, a surface integral for surface charges, and a volume integral for volume charges.

The electric field formula of Section 2.3.2 obtained by superposition of coulomb's law is easily re-obtained by taking the negative gradient of (10), recognizing that derivatives are to be taken with respect to field positions (x, y, z) while the integration is over source positions (x_Q, y_Q, z_Q). The del operator can thus be brought inside the integral and operates only on the quantity r_{QP}:

$$\mathbf{E}=-\nabla V=-\int_{\text{all } q}\frac{dq}{4\pi\varepsilon_0}\nabla\left(\frac{1}{r_{QP}}\right)$$

$$=\int_{\text{all } q}\frac{dq}{4\pi\varepsilon_0 r_{QP}^2}\mathbf{i}_{QP} \tag{11}$$

where we use the results of Section 2.4.1*b* for the gradient of the reciprocal distance.

2-5-4 Finite Length Line Charge

To demonstrate the usefulness of the potential function, consider the uniform distribution of line charge λ_0 of finite length $2L$ centered on the z axis in Figure 2-21. Distinguishing between the position of the charge element $dq = \lambda_0\, dz'$ at z' and the field point at coordinate z, the distance between source and field point is

$$r_{QP} = [r^2 + (z - z')^2]^{1/2} \tag{12}$$

Substituting into (10) yields

$$\begin{aligned} V &= \int_{-L}^{L} \frac{\lambda_0\, dz'}{4\pi\varepsilon_0[r^2 + (z - z')^2]^{1/2}} \\ &= \frac{\lambda_0}{4\pi\varepsilon_0} \ln\left(\frac{z - L + [r^2 + (z - L)^2]^{1/2}}{z + L + [r^2 + (z + L)^2]^{1/2}}\right) \\ &= \frac{\lambda_0}{4\pi\varepsilon_0}\left(\sinh^{-1}\frac{z - L}{r} - \sinh^{-1}\frac{z + L}{r}\right) \end{aligned} \tag{13}$$

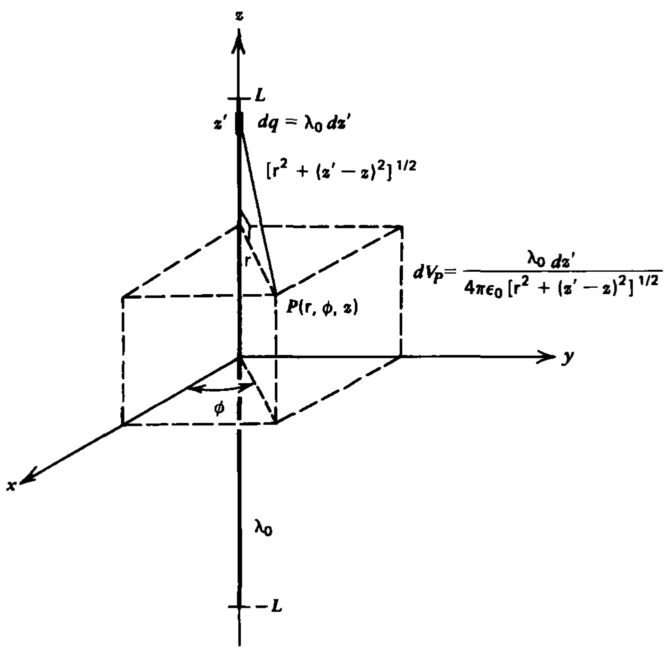


Figure 2-21 The potential from a finite length of line charge is obtained by adding the potentials due to each incremental line charge element.

The field components are obtained from (13) by taking the negative gradient of the potential:

$$E_z=-\frac{\partial V}{\partial z}=\frac{\lambda_0}{4\pi\varepsilon_0}\left(\frac{1}{[r^2+(z-L)^2]^{1/2}}-\frac{1}{[r^2+(z+L)^2]^{1/2}}\right)$$

$$E_r=-\frac{\partial V}{\partial r}=\frac{\lambda_0 r}{4\pi\varepsilon_0}\left(\frac{1}{[r^2+(z-L)^2]^{1/2}[z-L+[r^2+(z-L)^2]^{1/2}]}\right.$$
$$\left.-\frac{1}{[r^2+(z+L)^2]^{1/2}[z+L+[r^2+(z+L)^2]^{1/2}]}\right)$$
$$=-\frac{\lambda_0}{4\pi\varepsilon_0 r}\left(\frac{z-L}{[r^2+(z-L)^2]^{1/2}}-\frac{z+L}{[r^2+(z+L)^2]^{1/2}}\right) \quad (14)$$

As L becomes large, the field and potential approaches that of an infinitely long line charge:

$$\lim_{L\to\infty}\begin{cases}E_z=0\\ E_r=\dfrac{\lambda_0}{2\pi\varepsilon_0 r}\\ V=-\dfrac{\lambda_0}{2\pi\varepsilon_0}(\ln r-\ln 2L)\end{cases} \quad (15)$$

The potential has a constant term that becomes infinite when L is infinite. This is because the zero potential reference of (10) is at infinity, but when the line charge is infinitely long the charge at infinity is nonzero. However, this infinite constant is of no concern because it offers no contribution to the electric field.

Far from the line charge the potential of (13) approaches that of a point charge $2\lambda_0 L$:

$$\lim_{r^2=r^2+z^2\gg L^2} V=\frac{\lambda_0(2L)}{4\pi\varepsilon_0 r} \quad (16)$$

Other interesting limits of (14) are

$$\lim_{z=0}\begin{cases}E_z=0\\ E_r=\dfrac{\lambda_0 L}{2\pi\varepsilon_0 r(r^2+L^2)^{1/2}}\end{cases}$$

$$\lim_{r=0}\begin{cases}E_z=\dfrac{\lambda_0}{4\pi\varepsilon_0}\left(\dfrac{1}{|z-L|}-\dfrac{1}{|z+L|}\right)=\begin{cases}\dfrac{\pm\lambda_0 L}{2\pi\varepsilon_0(z^2-L^2)}, & z>L \\ & z<-L\\ \dfrac{\lambda_0 z}{2\pi\varepsilon_0(L^2-z^2)}, & -L\le z\le L\end{cases}\\ E_r=0\end{cases} \quad (17)$$

2-5-5 Charged Spheres

(a) Surface Charge

A sphere of radius R supports a uniform distribution of surface charge σ_0 with total charge $Q=\sigma_0 4\pi R^2$, as shown in Figure 2-22*a*. Each incremental surface charge element contributes to the potential as

$$dV=\frac{\sigma_0 R^2 \sin\theta\, d\theta\, d\phi}{4\pi\varepsilon_0 r_{QP}} \tag{18}$$

where from the law of cosines

$$r_{QP}^2=R^2+r^2-2rR\cos\theta \tag{19}$$

so that the differential change in r_{QP} about the sphere is

$$2r_{QP}\,dr_{QP}=2rR\sin\theta\, d\theta \tag{20}$$

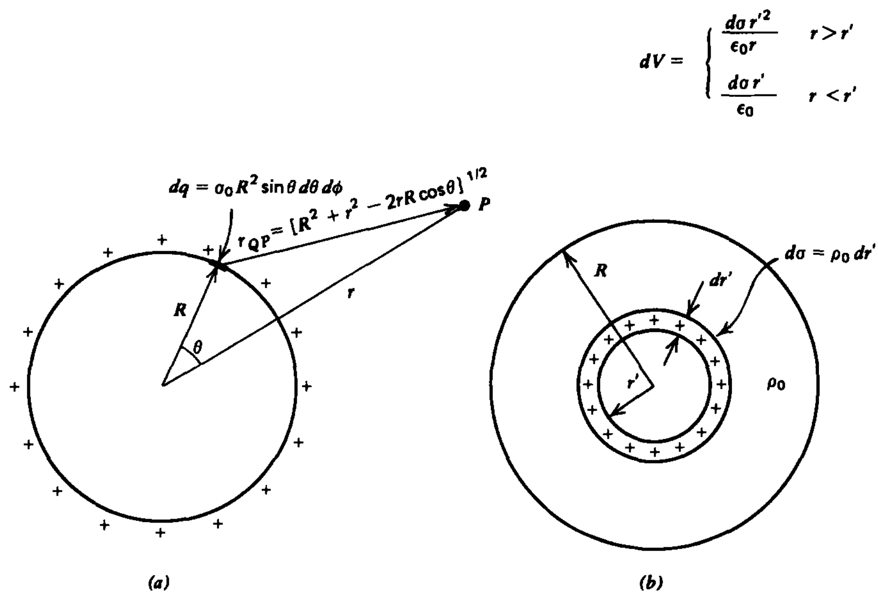


Figure 2-22 (*a*) A sphere of radius R supports a uniform distribution of surface charge σ_0. (*b*) The potential due to a uniformly volume charged sphere is found by summing the potentials due to differential sized shells.

Therefore, the total potential due to the whole charged sphere is

$$V=\int_{r_{QP}=|r-R|}^{r+R}\int_{\phi=0}^{2\pi}\frac{\sigma_0 R}{4\pi\varepsilon_0 r}\,dr_{QP}\,d\phi$$

$$=\frac{\sigma_0 R}{2\varepsilon_0 r}r_{QP}\Big|_{|r-R|}^{r+R}$$

$$=\begin{cases}\dfrac{\sigma_0 R^2}{\varepsilon_0 r}=\dfrac{Q}{4\pi\varepsilon_0 r}, & r>R\\[2ex] \dfrac{\sigma_0 R}{\varepsilon_0}=\dfrac{Q}{4\pi\varepsilon_0 R}, & r<R\end{cases} \tag{21}$$

Then, as found in Section 2.4.3*a* the electric field is

$$E_r=-\frac{\partial V}{\partial r}=\begin{cases}\dfrac{\sigma_0 R^2}{\varepsilon_0 r^2}=\dfrac{Q}{4\pi\varepsilon_0 r^2}, & r>R\\[2ex] 0 & r<R\end{cases} \tag{22}$$

Outside the sphere, the potential of (21) is the same as if all the charge Q were concentrated at the origin as a point charge, while inside the sphere the potential is constant and equal to the surface potential.

(b) Volume Charge

If the sphere is uniformly charged with density ρ_0 and total charge $Q=\frac{4}{3}\pi R^3\rho_0$, the potential can be found by breaking the sphere into differential size shells of thickness dr' and incremental surface charge $d\sigma=\rho_0\,dr'$. Then, integrating (21) yields

$$V=\begin{cases}\displaystyle\int_0^R\frac{\rho_0 r'^2}{\varepsilon_0 r}dr'=\frac{\rho_0 R^3}{3\varepsilon_0 r}=\frac{Q}{4\pi\varepsilon_0 r}, & r>R\\[2ex] \displaystyle\int_0^r\frac{\rho_0 r'^2}{\varepsilon_0 r}dr'+\int_r^R\frac{\rho_0 r'}{\varepsilon_0}dr'=\frac{\rho_0}{2\varepsilon_0}\left(R^2-\frac{r^2}{3}\right) & \\[2ex] \displaystyle\qquad=\frac{3Q}{8\pi\varepsilon_0 R^3}\left(R^2-\frac{r^2}{3}\right) & r<R\end{cases} \tag{23}$$

where we realized from (21) that for $r<R$ the interior shells have a different potential contribution than exterior shells.

Then, the electric field again agrees with Section 2.4.3*b*:

$$E_r=-\frac{\partial V}{\partial r}=\begin{cases}\dfrac{\rho_0 R^3}{3\varepsilon_0 r^2}=\dfrac{Q}{4\pi\varepsilon_0 r^2}, & r>R\\[2ex] \dfrac{\rho_0 r}{3\varepsilon_0}=\dfrac{Qr}{4\pi\varepsilon_0 R^3}, & r<R\end{cases} \tag{24}$$

(c) Two Spheres

Two conducting spheres with respective radii R_1 and R_2 have their centers a long distance D apart as shown in Figure 2-23. Different charges Q_1 and Q_2 are put on each sphere. Because $D \gg R_1+R_2$, each sphere can be treated as isolated. The potential on each sphere is then

$$V_1=\frac{Q_1}{4\pi\varepsilon_0 R_1}, \qquad V_2=\frac{Q_2}{4\pi\varepsilon_0 R_2} \tag{25}$$

If a wire is connected between the spheres, they are forced to be at the same potential:

$$V_0=\frac{q_1}{4\pi\varepsilon_0 R_1}=\frac{q_2}{4\pi\varepsilon_0 R_2} \tag{26}$$

causing a redistribution of charge. Since the total charge in the system must be conserved,

$$q_1+q_2=Q_1+Q_2 \tag{27}$$

Eq. (26) requires that the charges on each sphere be

$$q_1=\frac{R_1(Q_1+Q_2)}{R_1+R_2}, \qquad q_2=\frac{R_2(Q_1+Q_2)}{R_1+R_2} \tag{28}$$

so that the system potential is

$$V_0=\frac{Q_1+Q_2}{4\pi\varepsilon_0(R_1+R_2)} \tag{29}$$

Even though the smaller sphere carries less total charge, from (22) at $r=R$, where $E_r(R)=\sigma_0/\varepsilon_0$, we see that the surface electric field is stronger as the surface charge density is larger:

$$\begin{aligned} E_1(r=R_1)&=\frac{q_1}{4\pi\varepsilon_0 R_1^2}=\frac{Q_1+Q_2}{4\pi\varepsilon_0 R_1(R_1+R_2)}=\frac{V_0}{R_1} \\ E_2(r=R_2)&=\frac{q_2}{4\pi\varepsilon_0 R_2^2}=\frac{Q_1+Q_2}{4\pi\varepsilon_0 R_2(R_1+R_2)}=\frac{V_0}{R_2} \end{aligned} \tag{30}$$

For this reason, the electric field is always largest near corners and edges of equipotential surfaces, which is why

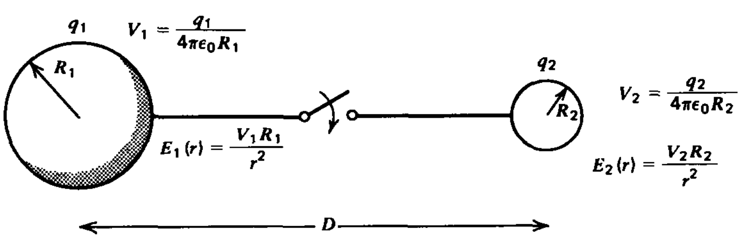


Figure 2-23 The charges on two spheres a long distance apart ($D \gg R_1+R_2$) must redistribute themselves when connected by a wire so that each sphere is at the same potential. The surface electric field is then larger at the smaller sphere.

sharp points must be avoided in high-voltage equipment. When the electric field exceeds a critical amount E_b, called the breakdown strength, spark discharges occur as electrons are pulled out of the surrounding medium. Air has a breakdown strength of $E_b \approx 3\times 10^6$ volts/m. If the two spheres had the same radius of 1 cm (10^{-2} m), the breakdown strength is reached when $V_0 \approx 30{,}000$ volts. This corresponds to a total system charge of $Q_1+Q_2 \approx 6.7\times 10^{-8}$ coul.

2-5-6 Poisson's and Laplace's Equations

The general governing equations for the free space electric field in integral and differential form are thus summarized as

$$\oint_S \varepsilon_0 \mathbf{E}\cdot d\mathbf{S} = \int_V \rho\, dV \Rightarrow \nabla\cdot\mathbf{E} = \rho/\varepsilon_0 \tag{31}$$

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = 0 \Rightarrow \nabla\times\mathbf{E} = 0 \Rightarrow \mathbf{E} = -\nabla V \tag{32}$$

The integral laws are particularly useful for geometries with great symmetry and with one-dimensional fields where the charge distribution is known. Often, the electrical potential of conducting surfaces are constrained by external sources so that the surface charge distributions, themselves sources of electric field are not directly known and are in part due to other charges by induction and conduction. Because of the coulombic force between charges, the charge distribution throughout space itself depends on the electric field and it is necessary to self-consistently solve for the equilibrium between the electric field and the charge distribution. These complications often make the integral laws difficult to use, and it becomes easier to use the differential form of the field equations. Using the last relation of (32) in Gauss's law of (31) yields a single equation relating the Laplacian of the potential to the charge density:

$$\nabla\cdot(\nabla V) = \nabla^2 V = -\rho/\varepsilon_0 \tag{33}$$

which is called Poisson's equation. In regions of zero charge ($\rho = 0$) this equation reduces to Laplace's equation, $\nabla^2 V = 0$.

2-6 THE METHOD OF IMAGES WITH LINE CHARGES AND CYLINDERS

2-6-1 Two Parallel Line Charges

The potential of an infinitely long line charge λ is given in Section 2.5.4 when the length of the line L is made very large. More directly, knowing the electric field of an infinitely long line charge from Section 2.3.3 allows us to obtain the potential by direct integration:

$$E_r = -\frac{\partial V}{\partial r} = \frac{\lambda}{2\pi\varepsilon_0 r} \Rightarrow V = -\frac{\lambda}{2\pi\varepsilon_0} \ln \frac{r}{r_0} \tag{1}$$

where r_0 is the arbitrary reference position of zero potential.

If we have two line charges of opposite polarity $\pm\lambda$ a distance $2a$ apart, we choose our origin halfway between, as in Figure 2-24a, so that the potential due to both charges is just the superposition of potentials of (1):

$$V = -\frac{\lambda}{2\pi\varepsilon_0} \ln \left(\frac{y^2 + (x+a)^2}{y^2 + (x-a)^2}\right)^{1/2} \tag{2}$$

where the reference potential point r_0 cancels out and we use Cartesian coordinates. Equipotential lines are then

$$\frac{y^2 + (x+a)^2}{y^2 + (x-a)^2} = e^{-4\pi\varepsilon_0 V/\lambda} = K_1 \tag{3}$$

where K_1 is a constant on an equipotential line. This relation is rewritten by completing the squares as

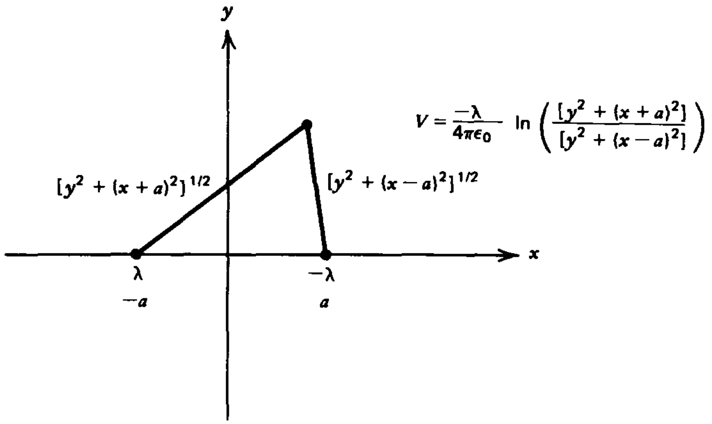
$$\left(x - \frac{a(1+K_1)}{K_1 - 1}\right)^2 + y^2 = \frac{4K_1 a^2}{(1-K_1)^2} \tag{4}$$

which we recognize as circles of radius $r = 2a\sqrt{K_1}/|1-K_1|$ with centers at $y=0, x = a(1+K_1)/(K_1 - 1)$, as drawn by dashed lines in Figure 2-24b. The value of $K_1 = 1$ is a circle of infinite radius with center at $x = \pm\infty$ and thus represents the $x=0$ plane. For values of K_1 in the interval $0 \leq K_1 \leq 1$ the equipotential circles are in the left half-plane, while for $1 \leq K_1 \leq \infty$ the circles are in the right half-plane.

The electric field is found from (2) as

$$\mathbf{E} = -\nabla V = \frac{\lambda}{2\pi\varepsilon_0}\left(\frac{-4axy\mathbf{i}_y + 2a(y^2 + a^2 - x^2)\mathbf{i}_x}{[y^2 + (x+a)^2][y^2 + (x-a)^2]}\right) \tag{5}$$

One way to plot the electric field distribution graphically is by drawing lines that are everywhere tangent to the electric field, called field lines or lines of force. These lines are everywhere perpendicular to the equipotential surfaces and tell us the direction of the electric field. The magnitude is proportional to the density of lines. For a single line charge, the field lines emanate radially. The situation is more complicated for the two line charges of opposite polarity in Figure 2-24 with the field lines always starting on the positive charge and terminating on the negative charge.



(a)

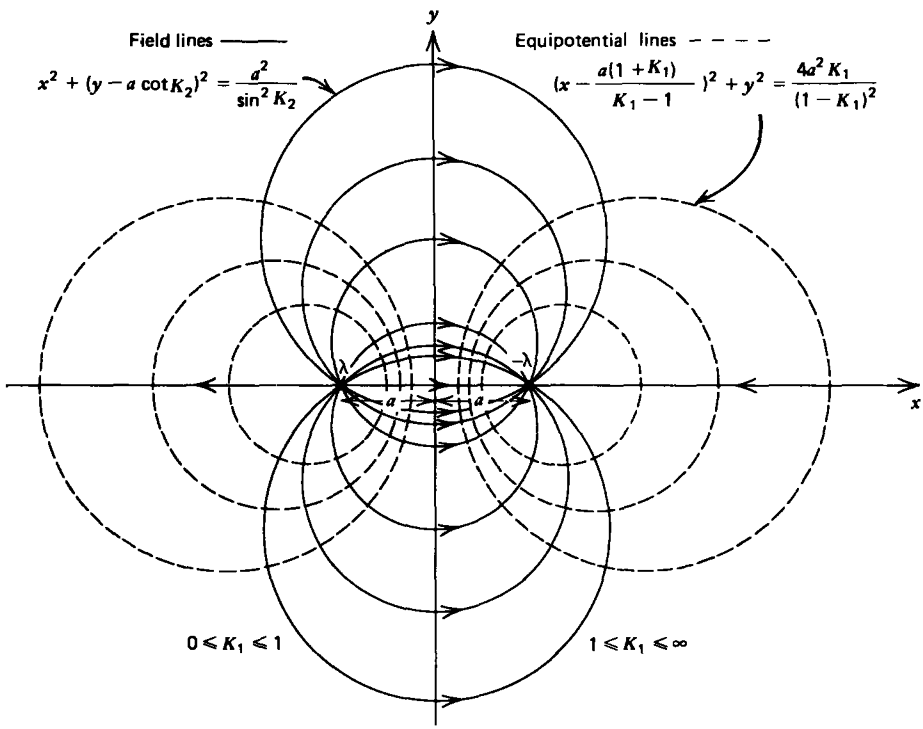


Figure 2-24 (*a*) Two parallel line charges of opposite polarity a distance $2a$ apart. (*b*) The equipotential (dashed) and field (solid) lines form a set of orthogonal circles.

For the field given by (5), the equation for the lines tangent to the electric field is

$$\frac{dy}{dx}=\frac{E_y}{E_x}=-\frac{2xy}{y^2+a^2-x^2}\Rightarrow\frac{d(x^2+y^2)}{a^2-(x^2+y^2)}+d(\ln y)=0 \qquad (6)$$

where the last equality is written this way so the expression can be directly integrated to

$$x^2+(y-a\cot K_2)^2=\frac{a^2}{\sin^2 K_2} \qquad (7)$$

where K_2 is a constant determined by specifying a single coordinate (x_0, y_0) along the field line of interest. The field lines are also circles of radius $a/\sin K_2$ with centers at $x=0, y=a\cot K_2$ as drawn by the solid lines in Figure 2-24*b*.

2-6-2 The Method of Images

(a) General properties

When a conductor is in the vicinity of some charge, a surface charge distribution is induced on the conductor in order to terminate the electric field, as the field within the equipotential surface is zero. This induced charge distribution itself then contributes to the external electric field subject to the boundary condition that the conductor is an equipotential surface so that the electric field terminates perpendicularly to the surface. In general, the solution is difficult to obtain because the surface charge distribution cannot be known until the field is known so that we can use the boundary condition of Section 2.4.6. However, the field solution cannot be found until the surface charge distribution is known.

However, for a few simple geometries, the field solution can be found by replacing the conducting surface by equivalent charges within the conducting body, called images, that guarantee that all boundary conditions are satisfied. Once the image charges are known, the problem is solved as if the conductor were not present but with a charge distribution composed of the original charges plus the image charges.

(b) Line Charge Near a Conducting Plane

The method of images can adapt a known solution to a new problem by replacing conducting bodies with an equivalent charge. For instance, we see in Figure 2-24*b* that the field lines are all perpendicular to the $x=0$ plane. If a conductor were placed along the $x=0$ plane with a single line charge λ at $x=-a$, the potential and electric field for $x<0$ is the same as given by (2) and (5).

A surface charge distribution is induced on the conducting plane in order to terminate the incident electric field as the field must be zero inside the conductor. This induced surface charge distribution itself then contributes to the external electric field for $x<0$ in exactly the same way as for a single image line charge $-\lambda$ at $x=+a$.

The force per unit length on the line charge λ is due only to the field from the image charge $-\lambda$;

$$\mathbf{f}=\lambda\mathbf{E}(-a,0)=\frac{\lambda^2}{2\pi\varepsilon_0(2a)}\mathbf{i}_x=\frac{\lambda^2}{4\pi\varepsilon_0 a}\mathbf{i}_x \tag{8}$$

From Section 2.4.6 we know that the surface charge distribution on the plane is given by the discontinuity in normal component of electric field:

$$\sigma(x=0)=-\varepsilon_0E_x(x=0)=\frac{-\lambda a}{\pi(y^2+a^2)} \tag{9}$$

where we recognize that the field within the conductor is zero. The total charge per unit length on the plane is obtained by integrating (9) over the whole plane:

$$\begin{aligned}\lambda_T&=\int_{-\infty}^{+\infty}\sigma(x=0)\,dy\\&=-\frac{\lambda a}{\pi}\int_{-\infty}^{+\infty}\frac{dy}{y^2+a^2}\\&=-\frac{\lambda a}{\pi}\frac{1}{a}\tan^{-1}\frac{y}{a}\bigg|_{-\infty}^{+\infty}\\&=-\lambda\end{aligned} \tag{10}$$

and just equals the image charge.

2-6-3 Line Charge and Cylinder

Because the equipotential surfaces of (4) are cylinders, the method of images also works with a line charge λ a distance D from the center of a conducting cylinder of radius R as in Figure 2-25. Then the radius R and distance a must fit (4) as

$$R=\frac{2a\sqrt{K_1}}{|1-K_1|},\qquad \pm a+\frac{a(1+K_1)}{K_1-1}=D \tag{11}$$

where the upper positive sign is used when the line charge is outside the cylinder, as in Figure 2-25*a*, while the lower negative sign is used when the line charge is within the cylinder, as in Figure 2-25*b*. Because the cylinder is chosen to be in the right half-plane, $1\le K_1\le\infty$, the unknown parameters K_1

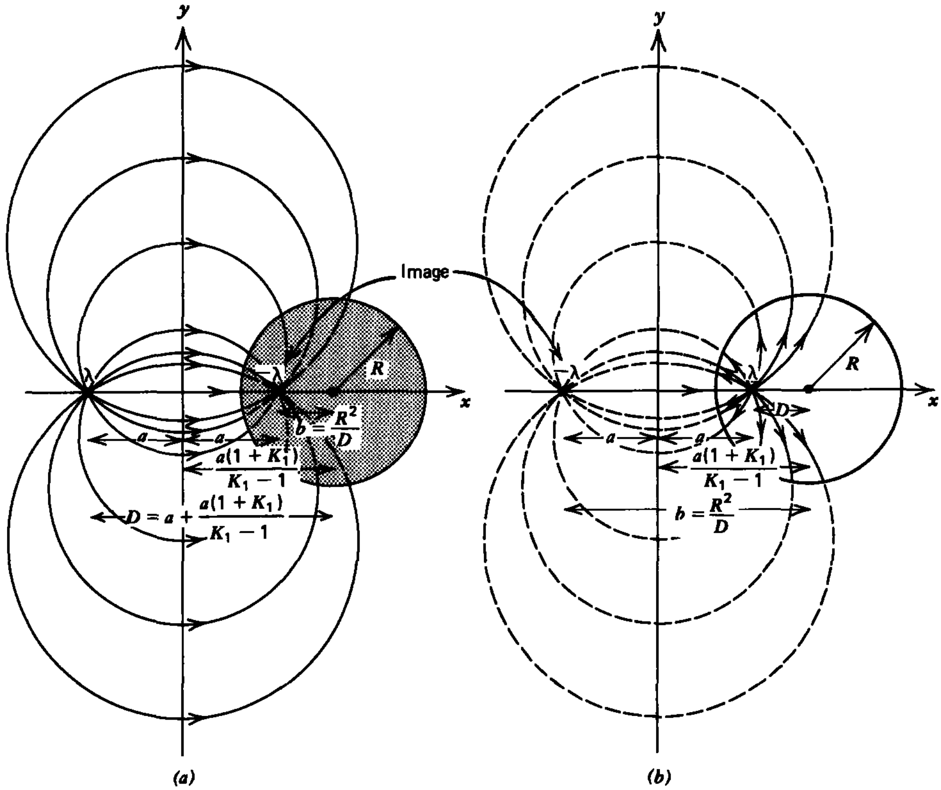


Figure 2-25 The electric field surrounding a line charge λ a distance D from the center of a conducting cylinder of radius R is the same as if the cylinder were replaced by an image charge $-\lambda$, a distance $b = R^2/D$ from the center. (*a*) Line charge outside cylinder. (*b*) Line charge inside cylinder.

and a are expressed in terms of the given values R and D from (11) as

$$K_1 = \left(\frac{D^2}{R^2}\right)^{\pm 1}, \qquad a = \pm\frac{D^2 - R^2}{2D} \tag{12}$$

For either case, the image line charge then lies a distance b from the center of the cylinder:

$$b = \frac{a(1+K_1)}{K_1 - 1} \mp a = \frac{R^2}{D} \tag{13}$$

being inside the cylinder when the inducing charge is outside ($R < D$), and vice versa, being outside the cylinder when the inducing charge is inside ($R > D$).

The force per unit length on the cylinder is then just due to the force on the image charge:

$$f_x = -\frac{\lambda^2}{2\pi\varepsilon_0(D-b)} = -\frac{\lambda^2 D}{2\pi\varepsilon_0(D^2-R^2)} \tag{14}$$

2-6-4 Two Wire Line

(a) Image Charges

We can continue to use the method of images for the case of two parallel equipotential cylinders of differing radii R_1 and R_2 having their centers a distance D apart as in Figure 2-26. We place a line charge λ a distance b_1 from the center of cylinder 1 and a line charge $-\lambda$ a distance b_2 from the center of cylinder 2, both line charges along the line joining the centers of the cylinders. We simultaneously treat the cases where the cylinders are adjacent, as in Figure 2-26*a*, or where the smaller cylinder is inside the larger one, as in Figure 2-26*b*.

The position of the image charges can be found using (13) realizing that the distance from each image charge to the center of the opposite cylinder is $D-b$ so that

$$b_1 = \frac{R_1^2}{D \mp b_2}, \qquad b_2 = \pm\frac{R_2^2}{D-b_1} \tag{15}$$

where the upper signs are used when the cylinders are adjacent and lower signs are used when the smaller cylinder is inside the larger one. We separate the two coupled equations in (15) into two quadratic equations in b_1 and b_2:

$$\begin{aligned} b_1^2 - \frac{[D^2-R_2^2+R_1^2]}{D}b_1 + R_1^2 = 0 \\ b_2^2 \mp \frac{[D^2-R_1^2+R_2^2]}{D}b_2 + R_2^2 = 0 \end{aligned} \tag{16}$$

with resulting solutions

$$\begin{aligned} b_2 &= \pm\frac{[D^2-R_1^2+R_2^2]}{2D} - \left[\left(\frac{D^2-R_1^2+R_2^2}{2D}\right)^2 - R_2^2\right]^{1/2} \\ b_1 &= \frac{[D^2+R_1^2-R_2^2]}{2D} \mp \left[\left(\frac{D^2+R_1^2-R_2^2}{2D}\right)^2 - R_1^2\right]^{1/2} \end{aligned} \tag{17}$$

We were careful to pick the roots that lay outside the region between cylinders. If the equal magnitude but opposite polarity image line charges are located at these positions, the cylindrical surfaces are at a constant potential.

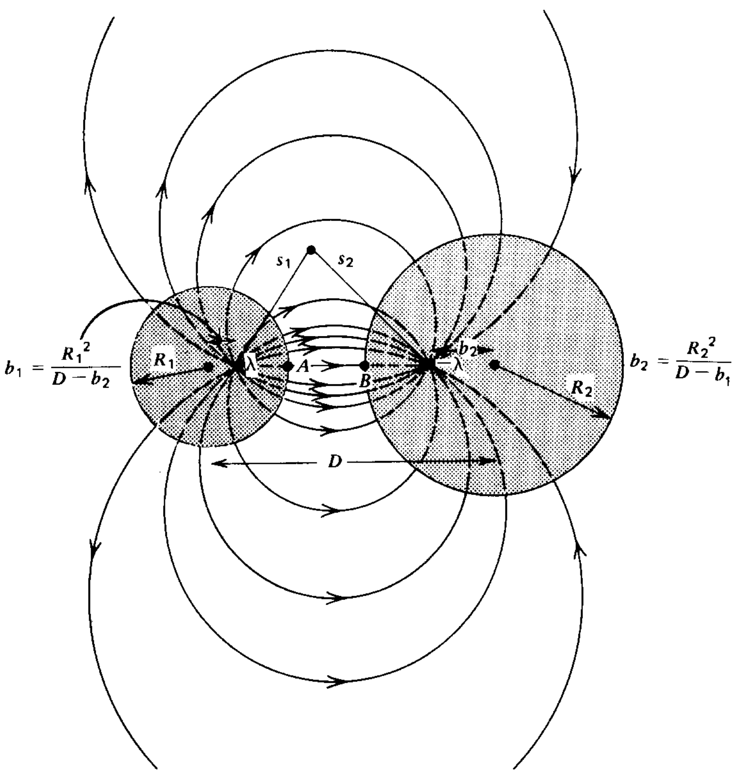


Figure 2-26 The solution for the electric field between two parallel conducting cylinders is found by replacing the cylinders by their image charges. The surface charge density is largest where the cylinder surfaces are closest together. This is called the proximity effect. (*a*) Adjacent cylinders. (*b*) Smaller cylinder inside the larger one.

(b) Force of Attraction

The attractive force per unit length on cylinder 1 is the force on the image charge λ due to the field from the opposite image charge $-\lambda$:

$$f_x = \frac{\lambda^2}{2\pi\varepsilon_0[\pm(D-b_1)-b_2]}$$

$$= \frac{\lambda^2}{4\pi\varepsilon_0\left[\left(\frac{D^2-R_1^2+R_2^2}{2D}\right)^2 - R_2^2\right]^{1/2}}$$

$$= \frac{\lambda^2}{4\pi\varepsilon_0\left[\left(\frac{D^2-R_2^2+R_1^2}{2D}\right)^2 - R_1^2\right]^{1/2}} \qquad (18)$$

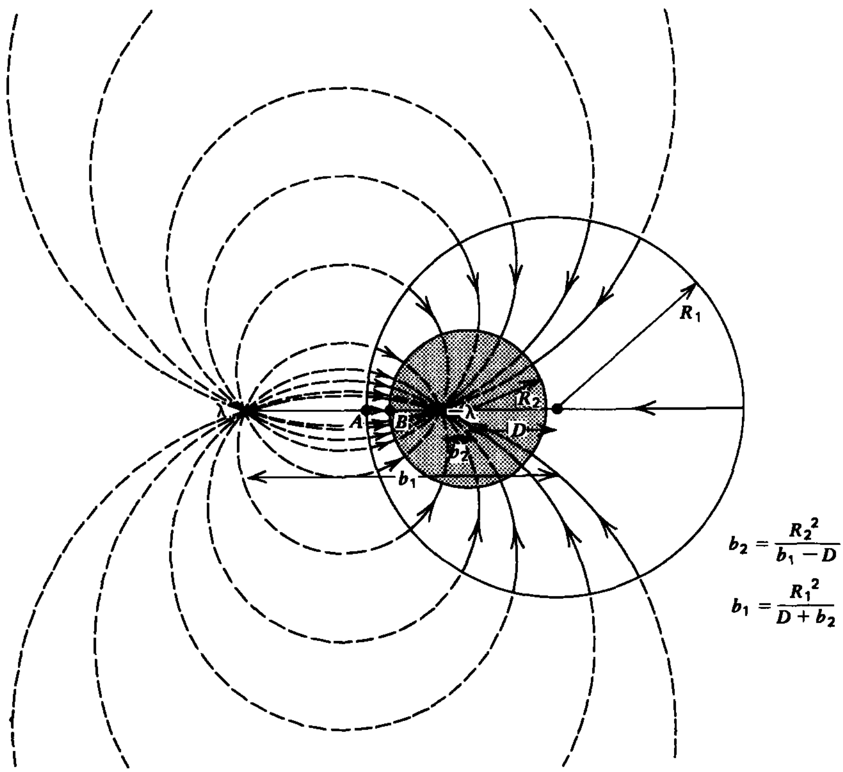


Fig. 2-26(*b*)

(c) Capacitance Per Unit Length

The potential of (2) in the region between the two cylinders depends on the distances from any point to the line charges:

$$V = -\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{s_1}{s_2} \tag{19}$$

To find the voltage difference between the cylinders we pick the most convenient points labeled A and B in Figure 2-26:

$$\begin{array}{ll} A & B \\ s_1 = \pm(R_1 - b_1) & s_1 = \pm(D - b_1 \mp R_2) \\ s_2 = \pm(D \mp b_2 - R_1) & s_2 = R_2 - b_2 \end{array} \tag{20}$$

although any two points on the surfaces could have been used. The voltage difference is then

$$V_1 - V_2 = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\pm\frac{(R_1 - b_1)(R_2 - b_2)}{(D \mp b_2 - R_1)(D - b_1 \mp R_2)}\right) \tag{21}$$

This expression can be greatly reduced using the relations

$$D \mp b_2 = \frac{R_1^2}{b_1}, \qquad D - b_1 = \pm \frac{R_2^2}{b_2} \tag{22}$$

to

$$\begin{aligned} V_1 - V_2 &= -\frac{\lambda}{2\pi\varepsilon_0} \ln \frac{b_1 b_2}{R_1 R_2} \\ &= \frac{\lambda}{2\pi\varepsilon_0} \ln \left\{ \pm \frac{[D^2 - R_1^2 - R_2^2]}{2R_1R_2} \right. \\ &\quad \left. + \left[\left(\frac{D^2 - R_1^2 - R_2^2}{2R_1R_2} \right)^2 - 1 \right]^{1/2} \right\} \end{aligned} \tag{23}$$

The potential difference $V_1 - V_2$ is linearly related to the line charge λ through a factor that only depends on the geometry of the conductors. This factor is defined as the capacitance per unit length and is the ratio of charge per unit length to potential difference:

$$\begin{aligned} C = \frac{\lambda}{V_1 - V_2} &= \frac{2\pi\varepsilon_0}{\ln \left\{ \pm \frac{[D^2 - R_1^2 - R_2^2]}{2R_1R_2} + \left[\left(\frac{D^2 - R_1^2 - R_2^2}{2R_1R_2} \right)^2 - 1 \right]^{1/2} \right\}} \\ &= \frac{2\pi\varepsilon_0}{\cosh^{-1} \left(\pm \frac{D^2 - R_1^2 - R_2^2}{2R_1R_2} \right)} \end{aligned} \tag{24}$$

where we use the identity*

$$\ln [y + (y^2 - 1)^{1/2}] = \cosh^{-1} y \tag{25}$$

We can examine this result in various simple limits. Consider first the case for adjacent cylinders ($D > R_1 + R_2$).

1. If the distance D is much larger than the radii,

$$\lim_{D \gg (R_1 + R_2)} C \approx \frac{2\pi\varepsilon_0}{\ln [D^2/(R_1R_2)]} = \frac{2\pi\varepsilon_0}{\cosh^{-1} [D^2/(2R_1R_2)]} \tag{26}$$

2. The capacitance between a cylinder and an infinite plane can be obtained by letting one cylinder have infinite radius but keeping finite the closest distance $s =$

* $y = \cosh x = \dfrac{e^x + e^{-x}}{2}$

$(e^x)^2 - 2ye^x + 1 = 0$

$e^x = y \pm (y^2 - 1)^{1/2}$

$x = \cosh^{-1} y = \ln [y \pm (y^2 - 1)^{1/2}]$

$D-R_1-R_2$ between cylinders. If we let R_1 become infinite, the capacitance becomes

$$\lim_{\substack{R_1\to\infty \\ D-R_1-R_2=s\ (\text{finite})}} C = \frac{2\pi\varepsilon_0}{\ln\left\{\frac{s+R_2}{R_2}+\left[\left(\frac{s+R_2}{R_2}\right)^2-1\right]^{1/2}\right\}} = \frac{2\pi\varepsilon_0}{\cosh^{-1}\left(\frac{s+R_2}{R_2}\right)} \quad (27)$$

3. If the cylinders are identical so that $R_1=R_2\equiv R$, the capacitance per unit length reduces to

$$\lim_{R_1=R_2\equiv R} C = \frac{\pi\varepsilon_0}{\ln\left\{\frac{D}{2R}+\left[\left(\frac{D}{2R}\right)^2-1\right]^{1/2}\right\}} = \frac{\pi\varepsilon_0}{\cosh^{-1}\frac{D}{2R}} \quad (28)$$

4. When the cylinders are concentric so that $D=0$, the capacitance per unit length is

$$\lim_{D=0} C = \frac{2\pi\varepsilon_0}{\ln(R_1/R_2)} = \frac{2\pi\varepsilon_0}{\cosh^{-1}[(R_1^2+R_2^2)/(2R_1R_2)]} \quad (29)$$

2-7 THE METHOD OF IMAGES WITH POINT CHARGES AND SPHERES

2-7-1 Point Charge and a Grounded Sphere

A point charge q is a distance D from the center of the conducting sphere of radius R at zero potential as shown in Figure 2-27a. We try to use the method of images by placing a single image charge q' a distance b from the sphere center along the line joining the center to the point charge q.

We need to find values of q' and b that satisfy the zero potential boundary condition at $r=R$. The potential at any point P outside the sphere is

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{s}+\frac{q'}{s'}\right) \quad (1)$$

where the distance from P to the point charges are obtained from the law of cosines:

$$\begin{aligned} s &= [r^2+D^2-2rD\cos\theta]^{1/2} \\ s' &= [b^2+r^2-2rb\cos\theta]^{1/2} \end{aligned} \quad (2)$$

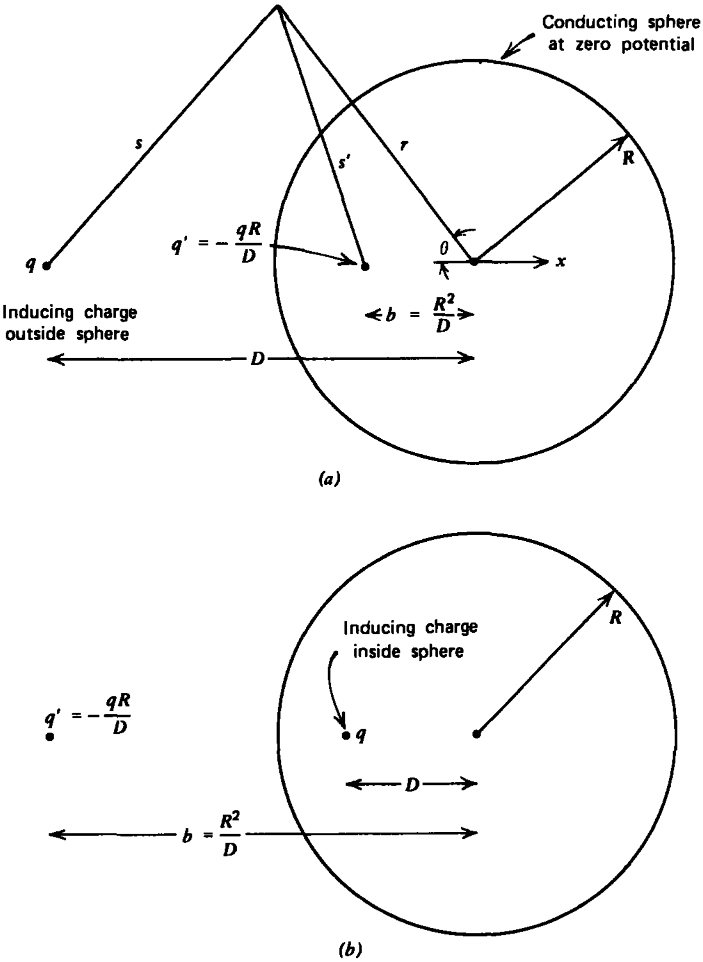


Figure 2-27 (*a*) The field due to a point charge q, a distance D outside a conducting sphere of radius R, can be found by placing a single image charge $-qR/D$ at a distance $b = R^2/D$ from the center of the sphere. (*b*) The same relations hold true if the charge q is inside the sphere but now the image charge is outside the sphere, since $D < R$.

At $r = R$, the potential in (1) must be zero so that q and q' must be of opposite polarity:

$$\left(\frac{q}{s}+\frac{q'}{s'}\right)\bigg|_{r=R} = 0 \Rightarrow \left(\frac{q}{s}\right)^2 = \left(\frac{q'}{s'}\right)^2\bigg|_{r=R} \tag{3}$$

where we square the equalities in (3) to remove the square roots when substituting (2),

$$q^2[b^2 + R^2 - 2Rb\cos\theta] = q'^2[R^2 + D^2 - 2RD\cos\theta] \tag{4}$$

Since (4) must be true for all values of θ, we obtain the following two equalities:

$$q^2(b^2+R^2)=q'^2(R^2+D^2)$$
$$q^2b=q'^2D \tag{5}$$

Eliminating q and q' yields a quadratic equation in b:

$$b^2-bD\left[1+\left(\frac{R}{D}\right)^2\right]+R^2=0 \tag{6}$$

with solution

$$b=\frac{D}{2}\left[1+\left(\frac{R}{D}\right)^2\right]\pm\sqrt{\left\{\frac{D}{2}\left[1+\left(\frac{R}{D}\right)^2\right]\right\}^2-R^2}$$
$$=\frac{D}{2}\left[1+\left(\frac{R}{D}\right)^2\right]\pm\sqrt{\left\{\frac{D}{2}\left[1-\left(\frac{R}{D}\right)^2\right]\right\}^2}$$
$$=\frac{D}{2}\left\{\left[1+\left(\frac{R}{D}\right)^2\right]\pm\left[1-\left(\frac{R}{D}\right)^2\right]\right\} \tag{7}$$

We take the lower negative root so that the image charge is inside the sphere with value obtained from using (7) in (5):

$$b=\frac{R^2}{D}, \qquad q'=-q\frac{R}{D} \tag{8}$$

remembering from (3) that q and q' have opposite sign. We ignore the $b=D$ solution with $q'=-q$ since the image charge must always be outside the region of interest. If we allowed this solution, the net charge at the position of the inducing charge is zero, contrary to our statement that the net charge is q.

The image charge distance b obeys a similar relation as was found for line charges and cylinders in Section 2.6.3. Now, however, the image charge magnitude does not equal the magnitude of the inducing charge because not all the lines of force terminate on the sphere. Some of the field lines emanating from q go around the sphere and terminate at infinity.

The force on the grounded sphere is then just the force on the image charge $-q'$ due to the field from q:

$$f_x=\frac{qq'}{4\pi\varepsilon_0(D-b)^2}=-\frac{q^2R}{4\pi\varepsilon_0D(D-b)^2}=-\frac{q^2RD}{4\pi\varepsilon_0(D^2-R^2)^2} \tag{9}$$

The electric field outside the sphere is found from (1) using (2) as

$$\mathbf{E}=-\nabla V=\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{s^3}[(r-D\cos\theta)\mathbf{i}_r+D\sin\theta\mathbf{i}_\theta]\right.$$
$$\left.+\frac{q'}{s'^3}[(r-b\cos\theta)\mathbf{i}_r+b\sin\theta\mathbf{i}_\theta]\right) \tag{10}$$

On the sphere where $s'=(R/D)s$, the surface charge distribution is found from the discontinuity in normal electric field as given in Section 2.4.6:

$$\sigma(r=R)=\varepsilon_0 E_r(r=R)=-\frac{q(D^2-R^2)}{4\pi R[R^2+D^2-2RD\cos\theta]^{3/2}} \tag{11}$$

The total charge on the sphere

$$q_T=\int_0^\pi \sigma(r=R)2\pi R^2\sin\theta\, d\theta$$
$$=-\frac{q}{2}R(D^2-R^2)\int_0^\pi \frac{\sin\theta\, d\theta}{[R^2+D^2-2RD\cos\theta]^{3/2}} \tag{12}$$

can be evaluated by introducing the change of variable

$$u=R^2+D^2-2RD\cos\theta, \qquad du=2RD\sin\theta\, d\theta \tag{13}$$

so that (12) integrates to

$$q_T=-\frac{q(D^2-R^2)}{4D}\int_{(D-R)^2}^{(D+R)^2}\frac{du}{u^{3/2}}$$
$$=-\frac{q(D^2-R^2)}{4D}\left(-\frac{2}{u^{1/2}}\right)\Bigg|_{(D-R)^2}^{(D+R)^2}=-\frac{qR}{D} \tag{14}$$

which just equals the image charge q'.

If the point charge q is inside the grounded sphere, the image charge and its position are still given by (8), as illustrated in Figure 2-27*b*. Since $D<R$, the image charge is now outside the sphere.

2-7-2 Point Charge Near a Grounded Plane

If the point charge is a distance a from a grounded plane, as in Figure 2-28*a*, we consider the plane to be a sphere of infinite radius R so that $D=R+a$. In the limit as R becomes infinite, (8) becomes

$$\lim_{\substack{R\to\infty\\ D=R+a}} q'=-q, \qquad b=\frac{R}{(1+a/R)}=R-a \tag{15}$$

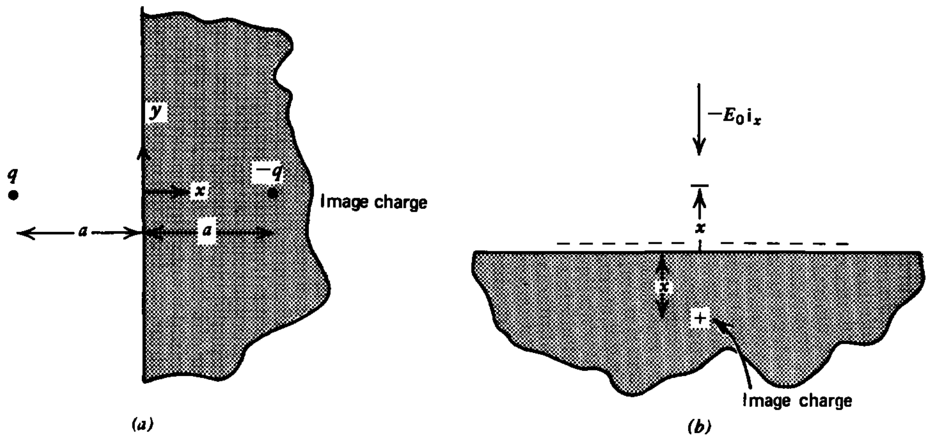


Figure 2-28 (*a*) A point charge q near a conducting plane has its image charge $-q$ symmetrically located behind the plane. (*b*) An applied uniform electric field causes a uniform surface charge distribution on the conducting plane. Any injected charge must overcome the restoring force due to its image in order to leave the electrode.

so that the image charge is of equal magnitude but opposite polarity and symmetrically located on the opposite side of the plane.

The potential at any point (x, y, z) outside the conductor is given in Cartesian coordinates as

$$V=\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{[(x+a)^2+y^2+z^2]^{1/2}}-\frac{1}{[(x-a)^2+y^2+z^2]^{1/2}}\right) \tag{16}$$

with associated electric field

$$\mathbf{E}=-\nabla V=\frac{q}{4\pi\varepsilon_0}\left(\frac{(x+a)\mathbf{i}_x+y\mathbf{i}_y+z\mathbf{i}_z}{[(x+a)^2+y^2+z^2]^{3/2}}-\frac{(x-a)\mathbf{i}_x+y\mathbf{i}_y+z\mathbf{i}_z}{[(x-a)^2+y^2+z^2]^{3/2}}\right) \tag{17}$$

Note that as required the field is purely normal to the grounded plane

$$E_y(x=0)=0, \qquad E_z(x=0)=0 \tag{18}$$

The surface charge density on the conductor is given by the discontinuity of normal **E**:

$$\begin{aligned}\sigma(x=0)&=-\varepsilon_0 E_x(x=0)\\&=-\frac{q}{4\pi}\frac{2a}{[y^2+z^2+a^2]^{3/2}}\\&=-\frac{qa}{2\pi(\mathrm{r}^2+a^2)^{3/2}};\ \mathrm{r}^2=y^2+z^2\end{aligned} \tag{19}$$

where the minus sign arises because the surface normal points in the negative x direction.

The total charge on the conducting surface is obtained by integrating (19) over the whole surface:

$$\begin{aligned} q_T &= \int_0^\infty \sigma(x=0)2\pi \mathrm{r}\, d\mathrm{r} \\ &= -qa\int_0^\infty \frac{\mathrm{r}\,d\mathrm{r}}{(\mathrm{r}^2+a^2)^{3/2}} \\ &= \frac{qa}{(\mathrm{r}^2+a^2)^{1/2}}\bigg|_0^\infty = -q \end{aligned} \tag{20}$$

As is always the case, the total charge on a conducting surface must equal the image charge.

The force on the conductor is then due only to the field from the image charge:

$$\mathbf{f} = -\frac{q^2}{16\pi\varepsilon_0 a^2}\mathbf{i}_x \tag{21}$$

This attractive force prevents charges from escaping from an electrode surface when an electric field is applied. Assume that an electric field $-E_0\mathbf{i}_x$ is applied perpendicular to the electrode shown in Figure (2-28*b*). A uniform negative surface charge distribution $\sigma = -\varepsilon_0 E_0$ as given in (2.4.6) arises to terminate the electric field as there is no electric field within the conductor. There is then an upwards Coulombic force on the surface charge, so why aren't the electrons pulled out of the electrode? Imagine an ejected charge $-q$ a distance x from the conductor. From (15) we know that an image charge $+q$ then appears at $-x$ which tends to pull the charge $-q$ back to the electrode with a force given by (21) with $a = x$ in opposition to the imposed field that tends to pull the charge away from the electrode. The total force on the charge $-q$ is then

$$f_x = qE_0 - \frac{q^2}{4\pi\varepsilon_0(2x)^2} \tag{22}$$

The force is zero at position x_c

$$f_x = 0 \Rightarrow x_c = \left[\frac{q}{16\pi\varepsilon_0 E_0}\right]^{1/2} \tag{23}$$

For an electron ($q = 1.6\times 10^{-19}$ coulombs) in a field of $E_0 = 10^6$ v/m, $x_c \approx 1.9\times 10^{-8}$ m. For smaller values of x the net force is negative tending to pull the charge back to the electrode. If the charge can be propelled past x_c by external forces, the imposed field will then carry the charge away from the electrode. If this external force is due to heating of the electrode, the process is called thermionic emission. High field emission even with a cold electrode occurs when the electric field E_0 becomes sufficiently large (on the order of 10^{10} v/m) that the coulombic force overcomes the quantum mechanical binding forces holding the electrons within the electrode.

2-7-3 Sphere With Constant Charge

If the point charge q is outside a conducting sphere $(D>R)$ that now carries a constant total charge Q_0, the induced charge is still $q'=-qR/D$. Since the total charge on the sphere is Q_0, we must find another image charge that keeps the sphere an equipotential surface and has value Q_0+qR/D. This other image charge must be placed at the center of the sphere, as in Figure 2-29*a*. The original charge q plus the image charge $q'=-qR/D$ puts the sphere at zero potential. The additional image charge at the center of the sphere raises the potential of the sphere to

$$V=\frac{Q_0+qR/D}{4\pi\varepsilon_0 R} \tag{24}$$

The force on the sphere is now due to the field from the point charge q acting on the two image charges:

$$\begin{aligned} f_x &= \frac{q}{4\pi\varepsilon_0}\left(-\frac{qR}{D(D-b)^2}+\frac{(Q_0+qR/D)}{D^2}\right) \\ &= \frac{q}{4\pi\varepsilon_0}\left(-\frac{qRD}{(D^2-R^2)^2}+\frac{(Q_0+qR/D)}{D^2}\right) \end{aligned} \tag{25}$$

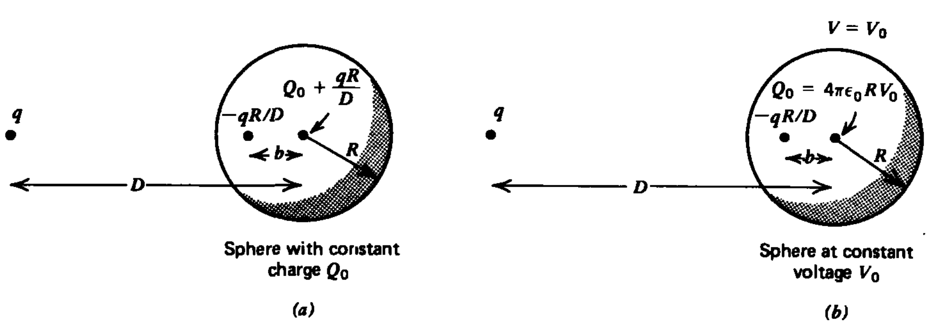


Figure 2-29 (*a*) If a conducting sphere carries a constant charge Q_0 or (*b*) is at a constant voltage V_0, an additional image charge is needed at the sphere center when a charge q is nearby.

2-7-4 Constant Voltage Sphere

If the sphere is kept at constant voltage V_0, the image charge $q' = -qR/D$ at distance $b = R^2/D$ from the sphere center still keeps the sphere at zero potential. To raise the potential of the sphere to V_0, another image charge,

$$Q_0 = 4\pi\varepsilon_0 R V_0 \tag{26}$$

must be placed at the sphere center, as in Figure 2-29*b*. The force on the sphere is then

$$f_x = \frac{q}{4\pi\varepsilon_0}\left(-\frac{qR}{D(D-b)^2} + \frac{Q_0}{D^2}\right) \tag{27}$$

PROBLEMS

Section 2.1

1. Faraday's "ice-pail" experiment is repeated with the following sequence of steps:

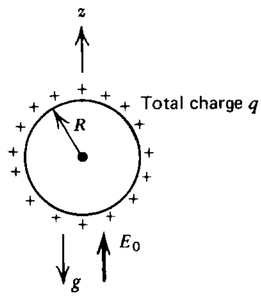
   (i) A ball with total charge Q is brought inside an insulated metal ice-pail without touching.
   (ii) The outside of the pail is momentarily connected to the ground and then disconnected so that once again the pail is insulated.
   (iii) Without touching the pail, the charged ball is removed.

   (a) Sketch the charge distribution on the inside and outside of the pail during each step.
   (b) What is the net charge on the pail after the charged ball is removed?

2. A sphere initially carrying a total charge Q is brought into momentary contact with an uncharged identical sphere.
   (a) How much charge is on each sphere?
   (b) This process is repeated for N identical initially uncharged spheres. How much charge is on each of the spheres including the original charged sphere?
   (c) What is the total charge in the system after the N contacts?

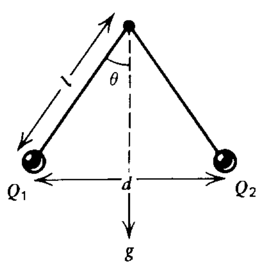
Section 2.2

3. The charge of an electron was first measured by Robert A. Millikan in 1909 by measuring the electric field necessary to levitate a small charged oil drop against its weight. The oil droplets were sprayed and became charged by frictional electrification.



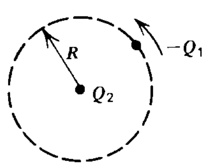
A spherical droplet of radius R and effective mass density ρ_m carries a total charge q in a gravity field g. What electric field $E_0\mathbf{i}_z$ will suspend the charged droplet? Millikan found by this method that all droplets carried integer multiples of negative charge $e = -1.6\times 10^{-19}$ coul.

4. Two small conducting balls, each of mass m, are at the end of insulating strings of length l joined at a point. Charges are



placed on the balls so that they are a distance d apart. A charge Q_1 is placed on ball 1. What is the charge Q_2 on ball 2?

5. A point charge $-Q_1$ of mass m travels in a circular orbit of radius R about a charge of opposite sign Q_2.



(a) What is the equilibrium angular speed of the charge $-Q_1$?

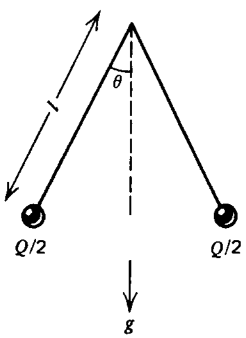
(b) This problem describes Bohr's one electron model of the atom if the charge $-Q_1$ is that of an electron and $Q_2 = Ze$ is the nuclear charge, where Z is the number of protons. According to the postulates of quantum mechanics the angular momentum L of the electron must be quantized,

$$L = mvR = nh/2\pi, \qquad n = 1, 2, 3, \cdots$$

where $h = 6.63\times 10^{-34}$ joule-sec is Planck's constant. What are the allowed values of R?

(c) For the hydrogen atom ($Z = 1$) what is the radius of the smallest allowed orbit and what is the electron's orbital velocity?

6. An electroscope measures charge by the angular deflection of two identical conducting balls suspended by an essentially weightless insulating string of length l. Each ball has mass M in the gravity field g and when charged can be considered a point charge.

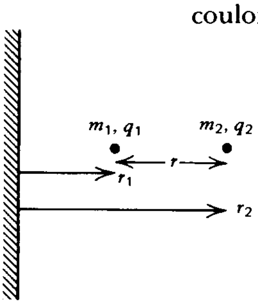


A total charge Q is deposited on the two balls of the electroscope. The angle θ from the normal obeys a relation of the form

$$\tan\theta \sin^2\theta = \text{const}$$

What is the constant?

7. Two point charges q_1 and q_2 in vacuum with respective masses m_1 and m_2 attract (or repel) each other via the coulomb force.



(a) Write a single differential equation for the distance between the charges $r = r_2 - r_1$. What is the effective mass of the charges? (**Hint:** Write Newton's law for each charge and take a mass-weighted difference.)

(b) If the two charges are released from rest at $t = 0$ when a distance r_0 from one another, what is their relative velocity $v = dr/dt$ as a function of r? **Hint:**

$$\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = v\frac{dv}{dr} = \frac{d}{dr}\left(\frac{1}{2}v^2\right)$$

(c) What is their position as a function of time? Separately consider the cases when the charges have the same or opposite polarity. **Hint:**

Let $$u=\sqrt{r}$$

$$\int \frac{u^2\,du}{\sqrt{a^2-u^2}} = -\frac{u}{2}\sqrt{a^2-u^2}+\frac{a^2}{2}\sin^{-1}\frac{u}{a}$$

$$\int \frac{u^2\,du}{\sqrt{u^2-a^2}} = \frac{u}{2}\sqrt{u^2-a^2}+\frac{a^2}{2}\ln\left(u+\sqrt{u^2-a^2}\right)$$

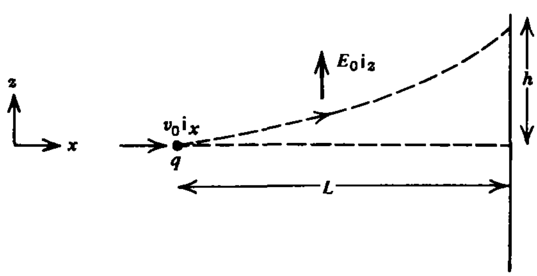
(d) If the charges are of opposite polarity, at what time will they collide? (**Hint:** If you get a negative value of time, check your signs of square roots in (b).)

(e) If the charges are taken out of the vacuum and placed in a viscous medium, the velocity rather than the acceleration is proportional to the force

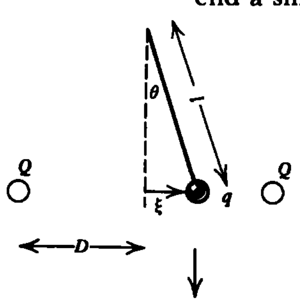
$$\beta_1\mathbf{v}_1=\mathbf{f}_1,\qquad \beta_2\mathbf{v}_2=\mathbf{f}_2$$

where β_1 and β_2 are the friction coefficients for each charge. Repeat parts (a)–(d) for this viscous dominated motion.

8. A charge q of mass m with initial velocity $\mathbf{v}=v_0\mathbf{i}_x$ is injected at $x=0$ into a region of uniform electric field $\mathbf{E}=E_0\mathbf{i}_z$. A screen is placed at the position $x=L$. At what height h does the charge hit the screen? Neglect gravity.



9. A pendulum with a weightless string of length l has on its end a small sphere with charge q and mass m. A distance D



away on either side of the pendulum mass are two fixed spheres each carrying a charge Q. The three spheres are of sufficiently small size that they can be considered as point charges and masses.

(a) Assuming the pendulum displacement ξ to be small ($\xi \ll D$), show that Newton's law can be approximately written as

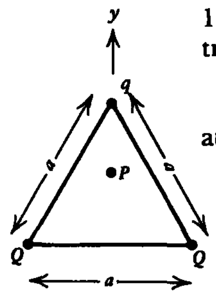
$$\frac{d^2\xi}{dt^2}+\omega_0^2\xi=0$$

What is ω_0^2? **Hint:**

$$\sin\theta\approx\frac{\xi}{l}, \quad \frac{1}{(D\pm\xi)^2}\approx\frac{1}{D^2}\mp\frac{2\xi}{D^3}$$

(b) At $t=0$ the pendulum is released from rest with $\xi=\xi_0$. What is the subsequent pendulum motion?

(c) For what values of qQ is the motion unbounded with time?

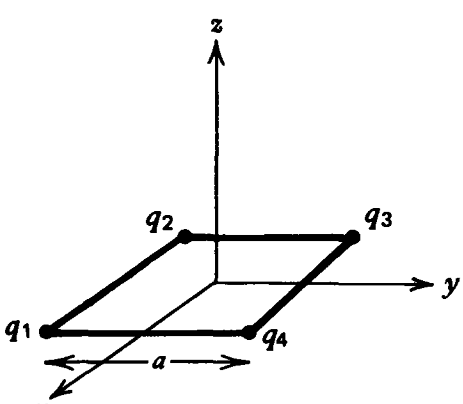


10. Charges Q, Q, and q lie on the corners of an equilateral triangle with sides of length a.

(a) What is the force on the charge q?

(b) What must q be for $\mathbf{E}$ to be zero half-way up the altitude at P?

11. Find the electric field along the z axis due to four equal magnitude point charges q placed on the vertices of a square with sides of length a in the xy plane centered at the origin



when:

(a) the charges have the same polarity, $q_1 = q_2 = q_3 = q_4 \equiv q$;
(b) the charges alternate in polarity, $q_1 = q_3 \equiv q$, $q_2 = q_4 \equiv -q$;
(c) the charges are $q_1 = q_2 \equiv q$, $q_3 = q_4 \equiv -q$.

Section 2.3
12. Find the total charge in each of the following distributions where a is a constant parameter:

(a) An infinitely long line charge with density

$$\lambda(z) = \lambda_0 e^{-|z|/a}$$

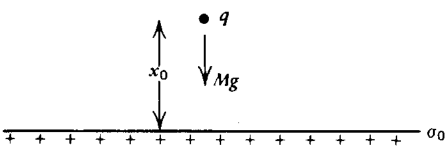
(b) A spherically symmetric volume charge distributed over all space

$$\rho(r) = \frac{\rho_0}{[1 + r/a]^4}$$

(Hint: Let $u = 1 + r/a$.)
(c) An infinite sheet of surface charge with density

$$\sigma(x, y) = \frac{\sigma_0 e^{-|x|/a}}{[1 + (y/b)^2]}$$

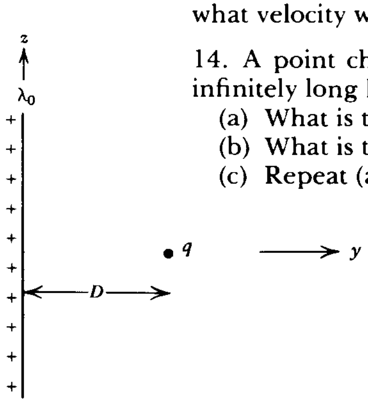
13. A point charge q with mass M in a gravity field g is released from rest a distance x_0 above a sheet of surface charge with uniform density σ_0.

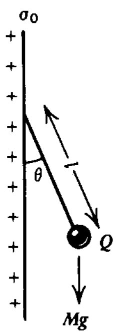


(a) What is the position of the charge as a function of time?
(b) For what value of σ_0 will the charge remain stationary?
(c) If σ_0 is less than the value of (b), at what time and with what velocity will the charge reach the sheet?

14. A point charge q at $z = 0$ is a distance D away from an infinitely long line charge with uniform density λ_0.
(a) What is the force on the point charge q?
(b) What is the force on the line charge?
(c) Repeat (a) and (b) if the line charge has a distribution

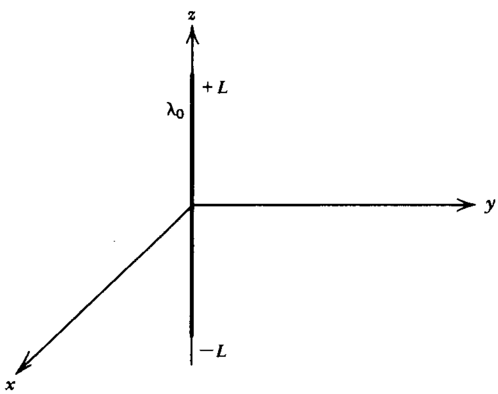
$$\lambda(z) = \frac{\lambda_0 |z|}{a}$$



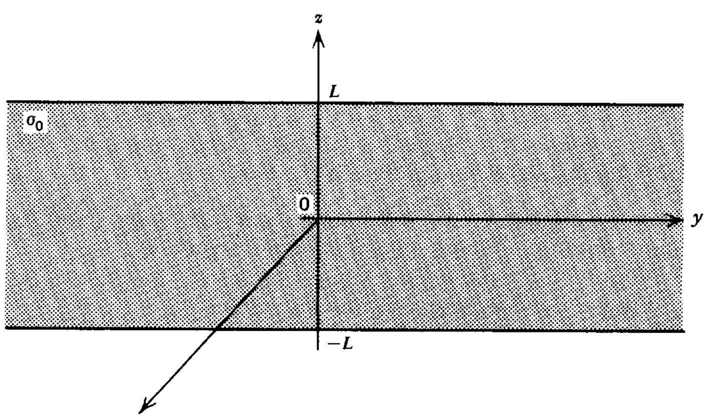


15. A small sphere of mass M in a gravity field g carrying a charge Q is connected by a massless string to a sheet of surface charge of the same polarity with density σ_0. What is the angle θ between the sheet and charge?

16. A line charge λ along the z axis extends over the interval $-L \le z \le L$.



(a)



(b)

(a) Find the electric field in the $z = 0$ plane.

(b) Using the results of (a) find the electric field in the $z = 0$ plane due to an infinite strip ($-\infty \le y \le \infty$) of height $2L$ with surface charge density σ_0. Check your results with the text for $L \to \infty$. **Hint:** Let $u = x^2 + y^2$

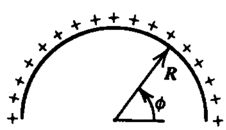
$$\int \frac{du}{u\sqrt{u-x^2}\sqrt{L^2+u}} = \frac{1}{Lx}\sin^{-1}\left(\frac{(L^2-x^2)u - 2L^2x^2}{u(L^2+x^2)}\right)$$

17. An infinitely long hollow semi-cylinder of radius R carries a uniform surface charge distribution σ_0.

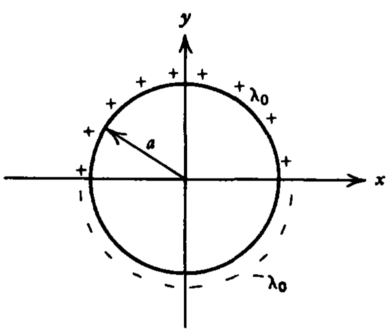
(a) What is the electric field along the axis of the cylinder?

(b) Use the results of (a) to find the electric field along the axis due to a semi-cylinder of volume charge ρ_0.

(c) Repeat (a) and (b) to find the electric field at the center of a uniformly surface or volume charged hemisphere.



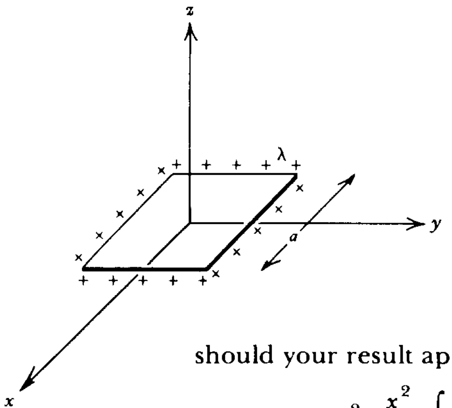
18. (a) Find the electric field along the z axis of a circular loop centered in the xy plane of radius a carrying a uniform line charge λ_0 for $y > 0$ and $-\lambda_0$ for $y < 0$.



(b) Use the results of (a) to find the electric field along the z axis of a circular disk of radius a carrying a uniform surface charge σ_0 for $y > 0$ and $-\sigma_0$ for $y < 0$.

19. (a) Find the electric field along the z axis due to a square loop with sides of length a centered about the z axis in the xy plane carrying a uniform line charge λ. What should your result approach for $z \gg a$?

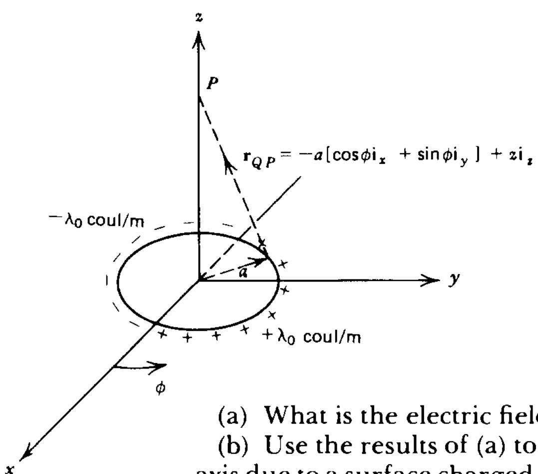
(b) Use the results of (a) to find the electric field along the z axis due to a square of uniform surface charge σ_0. What



should your result approach as $a \to \infty$? **Hint:** Let

$$u = z^2 + \frac{x^2}{4}, \quad \int \frac{du}{u\sqrt{2u - z^2}} = \frac{2}{|z|} \tan^{-1} \sqrt{\frac{2u - z^2}{z^2}}$$

20. A circular loop of radius a in the xy plane has a uniform line charge distribution λ_0 for $y > 0$ and $-\lambda_0$ for $y < 0$.



(a) What is the electric field along the z axis?

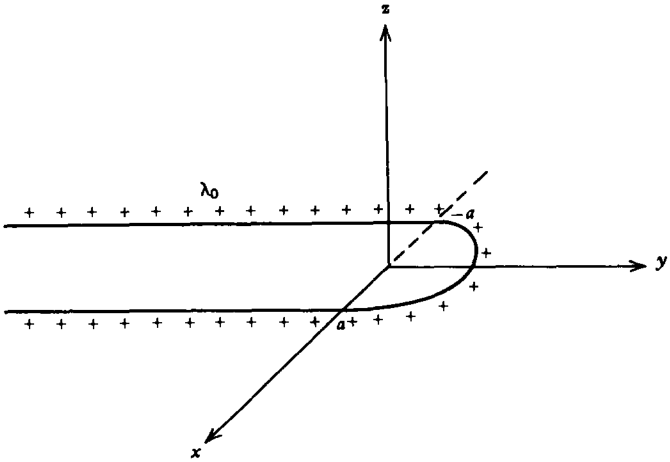
(b) Use the results of (a) to find the electric field along the z axis due to a surface charged disk, whose density is σ_0 for $y > 0$ and $-\sigma_0$ for $y < 0$. **Hint:**

$$\int \frac{r^2\, dr}{(r^2 + z^2)^{3/2}} = -\frac{r}{\sqrt{r^2 + z^2}} + \ln\left(r + \sqrt{r^2 + z^2}\right)$$

(c) Repeat (a) if the line charge has distribution $\lambda = \lambda_0 \sin\phi$.

(*d*) Repeat (b) if the surface charge has distribution $\sigma = \sigma_0 \sin\phi$.

21. An infinitely long line charge with density λ_0 is folded in half with both halves joined by a half-circle of radius a. What is the electric field along the z axis passing through the center



of the circle. **Hint:**

$$\int \frac{x\,dx}{[x^2+a^2]^{3/2}} = \frac{-1}{[x^2+a^2]^{1/2}}$$

$$\int \frac{dx}{[x^2+a^2]^{3/2}} = \frac{x}{a^2[x^2+a^2]^{1/2}}$$

$$\mathbf{i}_r = \cos\phi\,\mathbf{i}_x + \sin\phi\,\mathbf{i}_y$$

Section 2.4

22. Find the total charge enclosed within each of the following volumes for the given electric fields:

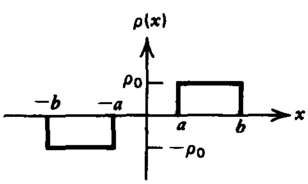
(a) $\mathbf{E} = Ar^2\mathbf{i}_r$ for a sphere of radius R;

(b) $\mathbf{E} = Ar^2\mathbf{i}_r$ for a cylinder of radius a and length L;

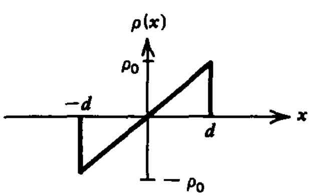
(c) $\mathbf{E} = A(x\mathbf{i}_x + y\mathbf{i}_y)$ for a cube with sides of length a having a corner at the origin.

23. Find the electric field everywhere for the following planar volume charge distributions:

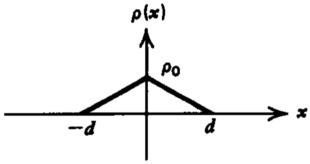
(a) $\rho(x) = \rho_0 e^{-|x|/a}, \quad -\infty \le x \le \infty$



(b) $\rho(x) = \begin{cases} -\rho_0, & -b \le x \le -a \\ \rho_0, & a \le x \le b \end{cases}$



(c) $\rho(x) = \dfrac{\rho_0 x}{d}, \quad -d \le x \le d$



(d) $\rho(x)=\begin{cases}\rho_0(1+x/d), & -d\le x\le 0\\ \rho_0(1-x/d), & 0\le x\le d\end{cases}$

24. Find the electric field everywhere for the following spherically symmetric volume charge distributions:

(a) $\rho(r)=\rho_0 e^{-r/a}, \quad 0\le r\le\infty$

$\left(\textbf{Hint: } \int r^2 e^{-r/a}\,dr=-a\,e^{-r/a}[r^2+2a^2(r/a+1)].\right)$

(b) $\rho(r)=\begin{cases}\rho_1, & 0\le r<R_1\\ \rho_2, & R_1<r<R_2\end{cases}$

(c) $\rho(r)=\rho_0 r/R, \quad 0<r<R$

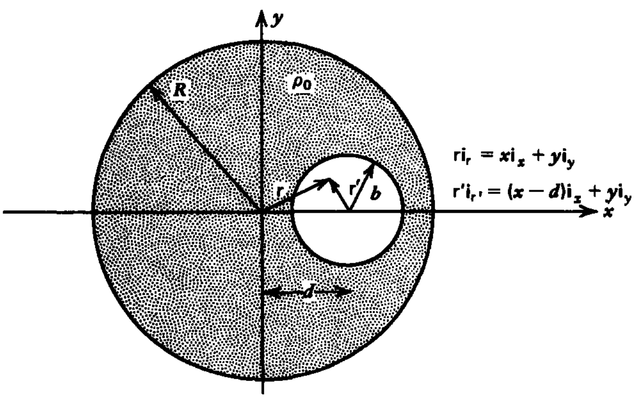
25. Find the electric field everywhere for the following cylindrically symmetric volume charge distributions:

(a) $\rho(\mathrm{r})=\rho_0\,e^{-\mathrm{r}/a}, \quad 0<\mathrm{r}<\infty$

$\left[\textbf{Hint: } \int \mathrm{r}e^{-\mathrm{r}/a}\,d\mathrm{r}=-a^2\,e^{-\mathrm{r}/a}\,(\mathrm{r}/a+1).\right]$

(b) $\rho(\mathrm{r})=\begin{cases}\rho_1, & 0<\mathrm{r}<a\\ \rho_2, & a<\mathrm{r}<b\end{cases}$

(c) $\rho(\mathrm{r})=\rho_0\mathrm{r}/a, \quad 0<\mathrm{r}<a$

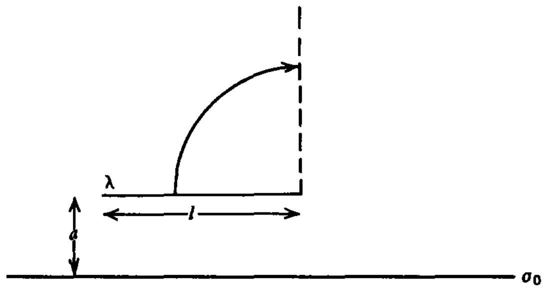


26. An infinitely long cylinder of radius R with uniform volume charge density ρ_0 has an off-axis hole of radius b with center a distance d away from the center of the cylinder.

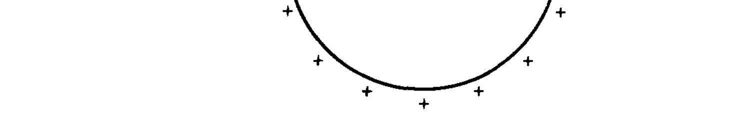
What is the electric field within the hole? (**Hint:** Replace the hole by the superposition of volume charge distributions of density ρ_0 and $-\rho_0$ and use the results of (27). Convert the cylindrical coordinates to Cartesian coordinates for ease of vector addition.)

Section 2.5

27. A line charge λ of length l lies parallel to an infinite sheet of surface charge σ_0. How much work is required to rotate the line charge so that it is vertical?



28. A point charge q of mass m is injected at infinity with initial velocity $v_0\mathbf{i}_x$ towards the center of a uniformly charged sphere of radius R. The total charge on the sphere Q is the same sign as q.



(a) What is the minimum initial velocity necessary for the point charge to collide with the sphere?

(b) If the initial velocity is half of the result in (a), how close does the charge get to the sphere?

29. Find the electric field and volume charge distributions for the following potential distributions:

(a) $V = Ax^2$

(b) $V = Axyz$

(c) $V = Ar^2 \sin\phi + Brz$

(d) $V = Ar^2 \sin\theta \cos\phi$

30. Which of the following vectors can be an electric field? If so, what is the volume charge density?

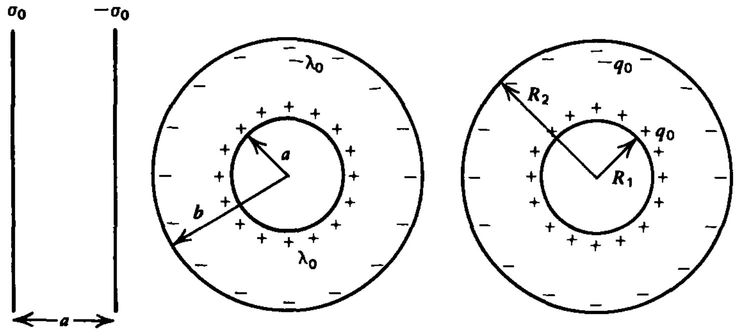
(a) $\mathbf{E} = ax^2y^2\mathbf{i}_x$

(b) $\mathbf{E} = a(\mathbf{i}_r \cos\theta - \mathbf{i}_\theta \sin\theta)$

(c) $\mathbf{E} = a(y\mathbf{i}_x - x\mathbf{i}_y)$

(d) $\mathbf{E} = (a/r^2)[\mathbf{i}_r(1+\cos\phi) + \mathbf{i}_\phi \sin\phi]$

31. Find the potential difference V between the following surface charge distributions:



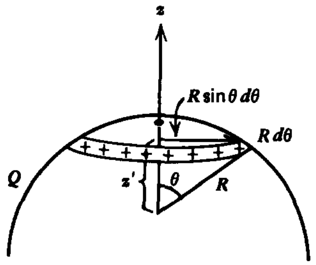
(a) (b) (c)

(a) Two parallel sheets of surface charge of opposite polarity $\pm\sigma_0$ and spacing a.

(b) Two coaxial cylinders of surface charge having infinite length and respective radii a and b. The total charge per unit length on the inner cylinder is λ_0 while on the outer cylinder is $-\lambda_0$.

(c) Two concentric spheres of surface charge with respective radii R_1 and R_2. The inner sphere carries a uniformly distributed surface charge with total charge q_0. The outer sphere has total charge $-q_0$.

32. A hemisphere of radius R has a uniformly distributed surface charge with total charge Q.



(a) Break the spherical surface into hoops of line charge of thickness $R\,d\theta$. What is the radius of the hoop, its height z', and its total incremental charge dq?

(b) What is the potential along the z axis due to this incremental charged hoop? Eliminate the dependence on θ and express all variables in terms of z', the height of the differential hoop of line charge.

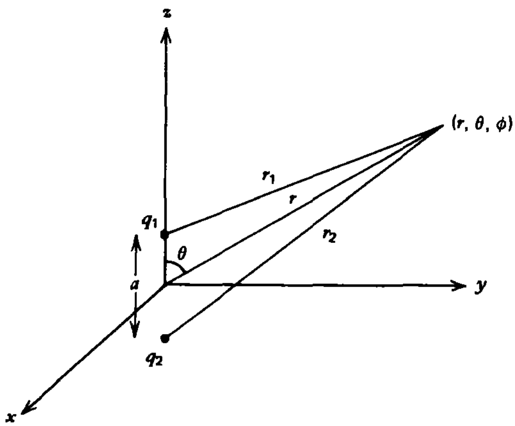
(c) What is the potential at any position along the z axis due to the entire hemisphere of surface charge? **Hint:**

$$\int \frac{dz'}{[a+bz']^{1/2}} = \frac{2\sqrt{a+bz'}}{b}$$

(d) What is the electric field along the z axis?

(e) If the hemisphere is uniformly charged throughout its volume with total charge Q, find the potential and electric field at all points along the z axis. (**Hint:** $\int r\sqrt{z^2+r^2}\,dr = \frac{1}{3}(z^2+r^2)^{3/2}$.)

33. Two point charges q_1 and q_2 lie along the z axis a distance a apart.



(a) Find the potential at the coordinate (r, θ, ϕ). (**Hint:** $r_1^2 = r^2 + (a/2)^2 - ar\cos\theta$.)

(b) What is the electric field?

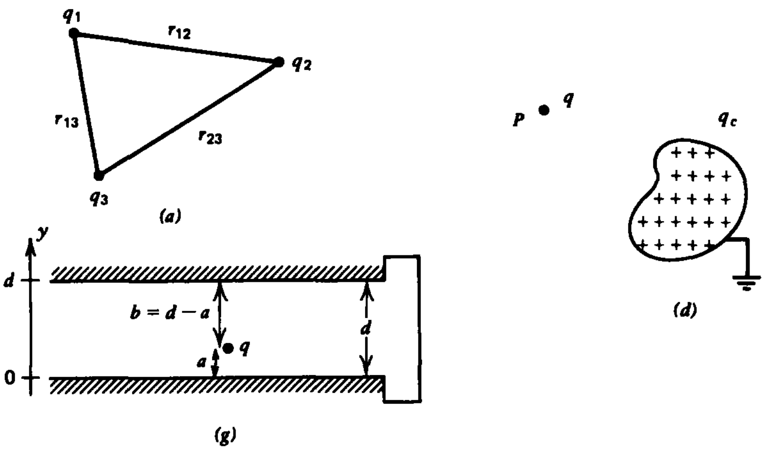
(c) An electric dipole is formed if $q_2 = -q_1$. Find an approximate expression for the potential and electric field for points far from the dipole, $r \gg a$.

(d) What is the equation of the field lines in this far field limit that is everywhere tangent to the electric field

$$\frac{dr}{r\,d\theta} = \frac{E_r}{E_\theta}$$

Find the equation of the field line that passes through the point $(r = r_0, \theta = \pi/2)$. (**Hint:** $\int \cot\theta\, d\theta = \ln \sin\theta$.)

34. (a) Find the potentials V_1, V_2, and V_3 at the location of each of the three-point charges shown.



(b) Now consider another set of point charges q_1', q_2', and q_3' at the same positions and calculate the potentials V_1', V_2', and V_3'. Verify by direct substitution that

$$q_1' V_1 + q_2' V_2 + q_3' V_3 = q_1 V_1' + q_2 V_2' + q_3 V_3'$$

The generalized result for any number of charges is called Green's reciprocity theorem,

$$\sum_{i=1}^{N} (q_i V_i' - q_i' V_i) = 0$$

(c) Show that Green's reciprocity theorem remains unchanged for perfect conductors as the potential on the conductor is constant. The q_i is then the total charge on the conductor.

(d) A charge q at the point P is in the vicinity of a zero potential conductor. It is known that if the conductor is charged to a voltage V_c, the potential at the point P in the absence of the point charge is V_p. Find the total charge q_c induced on the grounded conductor. (**Hint:** Let $q_1 = q$, $q_2 = q_c$, $V_2 = 0$, $q_1' = 0$, $V_1' = V_p$, $V_2' = V_c$.)

(e) If the conductor is a sphere of radius R and the point P is a distance D from the center of the sphere, what is q_c? Is this result related to the method of images?

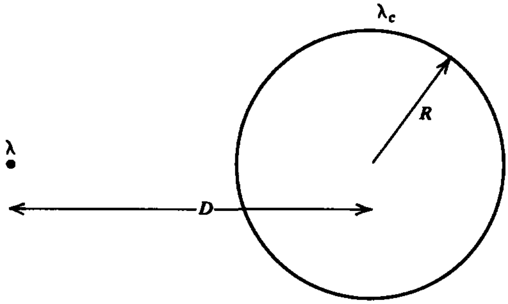
(f) A line charge λ is a distance D from the center of a grounded cylinder of radius a. What is the total charge per unit length induced on the cylinder?

(g) A point charge q is between two zero potential perfect conductors. What is the total charge induced on each conducting surface? (**Hint:** Try $q_1 = q$, $q_2 = q(y = 0)$, $q_3 = q(y = d)$, $V_2 = 0$, $V_3 = 0$, $q_1' = 0$, $V_2' = V_0$, $V_3' = 0$.)

(h) A point charge q travels at constant velocity v_0 between shorted parallel plate electrodes of spacing d. What is the short circuit current as a function of time?

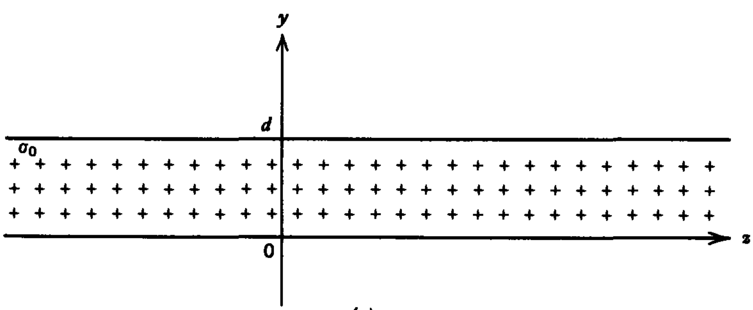
Section 2.6

35. An infinitely long line charge λ is a distance D from the center of a conducting cylinder of radius R that carries a total charge per unit length λ_c. What is the force per unit length on

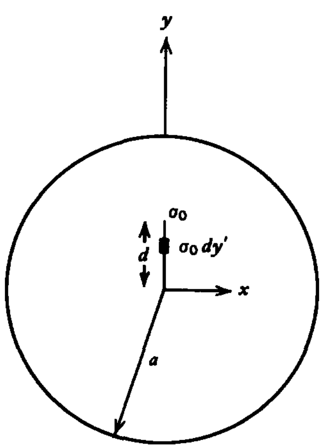


the cylinder? (**Hint:** Where can another image charge be placed with the cylinder remaining an equipotential surface?)

36. An infinitely long sheet of surface charge of width d and uniform charge density σ_0 is placed in the yz plane.



(a)



(b)

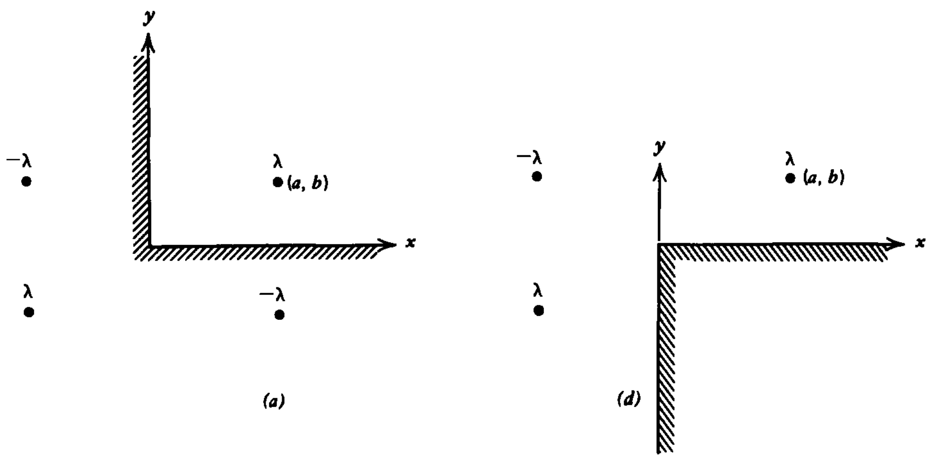
(a) Find the electric field everywhere in the yz plane. (**Hint:** Break the sheet into differential line charge elements $d\lambda = \sigma_0\, dy'$.)

(b) An infinitely long conducting cylinder of radius a surrounds the charged sheet that has one side along the axis of the cylinder. Find the image charge and its location due to an incremental line charge element $\sigma_0\, dy'$ at distance y'.

(c) What is the force per unit length on the cylinder? **Hint:**

$$\int \ln(1-cy')\, dy' = -\left(\frac{1-cy'}{c}\right)[\ln(1-cy')-1]$$

37. A line charge λ is located at coordinate (a, b) near a right-angled conducting corner.



(a) Verify that the use of the three image line charges shown satisfy all boundary conditions.

(b) What is the force per unit length on λ?

(c) What charge per unit length is induced on the surfaces $x = 0$ and $y = 0$?

(d) Now consider the inverse case when three line charges of alternating polarity $\pm\lambda$ are outside a conducting corner. What is the force on the conductor?

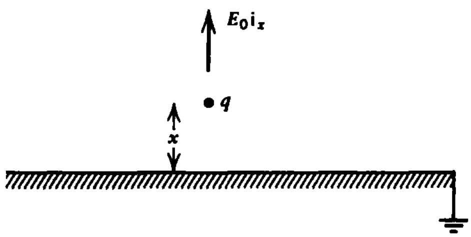
(e) Repeat (a)–(d) with point charges.

Section 2.7

38. A positive point charge q within a uniform electric field $E_0\mathbf{i}_x$ is a distance x from a grounded conducting plane.

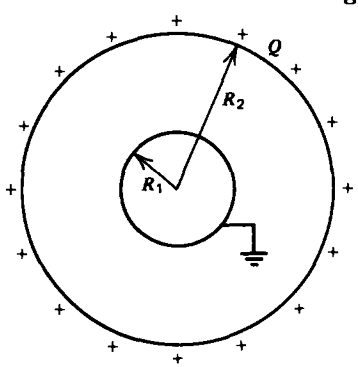
(a) At what value of x is the force on the charge equal to zero?

(b) If the charge is initially at a position equal to half the value found in (a), what minimum initial velocity is necessary for the charge to continue on to $x = +\infty$? (**Hint:** $E_x = -dV/dx$.)



(c) If $E_0=0$, how much work is necessary to move the point charge from $x=d$ to $x=+\infty$?

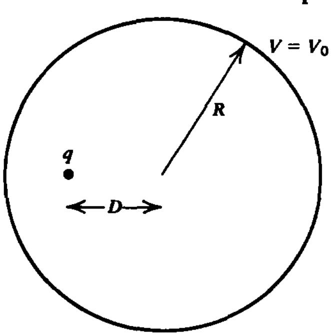
39. A sphere of radius R_2 having a uniformly distributed surface charge Q surrounds a grounded sphere of radius R_1.



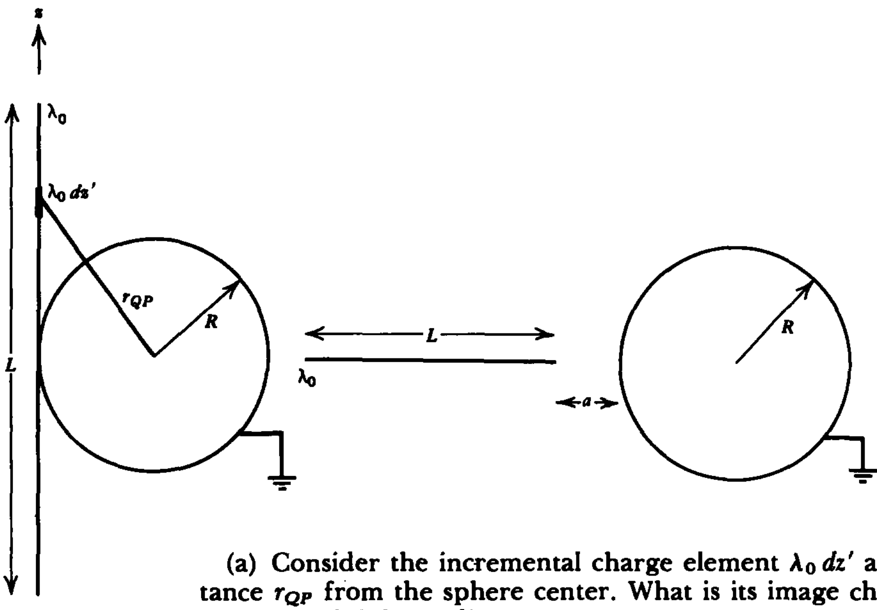
(a) What is the total charge induced on the grounded sphere? (**Hint:** Consider the image charge due to an incremental charge $dq=(Q/4\pi)\sin\theta\, d\theta\, d\phi$ at $r=R_2$.)

(b) What are the potential and electric field distributions everywhere?

40. A point charge q located a distance D $(D<R)$ from the center is within a conducting sphere of radius R that is at constant potential V_0. What is the force on q?



41. A line charge of length L with uniform density λ_0 is orientated the two ways shown with respect to a grounded sphere of radius R. For both cases:

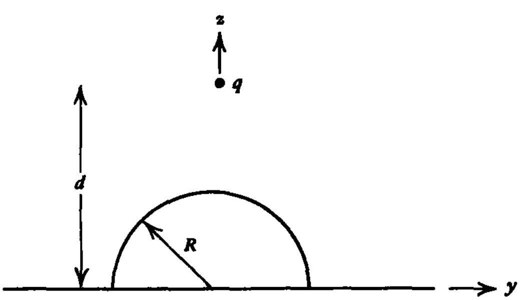


(a) Consider the incremental charge element $\lambda_0\,dz'$ a distance r_{QP} from the sphere center. What is its image charge and where is it located?

(b) What is the total charge induced on the sphere? **Hint:**

$$\int \frac{dz'}{\sqrt{R^2+z'^2}} = \ln\left(z'+\sqrt{R^2+z'^2}\right)$$

42. A conducting hemispherical projection of radius R is placed upon a ground plane of infinite extent. A point charge q is placed a distance d ($d>R$) above the center of the hemisphere.

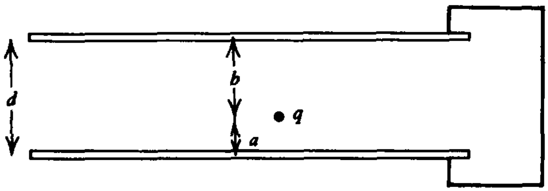


(a) What is the force on q? (**Hint:** Try placing three image charges along the z axis to make the plane and hemisphere have zero potential.)

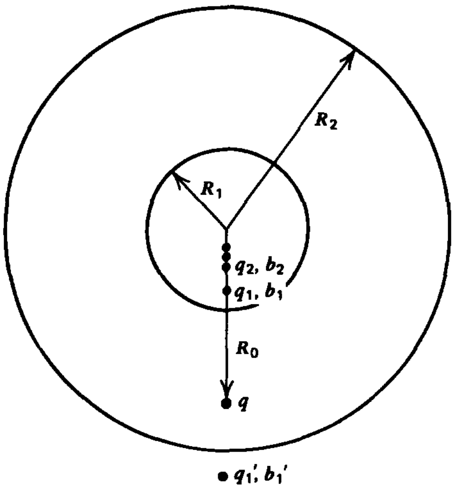
(b) What is the total charge induced on the hemisphere at $r=R$ and on the ground plane $|y|>R$? **Hint:**

$$\int \frac{r\,dr}{[r^2+d^2]^{3/2}} = \frac{-1}{\sqrt{r^2+d^2}}$$

43. A point charge q is placed between two parallel grounded conducting planes a distance d apart.



(a)



(c)

(a) The point charge q a distance a above the lower plane and a distance b below the upper conductor has symmetrically located image charges. However, each image charge itself has an image in the opposite conductor. Show that an infinite number of image charges are necessary. What are the locations of these image charges?

(b) Show that the total charge on each conductor cannot be found by this method as the resulting series is divergent.

(c) Now consider a point charge q, a radial distance R_0 from the center of two concentric grounded conducting spheres of radii R_1 and R_2. Show that an infinite number of image charges in each sphere are necessary where, if we denote the nth image charge in the smaller sphere as q_n a distance b_n from the center and the nth image charge in the outer sphere as q'_n a distance b'_n from the center, then

$$q_{n+1} = -\frac{R_1}{b'_n} q'_n, \qquad q'_{n+1} = -\frac{R_2}{b_n} q_n$$

$$b_{n+1} = \frac{R_1^2}{b'_n}, \qquad b'_{n+1} = \frac{R_2^2}{b_n}$$

(d) Show that the equations in (c) can be simplified to

$$q_{n+1} - q_{n-1}\left(\frac{R_1}{R_2}\right) = 0$$

$$b_{n+1} - b_{n-1}\left(\frac{R_1}{R_2}\right)^2 = 0$$

(e) Try power-law solutions

$$q_n = A\lambda^n, \qquad b_n = B\alpha^n$$

and find the characteristic values of λ and α that satisfy the equations in (d).

(f) Taking a linear combination of the solutions in (e), evaluate the unknown amplitude coefficients by substituting in values for $n = 1$ and $n = 2$. What are all the q_n and b_n?

(g) What is the total charge induced on the inner sphere?

(Hint: $\sum_{n=1}^{\infty} a^n = a/(1-a)$ for $a < 1$)

(h) Using the solutions of (f) with the difference relations of (c), find q'_n and b'_n.

(i) Show that $\sum_{n=1}^{\infty} q'_n$ is not a convergent series so that the total charge on the outer sphere cannot be found by this method.

(j) Why must the total induced charge on both spheres be $-q$? What then is the total induced charge on the outer sphere?

(k) Returning to our original problem in (a) and (b) of a point charge between parallel planes, let the radii of the spheres approach infinity such that the distances

$$d = R_2 - R_1, \qquad a = R_2 - R_0, \qquad b = R_0 - R_1$$

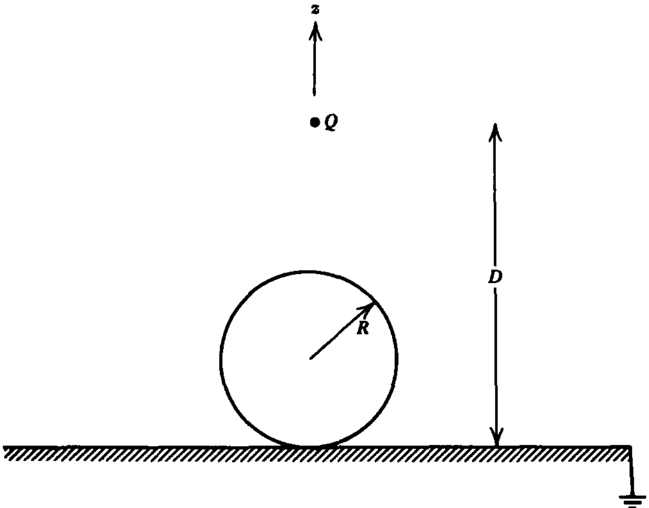
remains finite. What is the total charge induced on each plane conductor?

44. A point charge Q is a distance D above a ground plane. Directly below is the center of a small conducting sphere of radius R that rests on the plane.

(a) Find the first image charges and their positions in the sphere and in the plane.

(b) Now find the next image of each induced in the other. Show that two sets of image charges are induced on the sphere where each obey the difference equations

$$q_{n+1} = \frac{q_n R}{2R - b_n}, \qquad b_{n+1} = \frac{R^2}{2R - b_n}$$



(c) Eliminating the b_n, show that the governing difference equation is

$$\frac{1}{q_{n+1}}-\frac{2}{q_n}+\frac{1}{q_{n-1}}=0$$

Guess solutions of the form

$$P_n = 1/q_n = A\lambda^n$$

and find the allowed values of λ that satisfy the difference equation. (**Hint:** For double roots of λ the total solution is of the form $P_n = (A_1 + A_2 n)\lambda^n$.)

(d) Find all the image charges and their positions in the sphere and in the plane.

(e) Write the total charge induced on the sphere in the form

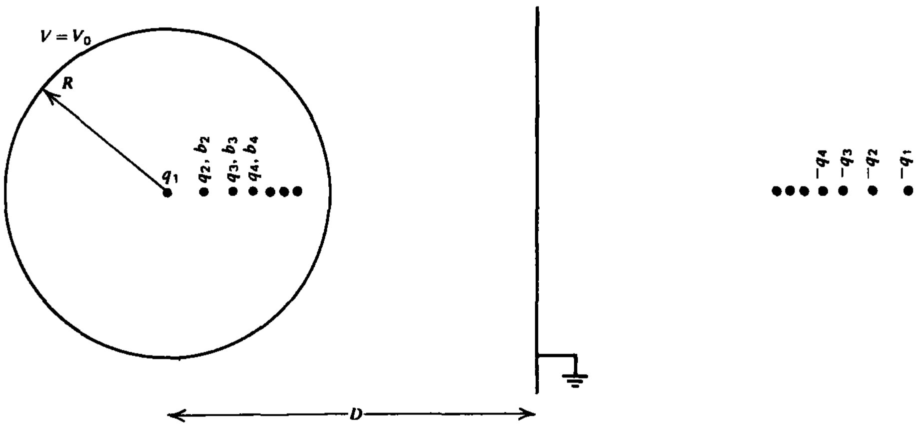
$$q_T = \sum_{n=1}^{\infty} \frac{A}{[1-an^2]}$$

What are A and a?

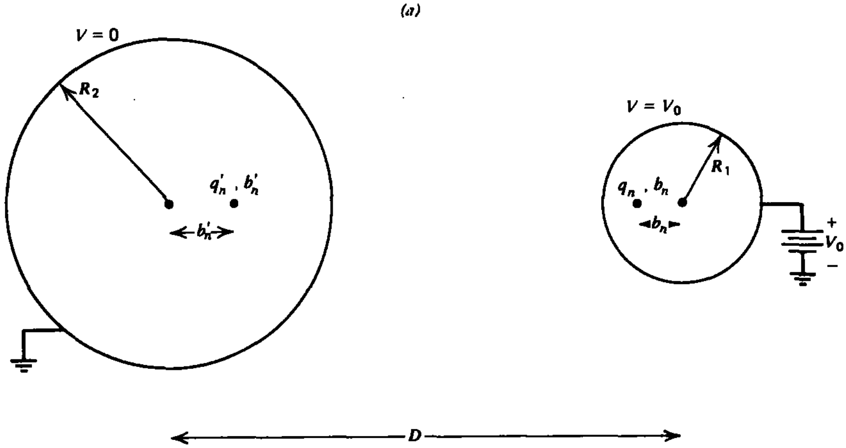
(f) We wish to generalize this problem to that of a sphere resting on the ground plane with an applied field $\mathbf{E} = -E_0\mathbf{i}_z$ at infinity. What must the ratio Q/D^2 be, such that as Q and D become infinite the field far from the sphere in the $\theta = \pi/2$ plane is $-E_0\mathbf{i}_z$?

(g) In this limit what is the total charge induced on the sphere? (**Hint:** $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$.)

45. A conducting sphere of radius R at potential V_0 has its center a distance D from an infinite grounded plane.



(a)



(f)

(a) Show that an infinite number of image charges in the plane and in the sphere are necessary to satsify the boundary conditions

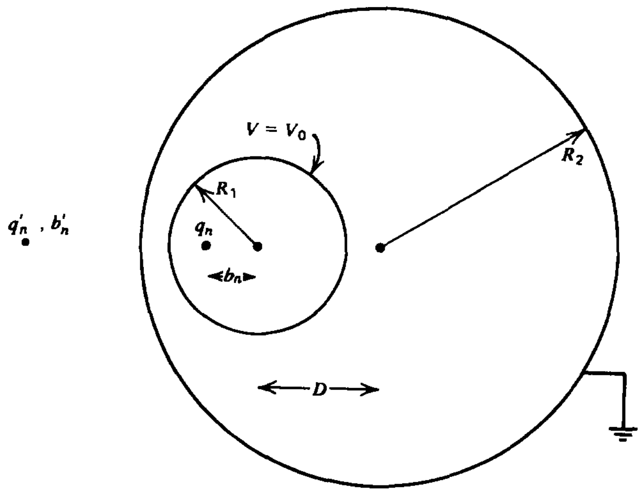
$$q_n = \frac{q_{n-1}R}{2D - b_{n-1}}, \qquad b_n = \frac{R^2}{2D - b_{n-1}}$$

What are q_1 and q_2?

(b) Show that the governing difference equation is

$$\frac{1}{q_{n-1}} - \frac{c}{q_n} + \frac{1}{q_{n+1}} = 0$$

What is c?



(f)

(c) Solve the difference equation in (b) assuming solutions of the form

$$P_n = 1/q_n = A\lambda^n$$

What values of λ satisfy (b)? **Hint:**

$$c/2 + \sqrt{(c/2)^2 - 1} = \frac{1}{c/2 - \sqrt{(c/2)^2 - 1}}$$

(d) What is the position of each image charge? What is the limiting position of the image charges as $n \to \infty$?

(e) Show that the capacitance (the ratio of the total charge on the sphere to the voltage V_0) can be written as an infinite series

$$C = C_0(\lambda^2 - 1)\left(\frac{1}{\lambda^2 - 1} + \frac{\lambda}{\lambda^4 - 1} + \frac{\lambda^2}{\lambda^6 - 1} + \frac{\lambda^3}{\lambda^8 - 1} + \cdots\right)$$

What are C_0 and λ?

(f) Show that the image charges and their positions for two spheres obey the difference equations

$$q_{n+1} = \mp\frac{q'_n R_1}{D - b'_n}, \qquad b_{n+1} = \pm\frac{R_1^2}{D - b'_n}$$

$$q'_n = -\frac{R_2 q_n}{D \mp b_n}, \qquad b'_n = \frac{R_2^2}{D \mp b_n}$$

where we use the upper signs for adjacent spheres and the lower signs when the smaller sphere of radius R_1 is inside the larger one.

(g) Show that the governing difference equation is of the form

$$P_{n+1} \mp cP_n + P_{n-1} = 0$$

What are P_n and c?

(h) Solve (g) assuming solutions of the form

$$P_n = A\lambda^n$$

(i) Show that the capacitance is of the form

$$C = C_0(1-\xi^2)\left(\frac{1}{1-\xi^2} + \frac{\lambda}{1-\xi^2\lambda^2} + \frac{\lambda^2}{1-\xi^4\lambda^4} + \cdots\right)$$

What are C_0, ξ, and λ?

(j) What is the capacitance when the two spheres are concentric so that $D = 0$. (**Hint:** $\sum_{n=0}^{\infty} a^n = 1/(1-a)$ for $a < 1$.)

chapter 3

polarization and conduction

The presence of matter modifies the electric field because even though the material is usually charge neutral, the field within the material can cause charge motion, called conduction, or small charge displacements, called polarization. Because of the large number of atoms present, 6.02×10^{23} per gram molecular weight (Avogadro's number), slight imbalances in the distribution have large effects on the fields inside and outside the materials. We must then self-consistently solve for the electric field with its effect on charge motion and redistribution in materials, with the charges. resultant effect back as another source of electric field.

3-1 POLARIZATION

In many electrically insulating materials, called dielectrics, electrons are tightly bound to the nucleus. They are not mobile, but if an electric field is applied, the negative cloud of electrons can be slightly displaced from the positive nucleus, as illustrated in Figure 3-1*a*. The material is then said to have an electronic polarization. Orientational polarizability as in Figure 3-1*b* occurs in polar molecules that do not share their

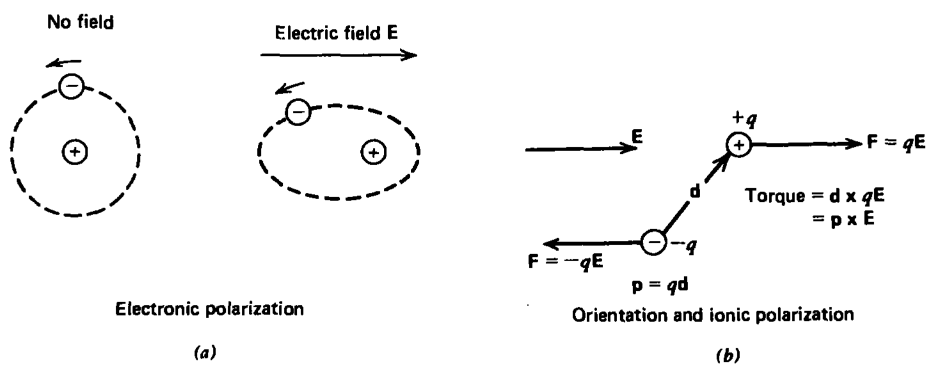


Figure 3-1 An electric dipole consists of two charges of equal magnitude but opposite sign, separated by a small vector distance **d**. (*a*) Electronic polarization arises when the average motion of the electron cloud about its nucleus is slightly displaced. (*b*) Orientation polarization arises when an asymmetric polar molecule tends to line up with an applied electric field. If the spacing **d** also changes, the molecule has ionic polarization.

electrons symmetrically so that the net positive and negative charges are separated. An applied electric field then exerts a torque on the molecule that tends to align it with the field. The ions in a molecule can also undergo slight relative displacements that gives rise to ionic polarizability.

The slightly separated charges for these cases form electric dipoles. Dielectric materials have a distribution of such dipoles. Even though these materials are charge neutral because each dipole contains an equal amount of positive and negative charges, a net charge can accumulate in a region if there is a local imbalance of positive or negative dipole ends. The net polarization charge in such a region is also a source of the electric field in addition to any other free charges.

3-1-1 The Electric Dipole

The simplest model of an electric dipole, shown in Figure 3-2*a*, has a positive and negative charge of equal magnitude q separated by a small vector displacement **d** directed from the negative to positive charge along the z axis. The electric potential is easily found at any point P as the superposition of potentials from each point charge alone:

$$V=\frac{q}{4\pi\varepsilon_0 r_+}-\frac{q}{4\pi\varepsilon_0 r_-} \tag{1}$$

The general potential and electric field distribution for any displacement **d** can be easily obtained from the geometry relating the distances r_+ and r_- to the spherical coordinates r and θ. By symmetry, these distances are independent of the angle ϕ. However, in dielectric materials the separation between charges are of atomic dimensions and so are very small compared to distances of interest far from the dipole. So, with r_+ and r_- much greater than the dipole spacing d, we approximate them as

$$\lim_{r\gg d}\begin{aligned} r_+ &\approx r-\frac{d}{2}\cos\theta \\ r_- &\approx r+\frac{d}{2}\cos\theta \end{aligned} \tag{2}$$

Then the potential of (1) is approximately

$$V\approx\frac{qd\cos\theta}{4\pi\varepsilon_0 r^2}=\frac{\mathbf{p}\cdot\mathbf{i}_r}{4\pi\varepsilon_0 r^2} \tag{3}$$

where the vector **p** is called the dipole moment and is defined as

$$\mathbf{p}=q\mathbf{d}\ \text{(coul-m)} \tag{4}$$

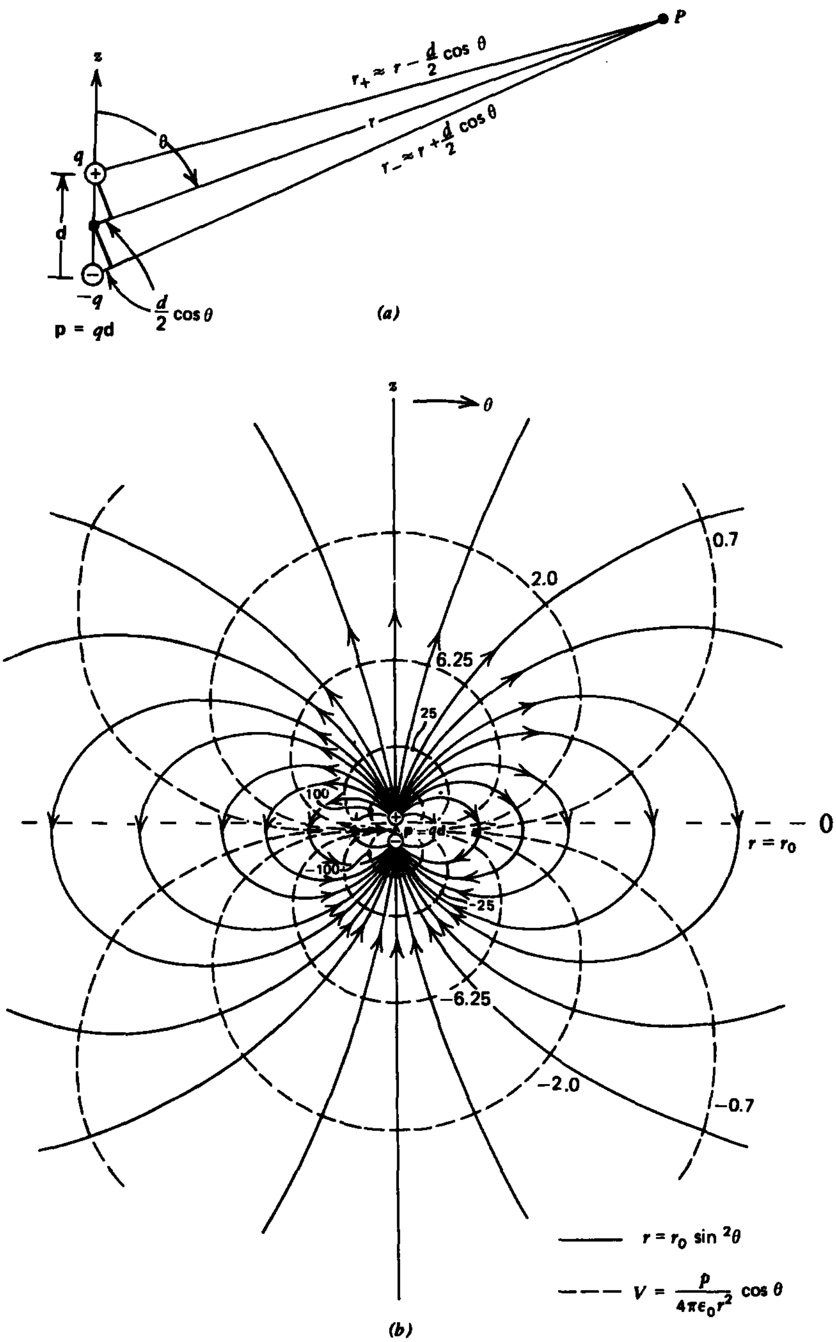


Figure 3-2 (*a*) The potential at any point P due to the electric dipole is equal to the sum of potentials of each charge alone. (*b*) The equi-potential (dashed) and field lines (solid) for a point electric dipole calibrated for $4\pi\varepsilon_0/p = 100$.

Because the separation of atomic charges is on the order of 1 Å(10^{-10} m) with a charge magnitude equal to an integer multiple of the electron charge ($q = 1.6\times 10^{-19}$ coul), it is convenient to express dipole moments in units of debyes defined as 1 debye $= 3.33\times 10^{-30}$ coul-m so that dipole moments are of order $p = 1.6\times 10^{-29}$ coul-m ≈ 4.8 debyes. The electric field for the point dipole is then

$$\mathbf{E} = -\nabla V = \frac{p}{4\pi\varepsilon_0 r^3}[2\cos\theta\mathbf{i}_r + \sin\theta\mathbf{i}_\theta] = \frac{3(\mathbf{p}\cdot\mathbf{i}_r)\mathbf{i}_r - \mathbf{p}}{4\pi\varepsilon_0 r^3} \tag{5}$$

the last expressions in (3) and (5) being coordinate independent. The potential and electric field drop off as a single higher power in r over that of a point charge because the net charge of the dipole is zero. As one gets far away from the dipole, the fields due to each charge tend to cancel. The point dipole equipotential and field lines are sketched in Figure 3-2*b*. The lines tangent to the electric field are

$$\frac{dr}{r\,d\theta} = \frac{E_r}{E_\theta} = 2\cot\theta \Rightarrow r = r_0 \sin^2\theta \tag{6}$$

where r_0 is the position of the field line when $\theta = \pi/2$. All field lines start on the positive charge and terminate on the negative charge.

If there is more than one pair of charges, the definition of dipole moment in (4) is generalized to a sum over all charges,

$$\mathbf{p} = \sum_{\text{all charges}} q_i \mathbf{r}_i \tag{7}$$

where $\mathbf{r}_i$ is the vector distance from an origin to the charge q_i as in Figure 3-3. When the net charge in the system is zero ($\sum q_i = 0$), the dipole moment is independent of the choice of origins for if we replace $\mathbf{r}_i$ in (7) by $\mathbf{r}_i + \mathbf{r}_0$, where $\mathbf{r}_0$ is the constant vector distance between two origins:

$$\begin{aligned}\mathbf{p} &= \sum q_i(\mathbf{r}_i + \mathbf{r}_0)\\ &= \sum q_i \mathbf{r}_i + \mathbf{r}_0 \cancelto{0}{\sum q_i}\\ &= \sum q_i \mathbf{r}_i\end{aligned} \tag{8}$$

The result is unchanged from (7) as the constant $\mathbf{r}_0$ could be taken outside the summation.

If we have a continuous distribution of charge (7) is further generalized to

$$\mathbf{p} = \int_{\text{all } q} \mathbf{r}\,dq \tag{9}$$

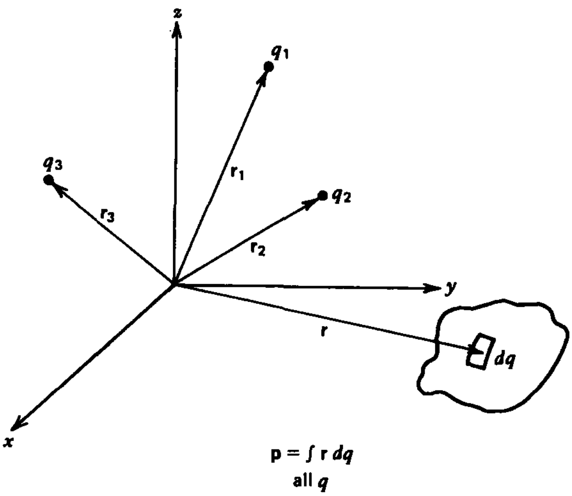


Figure 3-3 The dipole moment can be defined for any distribution of charge. If the net charge in the system is zero, the dipole moment is independent of the location of the origin.

Then the potential and electric field far away from any dipole distribution is given by the coordinate independent expressions in (3) and (5) where the dipole moment **p** is given by (7) and (9).

3-1-2 Polarization Charge

We enclose a large number of dipoles within a dielectric medium with the differential-sized rectangular volume $\Delta x\,\Delta y\,\Delta z$ shown in Figure 3-4*a*. All totally enclosed dipoles, being charge neutral, contribute no net charge within the volume. Only those dipoles within a distance $\mathbf{d}\cdot\mathbf{n}$ of each surface are cut by the volume and thus contribute a net charge where $\mathbf{n}$ is the unit normal to the surface at each face, as in Figure 3-4*b*. If the number of dipoles per unit volume is N, it is convenient to define the number density of dipoles as the polarization vector $\mathbf{P}$:

$$\mathbf{P}=N\mathbf{p}=Nq\mathbf{d} \tag{10}$$

The net charge enclosed near surface 1 is

$$dq_1=(Nqd_x)|_x\,\Delta y\,\Delta z=P_x(x)\,\Delta y\,\Delta z \tag{11}$$

while near the opposite surface 2

$$dq_2=-(Nqd_x)|_{x+\Delta x}\,\Delta y\,\Delta z=-P_x(x+\Delta x)\,\Delta y\,\Delta z \tag{12}$$

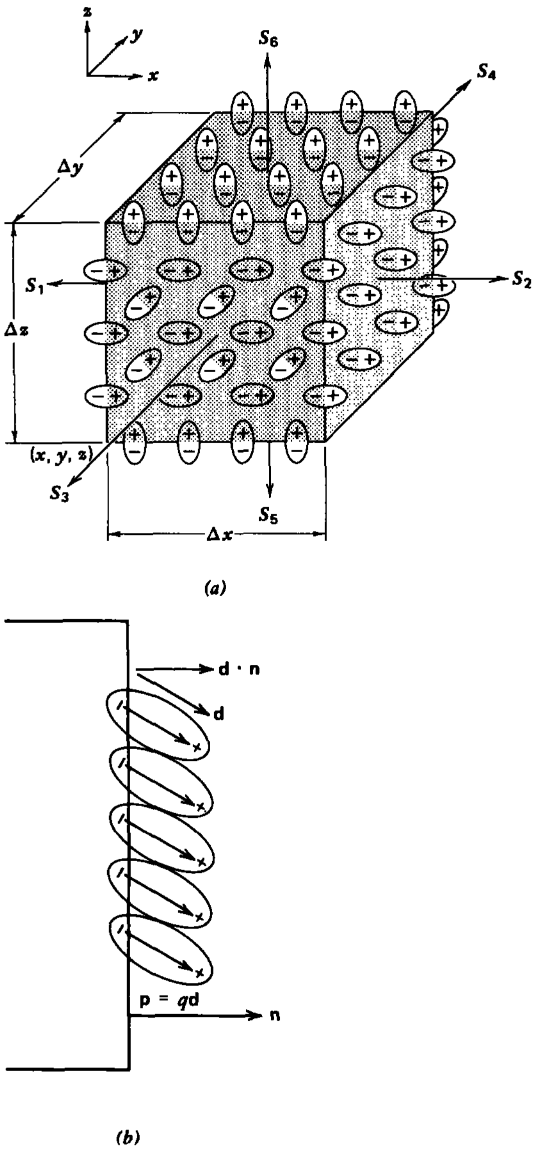


Figure 3-4 (*a*) The net charge enclosed within a differential-sized volume of dipoles has contributions only from the dipoles that are cut by the surfaces. All totally enclosed dipoles contribute no net charge. (*b*) Only those dipoles within a distance $\mathbf{d}\cdot\mathbf{n}$ of the surface are cut by the volume.

where we assume that Δy and Δz are small enough that the polarization $\mathbf{P}$ is essentially constant over the surface. The polarization can differ at surface 1 at coordinate x from that at surface 2 at coordinate $x+\Delta x$ if either the number density

N, the charge q, or the displacement $\mathbf{d}$ is a function of x. The difference in sign between (11) and (12) is because near S_1 the positive charge is within the volume, while near S_2 negative charge remains in the volume. Note also that only the component of $\mathbf{d}$ normal to the surface contributes to the volume of net charge.

Similarly, near the surfaces S_3 and S_4 the net charge enclosed is

$$\begin{aligned} dq_3 &= (Nqd_y)|_y \,\Delta x\,\Delta z = P_y(y)\,\Delta x\,\Delta z \\ dq_4 &= -(Nqd_y)|_{y+\Delta y}\,\Delta x\,\Delta z = -P_y(y+\Delta y)\,\Delta x\,\Delta z \end{aligned} \tag{13}$$

while near the surfaces S_5 and S_6 with normal in the z direction the net charge enclosed is

$$\begin{aligned} dq_5 &= (Nqd_z)|_z \,\Delta x\,\Delta y = P_z(z)\,\Delta x\,\Delta y \\ dq_6 &= -(Nqd_z)|_{z+\Delta z}\,\Delta x\,\Delta y = -P_z(z+\Delta z)\,\Delta x\,\Delta y \end{aligned} \tag{14}$$

The total charge enclosed within the volume is the sum of (11)–(14):

$$\begin{aligned} dq_T &= dq_1 + dq_2 + dq_3 + dq_4 + dq_5 + dq_6 \\ &= \left(\frac{P_x(x) - P_x(x+\Delta x)}{\Delta x} + \frac{P_y(y) - P_y(y+\Delta y)}{\Delta y}\right. \\ &\quad \left. + \frac{P_z(z) - P_z(z+\Delta z)}{\Delta z}\right)\Delta x\,\Delta y\,\Delta z \end{aligned} \tag{15}$$

As the volume shrinks to zero size, the polarization terms in (15) define partial derivatives so that the polarization volume charge density is

$$\rho_p = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \frac{dq_T}{\Delta x\,\Delta y\,\Delta z} = -\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}\right) = -\nabla \cdot \mathbf{P} \tag{16}$$

This volume charge is also a source of the electric field and needs to be included in Gauss's law

$$\nabla \cdot (\varepsilon_0 \mathbf{E}) = \rho_f + \rho_p = \rho_f - \nabla \cdot \mathbf{P} \tag{17}$$

where we subscript the free charge ρ_f with the letter f to distinguish it from the polarization charge ρ_p. The total polarization charge within a region is obtained by integrating (16) over the volume,

$$q_p = \int_V \rho_p\, dV = -\int_V \nabla \cdot \mathbf{P}\, dV = -\oint_S \mathbf{P} \cdot d\mathbf{S} \tag{18}$$

where we used the divergence theorem to relate the polarization charge to a surface integral of the polarization vector.

3-1-3 The Displacement Field

Since we have no direct way of controlling the polarization charge, it is convenient to cast Gauss's law only in terms of free charge by defining a new vector **D** as

$$\mathbf{D}=\varepsilon_0\mathbf{E}+\mathbf{P} \tag{19}$$

This vector **D** is called the displacement field because it differs from $\varepsilon_0\mathbf{E}$ due to the slight charge displacements in electric dipoles. Using (19), (17) can be rewritten as

$$\nabla\cdot(\varepsilon_0\mathbf{E}+\mathbf{P})=\nabla\cdot\mathbf{D}=\rho_f \tag{20}$$

where ρ_f only includes the free charge and not the bound polarization charge. By integrating both sides of (20) over a volume and using the divergence theorem, the new integral form of Gauss's law is

$$\int_V \nabla\cdot\mathbf{D}\,dV=\oint_S \mathbf{D}\cdot d\mathbf{S}=\int_V \rho_f\,dV \tag{21}$$

In free space, the polarization **P** is zero so that $\mathbf{D}=\varepsilon_0\mathbf{E}$ and (20)–(21) reduce to the free space laws used in Chapter 2.

3-1-4 Linear Dielectrics

It is now necessary to find the constitutive law relating the polarization **P** to the applied electric field **E**. An accurate discussion would require the use of quantum mechanics, which is beyond the scope of this text. However, a simplified classical model can be used to help us qualitatively understand the most interesting case of a linear dielectric.

(a) Polarizability

We model the atom as a fixed positive nucleus with a surrounding uniform spherical negative electron cloud, as shown in Figure 3-5*a*. In the absence of an applied electric field, the dipole moment is zero because the center of charge for the electron cloud is coincident with the nucleus. More formally, we can show this using (9), picking our origin at the position of the nucleus:

$$\mathbf{p}=Q\cancelto{0}{(0)}-\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{R_0}\mathbf{i}_r\rho_0 r^3\sin\theta\,dr\,d\theta\,d\phi \tag{22}$$

Since the radial unit vector $\mathbf{i}_r$ changes direction in space, it is necessary to use Table 1-2 to write $\mathbf{i}_r$ in terms of the constant Cartesian unit vectors:

$$\mathbf{i}_r=\sin\theta\cos\phi\,\mathbf{i}_x+\sin\theta\sin\phi\,\mathbf{i}_y+\cos\theta\,\mathbf{i}_z \tag{23}$$

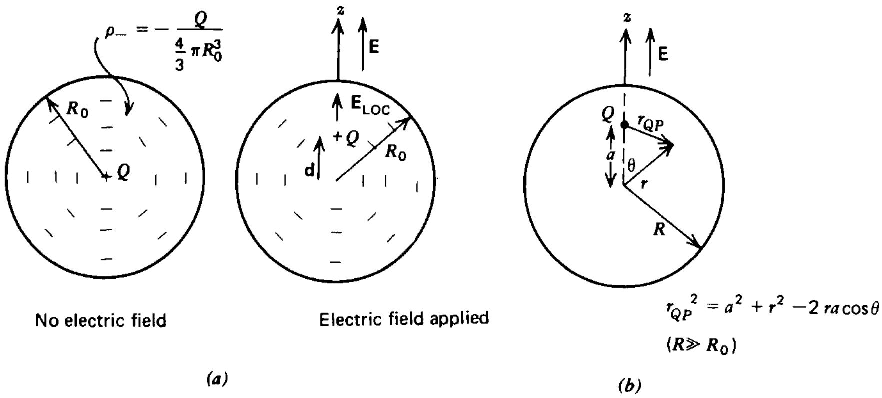


Figure 3-5 (*a*) A simple atomic classical model has a negative spherical electron cloud of small radius R_0 centered about a positive nucleus when no external electric field is present. An applied electric field tends to move the positive charge in the direction of the field and the negative charge in the opposite direction creating an electric dipole. (*b*) The average electric field within a large sphere of radius R ($R \gg R_0$) enclosing many point dipoles is found by superposing the average fields due to each point charge.

When (23) is used in (22) the x and y components integrate to zero when integrated over ϕ, while the z component is zero when integrated over θ so that $\mathbf{p} = 0$.

An applied electric field tends to push the positive charge in the direction of the field and the negative charge in the opposite direction causing a slight shift $\mathbf{d}$ between the center of the spherical cloud and the positive nucleus, as in Figure 3-5*a*. Opposing this movement is the attractive coulombic force. Considering the center of the spherical cloud as our origin, the self-electric field within the cloud is found from Section 2.4.3*b* as

$$E_r = -\frac{Qr}{4\pi\varepsilon_0 R_0^3} \tag{24}$$

In equilibrium the net force $\mathbf{F}$ on the positive charge is zero,

$$\mathbf{F} = Q\left(\mathbf{E}_{\text{Loc}} - \frac{Q\mathbf{d}}{4\pi\varepsilon_0 R_0^3}\right) = 0 \tag{25}$$

where we evaluate (24) at $r = d$ and $\mathbf{E}_{\text{Loc}}$ is the local polarizing electric field acting on the dipole. From (25) the equilibrium dipole spacing is

$$\mathbf{d} = \frac{4\pi\varepsilon_0 R_0^3}{Q}\mathbf{E}_{\text{Loc}} \tag{26}$$

so that the dipole moment is written as

$$\mathbf{p} = Q\mathbf{d} = \alpha\mathbf{E}_{\text{Loc}}, \qquad \alpha = 4\pi\varepsilon_0 R_0^3 \tag{27}$$

where α is called the polarizability.

(b) The Local Electric Field

If this dipole were isolated, the local electric field would equal the applied macroscopic field. However, a large number density N of neighboring dipoles also contributes to the polarizing electric field. The electric field changes drastically from point to point within a small volume containing many dipoles, being equal to the superposition of fields due to each dipole given by (5). The macroscopic field is then the average field over this small volume.

We calculate this average field by first finding the average field due to a single point charge Q a distance a along the z axis from the center of a spherical volume with radius R much larger than the radius of the electron cloud ($R \gg R_0$) as in Figure 3-5b. The average field due to this charge over the spherical volume is

$$<\mathbf{E}> = \frac{1}{\frac{4}{3}\pi R^3}\int_{r=0}^{R}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{Q(r\mathbf{i}_r - a\mathbf{i}_z)r^2\sin\theta\,dr\,d\theta\,d\phi}{4\pi\varepsilon_0[a^2+r^2-2ra\cos\theta]^{3/2}} \tag{28}$$

where we used the relationships

$$r_{QP}^2 = a^2+r^2-2ra\cos\theta, \qquad \mathbf{r}_{QP} = r\mathbf{i}_r - a\mathbf{i}_z \tag{29}$$

Using (23) in (28) again results in the x and y components being zero when integrated over ϕ. Only the z component is now nonzero:

$$<E_z> = \frac{Q}{\frac{4}{3}\pi R^3}\frac{2\pi}{(4\pi\varepsilon_0)}\int_{\theta=0}^{\pi}\int_{r=0}^{R}\frac{r^3(\cos\theta - a/r)\sin\theta\,dr\,d\theta}{[a^2+r^2-2ra\cos\theta]^{3/2}} \tag{30}$$

We introduce the change of variable from θ to u

$$u = r^2+a^2-2ar\cos\theta, \qquad du = 2ar\sin\theta\,d\theta \tag{31}$$

so that (30) can be integrated over u and r. Performing the u integration first we have

$$\begin{aligned}<E_z> &= \frac{3Q}{8\pi R^3\varepsilon_0}\int_{r=0}^{R}\int_{(r-a)^2}^{(r+a)^2}\frac{r}{4a^2}\frac{(r^2-a^2-u)}{u^{3/2}}\,dr\,du\\ &= \frac{3Q}{8\pi R^3\varepsilon_0}\int_{r=0}^{R}\frac{r}{4a^2}\left(-\frac{2(r^2-a^2+u)}{u^{1/2}}\right)\Bigg|_{u=(r-a)^2}^{(r+a)^2}dr\\ &= -\frac{3Q}{8\pi R^3\varepsilon_0 a^2}\int_{r=0}^{R}r^2\left(1-\frac{r-a}{|r-a|}\right)dr\end{aligned} \tag{32}$$

We were careful to be sure to take the positive square root in the lower limit of u. Then for $r > a$, the integral is zero so that the integral limits over r range from 0 to a:

$$<E_z> = -\frac{3Q}{8\pi R^3 \varepsilon_0 a^2}\int_{r=0}^{a} 2r^2\,dr = \frac{-Qa}{4\pi\varepsilon_0 R^3} \tag{33}$$

To form a dipole we add a negative charge $-Q$, a small distance d below the original charge. The average electric field due to the dipole is then the superposition of (33) for both charges:

$$<E_z> = -\frac{Q}{4\pi\varepsilon_0 R^3}[a-(a-d)] = -\frac{Qd}{4\pi\varepsilon_0 R^3} = -\frac{p}{4\pi\varepsilon_0 R^3} \tag{34}$$

If we have a number density N of such dipoles within the sphere, the total number of dipoles enclosed is $\frac{4}{3}\pi R^3 N$ so that superposition of (34) gives us the average electric field due to all the dipoles in terms of the polarization vector $\mathbf{P} = N\mathbf{p}$:

$$<\mathbf{E}> = -\frac{\frac{4}{3}\pi R^3 N\mathbf{p}}{4\pi\varepsilon_0 R^3} = -\frac{\mathbf{P}}{3\varepsilon_0} \tag{35}$$

The total macroscopic field is then the sum of the local field seen by each dipole and the average resulting field due to all the dipoles

$$\mathbf{E} = <\mathbf{E}> + \mathbf{E}_{\text{Loc}} = -\frac{\mathbf{P}}{3\varepsilon_0} + \mathbf{E}_{\text{Loc}} \tag{36}$$

so that the polarization $\mathbf{P}$ is related to the macroscopic electric field from (27) as

$$\mathbf{P} = N\mathbf{p} = N\alpha\mathbf{E}_{\text{Loc}} = N\alpha\left(\mathbf{E} + \frac{\mathbf{P}}{3\varepsilon_0}\right) \tag{37}$$

which can be solved for $\mathbf{P}$ as

$$\mathbf{P} = \frac{N\alpha}{1 - N\alpha/3\varepsilon_0}\mathbf{E} = \chi_e\varepsilon_0\mathbf{E}, \qquad \chi_e = \frac{N\alpha/\varepsilon_0}{1 - N\alpha/3\varepsilon_0} \tag{38}$$

where we introduce the electric susceptibility χ_e as the proportionality constant between $\mathbf{P}$ and $\varepsilon_0\mathbf{E}$. Then, use of (38) in (19) relates the displacement field $\mathbf{D}$ linearly to the electric field:

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon_0(1+\chi_e)\mathbf{E} = \varepsilon_0\varepsilon_r\mathbf{E} = \varepsilon\mathbf{E} \tag{39}$$

where $\varepsilon_r = 1 + \chi_e$ is called the relative dielectric constant and $\varepsilon = \varepsilon_r\varepsilon_0$ is the permittivity of the dielectric, also simply called the dielectric constant. In free space the susceptibility is zero ($\chi_e = 0$) so that $\varepsilon_r = 1$ and the permittivity is that of free space, $\varepsilon = \varepsilon_0$. The last relation in (39) is usually the most convenient to use as all the results of Chapter 2 are also correct within linear dielectrics if we replace ε_0 by ε. Typical values of relative permittivity are listed in Table 3-1 for various common substances. High dielectric constant materials are usually composed of highly polar molecules.

Table 3-1 The relative permittivity for various common substances at room temperature

	$\varepsilon_r = \varepsilon/\varepsilon_0$
Carbon Tetrachloride[a]	2.2
Ethanol[a]	24
Methanol[a]	33
n-Hexane[a]	1.9
Nitrobenzene[a]	35
Pure Water[a]	80
Barium Titanate[b] (with 20% Strontium Titanate)	>2100
Borosilicate Glass[b]	4.0
Ruby Mica (Muscovite)[b]	5.4
Polyethylene[b]	2.2
Polyvinyl Chloride[b]	6.1
Teflon[b] (Polytetrafluorethylene)	2.1
Plexiglas[b]	3.4
Paraffin Wax[b]	2.2

[a] *From* Lange's Handbook of Chemistry, *10th ed., McGraw-Hill, New York, 1961, pp. 1234–37.*
[b] *From A. R. von Hippel (Ed.) Dielectric Materials and Applications, M.I.T., Cambridge, Mass., 1966, pp. 301–370*

The polarizability and local electric field were only introduced so that we could relate microscopic and macroscopic fields. For most future problems we will describe materials by their permittivity ε because this constant is most easily measured. The polarizability is then easily found as

$$\varepsilon-\varepsilon_0=\frac{N\alpha}{1-N\alpha/3\varepsilon_0}\Rightarrow\frac{N\alpha}{3\varepsilon_0}=\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0} \tag{40}$$

It then becomes simplest to work with the field vectors **D** and **E**. The polarization can always be obtained if needed from the definition

$$\mathbf{P}=\mathbf{D}-\varepsilon_0\mathbf{E}=(\varepsilon-\varepsilon_0)\mathbf{E} \tag{41}$$

EXAMPLE 3-1 POINT CHARGE WITHIN A DIELECTRIC SPHERE

Find all the fields and charges due to a point charge q within a linear dielectric sphere of radius R and permittivity ε surrounded by free space, as in Figure 3-6.

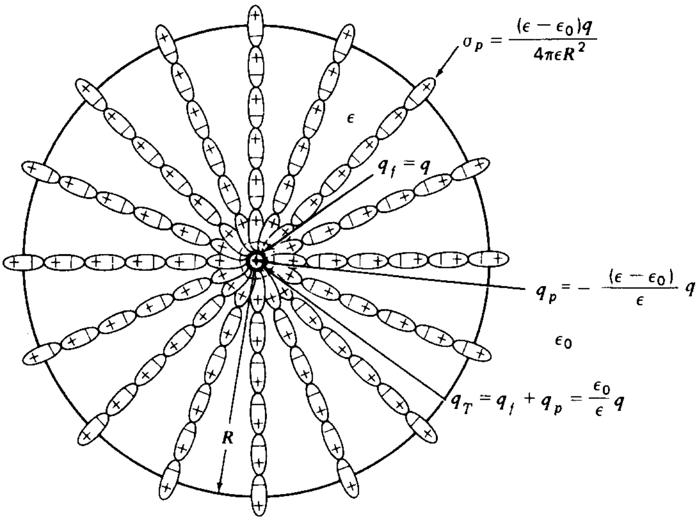


Figure 3-6 The electric field due to a point charge within a dielectric sphere is less than the free space field because of the partial neutralization of the point charge by the accumulation of dipole ends of opposite charge. The total polarization charge on the sphere remains zero as an equal magnitude but opposite sign polarization charge appears at the spherical interface.

SOLUTION

Applying Gauss's law of (21) to a sphere of any radius r whether inside or outside the sphere, the enclosed free charge is always q:

$$\oint_S \mathbf{D} \cdot \mathbf{dS} = D_r 4\pi r^2 = q \Rightarrow D_r = \frac{q}{4\pi r^2} \quad \text{all } r$$

The electric field is then discontinuous at $r = R$,

$$E_r = \begin{cases} \dfrac{D_r}{\varepsilon} = \dfrac{q}{4\pi\varepsilon r^2} & r < R \\ \dfrac{D_r}{\varepsilon_0} = \dfrac{q}{4\pi\varepsilon_0 r^2}, & r > R \end{cases}$$

due to the abrupt change of permittivities. The polarization field is

$$P_r = D_r - \varepsilon_0 E_r = \begin{cases} \dfrac{(\varepsilon - \varepsilon_0)q}{4\pi\varepsilon r^2}, & r < R \\ 0, & r > R \end{cases}$$

The volume polarization charge ρ_P is zero everywhere,

$$\rho_p = -\nabla \cdot \mathbf{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 P_r) = 0, \quad 0 < r < R$$

except at $r = 0$ where a point polarization charge is present, and at $r = R$ where we have a surface polarization charge found by using (18) for concentric Gaussian spheres of radius r inside and outside the dielectric sphere:

$$q_p = -\oint_S \mathbf{P} \cdot \mathbf{dS} = -P_r 4\pi r^2 = \begin{cases} -(\varepsilon - \varepsilon_0)q/\varepsilon, & r < R \\ 0, & r > R \end{cases}$$

We know that for $r < R$ this polarization charge must be a point charge at the origin as there is no volume charge contribution yielding a total point charge at the origin:

$$q_T = q_p + q = \frac{\varepsilon_0}{\varepsilon} q$$

This reduction of net charge is why the electric field within the sphere is less than the free space value. The opposite polarity ends of the dipoles are attracted to the point charge, partially neutralizing it. The total polarization charge enclosed by the sphere with $r > R$ is zero as there is an opposite polarity surface polarization charge at $r = R$ with density,

$$\sigma_p = \frac{(\varepsilon - \varepsilon_0)q}{4\pi\varepsilon R^2}$$

The total surface charge $\sigma_p 4\pi R^2 = (\varepsilon - \varepsilon_0)q/\varepsilon$ is equal in magnitude but opposite in sign to the polarization point charge at the origin. The total polarization charge always sums to zero.

3-1-5 Spontaneous Polarization

(a) Ferro-electrics

Examining (38) we see that when $N\alpha/3\varepsilon_0 = 1$ the polarization can be nonzero even if the electric field is zero. We can just meet this condition using the value of polarizability in (27) for electronic polarization if the whole volume is filled with contacting dipole spheres of the type in Figure 3-5a so that we have one dipole for every volume of $\frac{4}{3}\pi R_0^3$. Then any slight fluctuation in the local electric field increases the polarization, which in turn increases the local field resulting in spontaneous polarization so that all the dipoles over a region are aligned. In a real material dipoles are not so densely packed. Furthermore, more realistic statistical models including thermally induced motions have shown that most molecules cannot meet the conditions for spontaneous polarization.

However, some materials have sufficient additional contributions to the polarizabilities due to ionic and orientational polarization that the condition for spontaneous polarization is met. Such materials are called ferro-electrics, even though they are not iron compounds, because of their similarity in behavior to iron compound ferro-magnetic materials, which we discuss in Section 5.5.3*c*. Ferro-electrics are composed of permanently polarized regions, called domains, as illustrated in Figure 3-7*a*. In the absence of an electric field, these domains are randomly distributed so that the net macroscopic polarization field is zero. When an electric field is applied, the dipoles tend to align with the field so that domains with a polarization component along the field grow at the expense of nonaligned domains. Ferro-electrics typically have very high permittivities such as barium titanate listed in Table 3-1.

The domains do not respond directly with the electric field as domain orientation and growth is not a reversible process but involves losses. The polarization **P** is then nonlinearly related to the electric field **E** by the hysteresis curve shown in Figure 3-8. The polarization of an initially unpolarized sample increases with electric field in a nonlinear way until the saturation value $\mathbf{P}_{sat}$ is reached when all the domains are completely aligned with the field. A further increase in **E** does not increase **P** as all the dipoles are completely aligned.

As the field decreases, the polarization does not retrace its path but follows a new path as the dipoles tend to stick to their previous positions. Even when the electric field is zero, a

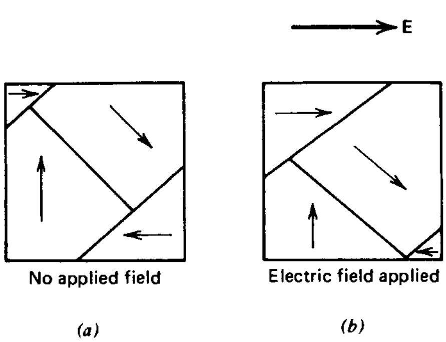


Figure 3-7 (*a*) In the absence of an applied electric field, a ferro-electric material generally has randomly distributed permanently polarized domains. Over a macroscopic volume, the net polarization due to all the domains is zero. (*b*) When an electric field is applied, domains with a polarization component in the direction of the field grow at the expense of nonaligned domains so that a net polarization can result.

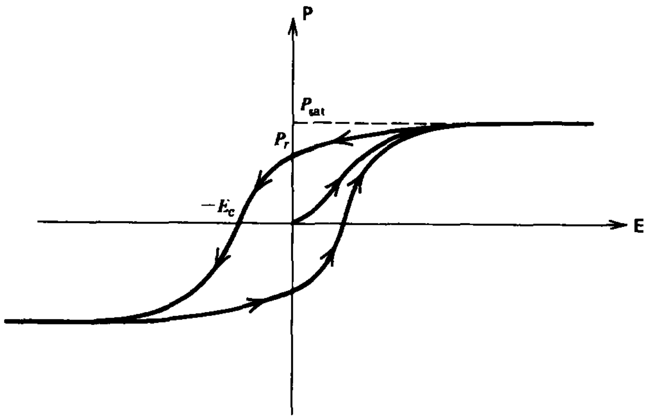


Figure 3-8 A typical ferro-electric hysteresis curve shows a saturation value $\mathbf{P}_{sat}$ when all the domains align with the field, a remanent polarization $\mathbf{P}_r$ when the electric field is removed, and a negative coercive electric field $-\mathbf{E}_c$, necessary to bring the polarization back to zero.

remanent polarization $\mathbf{P}_r$ remains. To bring the polarization to zero requires a negative coercive field $-E_c$. Further magnitude increases in negative electric field continues the symmetric hysteresis loop until a negative saturation is reached where all the dipoles have flipped over. If the field is now brought to zero and continued to positive field values, the whole hysteresis curve is traversed.

(b) Electrets

There are a class of materials called electrets that also exhibit a permanent polarization even in the absence of an applied electric field. Electrets are typically made using certain waxes or plastics that are heated until they become soft. They are placed within an electric field, tending to align the dipoles in the same direction as the electric field, and then allowed to harden. The dipoles are then frozen in place so that even when the electric field is removed a permanent polarization remains.

Other interesting polarization phenomena are:

Electrostriction—slight change in size of a dielectric due to the electrical force on the dipoles.

Piezo-electricity—when the electrostrictive effect is reversible so that a mechanical strain also creates a field.

Pyro-electricity—induced polarization due to heating or cooling.

3-2 CONDUCTION

3-2-1 Conservation of Charge

In contrast to dielectrics, most metals have their outermost band of electrons only weakly bound to the nucleus and are free to move in an applied electric field. In electrolytic solutions, ions of both sign are free to move. The flow of charge, called a current, is defined as the total charge flowing through a surface per unit time. In Figure 3-9*a* a single species of free charge with density ρ_{fi} and velocity $\mathbf{v}_i$ flows through a small differential sized surface $\mathbf{dS}$. The total charge flowing through this surface in a time Δt depends only on the velocity component perpendicular to the surface:

$$\Delta Q_i = \rho_{fi}\Delta t \mathbf{v}_i \cdot \mathbf{dS} \tag{1}$$

The tangential component of the velocity parallel to the surface $\mathbf{dS}$ only results in charge flow along the surface but not through it. The total differential current through $\mathbf{dS}$ is then defined as

$$dI_i = \frac{\Delta Q_i}{\Delta t} = \rho_{fi}\mathbf{v}_i \cdot \mathbf{dS} = \mathbf{J}_{fi} \cdot \mathbf{dS} \text{ ampere} \tag{2}$$

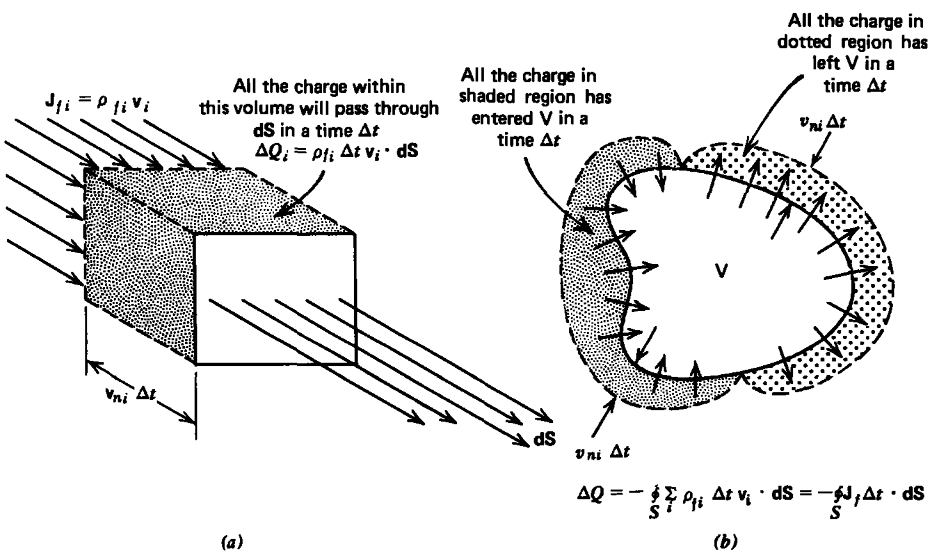


Figure 3-9 The current through a surface is defined as the number of charges per second passing through the surface. (*a*) The current is proportional to the component of charge velocity perpendicular to the surface. (*b*) The net change of total charge within a volume is equal to the difference of the charge entering to that leaving in a small time Δt.

where the free current density of these charges $\mathbf{J}_{fi}$ is a vector and is defined as

$$\mathbf{J}_{fi} = \rho_{fi}\mathbf{v}_i \text{ amp/m}^2 \tag{3}$$

If there is more than one type of charge carrier, the net charge density is equal to the algebraic sum of all the charge densities, while the net current density equals the vector sum of the current densities due to each carrier:

$$\rho_f = \sum \rho_{fi}, \quad \mathbf{J}_f = \sum \rho_{fi}\mathbf{v}_i \tag{4}$$

Thus, even if we have charge neutrality so that $\rho_f = 0$, a net current can flow if the charges move with different velocities. For example, two oppositely charged carriers with densities $\rho_1 = -\rho_2 \equiv \rho_0$ moving with respective velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ have

$$\rho_f = \rho_1 + \rho_2 = 0, \qquad \mathbf{J}_f = \rho_1\mathbf{v}_1 + \rho_2\mathbf{v}_2 = \rho_0(\mathbf{v}_1 - \mathbf{v}_2) \tag{5}$$

With $\mathbf{v}_1 \neq \mathbf{v}_2$ a net current flows with zero net charge. This is typical in metals where the electrons are free to flow while the oppositely charged nuclei remain stationary.

The total current I, a scalar, flowing through a macroscopic surface S, is then just the sum of the total differential currents of all the charge carriers through each incremental size surface element:

$$I = \int_S \mathbf{J}_f \cdot \mathbf{dS} \tag{6}$$

Now consider the charge flow through the closed volume V with surface S shown in Figure 3-9*b*. A time Δt later, that charge within the volume near the surface with the velocity component outward will leave the volume, while that charge just outside the volume with a velocity component inward will just enter the volume. The difference in total charge is transported by the current:

$$\begin{aligned}\Delta Q &= \int_V [\rho_f(t+\Delta t) - \rho_f(t)]\, dV \\ &= -\oint_S \sum \rho_{fi}\mathbf{v}_i \Delta t \cdot \mathbf{dS} = -\oint_S \mathbf{J}_f \Delta t \cdot \mathbf{dS}\end{aligned} \tag{7}$$

The minus sign on the right is necessary because when $\mathbf{v}_i$ is in the direction of $\mathbf{dS}$, charge has left the volume so that the enclosed charge decreases. Dividing (7) through by Δt and taking the limit as $\Delta t \to 0$, we use (3) to derive the integral conservation of charge equation:

$$\oint_S \mathbf{J}_f \cdot \mathbf{dS} + \int_V \frac{\partial \rho_f}{\partial t}\, dV = 0 \tag{8}$$

Using the divergence theorem, the surface integral can be converted to a volume integral:

$$\int_V \left[\nabla \cdot \mathbf{J}_f + \frac{\partial \rho_f}{\partial t}\right] dV = 0 \Rightarrow \nabla \cdot \mathbf{J}_f + \frac{\partial \rho_f}{\partial t} = 0 \qquad (9)$$

where the differential point form is obtained since the integral must be true for any volume so that the bracketed term must be zero at each point. From Gauss's law ($\nabla \cdot \mathbf{D} = \rho_f$) (8) and (9) can also be written as

$$\oint_S \left(\mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot \mathbf{dS} = 0, \qquad \nabla \cdot \left(\mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}\right) = 0 \qquad (10)$$

where $\mathbf{J}_f$ is termed the conduction current density and $\partial \mathbf{D}/\partial t$ is called the displacement current density.

This is the field form of Kirchoff's circuit current law that the algebraic sum of currents at a node sum to zero. Equation (10) equivalently tells us that the net flux of total current, conduction plus displacement, is zero so that all the current that enters a surface must leave it. The displacement current does not involve any charge transport so that time-varying current can be transmitted through space without charge carriers. Under static conditions, the displacement current is zero.

3-2-2 Charged Gas Conduction Models

(a) Governing Equations.

In many materials, including good conductors like metals, ionized gases, and electrolytic solutions as well as poorer conductors like lossy insulators and semiconductors, the charge carriers can be classically modeled as an ideal gas within the medium, called a plasma. We assume that we have two carriers of equal magnitude but opposite sign $\pm q$ with respective masses $m_\pm$ and number densities $n_\pm$. These charges may be holes and electrons in a semiconductor, oppositely charged ions in an electrolytic solution, or electrons and nuclei in a metal. When an electric field is applied, the positive charges move in the direction of the field while the negative charges move in the opposite direction. These charges collide with the host medium at respective frequencies ν_+ and ν_-, which then act as a viscous or frictional dissipation opposing the motion. In addition to electrical and frictional forces, the particles exert a force on themselves through a pressure term due to thermal agitation that would be present even if the particles were uncharged. For an ideal gas the partial pressure p is

$$p = nkT \text{ Pascals } [\text{kg-s}^{-2}\text{-m}^{-1}] \qquad (11)$$

where n is the number density of charges, T is the absolute temperature, and $k = 1.38 \times 10^{-23}$ joule/°K is called Boltzmann's constant.

The net pressure force on the small rectangular volume shown in Figure 3-10 is

$$\mathbf{f}_p = \left(\frac{p(x-\Delta x) - p(x)}{\Delta x}\mathbf{i}_x + \frac{p(y) - p(y+\Delta y)}{\Delta y}\mathbf{i}_y + \frac{p(z) - p(z+\Delta z)}{\Delta z}\mathbf{i}_z\right)\Delta x\, \Delta y\, \Delta z \tag{12}$$

where we see that the pressure only exerts a net force on the volume if it is different on each opposite surface. As the volume shrinks to infinitesimal size, the pressure terms in (12) define partial derivatives so that the volume force density becomes

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \frac{\mathbf{f}_p}{\Delta x\, \Delta y\, \Delta z} = -\left(\frac{\partial p}{\partial x}\mathbf{i}_x + \frac{\partial p}{\partial y}\mathbf{i}_y + \frac{\partial p}{\partial z}\mathbf{i}_z\right) = -\nabla p \tag{13}$$

Then using (11)–(13), Newton's force law for each charge carrier within the small volume is

$$m_\pm \frac{\partial \mathbf{v}_\pm}{\partial t} = \pm q\mathbf{E} - m_\pm \nu_\pm \mathbf{v}_\pm - \frac{1}{n_\pm}\nabla(n_\pm kT) \tag{14}$$

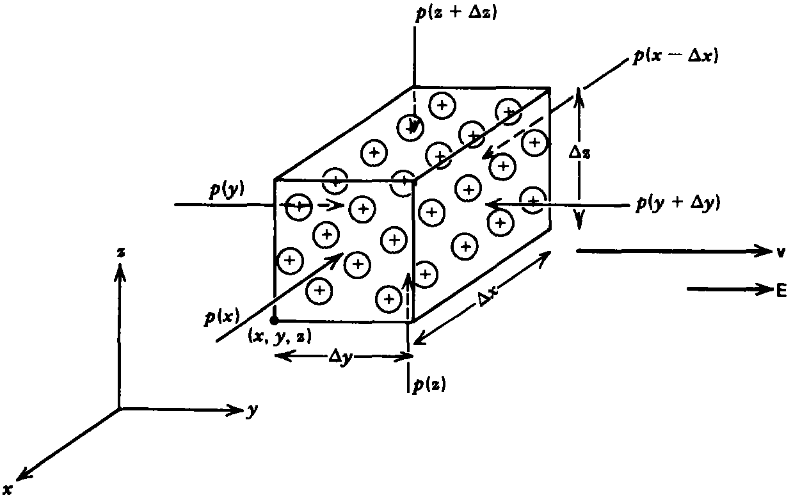


Figure 3-10 Newton's force law, applied to a small rectangular volume $\Delta x\, \Delta y\, \Delta z$ moving with velocity $\mathbf{v}$, enclosing positive charges with number density n. The pressure is the force per unit area acting normally inward on each surface and only contributes to the net force if it is different on opposite faces.

where the electric field $\mathbf{E}$ is due to the imposed field plus the field generated by the charges, as given by Gauss's law.

(b) Drift-Diffusion Conduction

Because in many materials the collision frequencies are typically on the order of $\nu \approx 10^{13}$ Hz, the inertia terms in (14) are often negligible. In this limit we can easily solve (14) for the velocity of each carrier as

$$\lim_{\partial \mathbf{v}_\pm/\partial t \ll \nu_\pm \mathbf{v}_\pm} \mathbf{v}_\pm = \frac{1}{m_\pm \nu_\pm}\left(\pm q\mathbf{E} - \frac{1}{n_\pm}\nabla(n_\pm kT)\right) \tag{15}$$

The charge and current density for each carrier are simply given as

$$\rho_\pm = \pm q n_\pm, \qquad \mathbf{J}_\pm = \rho_\pm \mathbf{v}_\pm = \pm q n_\pm \mathbf{v}_\pm \tag{16}$$

Multiplying (15) by the charge densities then gives us the constitutive law for each current as

$$\mathbf{J}_\pm = \pm q n_\pm \mathbf{v}_\pm = \pm \rho_\pm \mu_\pm \mathbf{E} - D_\pm \nabla \rho_\pm \tag{17}$$

where $\mu_\pm$ are called the particle mobilities and $D_\pm$ are their diffusion coefficients

$$\mu_\pm = \frac{q}{m_\pm \nu_\pm}\,[\text{A-kg}^{-1}\text{-s}^{-2}], \quad D_\pm = \frac{kT}{m_\pm \nu_\pm}\,[\text{m}^2\text{-s}^{-1}] \tag{18}$$

assuming that the system is at constant temperature. We see that the ratio $D_\pm/\mu_\pm$ for each carrier is the same having units of voltage, thus called the thermal voltage:

$$\frac{D_\pm}{\mu_\pm} = \frac{kT}{q}\ \text{volts}\ [\text{kg-m}^2\text{-A}^{-1}\text{-s}^{-3}] \tag{19}$$

This equality is known as Einstein's relation.

In equilibrium when the net current of each carrier is zero, (17) can be written in terms of the potential as ($\mathbf{E} = -\nabla V$)

$$\mathbf{J}_+ = \mathbf{J}_- = 0 = -\rho_\pm \mu_\pm \nabla V \mp D_\pm \nabla \rho_\pm \tag{20}$$

which can be rewritten as

$$\nabla\left[\pm\frac{\mu_\pm}{D_\pm} V + \ln \rho_\pm\right] = 0 \tag{21}$$

The bracketed term can then only be a constant, so the charge density is related to the potential by the Boltzmann distribution:

$$\rho_\pm = \pm \rho_0\, e^{\mp qV/kT} \tag{22}$$

where we use the Einstein relation of (19) and $\pm\rho_0$ is the equilibrium charge density of each carrier when $V = 0$ and are of equal magnitude because the system is initially neutral.

To find the spatial dependence of ρ and V we use (22) in Poisson's equation derived in Section 2.5.6:

$$\nabla^2 V = -\frac{(\rho_+ + \rho_-)}{\varepsilon} = -\frac{\rho_0}{\varepsilon}(e^{-qV/kT} - e^{qV/kT}) = \frac{2\rho_0}{\varepsilon}\sinh\frac{qV}{kT} \tag{23}$$

This equation is known as the Poisson–Boltzmann equation because the charge densities obey Boltzmann distributions.

Consider an electrode placed at $x = 0$ raised to the potential V_0 with respect to a zero potential at $x = \pm\infty$, as in Figure 3-11*a*. Because the electrode is long, the potential only varies

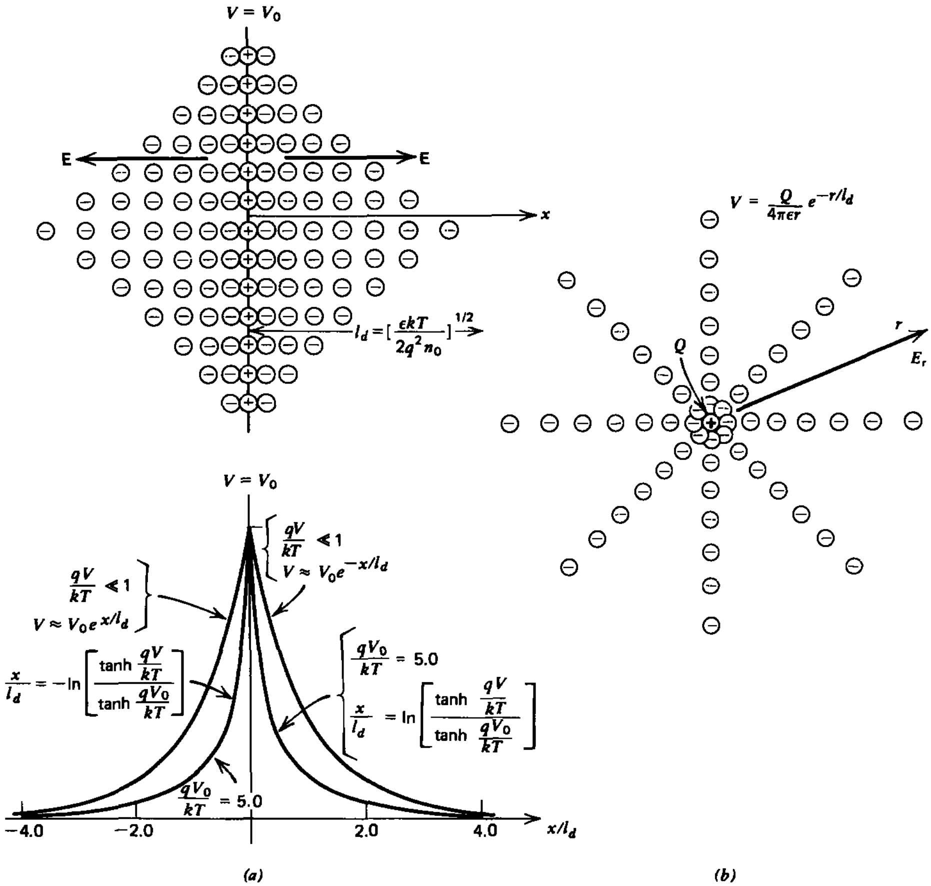


Figure 3-11 Opposite polarity mobile charges accumulate around any net charge inserted into a conductor described by the drift-diffusion equations, and tend to shield out its field for distances larger than the Debye length. (*a*) Electrode at potential V_0 with respect to a zero potential at $x = \pm\infty$. The spatial decay is faster for larger values of V_0. (*b*) Point charge.

with the x coordinate so that (23) becomes

$$\frac{d^2\tilde{V}}{dx^2}-\frac{1}{l_d^2}\sinh\tilde{V}=0,\qquad \tilde{V}=\frac{qV}{kT},\qquad l_d^2=\frac{\varepsilon kT}{2\rho_0 q}\tag{24}$$

where we normalize the voltage to the thermal voltage kT/q and l_d is called the Debye length.

If (24) is multiplied by $d\tilde{V}/dx$, it can be written as an exact differential:

$$\frac{d}{dx}\left[\frac{1}{2}\left(\frac{d\tilde{V}}{dx}\right)^2-\frac{\cosh\tilde{V}}{l_d^2}\right]=0\tag{25}$$

The bracketed term must then be a constant that is evaluated far from the electrode where the potential and electric field $\tilde{E}_x=-d\tilde{V}/dx$ are zero:

$$\frac{d\tilde{V}}{dx}=-\tilde{E}_x=\left[\frac{2}{l_d^2}(\cosh\tilde{V}-1)\right]^{1/2}=\mp\frac{2}{l_d}\sinh\frac{\tilde{V}}{2}\begin{cases}x>0\\x<0\end{cases}\tag{26}$$

The different signs taken with the square root are necessary because the electric field points in opposite directions on each side of the electrode. The potential is then implicitly found by direct integration as

$$\frac{\tanh(\tilde{V}/4)}{\tanh(\tilde{V}_0/4)}=e^{\mp x/l_d}\begin{cases}x>0\\x<0\end{cases}\tag{27}$$

The Debye length thus describes the characteristic length over which the applied potential exerts influence. In many materials the number density of carriers is easily of the order of $n_0\approx 10^{20}/\text{m}^3$, so that at room temperature ($T\approx 293°\text{K}$), l_d is typically 10^{-7} m.

Often the potentials are very small so that $qV/kT\ll 1$. Then, the hyperbolic terms in (27), as well as in the governing equation of (23), can be approximated by their arguments:

$$\nabla^2 V-\frac{V}{l_d^2}=0\tag{28}$$

This approximation is only valid when the potentials are much less than the thermal voltage kT/q, which at room temperature is about 25 mv. In this limit the solution of (27) shows that the voltage distribution decreases exponentially. At higher values of V_0, the decay is faster, as shown in Figure 3-11*a*.

If a point charge Q is inserted into the plasma medium, as in Figure 3-11*b*, the potential only depends on the radial distance r. In the small potential limit, (28) in spherical coordinates is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)-\frac{V}{l_d^2}=0\tag{29}$$

Realizing that this equation can be rewritten as

$$\frac{\partial^2}{\partial r^2}(rV)-\frac{(rV)}{l_d^2}=0 \tag{30}$$

we have a linear constant coefficient differential equation in the variable (rV) for which solutions are

$$rV=A_1\,e^{-r/l_d}+A_2\,e^{+r/l_d} \tag{31}$$

Because the potential must decay and not grow far from the charge, $A_2=0$ and the solution is

$$V=\frac{Q}{4\pi\varepsilon r}\,e^{-r/l_d} \tag{32}$$

where we evaluated A_1 by realizing that as $r\to 0$ the potential must approach that of an isolated point charge. Note that for small r the potential becomes very large and the small potential approximation is violated.

(c) Ohm's Law

We have seen that the mobile charges in a system described by the drift-diffusion equations accumulate near opposite polarity charge and tend to shield out its effect for distances larger than the Debye length. Because this distance is usually so much smaller than the characteristic system dimensions, most regions of space outside the Debye sheath are charge neutral with equal amounts of positive and negative charge density $\pm\rho_0$. In this region, the diffusion term in (17) is negligible because there are no charge density gradients. Then the total current density is proportional to the electric field:

$$\mathbf{J}=\mathbf{J}_++\mathbf{J}_-=\rho_0(\mathbf{v}_+-\mathbf{v}_-)=qn_0(\mu_++\mu_-)\mathbf{E}=\sigma\mathbf{E} \tag{33}$$

where σ [siemans/m $(\mathrm{m}^{-3}\text{-kg}^{-1}\text{-s}^3\text{-A}^2)$] is called the Ohmic conductivity and (33) is the point form of Ohm's law. Sometimes it is more convenient to work with the reciprocal conductivity $\rho_r=(1/\sigma)$ (ohm-m) called the resistivity. We will predominantly use Ohm's law to describe most media in this text, but it is important to remember that it is often not valid within the small Debye distances near charges. When Ohm's law is valid, the net charge is zero, thus giving no contribution to Gauss's law. Table 3-2 lists the Ohmic conductivities for various materials. We see that different materials vary over wide ranges in their ability to conduct charges.

The Ohmic conductivity of "perfect conductors" is large and is idealized to be infinite. Since all physical currents in (33) must remain finite, the electric field within the conductor is zero so that it imposes an equipotential surface:

$$\lim_{\sigma\to\infty} \mathbf{J}=\sigma\mathbf{E} \Rightarrow \begin{cases} \mathbf{E}=0 \\ V=\text{const} \\ \mathbf{J}=\text{finite} \end{cases} \tag{34}$$

Table 3-2 The Ohmic conductivity for various common substances at room temperature

	σ [siemen/m]
Silver[a]	6.3×10^{7}
Copper[a]	5.9×10^{7}
Gold[a]	4.2×10^{7}
Lead[a]	0.5×10^{7}
Tin[a]	0.9×10^{7}
Zinc[a]	1.7×10^{7}
Carbon[a]	7.3×10^{-4}
Mercury[b]	1.06×10^{6}
Pure Water[b]	4×10^{-6}
Nitrobenzene[b]	5×10^{-7}
Methanol[b]	4×10^{-5}
Ethanol[b]	1.3×10^{-7}
Hexane[b]	$<1\times 10^{-18}$

[a] *From* Handbook of Chemistry and Physics, *49th ed., The Chemical Rubber Co., 1968, p. E80.*
[b] *From* Lange's Handbook of Chemistry, *10th ed., McGraw-Hill, New York, 1961, pp. 1220–21.*

Throughout this text electrodes are generally assumed to be perfectly conducting and thus are at a constant potential. The external electric field must then be incident perpendicularly to the surface.

(d) Superconductors

One notable exception to Ohm's law is for superconducting materials at cryogenic temperatures. Then, with collisions negligible ($\nu_\pm=0$) and the absolute temperature low ($T\approx 0$), the electrical force on the charges is only balanced by their inertia so that (14) becomes simply

$$\frac{\partial \mathbf{v}_\pm}{\partial t}=\pm\frac{q}{m_\pm}\mathbf{E} \tag{35}$$

We multiply (35) by the charge densities that we assume to be constant so that the constitutive law relating the current density to the electric field is

$$\frac{\partial(\pm q n_\pm \mathbf{v}_\pm)}{\partial t} = \frac{\partial \mathbf{J}_\pm}{\partial t} = \frac{q^2 n_\pm}{m_\pm}\mathbf{E} = \omega_{p_\pm}^2 \varepsilon \mathbf{E}, \qquad \omega_{p_\pm}^2 = \frac{q^2 n_\pm}{m_\pm \varepsilon} \tag{36}$$

where $\omega_{p\pm}$ is called the plasma frequency for each carrier.

For electrons ($q = -1.6\times 10^{-19}$ coul, $m_- = 9.1\times 10^{-31}$ kg) of density $n_- \approx 10^{20}/\text{m}^3$ within a material with the permittivity of free space, $\varepsilon = \varepsilon_0 \approx 8.854\times 10^{-12}$ farad/m, the plasma frequency is

$$\omega_{p_-} = \sqrt{q^2 n_-/m_-\varepsilon} \approx 5.6\times 10^{11} \text{ radian/sec}$$
$$\Rightarrow f_{p_-} = \omega_{p_-}/2\pi \approx 9\times 10^{10} \text{ Hz} \tag{37}$$

If such a material is placed between parallel plate electrodes that are open circuited, the electric field and current density $\mathbf{J} = \mathbf{J}_+ + \mathbf{J}_-$ must be perpendicular to the electrodes, which we take as the x direction. If the electrode spacing is small compared to the width, the interelectrode fields far from the ends must then be x directed and be only a function of x. Then the time derivative of the charge conservation equation in (10) is

$$\frac{\partial}{\partial x}\left[\frac{\partial}{\partial t}(J_+ + J_-) + \varepsilon \frac{\partial^2 E}{\partial t^2}\right] = 0 \tag{38}$$

The bracketed term is just the time derivative of the total current density, which is zero because the electrodes are open circuited so that using (36) in (38) yields

$$\frac{\partial^2 E}{\partial t^2} + \omega_p^2 E = 0, \qquad \omega_p^2 = \omega_{p_+}^2 + \omega_{p_-}^2 \tag{39}$$

which has solutions

$$E = A_1 \sin \omega_p t + A_2 \cos \omega_p t \tag{40}$$

Any initial perturbation causes an oscillatory electric field at the composite plasma frequency ω_p. The charges then execute simple harmonic motion about their equilibrium position.

3-3 FIELD BOUNDARY CONDITIONS

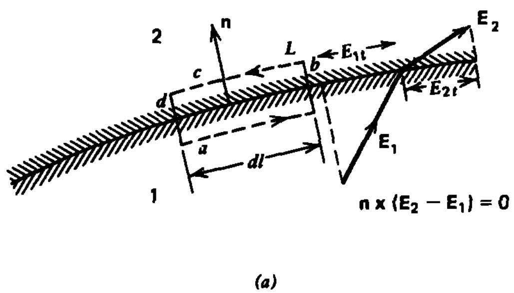
In many problems there is a surface of discontinuity separating dissimilar materials, such as between a conductor and a dielectric, or between different dielectrics. We must determine how the fields change as we cross the interface where the material properties change abruptly.

3-3-1 Tangential Component of E

We apply the line integral of the electric field around a contour of differential size enclosing the interface between dissimilar materials, as shown in Figure 3-12*a*. The long sections *a* and *c* of length **dl** are tangential to the surface and the short joining sections *b* and *d* are of zero length as the interface is assumed to have zero thickness. Applying the line integral of the electric field around this contour, from Section 2.5.6 we obtain

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = (E_{1t} - E_{2t})\, dl = 0 \tag{1}$$

where E_{1t} and E_{2t} are the components of the electric field tangential to the interface. We get no contribution from the normal components of field along sections *b* and *d* because the contour lengths are zero. The minus sign arises along *c* because the electric field is in the opposite direction of the contour traversal. We thus have that the tangential



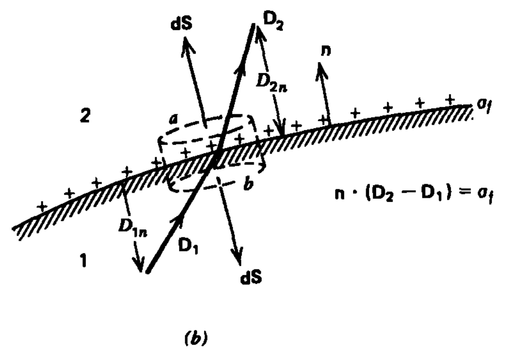


Figure 3-12 (*a*) Stokes' law applied to a line integral about an interface of discontinuity shows that the tangential component of electric field is continuous across the boundary. (b) Gauss's law applied to a pill-box volume straddling the interface shows that the normal component of displacement vector is discontinuous in the free surface charge density σ_f.

components of the electric field are continuous across the interface

$$E_{1t} = E_{2t} \Rightarrow \mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0 \tag{2}$$

where $\mathbf{n}$ is the interfacial normal shown in Figure 3-12*a*.

Within a perfect conductor the electric field is zero. Therefore, from (2) we know that the tangential component of $\mathbf{E}$ outside the conductor is also zero. Thus the electric field must always terminate perpendicularly to a perfect conductor.

3-3-2 Normal Component of D

We generalize the results of Section 2.4.6 to include dielectric media by again choosing a small Gaussian surface whose upper and lower surfaces of area dS are parallel to a surface charged interface and are joined by an infinitely thin cylindrical surface with zero area, as shown in Figure 3-12*b*. Then only faces *a* and *b* contribute to Gauss's law:

$$\oint_S \mathbf{D} \cdot \mathbf{dS} = (D_{2n} - D_{1n})\, dS = \sigma_f\, dS \tag{3}$$

where the interface supports a free surface charge density σ_f and D_{2n} and D_{1n} are the components of the displacement vector on either side of the interface in the direction of the normal $\mathbf{n}$ shown, pointing from region 1 to region 2. Reducing (3) and using more compact notation we have

$$D_{2n} - D_{1n} = \sigma_f, \quad \mathbf{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma_f \tag{4}$$

where the minus sign in front of $\mathbf{D}_1$ arises because the normal on the lower surface *b* is $-\mathbf{n}$. The normal components of the displacement vector are discontinuous if the interface has a surface charge density. If there is no surface charge ($\sigma_f = 0$), the normal components of $\mathbf{D}$ are continuous. If each medium has no polarization, (4) reduces to the free space results of Section 2.4.6.

At the interface between two different lossless dielectrics, there is usually no surface charge ($\sigma_f = 0$), unless it was deliberately placed, because with no conductivity there is no current to transport charge. Then, even though the normal component of the $\mathbf{D}$ field is continuous, the normal component of the electric field is discontinuous because the dielectric constant in each region is different.

At the interface between different conducting materials, free surface charge may exist as the current may transport charge to the surface discontinuity. Generally for such cases, the surface charge density is nonzero. In particular, if one region is a perfect conductor with zero internal electric field, the surface charge density on the surface is just equal to the normal component of **D** field at the conductor's surface,

$$\sigma_f = \mathbf{n} \cdot \mathbf{D} \tag{5}$$

where **n** is the outgoing normal from the perfect conductor.

3-3-3 Point Charge Above a Dielectric Boundary

If a point charge q within a region of permittivity ε_1 is a distance d above a planar boundary separating region I from region II with permittivity ε_2, as in Figure 3-13, the tangential component of **E** and in the absence of free surface charge the normal component of **D**, must be continuous across the interface. Let us try to use the method of images by placing an image charge q' at $y=-d$ so that the solution in region I is due to this image charge plus the original point charge q. The solution for the field in region II will be due to an image charge q'' at $y=d$, the position of the original point charge. Note that the appropriate image charge is always outside the region where the solution is desired. At this point we do not know if it is possible to satisfy the boundary conditions with these image charges; but we will try to find values of q' and q'' to do so.

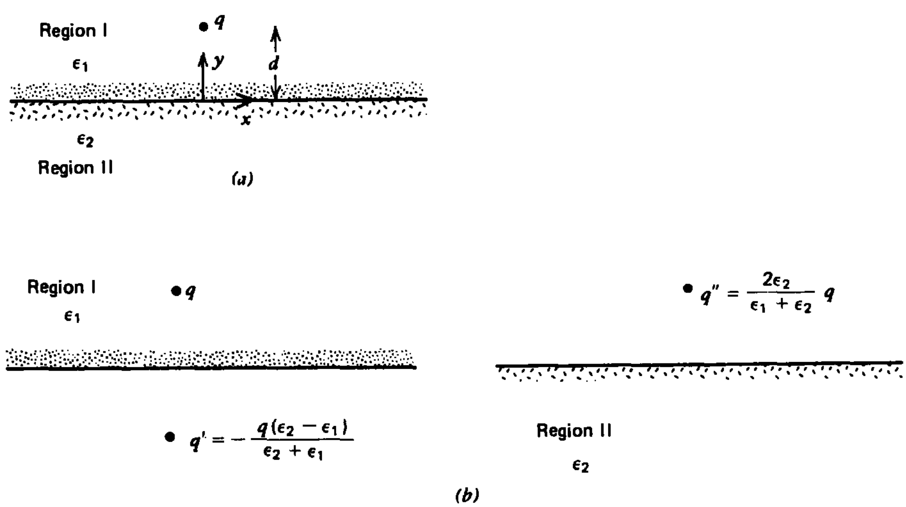


Figure 3-13 (*a*) A point charge q above a flat dielectric boundary requires different sets of image charges to solve for the fields in each region. (*b*) The field in region *I* is due to the original charge and the image charge q' while the field in region II is due only to image charge q''.

The potential in each region is

$$V_{\text{I}}=\frac{1}{4\pi\varepsilon_1}\left(\frac{q}{[x^2+(y-d)^2+z^2]^{1/2}}+\frac{q'}{[x^2+(y+d)^2+z^2]^{1/2}}\right),\quad y\geq 0$$

$$V_{\text{II}}=\frac{1}{4\pi\varepsilon_2}\frac{q''}{[x^2+(y-d)^2+z^2]^{1/2}},\quad y\leq 0 \tag{6}$$

with resultant electric field

$$E_{\text{I}}=-\nabla V_{\text{I}}$$

$$=\frac{1}{4\pi\varepsilon_1}\left(\frac{q[x\mathbf{i}_x+(y-d)\mathbf{i}_y+z\mathbf{i}_z]}{[x^2+(y-d)^2+z^2]^{3/2}}+\frac{q'[x\mathbf{i}_x+(y+d)\mathbf{i}_y+z\mathbf{i}_z]}{[x^2+(y+d)^2+z^2]^{3/2}}\right) \tag{7}$$

$$\mathbf{E}_{\text{II}}=-\nabla V_{\text{II}}=\frac{q''}{4\pi\varepsilon_2}\left(\frac{x\mathbf{i}_x+(y-d)\mathbf{i}_y+z\mathbf{i}_z}{[x^2+(y-d)^2+z^2]^{3/2}}\right)$$

To satisfy the continuity of tangential electric field at $y=0$ we have

$$\left.\begin{aligned}E_{x\text{I}}&=E_{x\text{II}}\\E_{z\text{I}}&=E_{z\text{II}}\end{aligned}\right\}\Rightarrow\frac{q+q'}{\varepsilon_1}=\frac{q''}{\varepsilon_2} \tag{8}$$

With no surface charge, the normal component of **D** must be continuous at $y=0$,

$$\varepsilon_1 E_{y\text{I}}=\varepsilon_2 E_{y\text{II}}\Rightarrow -q+q'=-q'' \tag{9}$$

Solving (8) and (9) for the unknown charges we find

$$q'=-\frac{(\varepsilon_2-\varepsilon_1)}{\varepsilon_1+\varepsilon_2}q$$

$$q''=\frac{2\varepsilon_2}{(\varepsilon_1+\varepsilon_2)}q \tag{10}$$

The force on the point charge q is due only to the field from image charge q':

$$\mathbf{f}=\frac{qq'}{4\pi\varepsilon_1(2d)^2}\mathbf{i}_y=-\frac{q^2(\varepsilon_2-\varepsilon_1)}{16\pi\varepsilon_1(\varepsilon_1+\varepsilon_2)d^2}\mathbf{i}_y \tag{11}$$

3-3-4 Normal Component of P and ε_0E

By integrating the flux of polarization over the same Gaussian pillbox surface, shown in Figure 3-12*b*, we relate the discontinuity in normal component of polarization to the surface polarization charge density σ_p using the relations from Section 3.1.2:

$$\oint_S \mathbf{P}\cdot d\mathbf{S} = -\int_S \sigma_p\, dS \Rightarrow P_{2n} - P_{1n} = -\sigma_p \Rightarrow \mathbf{n}\cdot(\mathbf{P}_2 - \mathbf{P}_1) = -\sigma_p \tag{12}$$

The minus sign in front of σ_p results because of the minus sign relating the volume polarization charge density to the divergence of **P**.

To summarize, polarization charge is the source of **P**, free charge is the source of **D**, and the total charge is the source of $\varepsilon_0\mathbf{E}$. Using (4) and (12), the electric field interfacial discontinuity is

$$\mathbf{n}\cdot(\mathbf{E}_2 - \mathbf{E}_1) = \frac{\mathbf{n}\cdot[(\mathbf{D}_2 - \mathbf{D}_1) - (\mathbf{P}_2 - \mathbf{P}_1)]}{\varepsilon_0} = \frac{\sigma_f + \sigma_p}{\varepsilon_0} \tag{13}$$

For linear dielectrics it is often convenient to lump polarization effects into the permittivity ε and never use the vector **P**, only **D** and **E**.

For permanently polarized materials, it is usually convenient to replace the polarization **P** by the equivalent polarization volume charge density and surface charge density of (12) and solve for **E** using the coulombic superposition integral of Section 2.3.2. In many dielectric problems, there is no volume polarization charge, but at surfaces of discontinuity a surface polarization charge is present as given by (12).

EXAMPLE 3-2 CYLINDER PERMANENTLY POLARIZED ALONG ITS AXIS

A cylinder of radius a and height L is centered about the z axis and has a uniform polarization along its axis, $\mathbf{P} = P_0\mathbf{i}_z$, as shown in Figure 3-14. Find the electric field **E** and displacement vector **D** everywhere on its axis.

SOLUTION

With a constant polarization **P**, the volume polarization charge density is zero:

$$\rho_p = -\nabla\cdot\mathbf{P} = 0$$

Since $\mathbf{P} = 0$ outside the cylinder, the normal component of **P** is discontinuous at the upper and lower surfaces yielding uniform surface polarization charges:

$$\sigma_p(z = L/2) = P_0, \quad \sigma_P(z = -L/2) = -P_0$$

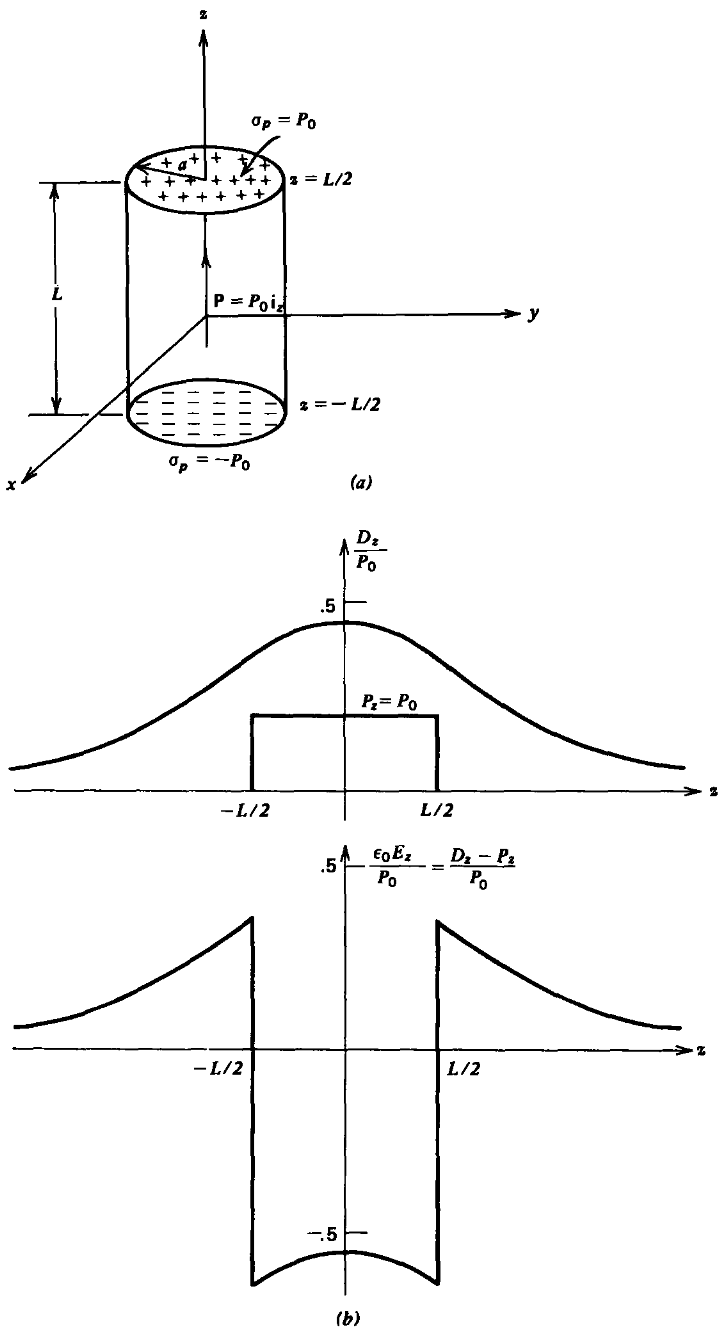


Figure 3-14 (*a*) The electric field due to a uniformly polarized cylinder of length L is the same as for two disks of surface charge of opposite polarity $\pm P_0$ at $z = L/2$. (*b*) The perpendicular displacement field D_z is continuous across the interfaces at $z = \pm L/2$ while the electric field E_z is discontinuous.

The solution for a single disk of surface charge was obtained in Section 2.3.5*b*. We superpose the results for the two disks taking care to shift the axial distance appropriately by $\pm L/2$ yielding the concise solution for the displacement field:

$$D_z = \frac{P_0}{2}\left(\frac{(z+L/2)}{[a^2+(z+L/2)^2]^{1/2}} - \frac{(z-L/2)}{[a^2+(z-L/2)^2]^{1/2}}\right)$$

The electric field is then

$$E_z = \begin{cases} D_z/\varepsilon_0, & |z| > L/2 \\ (D_z - P_0)/\varepsilon_0 & |z| < L/2 \end{cases}$$

These results can be examined in various limits. If the radius a becomes very large, the electric field should approach that of two parallel sheets of surface charge $\pm P_0$, as in Section 2.3.4*b*:

$$\lim_{a\to\infty} E_z = \begin{cases} 0, & |z| > L/2 \\ -P_0/\varepsilon_0, & |z| < L/2 \end{cases}$$

with a zero displacement field everywhere.

In the opposite limit, for large z ($z \gg a$, $z \gg L$) far from the cylinder, the axial electric field dies off as the dipole field with $\theta = 0$

$$\lim_{z\to\infty} E_z = \frac{p}{2\pi\varepsilon_0 z^3}, \quad p = P_0\pi a^2 L$$

with effective dipole moment p given by the product of the total polarization charge at $z = L/2$, $(P_0\pi a^2)$, and the length L.

3-3-5 Normal Component of J

Applying the conservation of total current equation in Section 3.2.1 to the same Gaussian pillbox surface in Figure 3-12*b* results in contributions again only from the upper and lower surfaces labeled "*a*" and "*b*":

$$\mathbf{n}\cdot\left(\mathbf{J}_2 - \mathbf{J}_1 + \frac{\partial}{\partial t}(\mathbf{D}_2 - \mathbf{D}_1)\right) = 0 \qquad (14)$$

where we assume that no surface currents flow along the interface. From (4), relating the surface charge density to the discontinuity in normal $\mathbf{D}$, this boundary condition can also be written as

$$\mathbf{n}\cdot(\mathbf{J}_2 - \mathbf{J}_1) + \frac{\partial \sigma_f}{\partial t} = 0 \qquad (15)$$

which tells us that if the current entering a surface is different from the current leaving, charge has accumulated at the interface. In the dc steady state the normal component of **J** is continuous across a boundary.

3-4 RESISTANCE

3-4-1 Resistance Between Two Electrodes

Two conductors maintained at a potential difference V within a conducting medium will each pass a total current I, as shown in Figure 3-15. By applying the surface integral form of charge conservation in Section 3.2.1 to a surface S' which surrounds both electrodes but is far enough away so that **J** and **D** are negligibly small, we see that the only nonzero current contributions are from the terminal wires that pass through the surface. These must sum to zero so that the

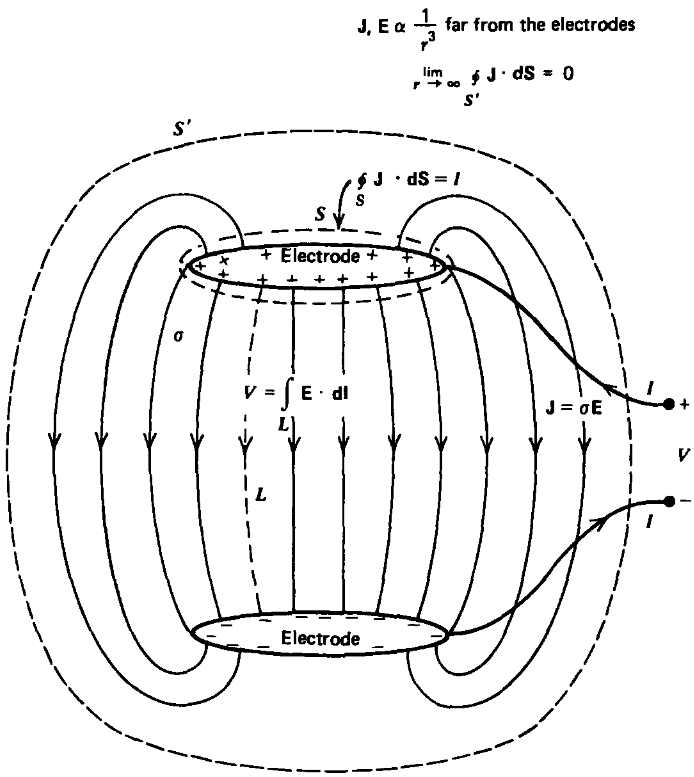


Figure 3-15 A voltage applied across two electrodes within an ohmic medium causes a current to flow into one electrode and out the other. The electrodes have equal magnitude but opposite polarity charges so that far away the fields die off as a dipole $\propto(1/r^3)$. Then, even though the surface S' is increasing as r^2, the flux of current goes to zero as $1/r$.

currents have equal magnitudes but flow in opposite directions. Similarly, applying charge conservation to a surface S just enclosing the upper electrode shows that the current I entering the electrode via the wire must just equal the total current (conduction plus displacement) leaving the electrode. This total current travels to the opposite electrode and leaves via the connecting wire.

The dc steady-state ratio of voltage to current between the two electrodes in Figure 3-15 is defined as the resistance:

$$R=\frac{V}{I}\text{ ohm }[\text{kg-m}^2\text{-s}^{-3}\text{-A}^{-2}] \tag{1}$$

For an arbitrary geometry, (1) can be expressed in terms of the fields as

$$R=\frac{\int_L \mathbf{E}\cdot d\mathbf{l}}{\oint_S \mathbf{J}\cdot d\mathbf{S}}=\frac{\int_L \mathbf{E}\cdot d\mathbf{l}}{\oint_S \sigma\mathbf{E}\cdot d\mathbf{S}} \tag{2}$$

where S is a surface completely surrounding an electrode and L is any path joining the two electrodes. Note that the field line integral is taken along the line from the high to low potential electrode so that the voltage difference V is equal to the positive line integral. From (2), we see that the resistance only depends on the geometry and conductivity σ and not on the magnitude of the electric field itself. If we were to increase the voltage by any factor, the field would also increase by this same factor everywhere so that this factor would cancel out in the ratio of (2). The conductivity σ may itself be a function of position.

3-4-2 Parallel Plate Resistor

Two perfectly conducting parallel plate electrodes of arbitrarily shaped area A and spacing l enclose a cylinder of material with Ohmic conductivity σ, as in Figure 3-16*a*. The current must flow tangential to the outer surface as the outside medium being free space has zero conductivity so that no current can pass through the interface. Because the tangential component of electric field is continuous, a field does exist in the free space region that decreases with increasing distance from the resistor. This three-dimensional field is difficult to calculate because it depends on three coordinates.

The electric field within the resistor is much simpler to calculate because it is perpendicular to the electrodes in the x direction. Gauss's law with no volume charge then tells us that

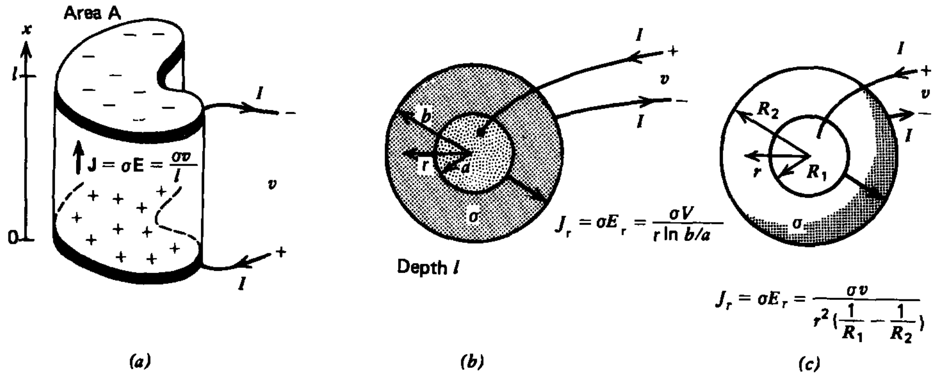


Figure 3-16 Simple resistor electrode geometries. (*a*) Parallel plates. (*b*) Coaxial cylinders. (*c*) Concentric spheres.

this field is constant:

$$\nabla \cdot (\varepsilon \mathbf{E}) = 0 \Rightarrow \frac{dE_x}{dx} = 0 \Rightarrow E_x = E_0 \tag{3}$$

However, the line integral of **E** between the electrodes must be the applied voltage v:

$$\int_0^l E_x\, dx = v \Rightarrow E_0 = v/l \tag{4}$$

The current density is then

$$\mathbf{J} = \sigma E_0 \mathbf{i}_x = (\sigma v/l)\mathbf{i}_x \tag{5}$$

so that the total current through the electrodes is

$$I = \oint_S \mathbf{J} \cdot \mathbf{dS} = (\sigma v/l)A \tag{6}$$

where the surface integral is reduced to a pure product because the constant current density is incident perpendicularly on the electrodes. The resistance is then

$$R = \frac{v}{I} = \frac{l}{\sigma A} = \frac{\text{spacing}}{\text{(conductivity) (electrode area)}} \tag{7}$$

Typical resistance values can vary over many orders of magnitude. If the electrodes have an area $A = 1\text{ cm}^2\ (10^{-4}\text{ m}^2)$ with spacing $l = 1\text{ mm}\ (10^{-3}\text{ m})$ a material like copper has a resistance $R \approx 0.17 \times 10^{-6}$ ohm while carbon would have a resistance $R \approx 1.4 \times 10^4$ ohm. Because of this large range of resistance values sub-units often used are micro-ohms $(1\ \mu\Omega = 10^{-6}\ \Omega)$, milli-ohms $(1\text{ m}\Omega = 10^{-3}\ \Omega)$, kilohm $(1\ k\Omega = 10^3\ \Omega)$, and megohms $(1\text{ M}\Omega = 10^6\ \Omega)$, where the symbol Ω is used to represent the unit of ohms.

Although the field outside the resistor is difficult to find, we do know that for distances far from the resistor the field approaches that of a point dipole due to the oppositely charged electrodes with charge density

$$\sigma_f(x=0) = -\sigma_f(x=l) = \varepsilon E_0 = \varepsilon v/l \tag{8}$$

and thus dipole moment

$$\mathbf{p} = -\sigma_f(x=0)Al\mathbf{i}_x = -\varepsilon A v \mathbf{i}_x \tag{9}$$

The minus sign arises because the dipole moment points from negative to positive charge. Note that (8) is only approximate because all of the external field lines in the free space region must terminate on the side and back of the electrodes giving further contributions to the surface charge density. Generally, if the electrode spacing l is much less than any of the electrode dimensions, this extra contribution is very small.

3-4-3 Coaxial Resistor

Two perfectly conducting coaxial cylinders of length l, inner radius a, and outer radius b are maintained at a potential difference v and enclose a material with Ohmic conductivity σ, as in Figure 3-16b. The electric field must then be perpendicular to the electrodes so that with no free charge Gauss's law requires

$$\nabla\cdot(\varepsilon\mathbf{E}) = 0 \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}(rE_r) = 0 \Rightarrow E_r = \frac{c}{r} \tag{10}$$

where c is an integration constant found from the voltage condition

$$\int_a^b E_r\,dr = c\ln r\Big|_a^b = v \Rightarrow c = \frac{v}{\ln(b/a)} \tag{11}$$

The current density is then

$$J_r = \sigma E_r = \frac{\sigma v}{r\ln(b/a)} \tag{12}$$

with the total current at any radius r being a constant

$$I = \int_{z=0}^{l}\int_{\phi=0}^{2\pi} J_r r\,d\phi\,dz = \frac{\sigma v 2\pi l}{\ln(b/a)} \tag{13}$$

so that the resistance is

$$R = \frac{v}{I} = \frac{\ln(b/a)}{2\pi\sigma l} \tag{14}$$

3-4-4 Spherical Resistor

We proceed in the same way for two perfectly conducting concentric spheres at a potential difference v with inner radius R_1 and outer radius R_2, as in Figure 3-16c. With no free charge, symmetry requires the electric field to be purely radial so that Gauss's law yields

$$\nabla\cdot(\varepsilon\mathbf{E})=0\Rightarrow\frac{1}{r^2}\frac{\partial}{\partial r}(r^2E_r)=0\Rightarrow E_r=\frac{c}{r^2} \tag{15}$$

where c is a constant found from the voltage condition as

$$\int_{R_1}^{R_2}E_r\,dr=-\frac{c}{r}\bigg|_{R_1}^{R_2}=v\Rightarrow c=\frac{v}{(1/R_1-1/R_2)} \tag{16}$$

The electric field and current density are inversely proportional to the square of the radius

$$J_r=\sigma E_r=\frac{\sigma v}{r^2(1/R_1-1/R_2)} \tag{17}$$

so that the current density is constant at any radius r

$$I=\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}J_r r^2\sin\theta\,d\theta\,d\phi=\frac{4\pi\sigma v}{(1/R_1-1/R_2)} \tag{18}$$

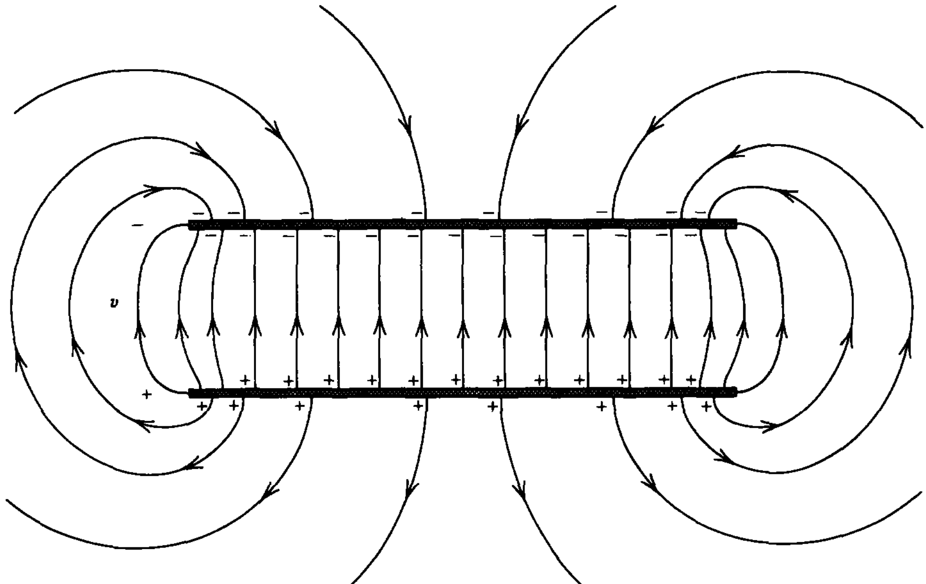
with resistance

$$\text{R}=\frac{v}{I}=\frac{(1/R_1-1/R_2)}{4\pi\sigma} \tag{19}$$

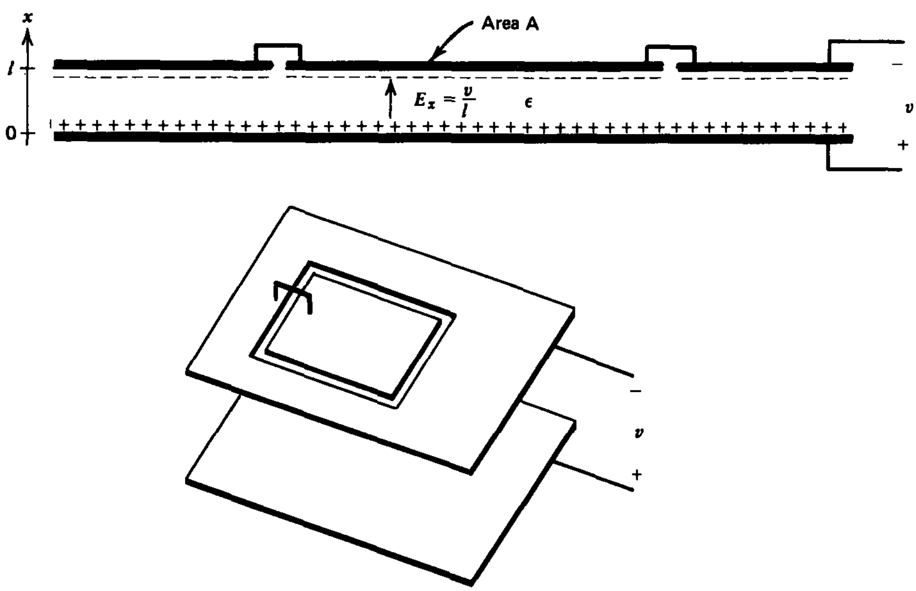
3-5 CAPACITANCE

3-5-1 Parallel Plate Electrodes

Parallel plate electrodes of finite size constrained to potential difference v enclose a dielectric medium with permittivity ε. The surface charge density does not distribute itself uniformly, as illustrated by the fringing field lines for infinitely thin parallel plate electrodes in Figure 3-17a. Near the edges the electric field is highly nonuniform decreasing in magnitude on the back side of the electrodes. Between the electrodes, far from the edges the electric field is uniform, being the same as if the electrodes were infinitely long. Fringing field effects can be made negligible if the electrode spacing l is much less than the depth d or width w. For more accurate work, end effects can be made even more negligible by using a guard ring encircling the upper electrode, as in Figure 3-17b. The guard ring is maintained at the same potential as the electrode, thus except for the very tiny gap, the field between



(a)



(b)

Figure 3-17 (*a*) Two infinitely thin parallel plate electrodes of finite area at potential difference v have highly nonuniform fields outside the interelectrode region. (*b*) A guard ring around one electrode removes end effects so that the field between the electrodes is uniform. The end effects now arise at the edge of the guard ring, which is far from the region of interest.

the electrodes is as if the end effects were very far away and not just near the electrode edges.

We often use the phrase "neglect fringing" to mean that the nonuniform field effects near corners and edges are negligible.

With the neglect of fringing field effects near the electrode ends, the electric field is perpendicular to the electrodes and related to the voltage as

$$\int_0^l E_x\,dx = v \Rightarrow E_x = v/l \tag{1}$$

The displacement vector is then proportional to the electric field terminating on each electrode with an equal magnitude but opposite polarity surface charge density given by

$$D_x = \varepsilon E_x = \sigma_f(x=0) = -\sigma_f(x=l) = \varepsilon v/l \tag{2}$$

The charge is positive where the voltage polarity is positive, and vice versa, with the electric field directed from the positive to negative electrode. The magnitude of total free charge on each electrode is

$$q_f = \sigma_f(x=0)A = \frac{\varepsilon A}{l}v \tag{3}$$

The capacitance C is defined as the magnitude of the ratio of total free charge on either electrode to the voltage difference between electrodes:

$$C = \frac{q_f}{v} = \frac{\varepsilon A}{l}$$

$$= \frac{\text{(permittivity) (electrode area)}}{\text{spacing}} \text{ farad } [\text{A}^2\text{-s}^4\text{-kg}^{-1}\text{-m}^{-2}] \tag{4}$$

Even though the system remains neutral, mobile electrons on the lower electrode are transported through the voltage source to the upper electrode in order to terminate the displacement field at the electrode surfaces, thus keeping the fields zero inside the conductors. Note that no charge is transported through free space. The charge transport between electrodes is due to work by the voltage source and results in energy stored in the electric field.

In SI units, typical capacitance values are very small. If the electrodes have an area of $A = 1\text{ cm}^2$ (10^{-4} m^2) with spacing of $l = 1$ mm (10^{-3} m), the free space capacitance is $C \approx 0.9 \times 10^{-12}$ farad. For this reason usual capacitance values are expressed in microfarads ($1\ \mu\text{f} = 10^{-6}$ farad), nanofarads ($1\text{ nf} = 10^{-9}$ farad), and picofarads ($1\text{ pf} = 10^{-12}$ farad).

With a linear dielectric of permittivity ε as in Figure 3-18a, the field of (1) remains unchanged for a given voltage but the charge on the electrodes and thus the capacitance increases with the permittivity, as given by (3). However, if the total free charge on each electrode were constrained, the voltage difference would decrease by the same factor.

These results arise because of the presence of polarization charges on the electrodes that partially cancel the free charge. The polarization vector within the dielectric-filled parallel plate capacitor is a constant

$$P_x = D_x - \varepsilon_0 E_x = (\varepsilon - \varepsilon_0)E_x = (\varepsilon - \varepsilon_0)v/l \tag{5}$$

so that the volume polarization charge density is zero. However, with zero polarization in the electrodes, there is a discontinuity in the normal component of polarization at the electrode surfaces. The boundary condition of Section 3.3.4 results in an equal magnitude but opposite polarity surface polarization charge density on each electrode, as illustrated in

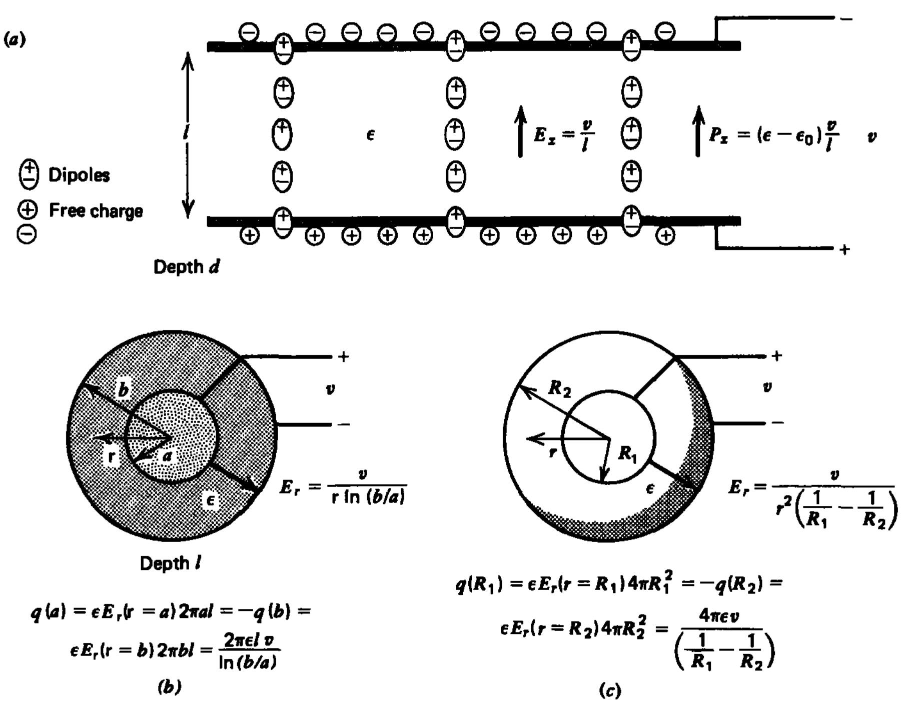


Figure 3-18 The presence of a dielectric between the electrodes increases the capacitance because for a given voltage additional free charge is needed on each electrode to overcome the partial neutralization of the attracted opposite polarity dipole ends. (a) Parallel plate electrodes. (b) Coaxial cylinders. (c) Concentric spheres.

Figure 3-18*a*:

$$\sigma_p(x=0) = -\sigma_p(x=l) = -P_x = -(\varepsilon - \varepsilon_0)v/l \tag{6}$$

Note that negative polarization charge appears on the positive polarity electrode and vice versa. This is because opposite charges attract so that the oppositely charged ends of the dipoles line up along the electrode surface partially neutralizing the free charge.

3-5-2 Capacitance for any Geometry

We have based our discussion around a parallel plate capacitor. Similar results hold for any shape electrodes in a dielectric medium with the capacitance defined as the magnitude of the ratio of total free charge on an electrode to potential difference. The capacitance is always positive by definition and for linear dielectrics is only a function of the geometry and dielectric permittivity and not on the voltage levels,

$$C = \frac{q_f}{v} = \frac{\oint_S \mathbf{D}\cdot d\mathbf{S}}{\int_L \mathbf{E}\cdot d\mathbf{l}} = \frac{\oint_S \varepsilon \mathbf{E}\cdot d\mathbf{S}}{\int_L \mathbf{E}\cdot d\mathbf{l}} \tag{7}$$

as multiplying the voltage by a constant factor also increases the electric field by the same factor so that the ratio remains unchanged.

The integrals in (7) are similar to those in Section 3.4.1 for an Ohmic conductor. For the same geometry filled with a homogenous Ohmic conductor or a linear dielectric, the resistance-capacitance product is a constant independent of the geometry:

$$RC = \frac{\int_L \mathbf{E}\cdot d\mathbf{l}}{\sigma \oint_S \mathbf{E}\cdot d\mathbf{S}} \frac{\varepsilon \oint_S \mathbf{E}\cdot d\mathbf{S}}{\int_L \mathbf{E}\cdot d\mathbf{l}} = \frac{\varepsilon}{\sigma} \tag{8}$$

Thus, for a given geometry, if either the resistance or capacitance is known, the other quantity is known immediately from (8). We can thus immediately write down the capacitance of the geometries shown in Figure 3-18 assuming the medium between electrodes is a linear dielectric with permittivity ε using the results of Sections 3.4.2–3.4.4:

$$\begin{aligned}
&\text{Parallel Plate} \quad R = \frac{l}{\sigma A} \Rightarrow C = \frac{\varepsilon A}{l} \\
&\text{Coaxial} \quad R = \frac{\ln (b/a)}{2\pi\sigma l} \Rightarrow C = \frac{2\pi\varepsilon l}{\ln (b/a)} \\
&\text{Spherical} \quad R = \frac{1/R_1 - 1/R_2}{4\pi\sigma} \Rightarrow C = \frac{4\pi\varepsilon}{(1/R_1 - 1/R_2)}
\end{aligned} \tag{9}$$

3-5-3 Current Flow Through a Capacitor

From the definition of capacitance in (7), the current to an electrode is

$$i=\frac{dq_f}{dt}=\frac{d}{dt}(Cv)=C\frac{dv}{dt}+v\frac{dC}{dt} \tag{10}$$

where the last term only arises if the geometry or dielectric permittivity changes with time. For most circuit applications, the capacitance is independent of time and (10) reduces to the usual voltage-current circuit relation.

In the capacitor of arbitrary geometry, shown in Figure 3-19, a conduction current i flows through the wires into the upper electrode and out of the lower electrode changing the amount of charge on each electrode, as given by (10). There is no conduction current flowing in the dielectric between the electrodes. As discussed in Section 3.2.1 the total current, displacement plus conduction, is continuous. Between the electrodes in a lossless capacitor, this current is entirely displacement current. The displacement field is itself related to the time-varying surface charge distribution on each electrode as given by the boundary condition of Section 3.3.2.

3-5-4 Capacitance of Two Contacting Spheres

If the outer radius R_2 of the spherical capacitor in (9) is put at infinity, we have the capacitance of an isolated sphere of radius R as

$$C=4\pi\varepsilon R \tag{11}$$

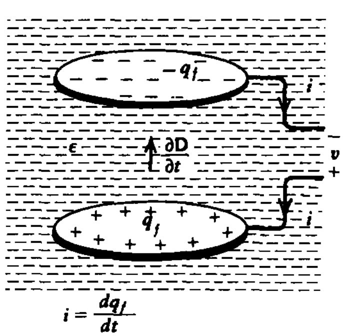


Figure 3-19 The conduction current i that travels through the connecting wire to an electrode in a lossless capacitor is transmitted through the dielectric medium to the opposite electrode via displacement current. No charge carriers travel through the lossless dielectric.

If the surrounding medium is free space ($\varepsilon = \varepsilon_0$) for $R = 1$ m, we have that $C \approx \frac{1}{9} \times 10^{-9}$ farad ≈ 111 pf.

We wish to find the self-capacitance of two such contacting spheres raised to a potential V_0, as shown in Figure 3-20. The capacitance is found by first finding the total charge on the two spheres. We can use the method of images by first placing an image charge $q_1 = Q = 4\pi\varepsilon R V_0$ at the center of each sphere to bring each surface to potential V_0. However, each of these charges will induce an image charge q_2 in the other sphere at distance b_2 from the center,

$$q_2 = -\frac{Q}{2}, \qquad b_2 = \frac{R^2}{D} = \frac{R}{2} \tag{12}$$

where we realize that the distance from inducing charge to the opposite sphere center is $D = 2R$. This image charge does not raise the potential of either sphere. Similarly, each of these image charges induces another image charge q_3 in the other sphere at disance b_3,

$$q_3 = -\frac{q_2 R}{D - b_2} = \frac{Q}{3}, \qquad b_3 = \frac{R^2}{D - b_2} = \frac{2}{3}R \tag{13}$$

which will induce a further image charge q_4, ad infinitum. An infinite number of image charges will be necessary, but with the use of difference equations we will be able to add all the image charges to find the total charge and thus the capacitance.

The nth image charge q_n and its distance from the center b_n are related to the $(n-1)$th images as

$$q_n = -\frac{q_{n-1} R}{D - b_{n-1}}, \qquad b_n = \frac{R^2}{D - b_{n-1}} \tag{14}$$

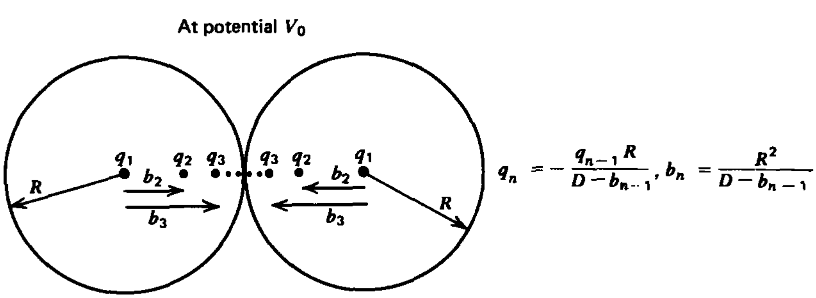


Figure 3-20 Two identical contacting spheres raised to a potential V_0 with respect to infinity are each described by an infinite number of image charges q_n each a distance b_n from the sphere center.

where $D=2R$. We solve the first relation for b_{n-1} as

$$D-b_{n-1}=-\frac{q_{n-1}}{q_n}R$$
$$b_n=\frac{q_n}{q_{n+1}}R+D \tag{15}$$

where the second relation is found by incrementing n in the first relation by 1. Substituting (15) into the second relation of (14) gives us a single equation in the q_n's:

$$\frac{q_nR}{q_{n+1}}+D=-\frac{Rq_n}{q_{n-1}}\Rightarrow\frac{1}{q_{n+1}}+\frac{2}{q_n}+\frac{1}{q_{n-1}}=0 \tag{16}$$

If we define the reciprocal charges as

$$p_n=1/q_n \tag{17}$$

then (16) becomes a homogeneous linear constant coefficient difference equation

$$p_{n+1}+2p_n+p_{n-1}=0 \tag{18}$$

Just as linear constant coefficient differential equations have exponential solutions, (18) has power law solutions of the form

$$p_n=A\lambda^n \tag{19}$$

where the characteristic roots λ, analogous to characteristic frequencies, are found by substitution back into (18),

$$\lambda^{n+1}+2\lambda^n+\lambda^{n-1}=0\Rightarrow\lambda^2+2\lambda+1=(\lambda+1)^2=0 \tag{20}$$

to yield a double root with $\lambda=-1$. Because of the double root, the superposition of both solutions is of the form

$$p_n=A_1(-1)^n+A_2n(-1)^n \tag{21}$$

similar to the behavior found in differential equations with double characteristic frequencies. The correctness of (21) can be verified by direct substitution back into (18). The constants A_1 and A_2 are determined from q_1 and q_2 as

$$\left.\begin{aligned}p_1&=1/Q=-A_1-A_2\\ p_2&=\frac{1}{q_2}=-\frac{2}{Q}=+A_1+2A_2\end{aligned}\right\}\Rightarrow\left\{\begin{aligned}A_1&=0\\ A_2&=-\frac{1}{Q}\end{aligned}\right. \tag{22}$$

so that the nth image charge is

$$q_n=\frac{1}{p_n}=\frac{1}{-(-1)^n n/Q}=\frac{-(-1)^nQ}{n} \tag{23}$$

The capacitance is then given as the ratio of the total charge on the two spheres to the voltage,

$$C=\frac{2\sum_{n=0}^{\infty}q_n}{V_0}=-\frac{2Q}{V_0}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}=+\frac{2Q}{V_0}[1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\cdots]$$
$$=8\pi\varepsilon R\ln 2 \tag{24}$$

where we recognize the infinite series to be the Taylor series expansion of $\ln(1+x)$ with $x=1$. The capacitance of two contacting spheres is thus $2\ln 2\approx 1.39$ times the capacitance of a single sphere given by (11).

The distance from the center to each image charge is obtained from (23) substituted into (15) as

$$b_n=\left(\frac{(-1)^n\,(n+1)}{n(-1)^{n+1}}+2\right)R=\frac{(n-1)}{n}R \tag{25}$$

We find the force of attraction between the spheres by taking the sum of the forces on each image charge on one of the spheres due to all the image charges on the other sphere. The force on the nth image charge on one sphere due to the mth image charge in the other sphere is

$$f_{nm}=\frac{-q_nq_m}{4\pi\varepsilon[2R-b_n-b_m]^2}=\frac{-Q_0^2(-1)^{n+m}}{4\pi\varepsilon R^2}\frac{nm}{(m+n)^2} \tag{26}$$

where we used (23) and (25). The total force on the left sphere is then found by summing over all values of m and n,

$$f=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}f_{nm}=\frac{-Q_0^2}{4\pi\varepsilon R^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+m}nm}{(n+m)^2}$$
$$=\frac{-Q_0^2}{4\pi\varepsilon R^2}\frac{1}{6}[\ln 2-\tfrac{1}{4}] \tag{27}$$

where the double series can be explicitly expressed.* The force is negative because the like charge spheres repel each other. If $Q_0=1$ coul with $R=1$ m, in free space this force is $f\approx 6.6\times 10^8$ nt, which can lift a mass in the earth's gravity field of 6.8×10^7 kg ($\approx 3\times 10^7$ lb).

3-6 LOSSY MEDIA

Many materials are described by both a constant permittivity ε and constant Ohmic conductivity σ. When such a material is placed between electrodes do we have a capacitor

* *See Albert D. Wheelon,* Tables of Summable Series and Integrals Involving Bessel Functions, *Holden Day, (1968) pp. 55, 56.*

or a resistor? We write the governing equations of charge conservation and Gauss's law with linear constitutive laws:

$$\nabla \cdot \mathbf{J}_f + \frac{\partial \rho_f}{\partial t} = 0, \quad \mathbf{J}_f = \sigma \mathbf{E} + \rho_f \mathbf{U} \tag{1}$$

$$\nabla \cdot \mathbf{D} = \rho_f, \quad \mathbf{D} = \varepsilon \mathbf{E} \tag{2}$$

We have generalized Ohm's law in (1) to include convection currents if the material moves with velocity $\mathbf{U}$. In addition to the conduction charges, any free charges riding with the material also contribute to the current. Using (2) in (1) yields a single partial differential equation in ρ_f:

$$\sigma \underbrace{(\nabla \cdot \mathbf{E})}_{\rho_f/\varepsilon} + \nabla \cdot (\rho_f \mathbf{U}) + \frac{\partial \rho_f}{\partial t} = 0 \Rightarrow \frac{\partial \rho_f}{\partial t} + \nabla \cdot (\rho_f \mathbf{U}) + \frac{\sigma}{\varepsilon} \rho_f = 0 \tag{3}$$

3-6-1 Transient Charge Relaxation

Let us first assume that the medium is stationary so that $\mathbf{U} = 0$. Then the solution to (3) for any initial possibly spatially varying charge distribution $\rho_0(x, y, z, t = 0)$ is

$$\rho_f = \rho_0(x, y, z, t = 0)\, e^{-t/\tau}, \quad \tau = \varepsilon/\sigma \tag{4}$$

where τ is the relaxation time. This solution is the continuum version of the resistance-capacitance (RC) decay time in circuits.

The solution of (4) tells us that at all positions within a conductor, any initial charge density dies off exponentially with time. It does not spread out in space. This is our justification of not considering any net volume charge in conducting media. If a system has no volume charge at $t = 0$ ($\rho_0 = 0$), it remains uncharged for all further time. Charge is transported through the region by the Ohmic current, but the net charge remains zero. Even if there happens to be an initial volume charge distribution, for times much longer than the relaxation time the volume charge density becomes negligibly small. In metals, τ is on the order of 10^{-19} sec, which is the justification of assuming the fields are zero within an electrode. Even though their large conductivity is not infinite, for times longer than the relaxation time τ, the field solutions are the same as if a conductor were perfectly conducting.

The question remains as to where the relaxed charge goes. The answer is that it is carried by the conduction current to surfaces of discontinuity where the conductivity abruptly changes.

3-6-2 Uniformly Charged Sphere

A sphere of radius R_2 with constant permittivity ε and Ohmic conductivity σ is uniformly charged up to the radius R_1 with charge density ρ_0 at time $t=0$, as in Figure 3-21. From R_1 to R_2 the sphere is initially uncharged so that it remains uncharged for all time. The sphere is surrounded by free space with permittivity ε_0 and zero conductivity.

From (4) we can immediately write down the volume charge distribution for all time,

$$\rho_f = \begin{cases} \rho_0 e^{-t/\tau}, & r < R_1 \\ 0, & r > R_1 \end{cases} \tag{5}$$

where $\tau = \varepsilon/\sigma$. The total charge on the sphere remains constant, $Q = \frac{4}{3}\pi R_1^3 \rho_0$, but the volume charge is transported by the Ohmic current to the interface at $r = R_2$ where it becomes a surface charge. Enclosing the system by a Gaussian surface with $r > R_2$ shows that the external electric field is time independent,

$$E_r = \frac{Q}{4\pi\varepsilon_0 r^2}, \quad r > R_2 \tag{6}$$

Similarly, applying Gaussian surfaces for $r < R_1$ and $R_1 < r < R_2$ yields

$$E_r = \begin{cases} \dfrac{\rho_0 r e^{-t/\tau}}{3\varepsilon} = \dfrac{Q r e^{-t/\tau}}{4\pi\varepsilon R_1^3}, & 0 < r < R_1 \\ \dfrac{Q e^{-t/\tau}}{4\pi\varepsilon r^2}, & R_1 < r < R_2 \end{cases} \tag{7}$$

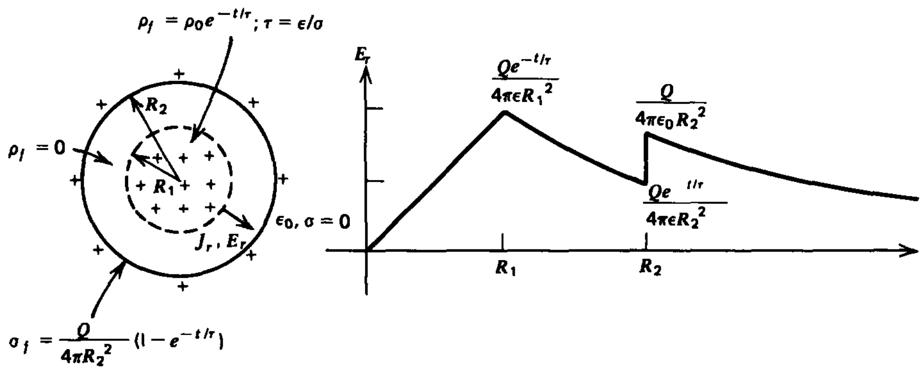


Figure 3-21 An initial volume charge distribution within an Ohmic conductor decays exponentially towards zero with relaxation time $\tau = \varepsilon/\sigma$ and appears as a surface charge at an interface of discontinuity. Initially uncharged regions are always uncharged with the charge transported through by the current.

The surface charge density at $r=R_2$ builds up exponentially with time:

$$\sigma_f(r=R_2)=\varepsilon_0 E_r(r=R_{2+})-\varepsilon E_r(r=R_{2-})$$
$$=\frac{Q}{4\pi R_2^2}(1-e^{-t/\tau}) \tag{8}$$

The charge is carried from the charged region ($r<R_1$) to the surface at $r=R_2$ via the conduction current with the charge density inbetween ($R_1<r<R_2$) remaining zero:

$$J_c=\sigma E_r=\begin{cases}\dfrac{\sigma Q r}{4\pi\varepsilon R_1^3}e^{-t/\tau}, & 0<r<R_1\\ \dfrac{\sigma Q\, e^{-t/\tau}}{4\pi\varepsilon r^2}, & R_1<r<R_2\\ 0, & r>R_2\end{cases} \tag{9}$$

Note that the total current, conduction plus displacement, is zero everywhere:

$$-J_c=J_d=\varepsilon\frac{\partial E_r}{\partial t}=\begin{cases}-\dfrac{Qr\sigma\, e^{-t/\tau}}{4\pi\varepsilon R_1^3}, & 0<r<R_1\\ -\dfrac{\sigma Q\, e^{-t/\tau}}{4\pi\varepsilon r^2}, & R_1<r<R_2\\ 0, & r>R_2\end{cases} \tag{10}$$

3-6-3 Series Lossy Capacitor

(a) Charging transient

To exemplify the difference between resistive and capacitive behavior we examine the case of two different materials in series stressed by a step voltage first turned on at $t=0$, as shown in Figure 3-22*a*. Since it takes time to charge up the interface, the interfacial surface charge cannot instantaneously change at $t=0$ so that it remains zero at $t=0_+$. With no surface charge density, the displacement field is continuous across the interface so that the solution at $t=0_+$ is the same as for two lossless series capacitors independent of the conductivities:

$$D_x=\varepsilon_1 E_1=\varepsilon_2 E_2 \tag{11}$$

The voltage constraint requires that

$$\int_0^{a+b} E_x\,dx=E_1 a+E_2 b=V \tag{12}$$

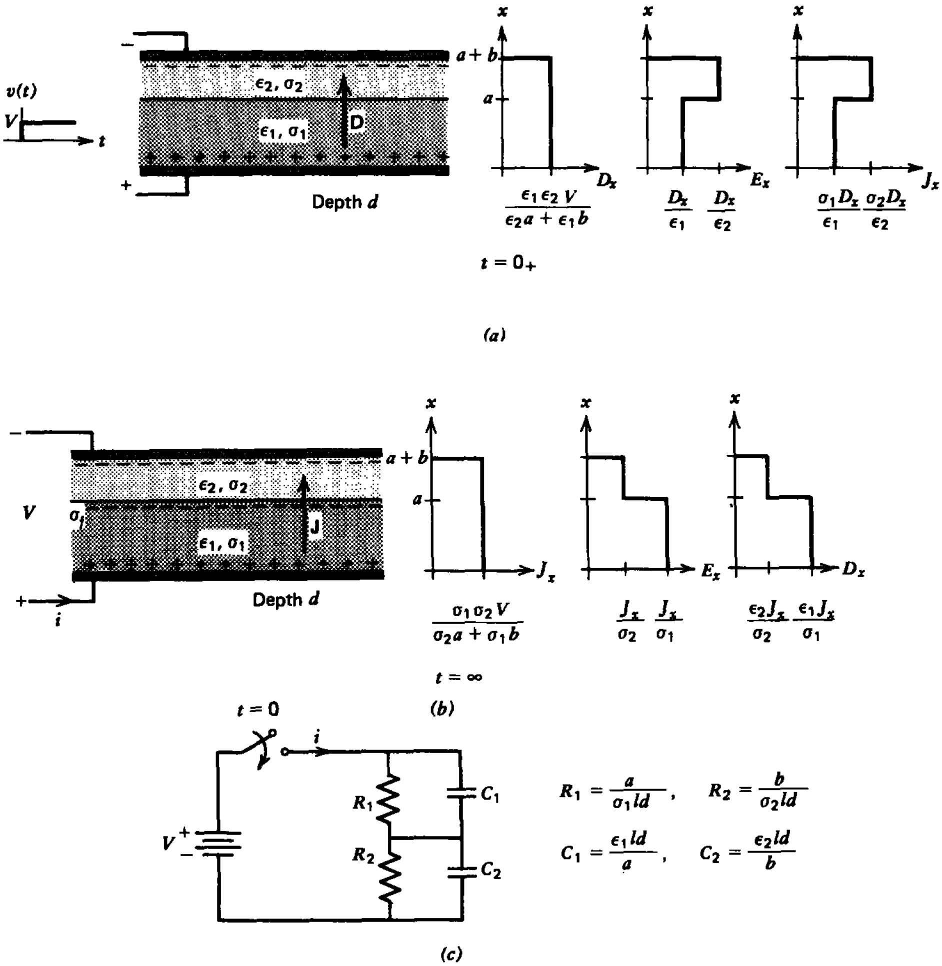


Figure 3-22 Two different lossy dielectric materials in series between parallel plate electrodes have permittivities and Ohmic conductivities that change abruptly across the interface. (*a*) At $t = 0_+$, right after a step voltage is applied, the interface is uncharged so that the displacement field is continuous with the solution the same as for two lossless dielectrics in series. (b) Since the current is discontinuous across the boundary between the materials, the interface will charge up. In the dc steady state the current is continuous. (c) Each region is equivalent to a resistor and capacitor in parallel.

so that the displacement field is

$$D_x(t = 0_+) = \frac{\varepsilon_1\varepsilon_2 V}{\varepsilon_2 a + \varepsilon_1 b} \tag{13}$$

The total current from the battery is due to both conduction and displacement currents. At $t = 0$, the displacement current is infinite (an impulse) as the displacement field instantaneously changes from zero to (13) to produce the surface charge on each electrode:

$$\sigma_f(x=0) = -\sigma_f(x=a+b) = D_x \tag{14}$$

After the voltage has been on a long time, the fields approach their steady-state values, as in Figure 3-22*b*. Because there are no more time variations, the current density must be continuous across the interface just the same as for two series resistors independent of the permittivities,

$$J_x(t\to\infty) = \sigma_1 E_1 = \sigma_2 E_2 = \frac{\sigma_1\sigma_2 V}{\sigma_2 a + \sigma_1 b} \tag{15}$$

where we again used (12). The interfacial surface charge is now

$$\sigma_f(x=a) = \varepsilon_2 E_2 - \varepsilon_1 E_1 = \frac{(\varepsilon_2\sigma_1 - \varepsilon_1\sigma_2)V}{\sigma_2 a + \sigma_1 b} \tag{16}$$

What we have shown is that for early times the system is purely capacitive, while for long times the system is purely resistive. The inbetween transient interval is found by using (12) with charge conservation applied at the interface:

$$\mathbf{n}\cdot\left(\mathbf{J}_2 - \mathbf{J}_1 + \frac{d}{dt}(\mathbf{D}_2 - \mathbf{D}_1)\right) = 0$$

$$\Rightarrow \sigma_2 E_2 - \sigma_1 E_1 + \frac{d}{dt}[\varepsilon_2 E_2 - \varepsilon_1 E_1] = 0 \tag{17}$$

With (12) to relate E_2 to E_1 we obtain a single ordinary differential equation in E_1,

$$\frac{dE_1}{dt} + \frac{E_1}{\tau} = \frac{\sigma_2 V}{\varepsilon_2 a + \varepsilon_1 b} \tag{18}$$

where the relaxation time is a weighted average of relaxation times of each material:

$$\tau = \frac{\varepsilon_1 b + \varepsilon_2 a}{\sigma_1 b + \sigma_2 a} \tag{19}$$

Using the initial condition of (13) the solutions for the fields are

$$\begin{aligned} E_1 &= \frac{\sigma_2 V}{\sigma_2 a + \sigma_1 b}(1 - e^{-t/\tau}) + \frac{\varepsilon_2 V}{\varepsilon_2 a + \varepsilon_1 b} e^{-t/\tau} \\ E_2 &= \frac{\sigma_1 V}{\sigma_2 a + \sigma_1 b}(1 - e^{-t/\tau}) + \frac{\varepsilon_1 V}{\varepsilon_2 a + \varepsilon_1 b} e^{-t/\tau} \end{aligned} \tag{20}$$

Note that as $t\to\infty$ the solutions approach those of (15). The interfacial surface charge is

$$\sigma_f(x=a)=\varepsilon_2E_2-\varepsilon_1E_1=\frac{(\varepsilon_2\sigma_1-\varepsilon_1\sigma_2)}{\sigma_2a+\sigma_1b}(1-e^{-t/\tau})V \quad (21)$$

which is zero at $t=0$ and agrees with (16) for $t\to\infty$.

The total current delivered by the voltage source is

$$i=\left(\sigma_1E_1+\varepsilon_1\frac{dE_1}{dt}\right)ld=\left(\sigma_2E_2+\varepsilon_2\frac{dE_2}{dt}\right)ld$$

$$=\left[\frac{\sigma_1\sigma_2}{\sigma_2a+\sigma_1b}+\left(\sigma_1-\frac{\varepsilon_1}{\tau}\right)\left(\frac{\varepsilon_2}{\varepsilon_2a+\varepsilon_1b}-\frac{\sigma_2}{\sigma_2a+\sigma_1b}\right)e^{-t/\tau}\right.$$

$$\left.+\frac{\varepsilon_1\varepsilon_2}{\varepsilon_2a+\varepsilon_1b}\delta(t)\right]ldV \quad (22)$$

where the last term is the impulse current that instantaneously puts charge on each electrode in zero time at $t=0$:

$$\delta(t)=\begin{cases}0, & t\neq 0\\ \infty, & t=0\end{cases}\Rightarrow\int_{0_-}^{0_+}\delta(t)\,dt=1$$

To reiterate, we see that for early times the capacitances dominate and that in the steady state the resistances dominate with the transition time depending on the relaxation times and geometry of each region. The equivalent circuit for the system is shown in Figure 3-22*c* as a series combination of a parallel resistor-capacitor for each region.

(b) Open Circuit

Once the system is in the dc steady state, we instantaneously open the circuit so that the terminal current is zero. Then, using (22) with $i=0$, we see that the fields decay independently in each region with the relaxation time of each region:

$$\begin{aligned}E_1&=\frac{\sigma_2V}{\sigma_2a+\sigma_1b}e^{-t/\tau_1}, \quad \tau_1=\frac{\varepsilon_1}{\sigma_1}\\ E_2&=\frac{\sigma_1V}{\sigma_2a+\sigma_1b}e^{-t/\tau_2}, \quad \tau_2=\frac{\varepsilon_2}{\sigma_2}\end{aligned} \quad (23)$$

The open circuit voltage and interfacial charge then decay as

$$\begin{aligned}V_{oc}&=E_1a+E_2b=\frac{V}{\sigma_2a+\sigma_1b}[\sigma_2a\,e^{-t/\tau_1}+\sigma_1b\,e^{-t/\tau_2}]\\ \sigma_f&=\varepsilon_2E_2-\varepsilon_1E_1=\frac{V}{\sigma_2a+\sigma_1b}[\varepsilon_2\sigma_1\,e^{-t/\tau_2}-\varepsilon_1\sigma_2\,e^{-t/\tau_1}]\end{aligned} \quad (24)$$

(c) Short Circuit

If the dc steady-state system is instead short circuited, we set $V=0$ in (12) and (18),

$$\begin{gathered} E_1 a + E_2 b = 0 \\ \frac{dE_1}{dt} + \frac{E_1}{\tau} = 0 \end{gathered} \tag{25}$$

where τ is still given by (19). Since at $t=0$ the interfacial surface charge cannot instantaneously change, the initial fields must obey the relation

$$\lim_{t=0} (\varepsilon_2 E_2 - \varepsilon_1 E_1) = -\left(\frac{\varepsilon_2 a}{b} + \varepsilon_1\right) E_1 = \frac{(\varepsilon_2 \sigma_1 - \varepsilon_1 \sigma_2) V}{\sigma_2 a + \sigma_1 b} \tag{26}$$

to yield the solutions

$$E_1 = -\frac{E_2 b}{a} = -\frac{(\varepsilon_2 \sigma_1 - \varepsilon_1 \sigma_2) b V}{(\varepsilon_2 a + \varepsilon_1 b)(\sigma_2 a + \sigma_1 b)} e^{-t/\tau} \tag{27}$$

The short circuit current and surface charge are then

$$\begin{gathered} i = \left[\left(\frac{\sigma_1 \varepsilon_2 - \varepsilon_1 \sigma_2}{\varepsilon_1 b + \varepsilon_2 a} \right)^2 \frac{abV}{(\sigma_2 a + \sigma_1 b)} e^{-t/\tau} - \frac{V \varepsilon_1 \varepsilon_2}{\varepsilon_2 a + \varepsilon_1 b} \delta(t) \right] ld \\ \sigma_f = \varepsilon_2 E_2 - \varepsilon_1 E_1 = \frac{(\varepsilon_2 \sigma_1 - \varepsilon_1 \sigma_2)}{\sigma_2 a + \sigma_1 b} V e^{-t/\tau} \end{gathered} \tag{28}$$

The impulse term in the current is due to the instantaneous change in displacement field from the steady-state values found from (15) to the initial values of (26).

(d) Sinusoidal Steady State

Now rather than a step voltage, we assume that the applied voltage is sinusoidal,

$$v(t) = V_0 \cos \omega t \tag{29}$$

and has been on a long time.

The fields in each region are still only functions of time and not position. It is convenient to use complex notation so that all quantities are written in the form

$$\begin{gathered} v(t) = \mathrm{Re}\,(V_0 e^{j\omega t}) \\ E_1(t) = \mathrm{Re}\,(\hat{E}_1 e^{j\omega t}), \quad E_2(t) = \mathrm{Re}\,(\hat{E}_2 e^{j\omega t}) \end{gathered} \tag{30}$$

Using carets above a term to designate a complex amplitude, the applied voltage condition of (12) requires

$$\hat{E}_1 a + \hat{E}_2 b = V_0 \tag{31}$$

while the interfacial charge conservation equation of (17) becomes

$$\sigma_2\hat{E}_2-\sigma_1\hat{E}_1+j\omega\,(\varepsilon_2\hat{E}_2-\varepsilon_1\hat{E}_1)=[\sigma_2+j\omega\varepsilon_2]\hat{E}_2 -[\sigma_1+j\omega\varepsilon_1]\hat{E}_1=0 \tag{32}$$

The solutions are

$$\frac{\hat{E}_1}{(j\omega\varepsilon_2+\sigma_2)}=\frac{\hat{E}_2}{(j\omega\varepsilon_1+\sigma_1)}=\frac{V_0}{[b(\sigma_1+j\omega\varepsilon_1)+a(\sigma_2+j\omega\varepsilon_2)]} \tag{33}$$

which gives the interfacial surface charge amplitude as

$$\hat{\sigma}_f=\varepsilon_2\hat{E}_2-\varepsilon_1\hat{E}_1=\frac{(\varepsilon_2\sigma_1-\varepsilon_1\sigma_2)V_0}{[b(\sigma_1+j\omega\varepsilon_1)+a(\sigma_2+j\omega\varepsilon_2)]} \tag{34}$$

As the frequency becomes much larger than the reciprocal relaxation times,

$$\omega \gg \frac{\sigma_1}{\varepsilon_1}, \quad \omega \gg \frac{\sigma_2}{\varepsilon_2} \tag{35}$$

the surface charge density goes to zero. This is because the surface charge cannot keep pace with the high-frequency alternations, and thus the capacitive component dominates. Thus, in experimental work charge accumulations can be prevented if the excitation frequencies are much faster than the reciprocal charge relaxation times.

The total current through the electrodes is

$$\begin{aligned}\hat{I}&=(\sigma_1+j\omega\varepsilon_1)\hat{E}_1 ld=(\sigma_2+j\omega\varepsilon_2)\hat{E}_2 ld\\ &=\frac{ld(\sigma_1+j\omega\varepsilon_1)(\sigma_2+j\omega\varepsilon_2)V_0}{[b(\sigma_1+j\omega\varepsilon_1)+a(\sigma_2+j\omega\varepsilon_2)]}\\ &=\frac{V_0}{\dfrac{R_2}{R_2C_2j\omega+1}+\dfrac{R_1}{R_1C_1j\omega+1}}\end{aligned} \tag{36}$$

with the last result easily obtained from the equivalent circuit in Figure 3-22*c*.

3-6-4 Distributed Systems

(a) Governing Equations

In all our discussions we have assumed that the electrodes are perfectly conducting so that they have no resistance and the electric field terminates perpendicularly. Consider now the parallel plate geometry shown in Figure 3-23*a*, where the electrodes have a large but finite conductivity σ_c. The electrodes are no longer equi-potential surfaces since as the current passes along the conductor an Ohmic *iR* drop results.

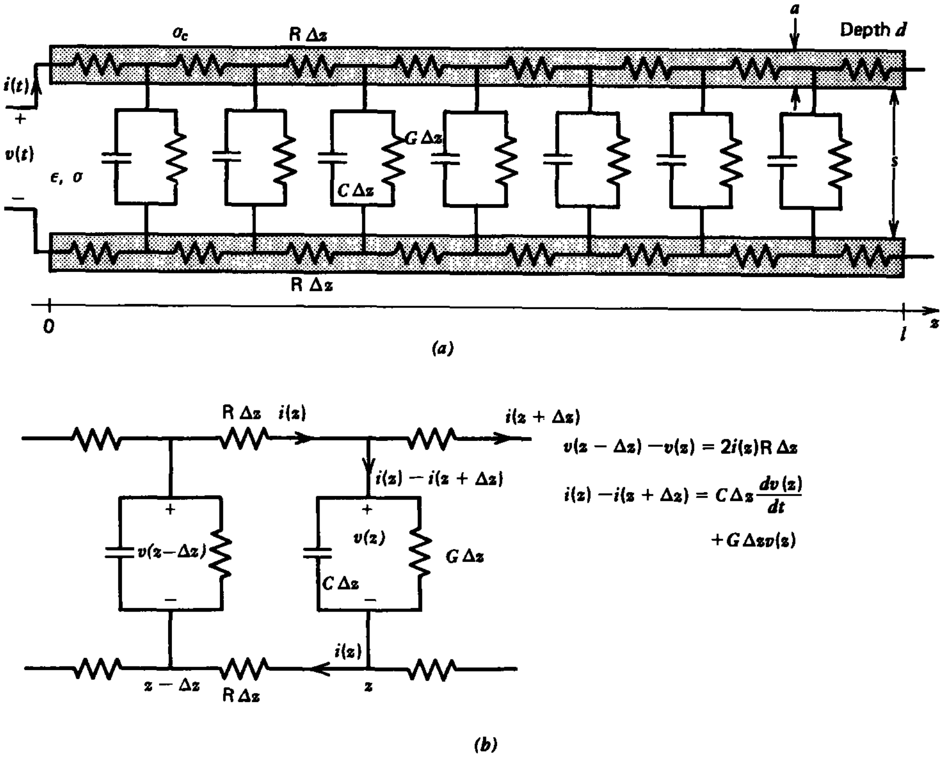


Figure 3-23 Lossy parallel plate electrodes with finite Ohmic conductivity σ_c enclose a lossy dielectric with permittivity ε and conductivity σ. (*a*) This system can be modeled by a distributed resistor-capacitor network. (*b*) Kirchoff's voltage and current laws applied to a section of length Δz allow us to describe the system by partial differential equations.

The current is also shunted through the lossy dielectric so that less current flows at the far end of the conductor than near the source. We can find approximate solutions by breaking the continuous system into many small segments of length Δz. The electrode resistance of this small section is

$$\mathrm{R}\,\Delta z=\frac{\Delta z}{\sigma_c a d} \tag{37}$$

where $\mathrm{R}=1/(\sigma_c a d)$ is just the resistance per unit length. We have shown in the previous section that the dielectric can be modeled as a parallel resistor-capacitor combination,

$$C\Delta z=\frac{\varepsilon d\,\Delta z}{s},\quad \frac{1}{G\Delta z}=\frac{s}{\sigma d\,\Delta z} \tag{38}$$

C is the capacitance per unit length and G is the conductance per unit length where the conductance is the reciprocal of the resistance. It is more convenient to work with the conductance because it is in parallel with the capacitance.

We apply Kirchoff's voltage and current laws for the section of equivalent circuit shown in Figure 3-23b:

$$\begin{aligned} v(z-\Delta z)-v(z)&=2i(z)\mathrm{R}\,\Delta z \\ i(z)-i(z+\Delta z)&=C\Delta z\frac{dv(z)}{dt}+G\Delta z v(z) \end{aligned} \tag{39}$$

The factor of 2 in the upper equation arises from the equal series resistances of the upper and lower conductors. Dividing through by Δz and taking the limit as Δz becomes infinitesimally small yields the partial differential equations

$$\begin{aligned} -\frac{\partial v}{\partial z}&=2i\mathrm{R} \\ -\frac{\partial i}{\partial z}&=C\frac{\partial v}{\partial t}+Gv \end{aligned} \tag{40}$$

Taking $\partial/\partial z$ of the upper equation allows us to substitute in the lower equation to eliminate i,

$$\frac{\partial^2 v}{\partial z^2}=2\mathrm{R}C\frac{\partial v}{\partial t}+2\mathrm{R}Gv \tag{41}$$

which is called a transient diffusion equation. Equations (40) and (41) are also valid for any geometry whose cross sectional area remains constant over its length. The 2R represents the series resistance per unit length of both electrodes, while C and G are the capacitance and conductance per unit length of the dielectric medium.

(b) Steady State

If a dc voltage V_0 is applied, the steady-state voltage is

$$\frac{d^2 v}{dz^2}-2\mathrm{R}Gv=0\Rightarrow v=A_1\sinh\sqrt{2\mathrm{R}G}z+A_2\cosh\sqrt{2\mathrm{R}G}z \tag{42}$$

where the constants are found by the boundary conditions at $z=0$ and $z=l$,

$$v(z=0)=V_0,\qquad i(z=l)=0 \tag{43}$$

We take the $z=l$ end to be open circuited. Solutions are

$$\begin{aligned} v(z)&=V_0\frac{\cosh\sqrt{2\mathrm{R}G}\,(z-l)}{\cosh\sqrt{2\mathrm{R}G}\,l} \\ i(z)&=-\frac{1}{2\mathrm{R}}\frac{dv}{dz}=V_0\sqrt{\frac{G}{2\mathrm{R}}}\frac{\sinh\sqrt{2\mathrm{R}G}(z-l)}{\cosh\sqrt{2\mathrm{R}G}\,l} \end{aligned} \tag{44}$$

(c) Transient Solution

If this dc voltage is applied as a step at $t=0$, it takes time for the voltage and current to reach these steady-state distributions. Because (41) is linear, we can use superposition and guess a solution for the voltage that is the sum of the steady-state solution in (44) and a transient solution that dies off with time:

$$v(z,t)=\frac{V_0\cosh\sqrt{2\mathrm{R}G}\,(z-l)}{\cosh\sqrt{2\mathrm{R}G}\,l}+\hat{v}(z)\,e^{-\alpha t} \tag{45}$$

At this point we do not know the function $\hat{v}(z)$ or α. Substituting the assumed solution of (45) back into (41) yields the ordinary differential equation

$$\frac{d^2\hat{v}}{dz^2}+p^2\hat{v}=0,\qquad p^2=2\mathrm{R}C\alpha-2\mathrm{R}G \tag{46}$$

which has the trigonometric solutions

$$\hat{v}(z)=a_1\sin pz+a_2\cos pz \tag{47}$$

Since the time-independent part of (45) already satisfies the boundary conditions at $z=0$, the transient part must be zero there so that $a_2=0$. The transient contribution to the current i, found from (40),

$$\begin{aligned} i(z,t)&=V_0\sqrt{\frac{G}{2\mathrm{R}}}\frac{\sinh\sqrt{2\mathrm{R}G}\,(z-l)}{\cosh\sqrt{2\mathrm{R}G}\,l}+\hat{\imath}(z)\,e^{-\alpha t}\\ \hat{\imath}(z)&=-\frac{1}{2\mathrm{R}}\frac{d\hat{v}(z)}{dz}=-\frac{pa_1}{2\mathrm{R}}\cos pz \end{aligned} \tag{48}$$

must still be zero at $z=l$, which means that pl must be an odd integer multiple of $\pi/2$,

$$pl=(2n+1)\frac{\pi}{2}\Rightarrow\alpha_n=\frac{1}{2\mathrm{R}C}\left((2n+1)\frac{\pi}{2l}\right)^2+\frac{G}{C},\quad n=0,1,2,\cdots \tag{49}$$

Since the boundary conditions allow an infinite number of values of α, the most general solution is the superposition of all allowed solutions:

$$v(z,t)=V_0\frac{\cosh\sqrt{2\mathrm{R}G}\,(z-l)}{\cosh\sqrt{2\mathrm{R}G}\,l}+\sum_{n=0}^{\infty}A_n\sin(2n+1)\frac{\pi z}{2l}e^{-\alpha_n t} \tag{50}$$

This solution satisfies the boundary conditions but not the initial conditions at $t=0$ when the voltage is first turned on. Before the voltage source is applied, the voltage distribution throughout the system is zero. It must remain zero right after being turned on otherwise the time derivative in (40) would be infinite, which requires nonphysical infinite currents. Thus we impose the initial condition

$$v(z, t=0)=0=V_0\frac{\cosh\sqrt{2RG}\,(z-l)}{\cosh\sqrt{2RG}\,l}+\sum_{n=0}^{\infty}A_n\sin(2n+1)\frac{\pi z}{2l}\tag{51}$$

We can solve for the amplitudes A_n by multiplying (51) through by $\sin(2m+1)\,\pi z/2l$ and then integrating over z from 0 to l:

$$\begin{aligned}0=&\frac{V_0}{\cosh\sqrt{2RG}\,l}\int_0^l\cosh\sqrt{2RG}\,(z-l)\sin(2m+1)\frac{\pi z}{2l}\,dz\\&+\int_0^l\sum_{n=0}^{\infty}A_n\sin(2n+1)\frac{\pi z}{2l}\sin(2m+1)\frac{\pi z}{2l}\,dz\end{aligned}\tag{52}$$

The first term is easily integrated by writing the hyperbolic cosine in terms of exponentials,* while the last term integrates to zero for all values of m not equal to n so that the amplitudes are

$$A_n=-\frac{1}{l^2}\frac{\pi V_0(2n+1)}{2RG+[(2n+1)\,\pi/2l]^2}\tag{53}$$

The total solutions are then

$$\begin{aligned}v(z,t)=&\frac{V_0\cosh\sqrt{2RG}\,(z-l)}{\cosh\sqrt{2RG}\,l}\\&-\frac{\pi V_0}{l^2}\sum_{n=0}^{\infty}\frac{(2n+1)\sin[(2n+1)(\pi z/2l)]\,e^{-\alpha_n t}}{2RG+[(2n+1)(\pi/2l)]^2}\end{aligned}$$

$$\begin{aligned}i(z,t)=&-\frac{1}{2R}\frac{\partial v}{\partial z}\\=&-\frac{V_0\sqrt{G/2R}\sinh\sqrt{2RG}\,(z-l)}{\cosh\sqrt{2RG}\,l}\\&+\frac{\pi^2V_0}{4l^3R}\sum_{n=0}^{\infty}\frac{(2n+1)^2\cos[(2n+1)(\pi z/2l)]\,e^{-\alpha_n t}}{2RG+[(2n+1)(\pi/2l)]^2}\end{aligned}\tag{54}$$

$$*\int\cosh a(z-l)\sin bz\,dz=\frac{1}{a^2+b^2}[a\sin bz\sinh a(z-l)-b\cos bz\cosh a(z-l)]$$

$$\int_0^l\sin(2n+1)bz\sin(2m+1)bz\,dz=\begin{cases}0 & m\neq n\\ l/2 & m=n\end{cases}$$

The fundamental time constant corresponds to the smallest value of α, which is when $n=0$:

$$\tau_0=\frac{1}{\alpha_0}=\frac{C}{G+\frac{1}{2R}\left(\frac{\pi}{2l}\right)^2} \tag{55}$$

For times long compared to τ_0 the system is approximately in the steady state. Because of the fast exponential decrease for times greater than zero, the infinite series in (54) can often be approximated by the first term. These solutions are plotted in Figure 3-24 for the special case where $G=0$. Then the voltage distribution builds up from zero to a constant value diffusing in from the left. The current near $z=0$ is initially very large. As time increases, with $G=0$, the current everywhere decreases towards a zero steady state.

3-6-5 Effects of Convection

We have seen that in a stationary medium any initial charge density decays away to a surface of discontinuity. We now wish to focus attention on a dc steady-state system of a conducting medium moving at constant velocity $U\mathbf{i}_x$, as in Figure 3-25. A source at $x=0$ maintains a constant charge density ρ_0. Then (3) in the dc steady state with constant

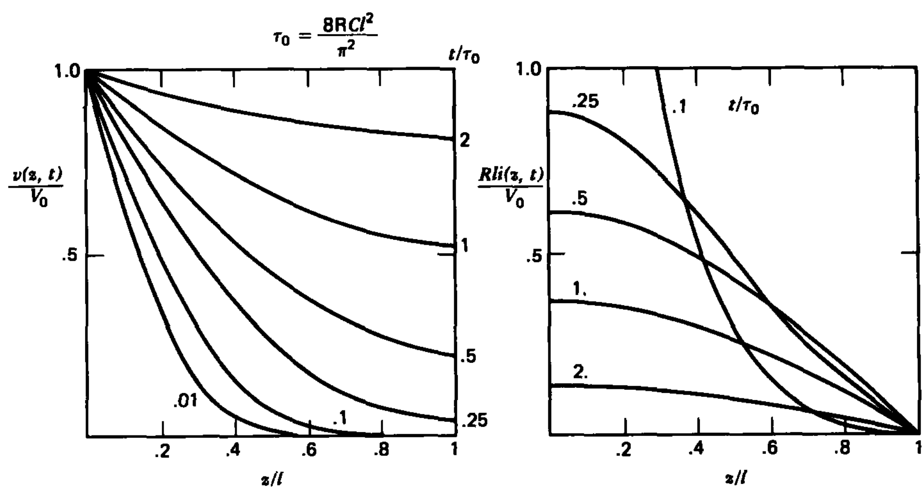


Figure 3-24 The transient voltage and current spatial distributions for various times for the lossy line in Figure 3-23a with $G=0$ for a step voltage excitation at $z=0$ with the $z=l$ end open circuited. The diffusion effects arise because of the lossy electrodes where the longest time constant is $\tau_0=8RCl^2/\pi^2$.

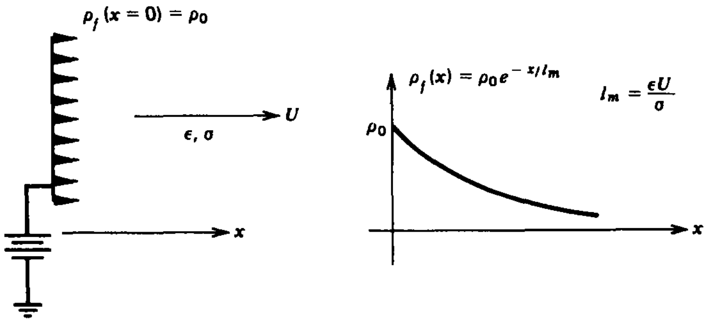


Figure 3-25 A moving conducting material with velocity $U\mathbf{i}_x$ tends to take charge injected at $x=0$ with it. The steady-state charge density decreases exponentially from the source.

velocity becomes

$$\frac{d\rho_f}{dx}+\frac{\sigma}{\varepsilon U}\rho_f=0 \tag{56}$$

which has exponentially decaying solutions

$$\rho_f=\rho_0\, e^{-x/l_m}, \qquad l_m=\frac{\varepsilon U}{\sigma} \tag{57}$$

where l_m represents a characteristic spatial decay length. If the system has cross-sectional area A, the total charge q in the system is

$$q=\int_0^\infty \rho_f A\, dx=\rho_0 l_m A \tag{58}$$

3-6-6 The Earth and its Atmosphere as a Leaky Spherical Capacitor*

In fair weather, at the earth's surface exists a dc electric field with approximate strength of 100 V/m directed radially toward the earth's center. The magnitude of the electric field decreases with height above the earth's surface because of the nonuniform electrical conductivity $\sigma(r)$ of the atmosphere approximated as

$$\sigma(r)=\sigma_0+a(r-R)^2 \text{ siemen/m} \tag{59}$$

where measurements have shown that

$$\begin{aligned}\sigma_0&\approx 3\times 10^{-14}\\ a&\approx .5\times 10^{-20}\end{aligned} \tag{60}$$

* *M. A. Uman, "The Earth and Its Atmosphere as a Leaky Spherical Capacitor,"* Am. J. Phys. *V. 42, Nov. 1974, pp. 1033–1035.*

and $R \approx 6\times 10^6$ meter is the earth's radius. The conductivity increases with height because of cosmic radiation in the lower atmosphere. Because of solar radiation the atmosphere acts as a perfect conductor above 50 km.

In the dc steady state, charge conservation of Section 3-2-1 with spherical symmetry requires

$$\nabla\cdot\mathbf{J}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2J_r)=0 \Rightarrow J_r=\sigma(r)E_r=\frac{C}{r^2} \tag{61}$$

where the constant of integration C is found by specifying the surface electric field $E_r(R)\approx -100$ V/m

$$J_r(r)=\frac{\sigma(R)E_r(R)R^2}{r^2} \tag{62}$$

At the earth's surface the current density is then

$$J_r(R)=\sigma(R)E_r(R)=\sigma_0E_r(R)\approx -3\times 10^{-12}\ \text{amp/m}^2 \tag{63}$$

The total current directed radially inwards over the whole earth is then

$$I=|J_r(R)4\pi R^2|\approx 1350\ \text{amp} \tag{64}$$

The electric field distribution throughout the atmosphere is found from (62) as

$$E_r(r)=\frac{J_r(r)}{\sigma(r)}=\frac{\sigma(R)E_r(R)R^2}{r^2\sigma(r)} \tag{65}$$

The surface charge density on the earth's surface is

$$\sigma_f(r=R)=\varepsilon_0E_r(R)\approx -8.85\times 10^{-10}\ \text{Coul/m}^2 \tag{66}$$

This negative surface charge distribution (remember: $E_r(r)<0$) is balanced by positive volume charge distribution throughout the atmosphere

$$\begin{aligned}\rho_f(r)&=\varepsilon_0\nabla\cdot\mathbf{E}=\frac{\varepsilon_0}{r^2}\frac{\partial}{\partial r}(r^2E_r)=\frac{\varepsilon_0\sigma(R)E_r(R)R^2}{r^2}\frac{d}{dr}\left(\frac{1}{\sigma(r)}\right)\\&=\frac{-\varepsilon_0\sigma(R)E_r(R)R^2}{r^2(\sigma(r))^2}2a(r-R)\end{aligned} \tag{67}$$

The potential difference between the upper atmosphere and the earth's surface is

$$\begin{aligned}V&=-\int_R^\infty E_r(r)dr\\&=-\sigma(R)E_r(R)R^2\int_R^\infty \frac{dr}{r^2[\sigma_0+a(r-R)^2]}\end{aligned}$$

$$= -\frac{\sigma(R)E_r(R)R^2}{a}\left\{-\frac{R}{\left(R^2+\frac{\sigma_0}{a}\right)^2}\ln\left[\frac{(r-R)^2+\frac{\sigma_0}{a}}{r^2}\right]\right.$$

$$\left.\left.-\frac{1}{r\left(R^2+\frac{\sigma_0}{a}\right)}+\frac{\left(R^2-\frac{\sigma_0}{a}\right)}{\sqrt{\frac{\sigma_0}{a}}\left(R^2+\frac{\sigma_0}{a}\right)^2}\tan^{-1}\frac{(r-R)}{\sqrt{\frac{\sigma_0}{a}}}\right\}\right|_{r=R}^{\infty}$$

$$= -\frac{\sigma(R)E_r(R)}{a\left(R^2+\frac{\sigma_0}{a}\right)^2}R^2\left\{R\ln\frac{\sigma_0}{aR^2}+\frac{\left(R^2+\frac{\sigma_0}{a}\right)}{R}+\frac{\frac{\pi}{2}\left(R^2-\frac{\sigma_0}{a}\right)}{\sqrt{\frac{\sigma_0}{a}}}\right\} \tag{68}$$

Using the parameters of (60), we see that $\sigma_0/a \ll R^2$ so that (68) approximately reduces to

$$V \approx -\frac{\sigma_0 E_r(R)}{aR^2}\left\{R\left(\ln\frac{\sigma_0}{aR^2}+1\right)+\frac{\pi R^2}{2\sqrt{\frac{\sigma_0}{a}}}\right\} \tag{69}$$

$$\approx 384{,}000 \text{ volts}$$

If the earth's charge were not replenished, the current flow would neutralize the charge at the earth's surface with a time constant of order

$$\tau = \frac{\varepsilon_0}{\sigma_0} \approx 300 \text{ seconds} \tag{70}$$

It is thought that localized stormy regions simultaneously active all over the world serve as "batteries" to keep the earth charged via negatively charged lightning to ground and corona at ground level, producing charge that moves from ground to cloud. This thunderstorm current must be upwards and balances the downwards fair weather current of (64).

3.7 FIELD-DEPENDENT SPACE CHARGE DISTRIBUTIONS

A stationary Ohmic conductor with constant conductivity was shown in Section 3-6-1 to not support a steady-state volume charge distribution. This occurs because in our classical Ohmic model in Section 3-2-2*c* one species of charge (e.g., electrons in metals) move relative to a stationary background species of charge with opposite polarity so that charge neutrality is maintained. However, if only one species of charge is injected into a medium, a net steady-state volume charge distribution can result.

Because of the electric force, this distribution of volume charge ρ_f contributes to and also in turn depends on the electric field. It now becomes necessary to simultaneously satisfy the coupled electrical and mechanical equations.

3-7-1 Space Charge Limited Vacuum Tube Diode

In vacuum tube diodes, electrons with charge $-e$ and mass m are boiled off the heated cathode, which we take as our zero potential reference. This process is called thermionic emission. A positive potential V_0 applied to the anode at $x=l$ accelerates the electrons, as in Figure 3-26. Newton's law for a particular electron is

$$m\frac{dv}{dt}=-eE=e\frac{dV}{dx} \tag{1}$$

In the dc steady state the velocity of the electron depends only on its position x so that

$$m\frac{dv}{dt}=m\frac{dv}{dx}\frac{dx}{dt}=mv\frac{dv}{dx}\Rightarrow\frac{d}{dx}\left(\tfrac{1}{2}mv^2\right)=\frac{d}{dx}(eV) \tag{2}$$

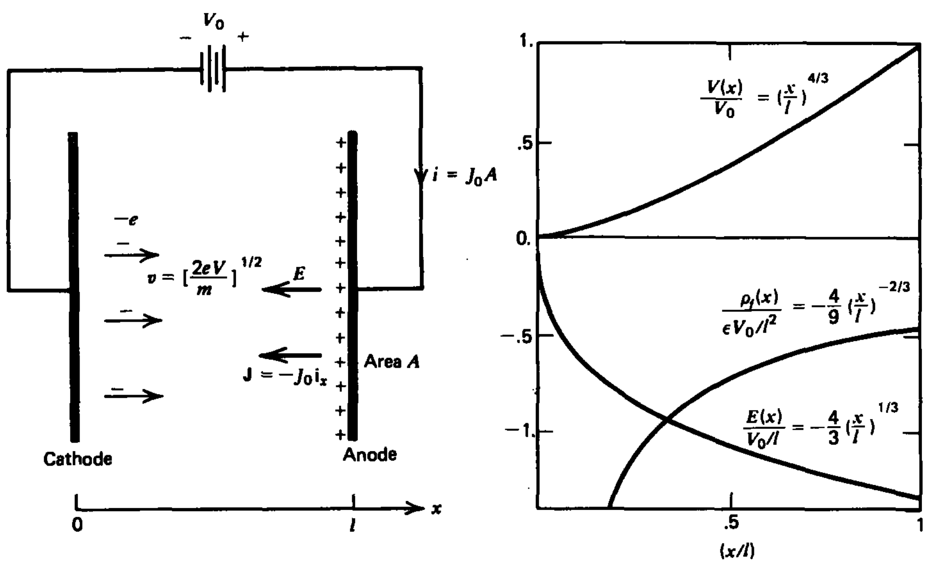


Figure 3-26 Space charge limited vacuum tube diode. (*a*) Thermionic injection of electrons from the heated cathode into vacuum with zero initial velocity. The positive anode potential attracts the electrons whose acceleration is proportional to the local electric field. (*b*) Steady-state potential, electric field, and volume charge distributions.

With this last equality, we have derived the energy conservation theorem

$$\frac{d}{dx}[\tfrac{1}{2}mv^2 - eV] = 0 \Rightarrow \tfrac{1}{2}mv^2 - eV = \text{const} \tag{3}$$

where we say that the kinetic energy $\frac{1}{2}mv^2$ plus the potential energy $-eV$ is the constant total energy. We limit ourselves here to the simplest case where the injected charge at the cathode starts out with zero velocity. Since the potential is also chosen to be zero at the cathode, the constant in (3) is zero. The velocity is then related to the electric potential as

$$v = \left(\frac{2e}{m}V\right)^{1/2} \tag{4}$$

In the time-independent steady state the current density is constant,

$$\nabla \cdot \mathbf{J} = 0 \Rightarrow \frac{dJ_x}{dx} = 0 \Rightarrow \mathbf{J} = -J_0\mathbf{i}_x \tag{5}$$

and is related to the charge density and velocity as

$$J_0 = -\rho_f v \Rightarrow \rho_f = -J_0\left(\frac{m}{2e}\right)^{1/2} V^{-1/2} \tag{6}$$

Note that the current flows from anode to cathode, and thus is in the negative x direction. This minus sign is incorporated in (5) and (6) so that J_0 is positive. Poisson's equation then requires that

$$\nabla^2 V = \frac{-\rho_f}{\varepsilon} \Rightarrow \frac{d^2V}{dx^2} = \frac{J_0}{\varepsilon}\left(\frac{m}{2e}\right)^{1/2} V^{-1/2} \tag{7}$$

Power law solutions to this nonlinear differential equation are guessed of the form

$$V = Bx^p \tag{8}$$

which when substituted into (7) yields

$$Bp(p-1)x^{p-2} = \frac{J_0}{\varepsilon}\left(\frac{m}{2e}\right)^{1/2} B^{-1/2}x^{-p/2} \tag{9}$$

For this assumed solution to hold for all x we require that

$$p - 2 = -\frac{p}{2} \Rightarrow p = \frac{4}{3} \tag{10}$$

which then gives us the amplitude B as

$$B = \left[\frac{9}{4}\frac{J_0}{\varepsilon}\left(\frac{m}{2e}\right)^{1/2}\right]^{2/3} \tag{11}$$

so that the potential is

$$V(x)=\left[\frac{9}{4}\frac{J_0}{\varepsilon}\left(\frac{m}{2e}\right)^{1/2}\right]^{2/3}x^{4/3} \tag{12}$$

The potential is zero at the cathode, as required, while the anode potential V_0 requires the current density to be

$$V(x=l)=V_0=\left[\frac{9}{4}\frac{J_0}{\varepsilon}\left(\frac{m}{2e}\right)^{1/2}\right]^{2/3}l^{4/3}$$

$$\Rightarrow J_0=\frac{4}{9}\frac{\varepsilon}{l^2}\left(\frac{2e}{m}\right)^{1/2}V_0^{3/2} \tag{13}$$

which is called the Langmuir–Child law.

The potential, electric field, and charge distributions are then concisely written as

$$\begin{aligned} V(x)&=V_0\left(\frac{x}{l}\right)^{4/3}\\ E(x)&=-\frac{dV(x)}{dx}=-\frac{4}{3}\frac{V_0}{l}\left(\frac{x}{l}\right)^{1/3}\\ \rho_f(x)&=\varepsilon\frac{dE(x)}{dx}=-\frac{4}{9}\varepsilon\frac{V_0}{l^2}\left(\frac{x}{l}\right)^{-2/3} \end{aligned} \tag{14}$$

and are plotted in Figure 3-26*b*. We see that the charge density at the cathode is infinite but that the total charge between the electrodes is finite,

$$q_T=\int_{x=0}^{l}\rho_f(x)A\,dx=-\frac{4}{3}\varepsilon\frac{V_0}{l}A \tag{15}$$

being equal in magnitude but opposite in sign to the total surface charge on the anode:

$$q_A=\sigma_f(x=l)A=-\varepsilon E(x=l)A=+\frac{4}{3}\varepsilon\frac{V_0}{l}A \tag{16}$$

There is no surface charge on the cathode because the electric field is zero there.

This displacement x of each electron can be found by substituting the potential distribution of (14) into (4),

$$v=\frac{dx}{dt}=\left(\frac{2eV_0}{m}\right)^{1/2}\left(\frac{x}{l}\right)^{2/3}\Rightarrow\frac{dx}{x^{2/3}}=\left(\frac{2eV_0}{ml^{4/3}}\right)^{1/2}dt \tag{17}$$

which integrates to

$$x=\frac{1}{27}\left(\frac{2eV_0}{ml^{4/3}}\right)^{3/2}t^3 \tag{18}$$

The charge transit time τ between electrodes is found by solving (18) with $x = l$:

$$\tau = 3l\left(\frac{m}{2eV_0}\right)^{1/2} \tag{19}$$

For an electron ($m = 9.1 \times 10^{-31}$ kg, $e = 1.6 \times 10^{-19}$ coul) with 100 volts applied across $l = 1$ cm (10^{-2} m) this time is $\tau \approx 5 \times 10^{-9}$ sec. The peak electron velocity when it reaches the anode is $v(x = l) \approx 6 \times 10^6$ m/sec, which is approximately 50 times less than the vacuum speed of light.

Because of these fast response times vacuum tube diodes are used in alternating voltage applications for rectification as current only flows when the anode is positive and as nonlinear circuit elements because of the three-halves power law of (13) relating current and voltage.

3-7-2 Space Charge Limited Conduction in Dielectrics

Conduction properties of dielectrics are often examined by injecting charge. In Figure 3-27, an electron beam with current density $\mathbf{J} = -J_0\mathbf{i}_x$ is suddenly turned on at $t = 0$.* In media, the acceleration of the charge is no longer proportional to the electric field. Rather, collisions with the medium introduce a frictional drag so that the velocity is proportional to the electric field through the electron mobility μ:

$$\mathbf{v} = -\mu\mathbf{E} \tag{20}$$

As the electrons penetrate the dielectric, the space charge front is a distance s from the interface where (20) gives us

$$ds/dt = -\mu E(s) \tag{21}$$

Although the charge density is nonuniformly distributed behind the wavefront, the total charge Q within the dielectric behind the wave front at time t is related to the current density as

$$J_0A = \rho_f\mu E_xA = -Q/t \Rightarrow Q = -J_0At \tag{22}$$

Gauss's law applied to the rectangular surface enclosing all the charge within the dielectric then relates the fields at the interface and the charge front to this charge as

$$\oint_S \varepsilon\mathbf{E}\cdot\mathbf{dS} = [\varepsilon E(s) - \varepsilon_0E(0)]A = Q = -J_0At \tag{23}$$

* *See P. K. Watson, J. M. Schneider, and H. R. Till, Electrohydrodynamic Stability of Space Charge Limited Currents In Dielectric Liquids. II. Experimental Study,* Phys. Fluids *13 (1970), p. 1955.*

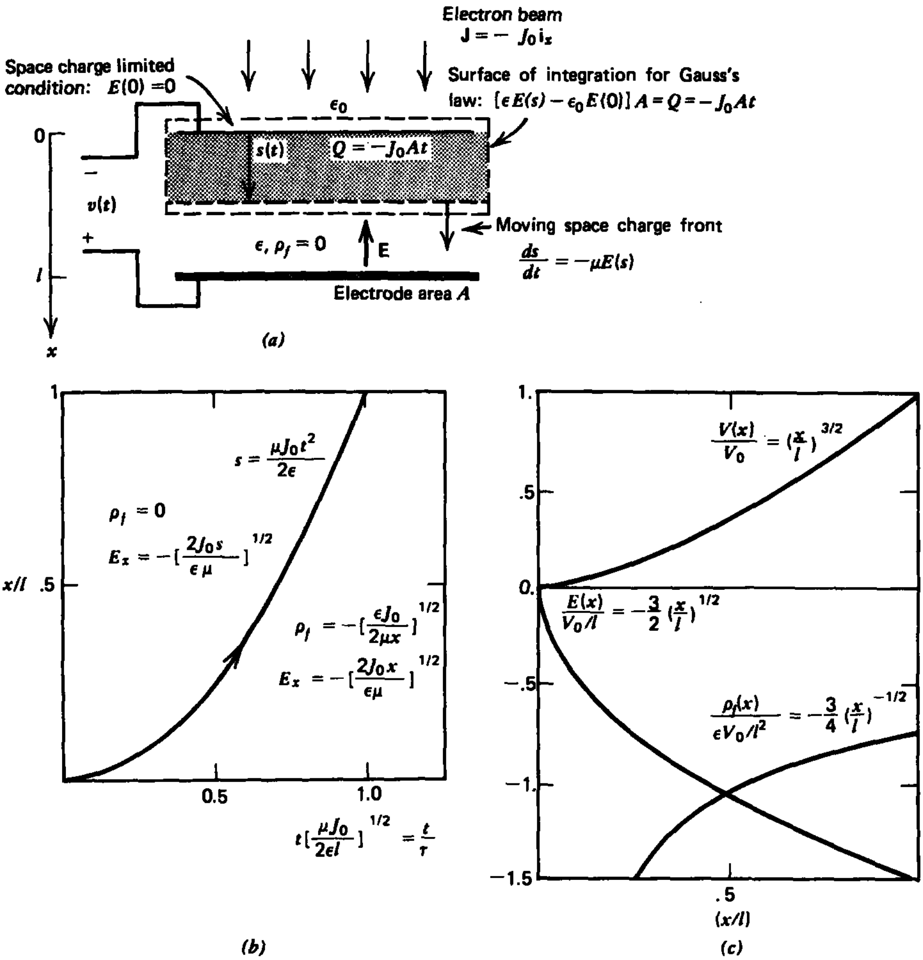


Figure 3-27 (a) An electron beam carrying a current $-J_0\mathbf{i}_x$ is turned on at $t=0$. The electrons travel through the dielectric with mobility μ. (b) The space charge front, at a distance s in front of the space charge limited interface at $x=0$, travels towards the opposite electrode. (c) After the transit time $t_c=[2\varepsilon l/\mu J_0]^{1/2}$ the steady-state potential, electric field, and space charge distributions.

The maximum current flows when $E(0)=0$, which is called space charge limited conduction. Then using (23) in (21) gives us the time dependence of the space charge front:

$$\frac{ds}{dt}=\frac{\mu J_0 t}{\varepsilon}\Rightarrow s(t)=\frac{\mu J_0 t^2}{2\varepsilon} \tag{24}$$

Behind the front Gauss's law requires

$$\frac{dE_x}{dx}=\frac{\rho_f}{\varepsilon}=\frac{J_0}{\varepsilon\mu E_x}\Rightarrow E_x\frac{dE_x}{dx}=\frac{J_0}{\varepsilon\mu} \tag{25}$$

while ahead of the moving space charge the charge density is zero so that the current is carried entirely by displacement current and the electric field is constant in space. The spatial distribution of electric field is then obtained by integrating (25) to

$$E_x = \begin{cases} -\sqrt{2J_0x/\varepsilon\mu}, & 0 \le x \le s(t) \\ -\sqrt{2J_0s/\varepsilon\mu}, & s(t) \le x \le l \end{cases} \tag{26}$$

while the charge distribution is

$$\rho_f = \varepsilon\frac{dE_x}{dx} = \begin{cases} -\sqrt{\varepsilon J_0/(2\mu x)}, & 0 \le x \le s(t) \\ 0, & s(t) \le x \le l \end{cases} \tag{27}$$

as indicated in Figure 3-27*b*.

The time dependence of the voltage across the dielectric is then

$$\begin{aligned} v(t) = \int_0^l E_x\,dx &= \int_0^{s(t)} \sqrt{\frac{2J_0x}{\varepsilon\mu}}\,dx + \int_{s(t)}^l \sqrt{\frac{2J_0s}{\varepsilon\mu}}\,dx \\ &= \frac{J_0lt}{\varepsilon} - \frac{\mu J_0^2t^3}{6\varepsilon^2}, \qquad s(t) \le l \end{aligned} \tag{28}$$

These transient solutions are valid until the space charge front s, given by (24), reaches the opposite electrode with $s = l$ at time

$$\tau = \sqrt{2\varepsilon l/\mu J_0} \tag{29}$$

Thereafter, the system is in the dc steady state with the terminal voltage V_0 related to the current density as

$$J_0 = \frac{9}{8}\frac{\varepsilon\mu V_0^2}{l^3} \tag{30}$$

which is the analogous Langmuir–Child's law for collision dominated media. The steady-state electric field and space charge density are then concisely written as

$$E_x = -\frac{3}{2}\frac{V_0}{l}\left(\frac{x}{l}\right)^{1/2}, \quad \rho_f = \varepsilon\frac{dE}{dx} = -\frac{3}{4}\frac{\varepsilon V_0}{l^2}\left(\frac{x}{l}\right)^{-1/2} \tag{31}$$

and are plotted in Figure 3-27*c*.

In liquids a typical ion mobility is of the order of $10^{-7}\,\text{m}^2/(\text{volt-sec})$ with a permittivity of $\varepsilon = 2\varepsilon_0 \approx 1.77\times10^{-11}$ farad/m. For a spacing of $l = 10^{-2}$ m with a potential difference of $V_0 = 10^4$ V the current density of (30) is $J_0 \approx 2\times10^{-4}\,\text{amp/m}^2$ with the transit time given by (29) $\tau \approx 0.133$ sec. Charge transport times in collison dominated media are much larger than in vacuum.

3-8 ENERGY STORED IN A DIELECTRIC MEDIUM

The work needed to assemble a charge distribution is stored as potential energy in the electric field because if the charges are allowed to move this work can be regained as kinetic energy or mechanical work.

3-8-1 Work Necessary to Assemble a Distribution of Point Charges

(a) Assembling the Charges

Let us compute the work necessary to bring three already existing free charges q_1, q_2, and q_3 from infinity to any position, as in Figure 3-28. It takes no work to bring in the first charge as there is no electric field present. The work necessary to bring in the second charge must overcome the field due to the first charge, while the work needed to bring in the third charge must overcome the fields due to both other charges. Since the electric potential developed in Section 2-5-3 is defined as the work per unit charge necessary to bring a point charge in from infinity, the total work necessary to bring in the three charges is

$$W=q_1(0)+q_2\left(\frac{q_1}{4\pi\varepsilon r_{12}}\right)+q_3\left(\frac{q_1}{4\pi\varepsilon r_{13}}+\frac{q_2}{4\pi\varepsilon r_{23}}\right) \qquad (1)$$

where the final distances between the charges are defined in Figure 3-28 and we use the permittivity ε of the medium. We can rewrite (1) in the more convenient form

$$W=\frac{1}{2}\left\{q_1\left[\frac{q_2}{4\pi\varepsilon r_{12}}+\frac{q_3}{4\pi\varepsilon r_{13}}\right]+q_2\left[\frac{q_1}{4\pi\varepsilon r_{12}}+\frac{q_3}{4\pi\varepsilon r_{23}}\right]+q_3\left[\frac{q_1}{4\pi\varepsilon r_{13}}+\frac{q_2}{4\pi\varepsilon r_{23}}\right]\right\} \qquad (2)$$

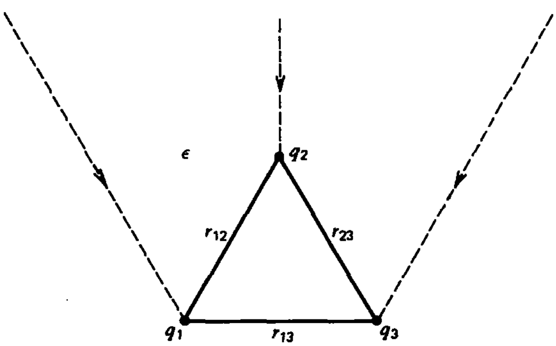


Figure 3-28 Three already existing point charges are brought in from an infinite distance to their final positions.

where we recognize that each bracketed term is just the potential at the final position of each charge and includes contributions from all the other charges, except the one located at the position where the potential is being evaluated:

$$W=\tfrac{1}{2}[q_1V_1+q_2V_2+q_3V_3] \tag{3}$$

Extending this result for any number N of already existing free point charges yields

$$W=\frac{1}{2}\sum_{n=1}^{N} q_nV_n \tag{4}$$

The factor of $\frac{1}{2}$ arises because the potential of a point charge at the time it is brought in from infinity is less than the final potential when all the charges are assembled.

(b) Binding Energy of a Crystal

One major application of (4) is in computing the largest contribution to the binding energy of ionic crystals, such as salt (NaCl), which is known as the Madelung electrostatic energy. We take a simple one-dimensional model of a crystal consisting of an infinitely long string of alternating polarity point charges $\pm q$ a distance a apart, as in Figure 3-29. The average work necessary to bring a positive charge as shown in Figure 3-29 from infinity to its position on the line is obtained from (4) as

$$W=\frac{1}{2}\frac{2q^2}{4\pi\varepsilon a}\left[-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\cdots\right] \tag{5}$$

The extra factor of 2 in the numerator is necessary because the string extends to infinity on each side. The infinite series is recognized as the Taylor series expansion of the logarithm

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots \tag{6}$$

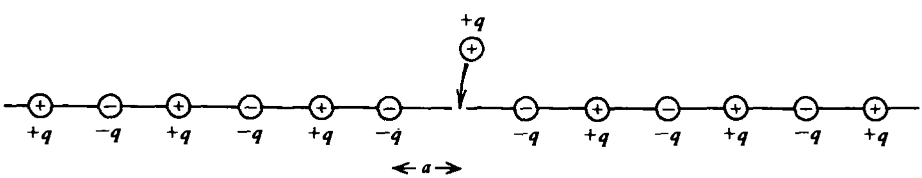


Figure 3-29 A one-dimensional crystal with alternating polarity charges $\pm q$ a distance a apart.

where $x=1$ so that*

$$W=\frac{-q^2}{4\pi\varepsilon a}\ln 2 \tag{7}$$

This work is negative because the crystal pulls on the charge as it is brought in from infinity. This means that it would take positive work to remove the charge as it is bound to the crystal. A typical ion spacing is about 3 Å (3×10^{-10} m) so that if q is a single proton ($q=1.6\times10^{-19}$ coul), the binding energy is $W\approx5.3\times10^{-19}$ joule. Since this number is so small it is usually more convenient to work with units of energy per unit electronic charge called electron volts (ev), which are obtained by dividing W by the charge on an electron so that, in this case, $W\approx3.3$ ev.

If the crystal was placed in a medium with higher permittivity, we see from (7) that the binding energy decreases. This is why many crystals are soluble in water, which has a relative dielectric constant of about 80.

3-8-2 Work Necessary to Form a Continuous Charge Distribution

Not included in (4) is the self-energy of each charge itself or, equivalently, the work necessary to assemble each point charge. Since the potential V from a point charge q is proportional to q, the self-energy is proportional q^2. However, evaluating the self-energy of a point charge is difficult because the potential is infinite at the point charge.

To understand the self-energy concept better it helps to model a point charge as a small uniformly charged spherical volume of radius R with total charge $Q=\frac{4}{3}\pi R^3\rho_0$. We assemble the sphere of charge from spherical shells, as shown in Figure 3-30, each of thickness dr_n and incremental charge $dq_n=4\pi r_n^2\,dr_n\rho_0$. As we bring in the nth shell to be placed at radius r_n the total charge already present and the potential there are

$$q_n=\frac{4}{3}\pi r_n^3\rho_0,\qquad V_n=\frac{q_n}{4\pi\varepsilon r_n}=\frac{r_n^2\rho_0}{3\varepsilon} \tag{8}$$

* *Strictly speaking, this series is only conditionally convergent for $x=1$ and its sum depends on the grouping of individual terms. If the series in* (6) *for $x=1$ is rewritten as*

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\cdots+\frac{1}{2k-1}-\frac{1}{4k-2}-\frac{1}{4k}\cdots,\quad k\geq 1$$

then its sum is $\frac{1}{2}\ln 2$. [See J. Pleines and S. Mahajan, On Conditionally Divergent Series and a Point Charge Between Two Parallel Grounded Planes, Am. J. Phys. *45 (1977) p. 868.*]

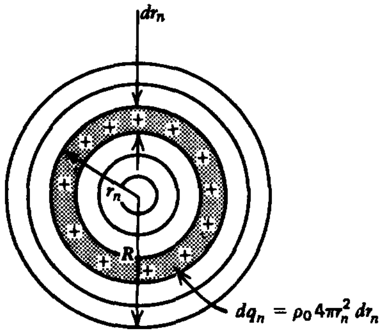


Figure 3-30 A point charge is modelled as a small uniformly charged sphere. It is assembled by bringing in spherical shells of differential sized surface charge elements from infinity.

so that the work required to bring in the nth shell is

$$dW_n = V_n\, dq_n = \frac{\rho_0^2 4\pi r_n^4}{3\varepsilon}\, dr_n \tag{9}$$

The total work necessary to assemble the sphere is obtained by adding the work needed for each shell:

$$W = \int dW_n = \int_0^R \frac{4\pi\rho_0^2 r^4}{3\varepsilon}\, dr = \frac{4\pi\rho_0^2 R^5}{15\varepsilon} = \frac{3Q^2}{20\pi\varepsilon R} \tag{10}$$

For a finite charge Q of zero radius the work becomes infinite. However, Einstein's theory of relativity tells us that this work necessary to assemble the charge is stored as energy that is related to the mass as

$$W = mc^2 = \frac{3Q^2}{20\pi\varepsilon R} \Rightarrow R = \frac{3Q^2}{20\pi\varepsilon mc^2} \tag{11}$$

which then determines the radius of the charge. For the case of an electron ($Q = 1.6\times 10^{-19}$ coul, $m = 9.1\times 10^{-31}$ kg) in free space ($\varepsilon = \varepsilon_0 = 8.854\times 10^{-12}$ farad/m), this radius is

$$\begin{aligned} R_{\text{electron}} &= \frac{3(1.6\times 10^{-19})^2}{20\pi(8.854\times 10^{-12})(9.1\times 10^{-31})(3\times 10^8)^2} \\ &\approx 1.69\times 10^{-15}\ \text{m} \end{aligned} \tag{12}$$

We can also obtain the result of (10) by using (4) where each charge becomes a differential element dq, so that the summation becomes an integration over the continuous free charge distribution:

$$W = \frac{1}{2}\int_{\text{all } q_f} V\, dq_f \tag{13}$$

For the case of the uniformly charged sphere, $dq_f = \rho_0\, dV$, the final potential within the sphere is given by the results of Section 2-5-5b:

$$V = \frac{\rho_0}{2\varepsilon}\left(R^2 - \frac{r^2}{3}\right) \tag{14}$$

Then (13) agrees with (10):

$$W = \frac{1}{2}\int \rho_0 V\, dV = \frac{4\pi\rho_0^2}{4\varepsilon}\int_0^R \left(R^2 - \frac{r^2}{3}\right) r^2\, dr = \frac{4\pi\rho_0^2 R^5}{15\varepsilon} = \frac{3Q^2}{20\pi\varepsilon R} \tag{15}$$

Thus, in general, we define (13) as the energy stored in the electric field, including the self-energy term. It differs from (4), which only includes interaction terms between different charges and not the infinite work necessary to assemble each point charge. Equation (13) is valid for line, surface, and volume charge distributions with the differential charge elements given in Section 2-3-1. Remember when using (4) and (13) that the zero reference for the potential is assumed to be at infinity. Adding a constant V_0 to the potential will change the energy unless the total charge in the system is zero

$$\begin{aligned} W &= \tfrac{1}{2}\int (V + V_0)\, dq_f \\ &= \tfrac{1}{2}\int V\, dq_f + \tfrac{1}{2} V_0 \int dq_f \overset{\nearrow 0}{} \\ &= \tfrac{1}{2}\int V\, dq_f \end{aligned} \tag{16}$$

3-8-3 Energy Density of the Electric Field

It is also convenient to express the energy W stored in a system in terms of the electric field. We assume that we have a volume charge distribution with density ρ_f. Then, $dq_f = \rho_f\, dV$, where ρ_f is related to the displacement field from Gauss's law:

$$W = \tfrac{1}{2}\int_V \rho_f V\, dV = \tfrac{1}{2}\int_V V(\nabla \cdot \mathbf{D})\, dV \tag{17}$$

Let us examine the vector expansion

$$\nabla \cdot (V\mathbf{D}) = (\mathbf{D}\cdot\nabla)V + V(\nabla\cdot\mathbf{D}) \Rightarrow V(\nabla\cdot\mathbf{D}) = \nabla\cdot(V\mathbf{D}) + \mathbf{D}\cdot\mathbf{E} \tag{18}$$

where $\mathbf{E} = -\nabla V$. Then (17) becomes

$$W = \tfrac{1}{2}\int_V \mathbf{D}\cdot\mathbf{E}\, dV + \tfrac{1}{2}\int_V \nabla\cdot(V\mathbf{D})\, dV \tag{19}$$

The last term on the right-hand side can be converted to a surface integral using the divergence theorem:

$$\int_V \nabla \cdot (V\mathbf{D})\, dV = \oint_S V\,\mathbf{D} \cdot \mathbf{dS} \tag{20}$$

If we let the volume V be of infinite extent so that the enclosing surface S is at infinity, the charge distribution that only extends over a finite volume looks like a point charge for which the potential decreases as $1/r$ and the displacement vector dies off as $1/r^2$. Thus the term, $V\mathbf{D}$ at best dies off as $1/r^3$. Then, even though the surface area of S increases as r^2, the surface integral tends to zero as r becomes infinite as $1/r$. Thus, the second volume integral in (19) approaches zero:

$$\lim_{r\to\infty} \int_V \nabla \cdot (V\mathbf{D})\, dV = \oint_S V\mathbf{D} \cdot \mathbf{dS} = 0 \tag{21}$$

This conclusion is not true if the charge distribution is of infinite extent, since for the case of an infinitely long line or surface charge, the potential itself becomes infinite at infinity because the total charge on the line or surface is infinite. However, for finite size charge distributions, which is always the case in reality, (19) becomes

$$\begin{aligned} W &= \tfrac{1}{2} \int_{\text{all space}} \mathbf{D} \cdot \mathbf{E}\, dV \\ &= \int_{\text{all space}} \tfrac{1}{2}\varepsilon E^2\, dV \end{aligned} \tag{22}$$

where the integration extends over all space. This result is true even if the permittivity ε is a function of position. It is convenient to define the energy density as the positive-definite quantity:

$$w = \tfrac{1}{2}\varepsilon E^2 \text{ joule/m}^3\ [\text{kg-m}^{-1}\text{-s}^{-2}] \tag{23}$$

where the total energy is

$$W = \int_{\text{all space}} w\, dV \tag{24}$$

Note that although (22) is numerically equal to (13), (22) implies that electric energy exists in those regions where a nonzero electric field exists even if no charge is present in that region, while (13) implies that electric energy exists only where the charge is nonzero. The answer as to where the energy is stored—in the charge distribution or in the electric field—is a matter of convenience since you cannot have one without the other. Numerically both equations yield the same answers but with contributions from different regions of space.

3-8-4 Energy Stored in Charged Spheres

(a) Volume Charge

We can also find the energy stored in a uniformly charged sphere using (22) since we know the electric field in each region from Section 2-4-3*b*. The energy density is then

$$w=\frac{\varepsilon}{2}E_r^2=\begin{cases}\dfrac{Q^2}{32\pi^2\varepsilon r^4}, & r>R\\[2ex] \dfrac{Q^2r^2}{32\pi^2\varepsilon R^6}, & r<R\end{cases} \tag{25}$$

with total stored energy

$$\begin{aligned}W&=\int_{\text{all space}} w\,dV\\ &=\frac{Q^2}{8\pi\varepsilon}\left(\int_0^R\frac{r^4}{R^6}\,dr+\int_R^\infty\frac{dr}{r^2}\right)=\frac{3}{20}\frac{Q^2}{\pi\varepsilon R}\end{aligned} \tag{26}$$

which agrees with (10) and (15).

(b) Surface Charge

If the sphere is uniformly charged on its surface $Q=4\pi R^2\sigma_0$, the potential and electric field distributions are

$$V(r)=\begin{cases}\dfrac{Q}{4\pi\varepsilon R}\\[2ex] \dfrac{Q}{4\pi\varepsilon r}\end{cases};\qquad E_r=\begin{cases}0, & r<R\\[2ex] \dfrac{Q}{4\pi\varepsilon r^2}, & r>R\end{cases} \tag{27}$$

Using (22) the energy stored is

$$W=\frac{\varepsilon}{2}\left(\frac{Q}{4\pi\varepsilon}\right)^2 4\pi\int_R^\infty\frac{dr}{r^2}=\frac{Q^2}{8\pi\varepsilon R} \tag{28}$$

This result is equally as easy obtained using (13):

$$\begin{aligned}W&=\tfrac{1}{2}\int_S\sigma_0 V(r=R)\,dS\\ &=\tfrac{1}{2}\sigma_0 V(r=R)4\pi R^2=\frac{Q^2}{8\pi\varepsilon R}\end{aligned} \tag{29}$$

The energy stored in a uniformly charged sphere is 20% larger than the surface charged sphere for the same total charge Q. This is because of the additional energy stored throughout the sphere's volume. Outside the sphere ($r>R$) the fields are the same as is the stored energy.

(c) Binding Energy of an Atom

In Section 3-1-4 we modeled an atom as a fixed positive point charge nucleus Q with a surrounding uniform spherical cloud of negative charge with total charge $-Q$, as in Figure 3-31. Potentials due to the positive point and negative volume charges are found from Section 2-5-5*b* as

$$V_+(r)=\frac{Q}{4\pi\varepsilon_0 r}$$

$$V_-(r)=\begin{cases}-\dfrac{3Q}{8\pi\varepsilon_0 R^3}\left(R^2-\dfrac{r^2}{3}\right), & r<R\\[2ex] -\dfrac{Q}{4\pi\varepsilon_0 r}, & r>R\end{cases} \tag{30}$$

The binding energy of the atom is easily found by superposition considering first the uniformly charged negative sphere with self-energy given in (10), (15), and (26) and then adding the energy of the positive point charge:

$$W=\frac{3Q^2}{20\pi\varepsilon_0 R}+Q[V_-(r=0)]=-\frac{9Q^2}{40\pi\varepsilon_0 R} \tag{31}$$

This is the work necessary to assemble the atom from charges at infinity. Once the positive nucleus is in place, it attracts the following negative charges so that the field does work on the charges and the work of assembly in (31) is negative. Equivalently, the magnitude of (31) is the work necessary for us to disassemble the atom by overcoming the attractive coulombic forces between the opposite polarity charges.

When alternatively using (4) and (13), we only include the potential of the negative volume charge at $r=0$ acting on the positive charge, while we include the total potential due to both in evaluating the energy of the volume charge. We do

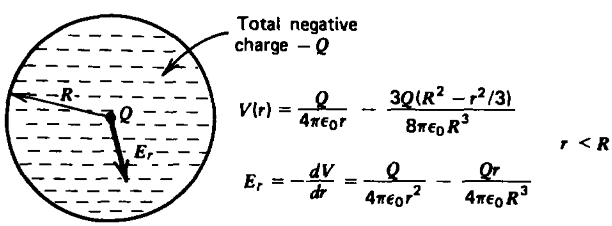


Figure 3-31 An atom can be modelled as a point charge Q representing the nucleus, surrounded by a cloud of uniformly distributed electrons with total charge $-Q$ within a sphere of radius R.

not consider the infinite self-energy of the point charge that would be included if we used (22):

$$W=\tfrac{1}{2}QV_-(r=0)-\tfrac{1}{2}\int_{r=0}^{R}[V_+(r)+V_-(r)]\frac{3Qr^2}{R^3}\,dr$$
$$=-\frac{3Q^2}{16\pi\varepsilon_0 R}-\frac{3Q^2}{8\pi\varepsilon_0 R^3}\int_0^R\left(r-\frac{3}{2}\frac{r^2}{R}+\frac{r^4}{2R^3}\right)dr$$
$$=-\frac{9Q^2}{40\pi\varepsilon_0 R} \tag{32}$$

3-8-5 Energy Stored in a Capacitor

In a capacitor all the charge resides on the electrodes as a surface charge. Consider two electrodes at voltage V_1 and V_2 with respect to infinity, and thus at voltage difference $V=V_2-V_1$, as shown in Figure 3-32. Each electrode carries opposite polarity charge with magnitude Q. Then (13) can be used to compute the total energy stored as

$$W=\tfrac{1}{2}\left[\int_{S_1}V_1\sigma_1\,dS_1+\int_{S_2}V_2\sigma_2\,dS_2\right] \tag{33}$$

Since each surface is an equipotential, the voltages V_1 and V_2 may be taken outside the integrals. The integral then reduces to the total charge $\pm Q$ on each electrode:

$$W=\tfrac{1}{2}\Big[V_1\int_{S_1}\underbrace{\sigma_1\,dS_1}_{-Q}+V_2\int_{S_2}\underbrace{\sigma_2\,dS_2}_{Q}\Big]$$
$$=\tfrac{1}{2}(V_2-V_1)Q=\tfrac{1}{2}QV \tag{34}$$

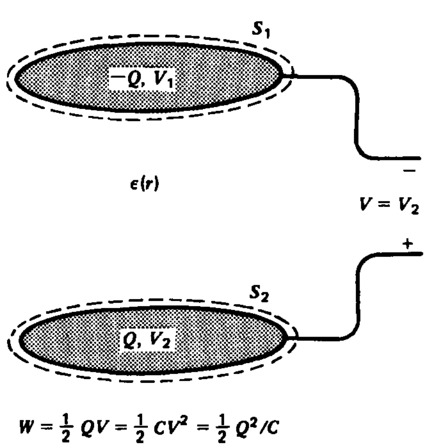


Figure 3-32 A capacitor stores energy in the electric field.

Since in a capacitor the charge and voltage are linearly related through the capacitance

$$Q=CV \tag{35}$$

the energy stored in the capacitor can also be written as

$$W=\tfrac{1}{2}QV=\tfrac{1}{2}CV^2=\tfrac{1}{2}\frac{Q^2}{C} \tag{36}$$

This energy is equivalent to (22) in terms of the electric field and gives us an alternate method to computing the capacitance if we know the electric field distribution.

EXAMPLE 3-3 CAPACITANCE OF AN ISOLATED SPHERE

A sphere of radius R carries a uniformly distributed surface charge Q. What is its capacitance?

SOLUTION

The stored energy is given by (28) or (29) so that (36) gives us the capacitance:

$$C=Q^2/2W=4\pi\varepsilon R$$

3.9 FIELDS AND THEIR FORCES

3-9-1 Force Per Unit Area on a Sheet of Surface Charge

A confusion arises in applying Coulomb's law to find the perpendicular force on a sheet of surface charge as the normal electric field is different on each side of the sheet. Using the over-simplified argument that half the surface charge resides on each side of the sheet yields the correct force

$$\mathbf{f}=\tfrac{1}{2}\int_S \sigma_f(\mathbf{E}_1+\mathbf{E}_2)\,d\mathbf{S} \tag{1}$$

where, as shown in Figure 3-33a, $\mathbf{E}_1$ and $\mathbf{E}_2$ are the electric fields on each side of the sheet. Thus, the correct field to use is the average electric field $\frac{1}{2}(\mathbf{E}_1+\mathbf{E}_2)$ across the sheet.

For the tangential force, the tangential components of $\mathbf{E}$ are continuous across the sheet ($E_{1t}=E_{2t}\equiv E_t$) so that

$$f_t=\tfrac{1}{2}\int_S \sigma_f(E_{1t}+E_{2t})\,dS=\int_S \sigma_f E_t\,dS \tag{2}$$

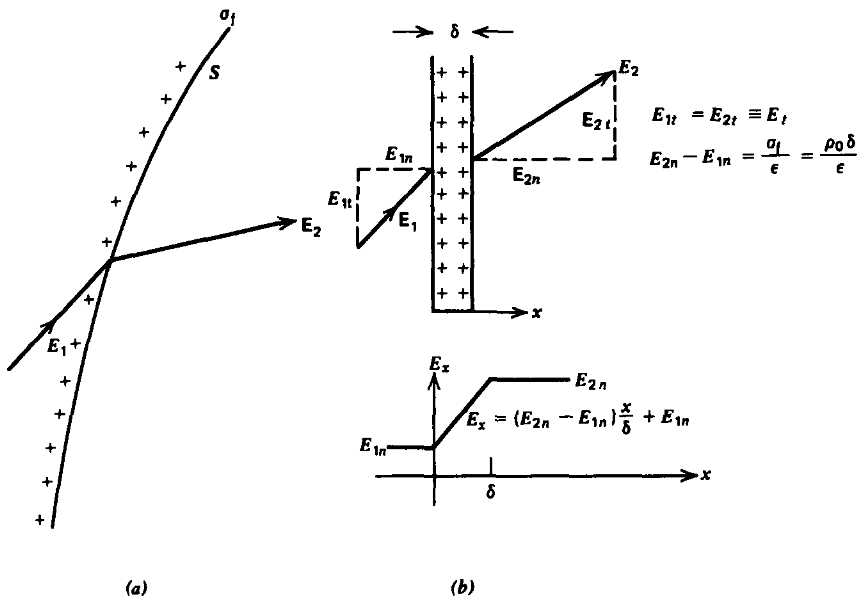


Figure 3-33 (*a*) The normal component of electric field is discontinuous across the sheet of surface charge. (*b*) The sheet of surface charge can be modeled as a thin layer of volume charge. The electric field then varies linearly across the volume.

The normal fields are discontinuous across the sheet so that the perpendicular force is

$$\sigma_f = \varepsilon(E_{2n} - E_{1n}) \Rightarrow f_n = \frac{1}{2}\int_S \varepsilon(E_{2n} - E_{1n})(E_{1n} + E_{2n})\, dS$$

$$= \frac{1}{2}\int_S \varepsilon(E_{2n}^2 - E_{1n}^2)\, dS \tag{3}$$

To be mathematically rigorous we can examine the field transition through the sheet more closely by assuming the surface charge is really a uniform volume charge distribution ρ_0 of very narrow thickness δ, as shown in Figure 3-33*b*. Over the small surface element dS, the surface appears straight so that the electric field due to the volume charge can then only vary with the coordinate x perpendicular to the surface. Then the point form of Gauss's law within the volume yields

$$\frac{dE_x}{dx} = \frac{\rho_0}{\varepsilon} \Rightarrow E_x = \frac{\rho_0 x}{\varepsilon} + \text{const} \tag{4}$$

The constant in (4) is evaluated by the boundary conditions on the normal components of electric field on each side of the sheet

$$E_x(x=0)=E_{1n}, \qquad E_x(x=\delta)=E_{2n} \tag{5}$$

so that the electric field is

$$E_x=(E_{2n}-E_{1n})\frac{x}{\delta}+E_{1n} \tag{6}$$

As the slab thickness δ becomes very small, we approach a sheet charge relating the surface charge density to the discontinuity in electric fields as

$$\lim_{\substack{\rho_0\to\infty\\ \delta\to 0}} \rho_0\delta=\sigma_f=\varepsilon(E_{2n}-E_{1n}) \tag{7}$$

Similarly the force per unit area on the slab of volume charge is

$$\begin{aligned} F_x&=\int_0^\delta \rho_0 E_x\,dx\\ &=\int_0^\delta \rho_0\left[(E_{2n}-E_{1n})\frac{x}{\delta}+E_{1n}\right]dx\\ &=\left[\rho_0(E_{2n}-E_{1n})\frac{x^2}{2\delta}+E_{1n}x\right]\Bigg|_0^\delta\\ &=\frac{\rho_0\delta}{2}(E_{1n}+E_{2n}) \end{aligned} \tag{8}$$

In the limit of (7), the force per unit area on the sheet of surface charge agrees with (3):

$$\lim_{\rho_0\delta=\sigma_f} F_x=\frac{\sigma_f}{2}(E_{1n}+E_{2n})=\frac{\varepsilon}{2}(E_{2n}^2-E_{1n}^2) \tag{9}$$

3-9-2 Forces on a Polarized Medium

(a) Force Density

In a uniform electric field there is no force on a dipole because the force on each charge is equal in magnitude but opposite in direction, as in Figure 3-34*a*. However, if the dipole moment is not aligned with the field there is an aligning torque given by $\mathbf{t}=\mathbf{p}\times\mathbf{E}$. The torque per unit volume $\mathbf{T}$ on a polarized medium with N dipoles per unit volume is then

$$\mathbf{T}=N\mathbf{t}=N\mathbf{p}\times\mathbf{E}=\mathbf{P}\times\mathbf{E} \tag{10}$$

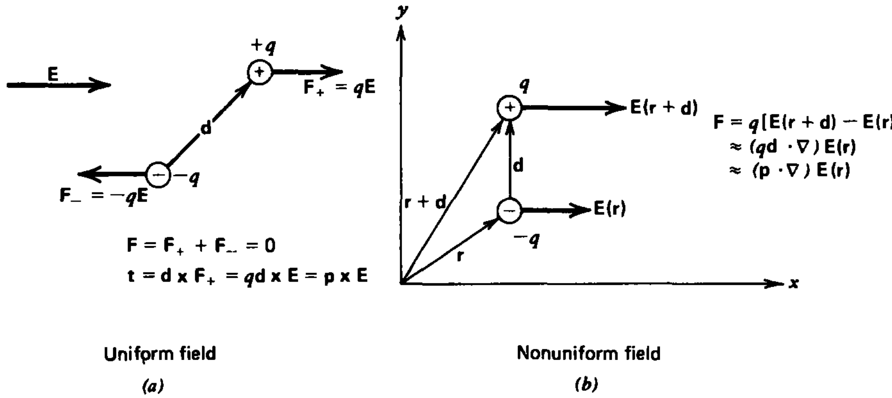


Figure 3-34 (*a*) A torque is felt by a dipole if its moment is not aligned with the electric field. In a uniform electric field there is no net force on a dipole because the force on each charge is equal in magnitude but opposite in direction. (*b*) There is a net force on a dipole only in a nonuniform field.

For a linear dielectric, this torque is zero because the polarization is induced by the field so that **P** and **E** are in the same direction.

A net force can be applied to a dipole if the electric field is different on each end, as in Figure 3-34*b*:

$$\mathbf{f} = -q[\mathbf{E}(\mathbf{r}) - \mathbf{E}(\mathbf{r}+\mathbf{d})] \tag{11}$$

For point dipoles, the dipole spacing **d** is very small so that the electric field at **r**+**d** can be expanded in a Taylor series as

$$\begin{aligned}\mathbf{E}(\mathbf{r}+\mathbf{d}) &\approx \mathbf{E}(\mathbf{r}) + d_x \frac{\partial}{\partial x}\mathbf{E}(\mathbf{r}) + d_y \frac{\partial}{\partial y}\mathbf{E}(\mathbf{r}) + d_z \frac{\partial}{\partial z}\mathbf{E}(\mathbf{r}) \\ &= \mathbf{E}(\mathbf{r}) + (\mathbf{d}\cdot\nabla)\mathbf{E}(\mathbf{r})\end{aligned} \tag{12}$$

Then the force on a point dipole is

$$\mathbf{f} = (q\mathbf{d}\cdot\nabla)\mathbf{E}(\mathbf{r}) = (\mathbf{p}\cdot\nabla)\mathbf{E}(\mathbf{r}) \tag{13}$$

If we have a distribution of such dipoles with number density N, the polarization force density is

$$\mathbf{F} = N\mathbf{f} = (N\mathbf{p}\cdot\nabla)\mathbf{E} = (\mathbf{P}\cdot\nabla)\mathbf{E} \tag{14}$$

Of course, if there is any free charge present we must also add the coulombic force density $\rho_f\mathbf{E}$.

(b) Permanently Polarized Medium

A permanently polarized material with polarization $P_0\mathbf{i}_y$ is free to slide between parallel plate electrodes, as is shown in Figure 3-35.

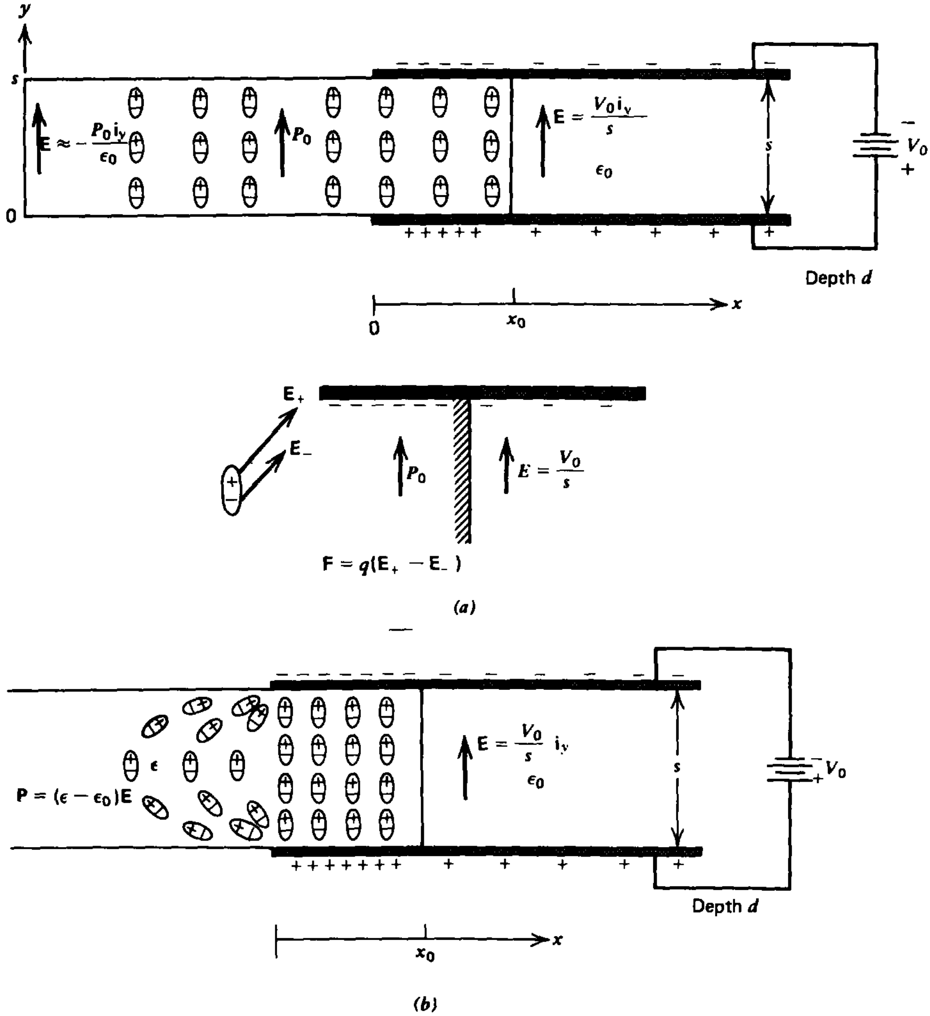


Figure 3-35 (*a*) A permanently polarized electret partially inserted into a capacitor has a force on it due to the Coulombic attraction between the dipole charges and the surface charge on the electrodes. The net force arises in the fringing field region as the end of the dipole further from the electrode edge feels a smaller electric field. Depending on the voltage magnitude and polarity, the electret can be pulled in or pushed out of the capacitor. (*b*) A linear dielectric is always attracted into a free space capacitor because of the net force on dipoles in the nonuniform field. The dipoles are now aligned with the electric field, no matter the voltage polarity.

We only know the electric field in the interelectrode region and from Example 3-2 far away from the electrodes:

$$E_y(x = x_0) = \frac{V_0}{s}, \qquad E_y(x = -\infty) = -\frac{P_0}{\varepsilon_0} \tag{15}$$

Unfortunately, neither of these regions contribute to the force because the electric field is uniform and (14) requires a field gradient for a force. The force arises in the fringing fields near the electrode edges where the field is nonuniform and, thus, exerts less of a force on the dipole end further from the electrode edges. At first glance it looks like we have a difficult problem because we do not know the fields where the force acts. However, because the electric field has zero curl,

$$\nabla\times\mathbf{E}=0\Rightarrow\frac{\partial E_x}{\partial y}=\frac{\partial E_y}{\partial x} \tag{16}$$

the x component of the force density can be written as

$$\begin{aligned}F_x&=P_y\frac{\partial E_x}{\partial y}\\&=P_y\frac{\partial E_y}{\partial x}\\&=\frac{\partial}{\partial x}(P_yE_y)-E_y\overset{0}{\cancel{\frac{\partial P_y}{\partial x}}}\end{aligned} \tag{17}$$

The last term in (17) is zero because $P_y=P_0$ is a constant. The total x directed force is then

$$\begin{aligned}f_x&=\int F_x\,dx\,dy\,dz\\&=\int_{x=-\infty}^{x_0}\int_{y=0}^{s}\int_{z=0}^{d}\frac{\partial}{\partial x}(P_yE_y)\,dx\,dy\,dz\end{aligned} \tag{18}$$

We do the x integration first so that the y and z integrations are simple multiplications as the fields at the limits of the x integration are independent of y and z:

$$f_x=P_0E_ysd\,\Big|_{x=-\infty}^{x_0}=P_0V_0d+\frac{P_0^2sd}{\varepsilon_0} \tag{19}$$

There is a force pulling the electret between the electrodes even if the voltage were zero due to the field generated by the surface charge on the electrodes induced by the electret. This force is increased if the imposed electric field and polarization are in the same direction. If the voltage polarity is reversed, the force is negative and the electret is pushed out if the magnitude of the voltage exceeds P_0s/ε_0.

(c) Linearly Polarized Medium

The problem is different if the slab is polarized by the electric field, as the polarization will then be in the direction of the electric field and thus have x and y components in the fringing fields near the electrode edges where the force arises, as in Figure 3-35*b*. The dipoles tend to line up as shown with the positive ends attracted towards the negative electrode and the negative dipole ends towards the positive electrode. Because the farther ends of the dipoles are in a slightly weaker field, there is a net force to the right tending to draw the dielectric into the capacitor.

The force density of (14) is

$$F_x = P_x \frac{\partial E_x}{\partial x} + P_y \frac{\partial E_x}{\partial y} = (\varepsilon - \varepsilon_0)\left(E_x \frac{\partial E_x}{\partial x} + E_y \frac{\partial E_x}{\partial y}\right) \tag{20}$$

Because the electric field is curl free, as given in (16), the force density is further simplified to

$$F_x = \frac{(\varepsilon - \varepsilon_0)}{2} \frac{\partial}{\partial x}(E_x^2 + E_y^2) \tag{21}$$

The total force is obtained by integrating (21) over the volume of the dielectric:

$$\begin{aligned} f_x &= \int_{x=-\infty}^{x_0} \int_{y=0}^{s} \int_{z=0}^{d} \frac{(\varepsilon - \varepsilon_0)}{2} \frac{\partial}{\partial x}(E_x^2 + E_y^2)\, dx\, dy\, dz \\ &= \frac{(\varepsilon - \varepsilon_0)sd}{2}(E_x^2 + E_y^2)\Big|_{x=-\infty}^{x_0} = \frac{(\varepsilon - \varepsilon_0)}{2} \frac{V_0^2 d}{s} \end{aligned} \tag{22}$$

where we knew that the fields were zero at $x = -\infty$ and uniform at $x = x_0$:

$$E_y(x_0) = V_0/s, \qquad E_x(x_0) = 0 \tag{23}$$

The force is now independent of voltage polarity and always acts in the direction to pull the dielectric into the capacitor if $\varepsilon > \varepsilon_0$.

3-9-3 Forces on a Capacitor

Consider a capacitor that has one part that can move in the x direction so that the capacitance depends on the coordinate x:

$$q = C(x)v \tag{24}$$

The current is obtained by differentiating the charge with respect to time:

$$\begin{aligned} i &= \frac{dq}{dt} = \frac{d}{dt}[C(x)v] = C(x)\frac{dv}{dt} + v\frac{dC(x)}{dt} \\ &= C(x)\frac{dv}{dt} + v\frac{dC(x)}{dx}\frac{dx}{dt} \end{aligned} \tag{25}$$

Note that this relation has an extra term over the usual circuit formula, proportional to the speed of the moveable member, where we expanded the time derivative of the capacitance by the chain rule of differentiation. Of course, if the geometry is fixed and does not change with time ($dx/dt = 0$), then (25) reduces to the usual circuit expression. The last term is due to the electro-mechanical coupling.

The power delivered to a time-dependent capacitance is

$$p = vi = v\frac{d}{dt}[C(x)v] \tag{26}$$

which can be expanded to the form

$$\begin{aligned} p &= \frac{d}{dt}[\tfrac{1}{2}C(x)v^2] + \tfrac{1}{2}v^2\frac{dC(x)}{dt} \\ &= \frac{d}{dt}[\tfrac{1}{2}C(x)v^2] + \tfrac{1}{2}v^2\frac{dC(x)}{dx}\frac{dx}{dt} \end{aligned} \tag{27}$$

where the last term is again obtained using the chain rule of differentiation. This expression can be put in the form

$$p = \frac{dW}{dt} + f_x\frac{dx}{dt} \tag{28}$$

where we identify the power p delivered to the capacitor as going into increasing the energy storage W and mechanical power $f_x\, dx/dt$ in moving a part of the capacitor:

$$W = \tfrac{1}{2}C(x)v^2, \qquad f_x = \tfrac{1}{2}v^2\frac{dC(x)}{dx} \tag{29}$$

Using (24), the stored energy and force can also be expressed in terms of the charge as

$$W = \frac{1}{2}\frac{q^2}{C(x)}, \qquad f_x = \frac{1}{2}\frac{q^2}{C^2(x)}\frac{dC(x)}{dx} = -\tfrac{1}{2}q^2\frac{d[1/C(x)]}{dx} \tag{30}$$

To illustrate the ease in using (29) or (30) to find the force, consider again the partially inserted dielectric in Figure 3-35*b*. The capacitance when the dielectric extends a distance x into the electrodes is

$$C(x) = \frac{\varepsilon x d}{s} + \varepsilon_0\frac{(l-x)d}{s} \tag{31}$$

so that the force on the dielectric given by (29) agrees with (22):

$$f_x = \tfrac{1}{2}V_0^2\frac{dC(x)}{dx} = \tfrac{1}{2}(\varepsilon - \varepsilon_0)\frac{V_0^2 d}{s} \tag{32}$$

Note that we neglected the fringing field contributions to the capacitance in (31) even though they are the physical origin of the force. The results agree because this extra capacitance does not depend on the position x of the dielectric when x is far from the electrode edges.

This method can only be used for linear dielectric systems described by (24). It is not valid for the electret problem treated in Section 3-9-2*b* because the electrode charge is not linearly related to the voltage, being in part induced by the electret.

EXAMPLE 3-4 FORCE ON A PARALLEL PLATE CAPACITOR

Two parallel, perfectly conducting electrodes of area A and a distance x apart are shown in Figure 3-36. For each of the following two configurations, find the force on the upper electrode in the x direction when the system is constrained to constant voltage V_0 or constant charge Q_0.

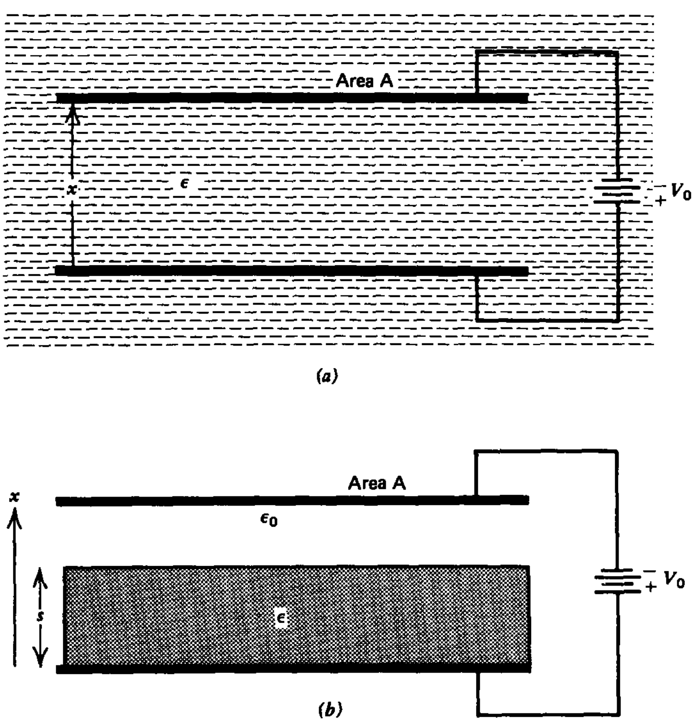


Figure 3-36 A parallel plate capacitor (a) immersed within a dielectric fluid or with (b) a free space region in series with a solid dielectric.

(a) Liquid Dielectric

The electrodes are immersed within a liquid dielectric with permittivity ε, as shown in Figure 3-36a.

SOLUTION

The capacitance of the system is

$$C(x)=\varepsilon A/x$$

so that the force from (29) for constant voltage is

$$f_x=\frac{1}{2}V_0^2\frac{dC(x)}{dx}=-\frac{1}{2}\frac{\varepsilon A V_0^2}{x^2}$$

The force being negative means that it is in the direction opposite to increasing x, in this case downward. The capacitor plates attract each other because they are oppositely charged and opposite charges attract. The force is independent of voltage polarity and gets infinitely large as the plate spacing approaches zero. The result is also valid for free space with $\varepsilon=\varepsilon_0$. The presence of the dielectric increases the attractive force.

If the electrodes are constrained to a constant charge Q_0 the force is then attractive but independent of x:

$$f_x=-\frac{1}{2}Q_0^2\frac{d}{dx}\frac{1}{C(x)}=-\frac{1}{2}\frac{Q_0^2}{\varepsilon A}$$

For both these cases, the numerical value of the force is the same because Q_0 and V_0 are related by the capacitance, but the functional dependence on x is different. The presence of a dielectric now decreases the force over that of free space.

(b) Solid Dielectric

A solid dielectric with permittivity ε of thickness s is inserted between the electrodes with the remainder of space having permittivity ε_0, as shown in Figure 3-36b.

SOLUTION

The total capacitance for this configuration is given by the series combination of capacitance due to the dielectric block and the free space region:

$$C(x)=\frac{\varepsilon\varepsilon_0 A}{\varepsilon_0 s+\varepsilon(x-s)}$$

The force on the upper electrode for constant voltage is

$$f_x = \tfrac{1}{2}V_0^2 \frac{d}{dx}C(x) = -\frac{\varepsilon^2\varepsilon_0 A V_0^2}{2[\varepsilon_0 s + \varepsilon(x-s)]^2}$$

If the electrode just rests on the dielectric so that $x = s$, the force is

$$f_x = -\frac{\varepsilon^2 A V_0^2}{2\varepsilon_0 s^2}.$$

This result differs from that of part (*a*) when $x = s$ by the factor $\varepsilon_r = \varepsilon/\varepsilon_0$ because in this case moving the electrode even slightly off the dielectric leaves a free space region in between. In part (*a*) no free space gap develops as the liquid dielectric fills in the region, so that the dielectric is always in contact with the electrode. The total force on the electrode-dielectric interface is due to both free and polarization charge.

With the electrodes constrained to constant charge, the force on the upper electrode is independent of position and also independent of the permittivity of the dielectric block:

$$f_x = -\tfrac{1}{2}Q_0^2 \frac{d}{dx}\frac{1}{C(x)} = -\frac{1}{2}\frac{Q_0^2}{\varepsilon_0 A}$$

3-10 ELECTROSTATIC GENERATORS

3-10-1 Van de Graaff Generator

In the 1930s, reliable means of generating high voltages were necessary to accelerate charged particles in atomic studies. In 1931, Van de Graaff developed an electrostatic generator where charge is sprayed onto an insulating moving belt that transports this charge onto a conducting dome, as illustrated in Figure 3-37a. If the dome was considered an isolated sphere of radius R, the capacitance is given as $C = 4\pi\varepsilon_0 R$. The transported charge acts as a current source feeding this capacitance, as in Figure 3-37*b*, so that the dome voltage builds up linearly with time:

$$i = C\frac{dv}{dt} \Rightarrow v = \frac{i}{C}t \qquad (1)$$

This voltage increases until the breakdown strength of the surrounding atmosphere is reached, whereupon a spark discharge occurs. In air, the electric field breakdown strength E_b is 3×10^6 V/m. The field near the dome varies as $E_r = VR/r^2$, which is maximum at $r = R$, which implies a maximum voltage of $V_{max} = E_b R$. For $V_{max} = 10^6$ V, the radius of the sphere

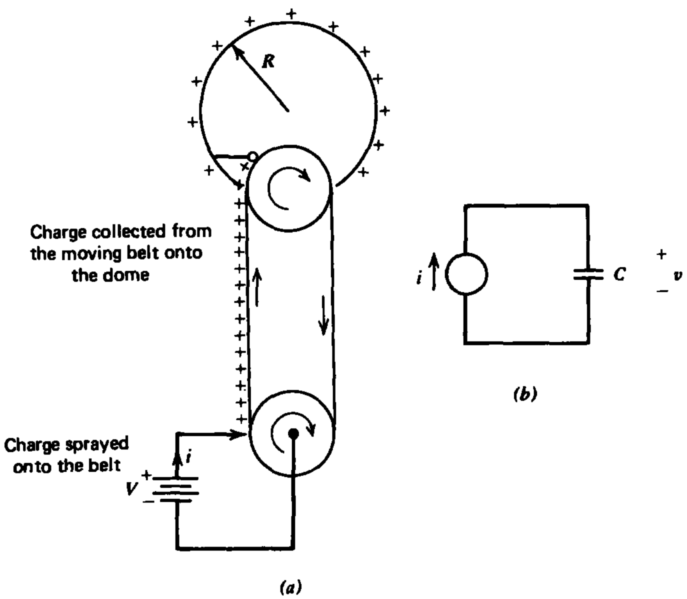


Figure 3-37 (*a*) A Van de Graaff generator consists of a moving insulating belt that transports injected charge onto a conducting dome which can thus rise to very high voltages, easily in excess of a million volts. (*b*) A simple equivalent circuit consists of the convecting charge modeled as a current source charging the capacitance of the dome.

must be $R \approx \frac{1}{3}$ m so that the capacitance is $C \approx 37$ pf. With a charging current of one microampere, it takes $t \approx 37$ sec to reach this maximum voltage.

3-10-2 Self-Excited Electrostatic Induction Machines

In the Van de Graaff generator, an external voltage source is necessary to deposit charge on the belt. In the late 1700s, self-excited electrostatic induction machines were developed that did not require any external electrical source. To understand how these devices work, we modify the Van de Graaff generator configuration, as in Figure 3-38a, by putting conducting segments on the insulating belt. Rather than spraying charge, we place an electrode at voltage V with respect to the lower conducting pulley so that opposite polarity charge is induced on the moving segments. As the segments move off the pulley, they carry their charge with them. So far, this device is similar to the Van de Graaff generator using induced charge rather than sprayed charge and is described by the same equivalent circuit where the current source now depends on the capacitance C_i between the inducing electrode and the segmented electrodes, as in Figure 3-38*b*.

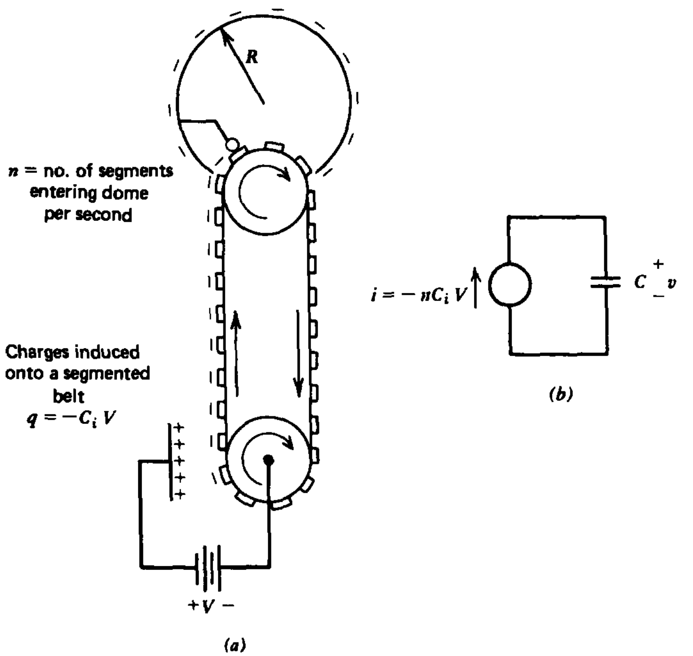


Figure 3-38 A modified Van de Graaff generator as an electrostatic induction machine. (*a*) Here charges are induced onto a segmented belt carrying insulated conductors as the belt passes near an electrode at voltage V. (*b*) Now the current source feeding the capacitor equivalent circuit depends on the capacitance C_i between the electrode and the belt.

Now the early researchers cleverly placed another induction machine nearby as in Figure 3-39*a*. Rather than applying a voltage source, because one had not been invented yet, they electrically connected the dome of each machine to the inducer electrode of the other. The induced charge on one machine was proportional to the voltage on the other dome. Although no voltage is applied, any charge imbalance on an inducer electrode due to random noise or stray charge will induce an opposite charge on the moving segmented belt that carries this charge to the dome of which some appears on the other inducer electrode where the process is repeated with opposite polarity charge. The net effect is that the charge on the original inducer has been increased.

More quantitatively, we use the pair of equivalent circuits in Figure 3-39*b* to obtain the coupled equations

$$-nC_iv_1 = C\frac{dv_2}{dt}, \qquad -nC_iv_2 = C\frac{dv_1}{dt} \tag{2}$$

where n is the number of segments per second passing through the dome. All voltages are referenced to the lower pulleys that are electrically connected together. Because these

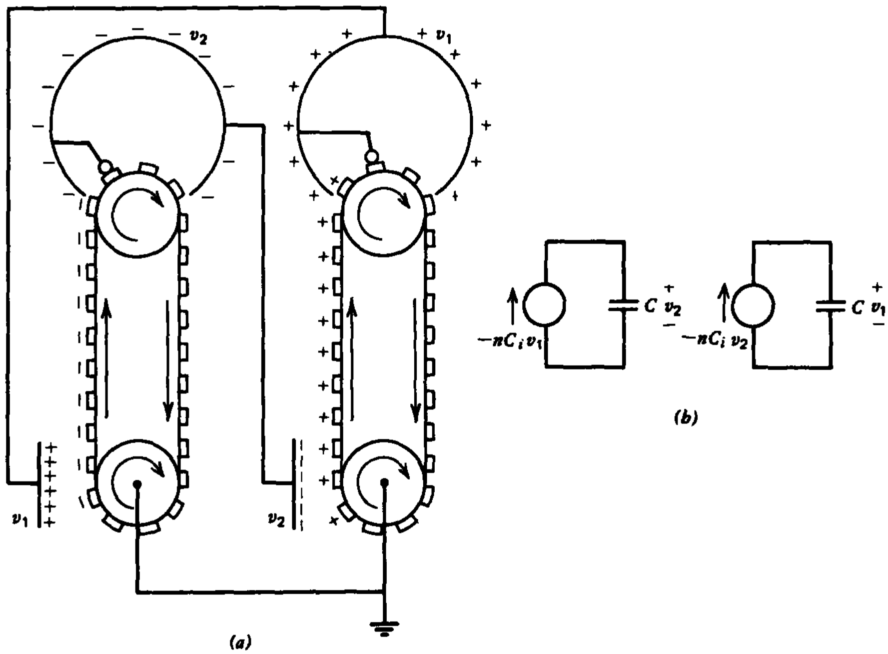


Figure 3-39 (a) A pair of coupled self-excited electrostatic induction machines generate their own inducing voltage. (b) The system is described by two simple coupled circuits.

are linear constant coefficient differential equations, the solutions must be exponentials:

$$v_1 = \hat{V}_1 e^{st}, \qquad v_2 = \hat{V}_2 e^{st} \tag{3}$$

Substituting these assumed solutions into (2) yields the following characteristic roots:

$$s^2 = \left(\frac{nC_i}{C}\right)^2 \Rightarrow s = \pm\frac{nC_i}{C} \tag{4}$$

so that the general solution is

$$\begin{aligned} v_1 &= A_1 e^{(nC_i/C)t} + A_2 e^{-(nC_i/C)t} \\ v_2 &= -A_1 e^{(nC_i/C)t} + A_2 e^{-(nC_i/C)t} \end{aligned} \tag{5}$$

where A_1 and A_2 are determined from initial conditions.

The negative root of (4) represents the uninteresting decaying solutions while the positive root has solutions that grow unbounded with time. This is why the machine is self-excited. Any initial voltage perturbation, no matter how small, increases without bound until electrical breakdown is reached. Using representative values of $n = 10$, $C_i = 2$ pf, and $C = 10$ pf, we have that $s = \pm 2$ so that the time constant for voltage build-up is about one-half second.

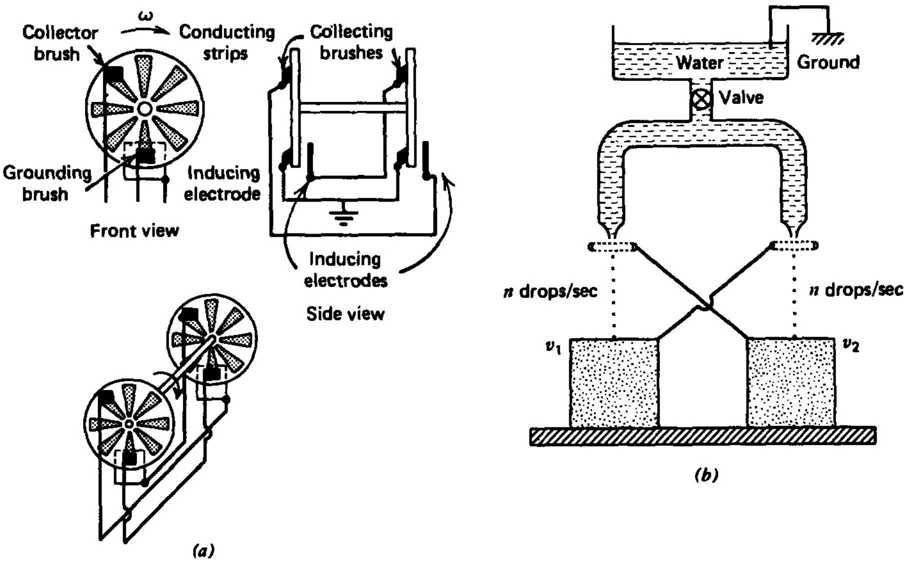


Figure 3-40 Other versions of self-excited electrostatic induction machines use (*a*) rotating conducting strips (Wimshurst machine) or (*b*) falling water droplets (Lord Kelvin's water dynamo). These devices are also described by the coupled equivalent circuits in Figure 3-39*b*.

The early electrical scientists did not use a segmented belt but rather conducting disks embedded in an insulating wheel that could be turned by hand, as shown for the Wimshurst machine in Figure 3-40*a*. They used the exponentially growing voltage to charge up a capacitor called a Leyden jar (credited to scientists from Leyden, Holland), which was a glass bottle silvered on the inside and outside to form two electrodes with the glass as the dielectric.

An analogous water drop dynamo was invented by Lord Kelvin (then Sir W. Thomson) in 1861, which replaced the rotating disks by falling water drops, as in Figure 3-40*b*. All these devices are described by the coupled equivalent circuits in Figure 3-39*b*.

3-10-3 Self-Excited Three-Phase Alternating Voltages

In 1967, Euerle* modified Kelvin's original dynamo by adding a third stream of water droplets so that three-phase

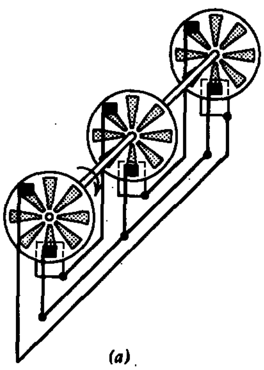
* *W. C. Euerle, "A Novel Method of Generating Polyphase Power," M.S. Thesis, Massachusetts Institute of Technology, 1967. See also J. R. Melcher, Electric Fields and Moving Media,* IEEE Trans. Education ***E-17*** *(1974), pp. 100–110, and the film by the same title produced for the National Committee on Electrical Engineering Films by the Educational Development Center, 39 Chapel St., Newton, Mass. 02160.*

alternating voltages were generated. The analogous three-phase Wimshurst machine is drawn in Figure 3-41*a* with equivalent circuits in Figure 3-41*b*. Proceeding as we did in (2) and (3),

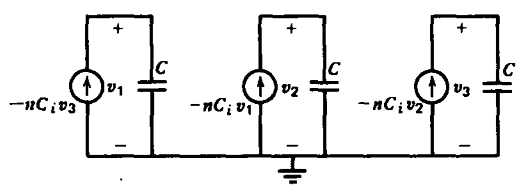
$$\begin{aligned} -nC_iv_1 &= C\frac{dv_2}{dt}, \qquad v_1=\hat{V}_1 e^{st} \\ -nC_iv_2 &= C\frac{dv_3}{dt}, \qquad v_2=\hat{V}_2 e^{st} \\ -nC_iv_3 &= C\frac{dv_1}{dt}, \qquad v_3=\hat{V}_3 e^{st} \end{aligned} \tag{6}$$

equation (6) can be rewritten as

$$\begin{bmatrix} nC_i & Cs & 0 \\ 0 & nC_i & Cs \\ Cs & 0 & nC_i \end{bmatrix} \begin{bmatrix} \hat{V}_1 \\ \hat{V}_2 \\ \hat{V}_3 \end{bmatrix} = 0 \tag{7}$$



(a)



(b)

Figure 3-41 (*a*) Self-excited three-phase ac Wimshurst machine. (*b*) The coupled equivalent circuit is valid for any of the analogous machines discussed.

which requires that the determinant of the coefficients of $\hat{V}_1$, $\hat{V}_2$, and $\hat{V}_3$ be zero:

$$(nC_i)^3+(Cs)^3=0 \Rightarrow s=\left(\frac{nC_i}{C}\right)^{1/3}(-1)^{1/3}$$

$$=\left(\frac{nC_i}{C}\right)^{1/3} e^{j(\pi/3)(2r-1)}, \qquad r=1,2,3 \tag{8}$$

$$\Rightarrow s_1=-\frac{nC_i}{C}$$

$$s_{2,3}=\frac{nC_i}{2C}[1\pm\sqrt{3}\,j]$$

where we realized that $(-1)^{1/3}$ has three roots in the complex plane. The first root is an exponentially decaying solution, but the other two are complex conjugates where the positive real part means exponential growth with time while the imaginary part gives the frequency of oscillation. We have a self-excited three-phase generator as each voltage for the unstable modes is 120° apart in phase from the others:

$$\frac{\hat{V}_2}{\hat{V}_1}=\frac{\hat{V}_3}{\hat{V}_2}=\frac{\hat{V}_1}{\hat{V}_3}=-\frac{nC_i}{Cs_{2,3}}=-\tfrac{1}{2}(1\pm\sqrt{3}j)=e^{\pm j(2/3)\pi} \tag{9}$$

Using our earlier typical values following (5), we see that the oscillation frequencies are very low, $f=(1/2\pi)\,\mathrm{Im}\,(s)\approx 0.28$ Hz.

3-10-4 Self-Excited Multi-frequency Generators

If we have N such generators, as in Figure 3-42, with the last one connected to the first one, the kth equivalent circuit yields

$$-nC_i\hat{V}_k=Cs\hat{V}_{k+1} \tag{10}$$

This is a linear constant coefficient difference equation. Analogously to the exponential time solutions in (3) valid for linear constant coefficient differential equations, solutions to (10) are of the form

$$\hat{V}_k=A\lambda^k \tag{11}$$

where the characteristic root λ is found by substitution back into (10) to yield

$$-nC_iA\lambda^k=CsA\lambda^{k+1} \Rightarrow \lambda=-nC_i/Cs \tag{12}$$

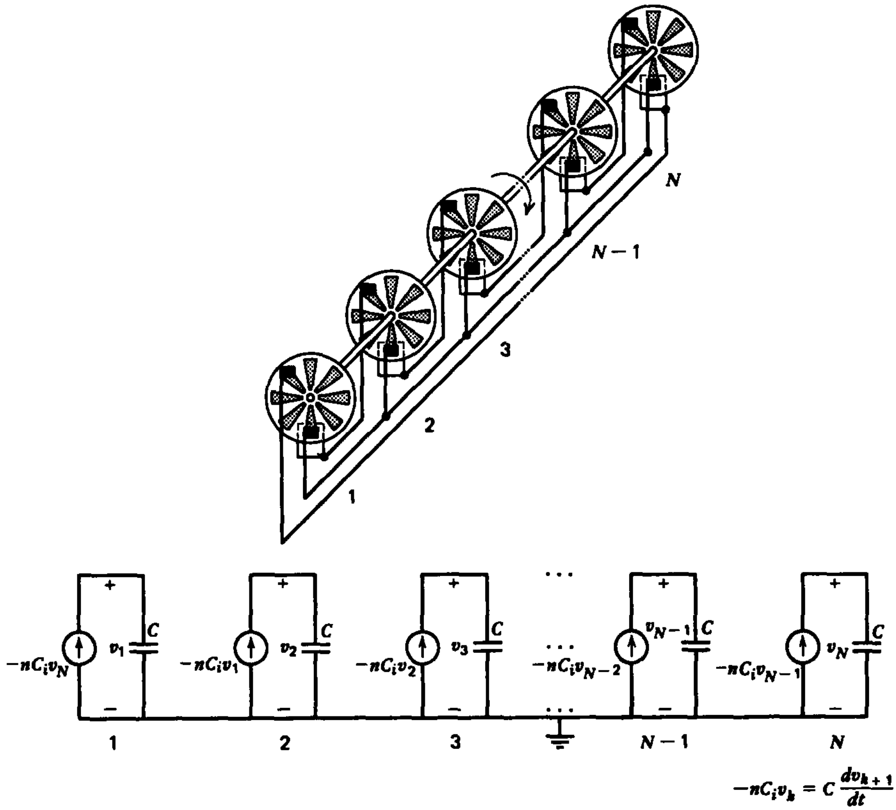


Figure 3-42 Multi-frequency, polyphase self-excited Wimshurst machine with equivalent circuit.

Since the last generator is coupled to the first one, we must have that

$$V_{N+1}=V_1 \Rightarrow \lambda^{N+1}=\lambda^1$$

$$\Rightarrow \lambda^N = 1$$

$$\Rightarrow \lambda = 1^{1/N} = e^{j2\pi r/N}, \qquad r=1,2,3,\ldots,N \tag{13}$$

where we realize that unity has N complex roots.

The system natural frequencies are then obtained from (12) and (13) as

$$s=-\frac{nC_i}{C\lambda}=-\frac{nC_i}{C}e^{-j2\pi r/N}, \qquad r=1,2,\ldots,N \tag{14}$$

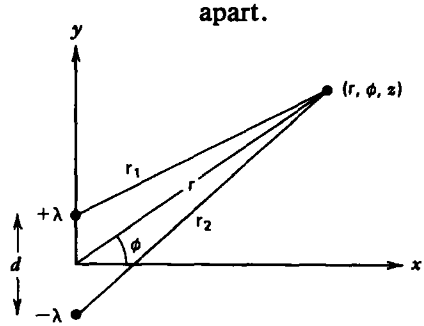
We see that for $N=2$ and $N=3$ we recover the results of (4) and (8). All the roots with a positive real part of s are unstable and the voltages spontaneously build up in time with oscillation frequencies ω_0 given by the imaginary part of s.

$$\omega_0=|\mathrm{Im}\,(s)|=\frac{nC_i}{C}|\sin 2\pi r/N| \tag{15}$$

PROBLEMS

Section 3-1

1. A two-dimensional dipole is formed by two infinitely long parallel line charges of opposite polarity $\pm\lambda$ a small distance $d\mathbf{i}_y$ apart.

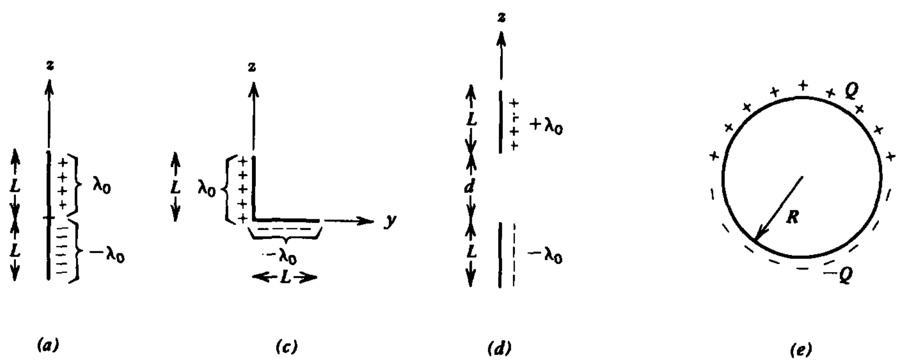


(a) What is the potential at any coordinate (r, ϕ, z)?

(b) What are the potential and electric field far from the dipole $(r \gg d)$? What is the dipole moment per unit length?

(c) What is the equation of the field lines?

2. Find the dipole moment for each of the following charge distributions:



(a) Two uniform colinear opposite polarity line charges $\pm\lambda_0$ each a small distance L along the z axis.

(b) Same as (a) with the line charge distribution as

$$\lambda(z) = \begin{cases} \lambda_0(1 - z/L), & 0 < z < L \\ -\lambda_0(1 + z/L), & -L < z < 0 \end{cases}$$

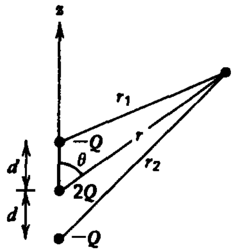
(c) Two uniform opposite polarity line charges $\pm\lambda_0$ each of length L but at right angles.

(d) Two parallel uniform opposite polarity line charges $\pm\lambda_0$ each of length L a distance $d\mathbf{i}_z$ apart.

(e) A spherical shell with total uniformly distributed surface charge Q on the upper half and $-Q$ on the lower half. (**Hint:** $\mathbf{i}_r = \sin\theta\cos\phi\,\mathbf{i}_x + \sin\theta\sin\phi\,\mathbf{i}_y + \cos\theta\,\mathbf{i}_z$.)

(f) A spherical volume with total uniformly distributed volume charge of Q in the upper half and $-Q$ on the lower half. (**Hint:** Integrate the results of (e).)

3. The linear quadrapole consists of two superposed dipoles along the z axis. Find the potential and electric field for distances far away from the charges ($r \gg d$).



$$\frac{1}{r_1} \approx \frac{1}{r}\left\{1 + \frac{d}{r}\cos\theta - \frac{1}{2}\left(\frac{d}{r}\right)^2(1 - 3\cos^2\theta)\right\}$$

$$\frac{1}{r_2} \approx \frac{1}{r}\left\{1 - \frac{d}{r}\cos\theta - \frac{1}{2}\left(\frac{d}{r}\right)^2(1 - 3\cos^2\theta)\right\}$$

Linear quadrapole

4. Model an atom as a fixed positive nucleus of charge Q with a surrounding spherical negative electron cloud of nonuniform charge density:

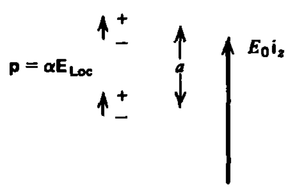
$$\rho = -\rho_0(1 - r/R_0), \qquad r < R_0$$

(a) If the atom is neutral, what is ρ_0?

(b) An electric field is applied with local field $\mathbf{E}_{Loc}$ causing a slight shift $\mathbf{d}$ between the center of the spherical cloud and the positive nucleus. What is the equilibrium dipole spacing?

(c) What is the approximate polarizability α if $9\varepsilon_0 E_{Loc}/(\rho_0 R_0) \ll 1$?

5. Two colinear dipoles with polarizability α are a distance a apart along the z axis. A uniform field $E_0\mathbf{i}_z$ is applied.



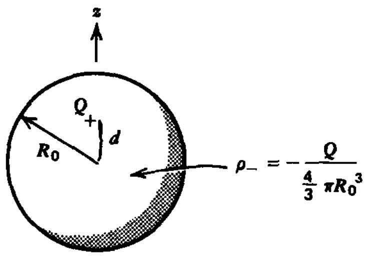
(a) What is the total local field seen by each dipole?

(b) Repeat (a) if we have an infinite array of dipoles with constant spacing a. (**Hint:** $\sum_{n=1}^{\infty} 1/n^3 \approx 1.2$.)

(c) If we assume that we have one such dipole within each volume of a^3, what is the permittivity of the medium?

6. A dipole is modeled as a point charge Q surrounded by a spherical cloud of electrons with radius R_0. Then the local

field $\mathbf{E}_{\text{Loc}}$ differs from the applied field $\mathbf{E}$ by the field due to the dipole itself. Since $\mathbf{E}_{\text{dip}}$ varies within the spherical cloud, we use the average field within the sphere.



(a) Using the center of the cloud as the origin, show that the dipole electric field within the cloud is

$$\mathbf{E}_{\text{dip}} = -\frac{Qr\mathbf{i}_r}{4\pi\varepsilon_0 R_0^3} + \frac{Q(r\mathbf{i}_r - d\mathbf{i}_z)}{4\pi\varepsilon_0[d^2 + r^2 - 2rd\cos\theta]^{3/2}}$$

(b) Show that the average x and y field components are zero. (**Hint:** $\mathbf{i}_r = \sin\theta\cos\phi\mathbf{i}_x + \sin\theta\sin\phi\mathbf{i}_y + \cos\theta\mathbf{i}_z$.)

(c) What is the average z component of the field? (**Hint:** Change variables to $u = r^2 + d^2 - 2rd\cos\theta$ and remember $\sqrt{(r-d)^2} = |r-d|$.)

(d) If we have one dipole within every volume of $\frac{4}{3}\pi R_0^3$, how is the polarization $\mathbf{P}$ related to the applied field $\mathbf{E}$?

7. Assume that in the dipole model of Figure 3-5a the mass of the positive charge is so large that only the electron cloud moves as a solid mass m.

(a) The local electric field is $\mathbf{E}_0$. What is the dipole spacing?

(b) At $t = 0$, the local field is turned off ($\mathbf{E}_0 = 0$). What is the subsequent motion of the electron cloud?

(c) What is the oscillation frequency if Q has the charge and mass of an electron with $R_0 = 10^{-10}$ m?

(d) In a real system there is always some damping that we take to be proportional to the velocity ($\mathbf{f}_{\text{damping}} = -\beta\mathbf{v}$). What is the equation of motion of the electron cloud for a sinusoidal electric field $\text{Re}(\hat{E}_0 e^{j\omega t})$?

(e) Writing the driven displacement of the dipole as

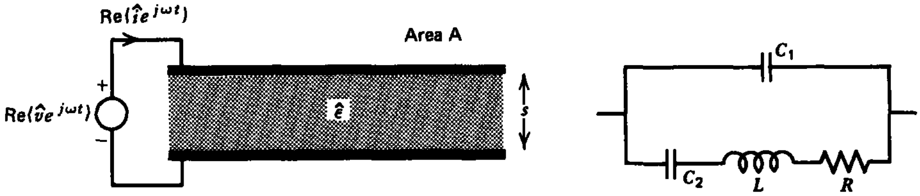
$$d = \text{Re}(\hat{d}e^{j\omega t}).$$

what is the complex polarizability $\hat{\alpha}$, where $\hat{p} = Q\hat{d} = \hat{\alpha}E_0$?

(f) What is the complex dielectric constant $\hat{\varepsilon} = \varepsilon_r + j\varepsilon_i$ of the system? (**Hint:** Define $\omega_p^2 = Q^2N/(m\varepsilon_0)$.)

(g) Such a dielectric is placed between parallel plate electrodes. Show that the equivalent circuit is a series R, L, C shunted by a capacitor. What are C_1, C_2, L, and R?

(h) Consider the limit where the electron cloud has no mass ($m = 0$). With the frequency ω as a parameter show that

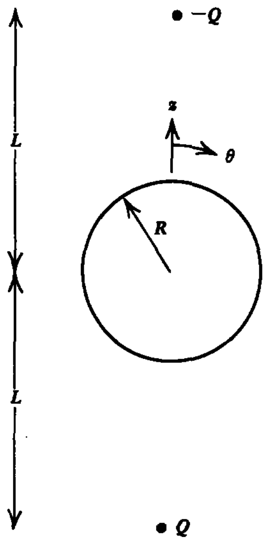


(g)

a plot of ε_r versus ε_i is a circle. Where is the center of the circle and what is its radius? Such a diagram is called a Cole-Cole plot.

(i) What is the maximum value of ε_i and at what frequency does it occur?

8. Two point charges of opposite sign $\pm Q$ are a distance L above and below the center of a grounded conducting sphere of radius R.



(a) What is the electric field everywhere along the z axis and in the $\theta = \pi/2$ plane? (**Hint:** Use the method of images.)

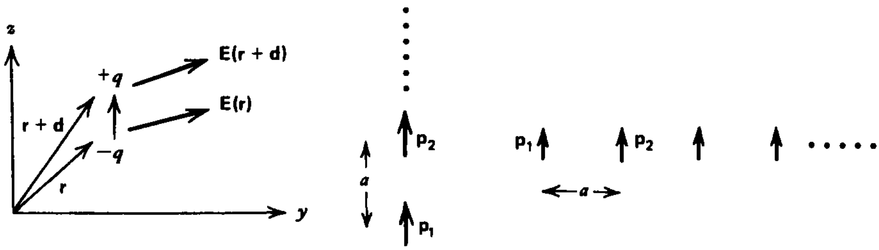
(b) We would like this problem to model the case of a conducting sphere in a uniform electric field by bringing the point charges $\pm Q$ out to infinity ($L \to \infty$). What must the ratio Q/L^2 be such that the field far from the sphere in the $\theta = \pi/2$ plane is $E_0\mathbf{i}_z$?

(c) In this limit, what is the electric field everywhere?

9. A dipole with moment $\mathbf{p}$ is placed in a nonuniform electric field.

(a) Show that the force on a dipole is

$$\mathbf{f} = (\mathbf{p} \cdot \nabla)\mathbf{E}$$

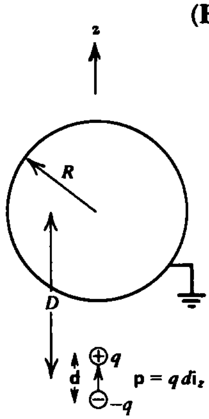


(a) (b), (c)

(b) Find the force on dipole 1 due to dipole 2 when the two dipoles are colinear, or are adjacent a distance a apart.

(c) Find the force on dipole 1 if it is the last dipole in an infinite array of identical colinear or adjacent dipoles with spacing a. (**Hint:** $\sum_{n=1}^{\infty} 1/n^4 = \pi^4/90$.)

10. A point dipole with moment $p\mathbf{i}_z$ is a distance D from the center of a grounded sphere of radius R.
(**Hint:** $d \ll D$.)



(a) What is the induced dipole moment of the sphere?

(b) What is the electric field everywhere along the z axis?

(c) What is the force on the sphere? (**Hint:** See Problem 9*a*.)

Section 3-2

11. Find the potential, electric field, and charge density distributions for each of the following charges placed within a medium of infinite extent, described by drift-diffusion conduction in the limit when the electrical potential is much less than the thermal voltage ($qV/kT \ll 1$):

(a) Sheet of surface charge σ_f placed at $x = 0$.
(b) Infinitely long line charge with uniform density λ. (**Hint:** Bessel's equation results.)
(c) Conducting sphere of radius R carrying a total surface charge Q.

12. Two electrodes at potential $\pm V_0/2$ located at $x = \pm l$ enclose a medium described by drift-diffusion conduction for two oppositely charged carriers, where $qV_0/kT \ll 1$.

(a) Find the approximate solutions of the potential, electric field, and charge density distributions. What is the charge polarity near each electrode?

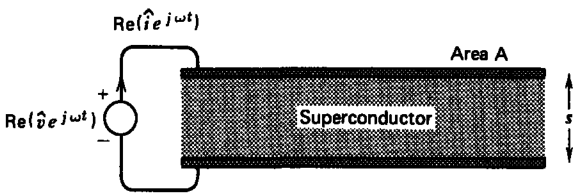
(b) What is the total charge per unit area within the volume of the medium and on each electrode?

13. (a) Neglecting diffusion effects but including charge inertia and collisions, what is the time dependence of the velocity of charge carriers when an electric field $E_0\mathbf{i}_x$ is instantaneously turned on at $t = 0$?

(b) After the charge carriers have reached their steady-state velocity, the electric field is suddenly turned off. What is their resulting velocity?

(c) This material is now placed between parallel plate electrodes of area A and spacing s. A sinusoidal voltage is applied $\text{Re}(V_0 e^{j\omega t})$. What is the equivalent circuit?

14. Parallel plate electrodes enclose a superconductor that only has free electrons with plasma frequency ω_{pe}.



(a) What is the terminal current when a sinusoidal voltage is applied?

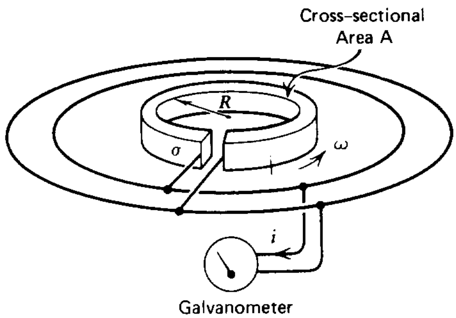
(b) What is the equivalent circuit?

15. A conducting ring of radius R is rotated at constant angular speed. The ring has Ohmic conductivity σ and cross sectional area A. A galvanometer is connected to the ends of the ring to indicate the passage of any charge. The connection is made by slip rings so that the rotation of the ring is unaffected by the galvanometer. The ring is instantly stopped, but the electrons within the ring continue to move a short time until their momentum is dissipated by collisions. For a particular electron of charge q and mass m conservation of momentum requires

$$\Delta(mv) = \int F\,dt$$

where $F = qE$ is the force on the electron.

(a) For the Ohmic conductor, relate the electric field to the current in the wire.

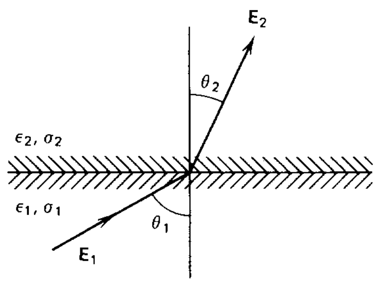


(b) When the ring is instantly stopped, what is the charge Q through the galvanometer? (**Hint:** $Q=\int i\,dt$. This experiment is described by R. C. Tolman and T. D. Stewart, *Phys. Rev.* **8**, No. 2 (1916), p. 97.)

(c) If the ring is an electron superconductor with plasma frequency ω_p, what is the resulting current in the loop when it stops?

Section 3.3

16. An electric field with magnitude E_1 is incident upon the interface between two materials at angle θ_1 from the normal. For each of the following material properties find the magnitude and direction of the field $\mathbf{E}_2$ in region 2.



(a) Lossless dielectrics with respective permittivities ε_1 and ε_2. There is no interfacial surface charge.

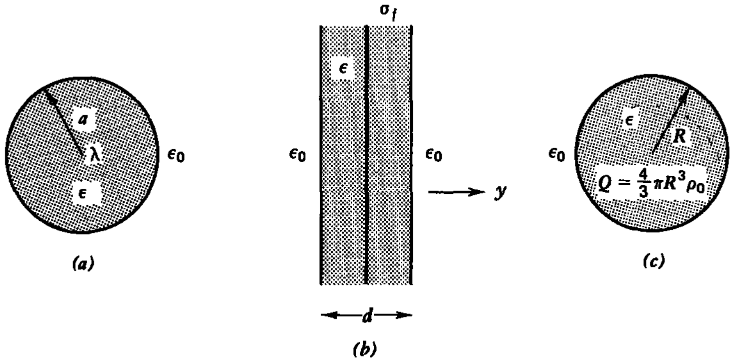
(b) Ohmic materials with respective conductivities σ_1 and σ_2 in the dc steady state. What is the free surface charge density σ_f on the interface?

(c) Lossy dielectrics $(\varepsilon_1, \sigma_1)$ and $(\varepsilon_2, \sigma_2)$ with a sinusoidally varying electric field

$$E_1 = \text{Re}\,(\hat{E}_1 e^{j\omega t})$$

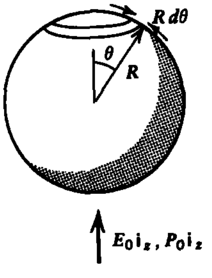
What is the free surface charge density σ_f on the interface?

17. Find the electric, displacement, and polarization fields and the polarization charge everywhere for each of the following configurations:



(a) An infinitely long line charge λ placed at the center of a dielectric cylinder of radius a and permittivity ε.
(b) A sheet of surface charge σ_f placed at the center of a dielectric slab with permittivity ε and thickness d.
(c) A uniformly charged dielectric sphere with permittivity ε and radius R carrying a total free charge Q.

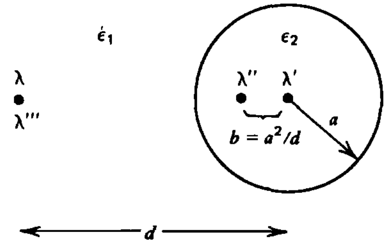
18. Lorentz calculated the local field acting on a dipole due to a surrounding uniformly polarized medium stressed by a macroscopic field $E_0\mathbf{i}_z$ by encircling the dipole with a small spherical free space cavity of radius R.



(a) If the medium outside the cavity has polarization $P_0\mathbf{i}_z$, what is the surface polarization charge on the spherical interface? (**Hint:** $\mathbf{i}_z = \mathbf{i}_r \cos\theta - \mathbf{i}_\theta \sin\theta$)
(b) Break this surface polarization charge distribution into hoop line charge elements of thickness $d\theta$. What is the total charge on a particular shell at angle θ?
(c) What is the electric field due to this shell at the center of the sphere where the dipole is?
(d) By integrating over all shells, find the total electric field acting on the dipole. This is called the Lorentz field. (**Hint:** Let $u = \cos\theta$).

19. A line charge λ within a medium of permittivity ε_1 is outside a dielectric cylinder of radius a and permittivity ε_2.

The line charge is parallel to the cylinder axis and a distance d from it.



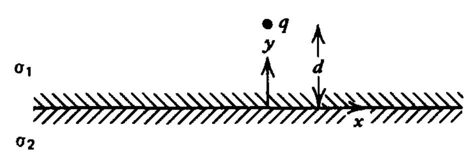
(a) Try using the method of images by placing a line charge λ' at the center and another image λ'' within the cylinder at distance $b = a^2/d$ from the axis along the line joining the axis to the line charge. These image charges together with the original line charge will determine the electric field outside the cylinder. Put another line charge λ''' at the position of the original line charge to determine the field within the cylinder. What values of λ', λ'', and λ''' satisfy the boundary conditions?

(b) Check your answers with that of Section 3-3-3 in the limit as the radius of the cylinder becomes large so that it looks like a plane.

(c) What is the force per unit length on the line charge λ?

(d) Repeat (a)–(c) when the line charge λ is within the dielectric cylinder.

20. A point charge q is a distance d above a planar boundary separating two Ohmic materials with respective conductivities σ_1 and σ_2.



(a) What steady-state boundary conditions must the electric field satisfy?

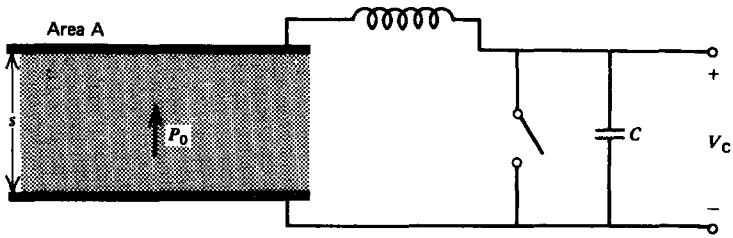
(b) What image charge configuration will satisfy these boundary conditions? (**Hint:** See Section 3-3-3.)

(c) What is the force on q?

21. The polarization of an electret is measured by placing it between parallel plate electrodes that are shorted together.

(a) What is the surface charge on the upper electrode?

(b) The switch is then opened and the upper electrode is taken far away from the electret. What voltage is measured across the capacitor?

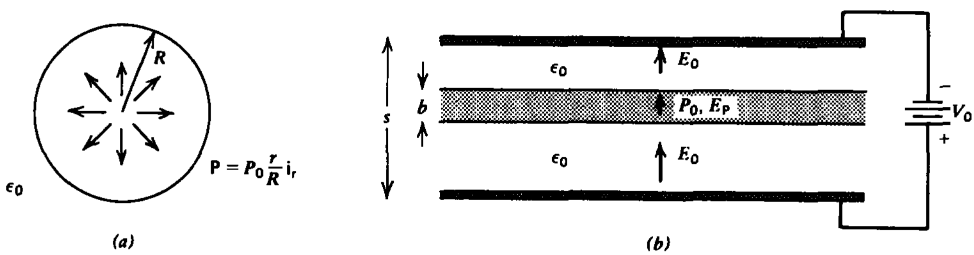


22. A cylinder of radius a and height L as in Figure 3-14, has polarization

$$\mathbf{P}=\frac{P_0 z}{L}\mathbf{i}_z$$

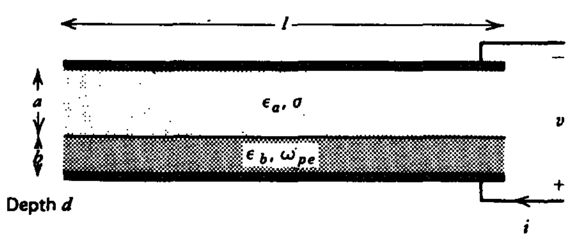
(a) What is the polarization charge distribution?
(b) Find the electric and displacement fields everywhere along the z axis. (**Hint:** Use the results of Sections 2-3-5b and 2-3-5d.)

23. Find the electric field everywhere for the following permanently polarized structures which do not support any free charge:



(a) Sphere of radius R with polarization $\mathbf{P}=(P_0 r/R)\mathbf{i}_r$.
(b) Permanently polarized slab $P_0\mathbf{i}_x$ of thickness b placed between parallel plate electrodes in free space at potential difference V_0.

24. Parallel plate electrodes enclose the series combination of an Ohmic conductor of thickness a with conductivity σ and a superconductor that only has free electrons with plasma



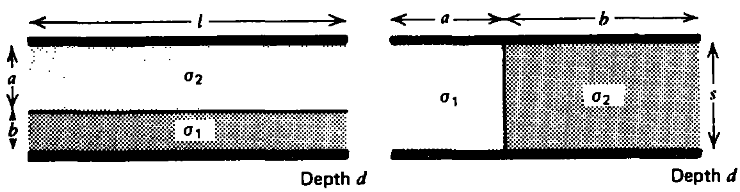
frequency ω_{pe}. What is the time dependence of the terminal current, the electric field in each region, and the surface charge at the interface separating the two conductors for each of the following terminal constraints:

(a) A step voltage V_0 is applied at $t=0$. For what values of ω_{pe} are the fields critically damped?

(b) A sinusoidal voltage $v(t)=V_0 \cos \omega t$ has been applied for a long time.

Section 3-4

25. Find the series and parallel resistance between two materials with conductivities σ_1 and σ_2 for each of the following electrode geometries:

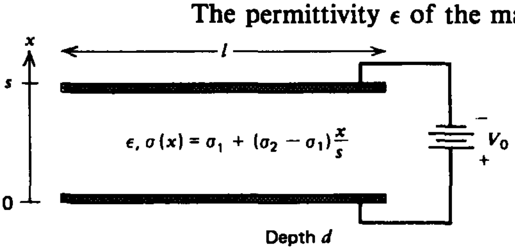


(a)

(b) and (c)

(a) Parallel plates.
(b) Coaxial cylinders.
(c) Concentric spheres.

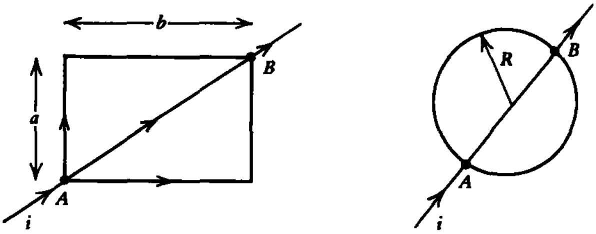
26. A pair of parallel plate electrodes at voltage difference V_0 enclose an Ohmic material whose conductivity varies linearly from σ_1 at the lower electrode to σ_2 at the upper electrode. The permittivity ϵ of the material is a constant.



(a) Find the fields and the resistance.
(b) What are the volume and surface charge distributions?

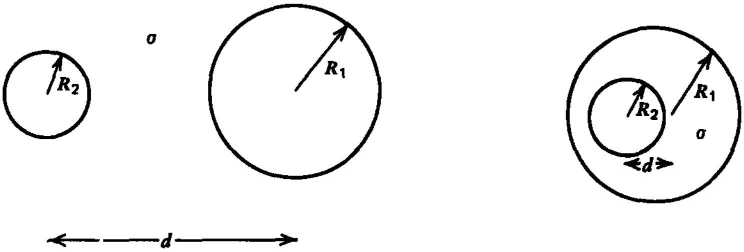
(c) What is the total volume charge in the system and how is it related to the surface charge on the electrodes?

27. A wire of Ohmic conductivity σ and cross sectional area A is twisted into the various shapes shown. What is the resistance R between the points A and B for each of the configurations?



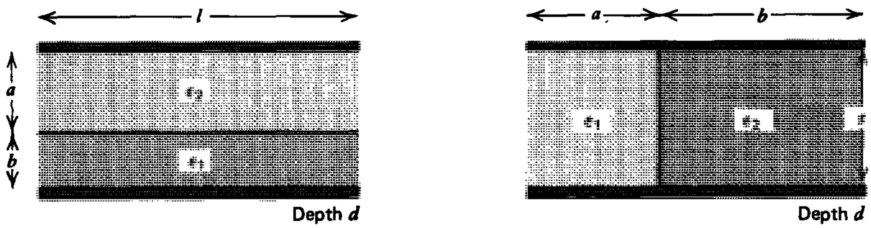
Section 3-5

28. Two conducting cylinders of length l and differing radii R_1 and R_2 within an Ohmic medium with conductivity σ have their centers a distance d apart. What is the resistance between cylinders when they are adjacent and when the smaller one is inside the larger one? (**Hint:** See Section 2-6-4*c*.)

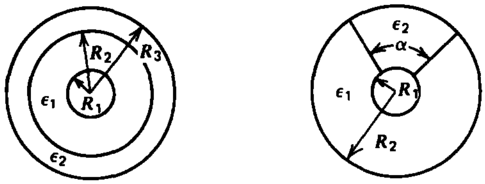


29. Find the series and parallel capacitance for each of the following geometries:

(a) Parallel plate.
(b) Coaxial cylinders.
(c) Concentric spheres.

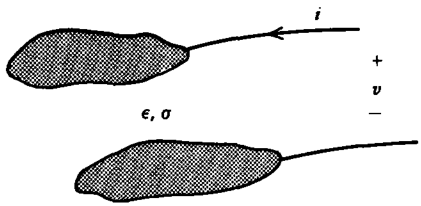


(a)



(Depth l for cylinders)
(*b*), (*c*)

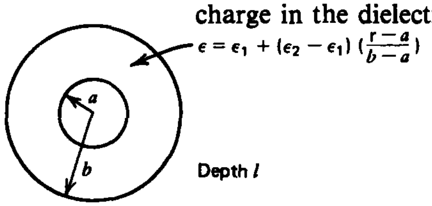
30. Two arbitrarily shaped electrodes are placed within a medium of constant permittivity ε and Ohmic conductivity σ. When a dc voltage V is applied across the system, a current I flows.



(a) What is the current $i(t)$ when a sinusoidal voltage $\text{Re}\,(V_0 e^{j\omega t})$ is applied?

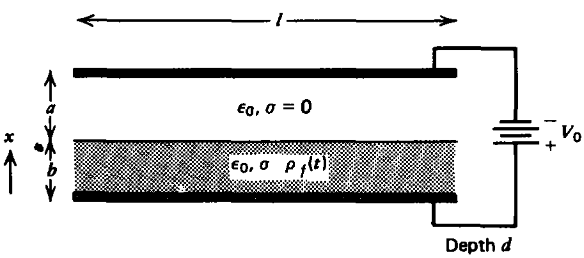
(b) What is the equivalent circuit of the system?

31. Concentric cylindrical electrodes of length l with respective radii a and b enclose an Ohmic material whose permittivity varies linearly with radius from ε_1 at the inner cylinder to ϵ_2 at the outer. What is the capacitance? There is no volume charge in the dielectric.



Section 3.6

32. A lossy material with the permittivity ε_0 of free space and conductivity σ partially fills the region between parallel plate electrodes at constant potential difference V_0 and is initially



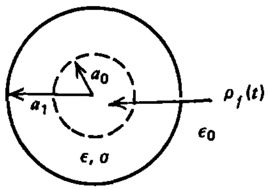
uniformly charged with density ρ_0 at $t=0$ with zero surface charge at $x=b$. What is the time dependence of the following:

(a) the electric field in each region? (**Hint:** See Section 3-3-5.)

(b) the surface charge at $x=b$?

(c) the force on the conducting material?

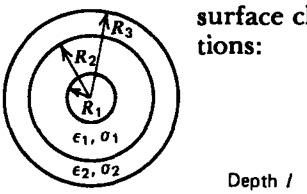
33. An infinitely long cylinder of radius a_1, permittivity ε, and conductivity σ is nonuniformly charged at $t=0$:



$$\rho_f(t=0)=\begin{cases}\rho_0 r/a_0, & 0<r<a_0\\ 0, & r>a_0\end{cases}$$

What is the time dependence of the electric field everywhere and the surface charge at $r=a_1$?

34. Concentric cylindrical electrodes enclose two different media in series. Find the electric field, current density, and surface charges everywhere for each of the following conditions:



(a) at $t=0_+$ right after a step voltage V_0 is applied to the initially unexcited system;

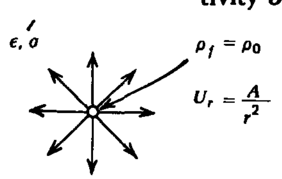
(b) at $t=\infty$ when the fields have reached their dc steady-state values;

(c) during the in-between transient interval. (What is the time constant τ?);

(d) a sinusoidal voltage $V_0 \cos \omega t$ is applied and has been on a long time;

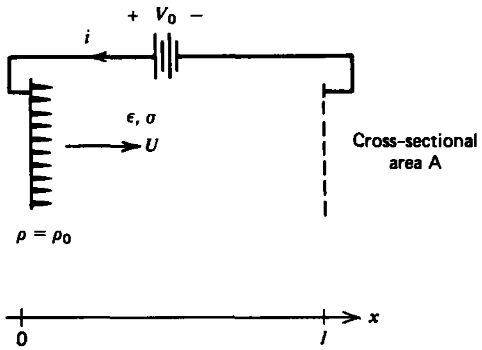
(e) what is the equivalent circuit for this system?

35. A fluid flow emanates radially from a point outlet with velocity distribution $U_r = A/r^2$. The fluid has Ohmic conductivity σ and permittivity ε. An external source maintains the



charge density ρ_0 at $r=0$. What are the steady-state charge and electric field distributions throughout space?

36. Charge maintained at constant density ρ_0 at $x=0$ is carried away by a conducting fluid travelling at constant velocity $U\mathbf{i}_x$ and is collected at $x=l$.



(a) What are the field and charge distributions within the fluid if the electrodes are at potential difference V_0?

(b) What is the force on the fluid?

(c) Repeat (a) and (b) if the voltage source is replaced by a load resistor R_L.

37. A dc voltage has been applied a long time to an open circuited resistive-capacitive structure so that the voltage and current have their steady-state distributions as given by (44). Find the resulting discharging transients for voltage and current if at $t=0$ the terminals at $z=0$ are suddenly:

(a) open circuited. **Hint:**

$$\int_0^l \sinh a(z-l)\sin\left(\frac{m\pi z}{l}\right)dz = -\frac{m\pi}{l}\frac{\sinh al}{[a^2+(m\pi/l)^2]}$$

(b) Short circuited. **Hint:**

$$\int_0^l \cosh a(z-l)\sin\left(\frac{(2n+1)\pi z}{2l}\right)dz = \frac{(2n+1)\pi\cosh al}{2l\left[a^2+\left[\frac{(2n+1)\pi}{2l}\right]^2\right]}$$

38. At $t=0$ a distributed resistive line as described in Section 3-6-4 has a step dc voltage V_0 applied at $z=0$. The other end at $z=l$ is short circuited.

(a) What are the steady-state voltage and current distributions?

(b) What is the time dependence of the voltage and current during the transient interval? **Hint:**

$$\int_0^l \sinh a(z-l)\sin\left(\frac{m\pi z}{l}\right)dz = -\frac{m\pi\sinh al}{l\left[a^2+(m\pi/l)^2\right]}$$

39. A distributed resistive line is excited at $z=0$ with a sinusoidal voltage source $v(t)=V_0\cos\omega t$ that has been on for a long time. The other end at $z=l$ is either open or short circuited.

(a) Using complex phasor notation of the form

$$v(z,t)=\operatorname{Re}\,(\hat{v}(z)e^{j\omega t})$$

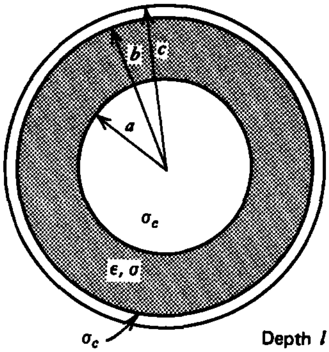
find the sinusoidal steady-state voltage and current distributions for each termination.

(b) What are the complex natural frequencies of the system?

(c) How much time average power is delivered by the source?

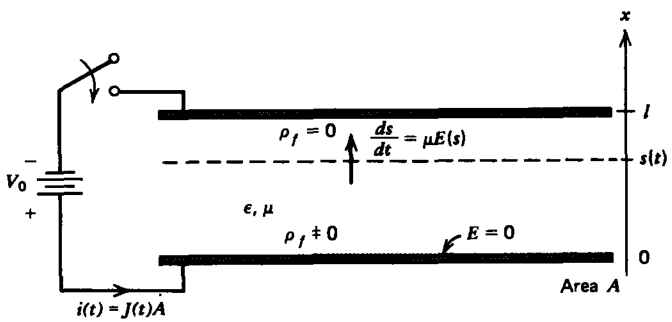
40. A lossy dielectric with permittivity ε and Ohmic conductivity σ is placed between coaxial cylindrical electrodes with large Ohmic conductivity σ_c and length l.

What is the series resistance per unit length 2R of the electrodes, and the capacitance C and conductance G per unit length of the dielectric?



Section 3.7

41. Two parallel plate electrodes of spacing l enclosing a dielectric with permittivity ε are stressed by a step voltage at $t=0$. Positive charge is then injected at $t=0$ from the lower electrode with mobility μ and travels towards the opposite electrode.



(a) Using the charge conservation equation of Section 3-2-1, show that the governing equation is

$$\frac{\partial E}{\partial t}+\mu E\frac{\partial E}{\partial x}=\frac{J(t)}{\varepsilon}$$

where $J(t)$ is the current per unit electrode area through the terminal wires. This current does not depend on x.

(b) By integrating (a) between the electrodes, relate the current $J(t)$ solely to the voltage and the electric field at the two electrodes.

(c) For space charge limited conditions ($E(x=0)=0$), find the time dependence of the electric field at the other electrode $E(x=l, t)$ before the charge front reaches it. (**Hint:** With constant voltage, $J(t)$ from (b) only depends on $E(x=l, t)$. Using (a) at $x=l$ with no charge, $\partial E/\partial x=0$, we have a single differential equation in $E(x=l, t)$.)

(d) What is the electric field acting on the charge front? (**Hint:** There is no charge ahead of the front.)

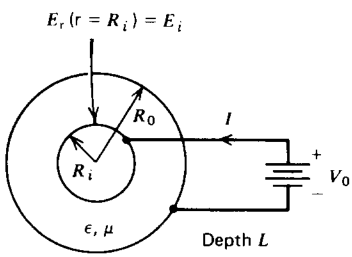
(e) What is the position of the front $s(t)$ as a function of time?

(f) At what time does the front reach the other electrode?

(g) What are the steady-state distribution of potential, electric field, and charge density? What is the steady-state current density $J(t\to\infty)$?

(h) Repeat (g) for nonspace charge limited conditions when the emitter electric field $E(x=0)=E_0$ is nonzero.

42. In a coaxial cylindrical geometry of length L, the inner electrode at $r=R_i$ is a source of positive ions with mobility μ in the dielectric medium. The inner cylinder is at a dc voltage V_0 with respect to the outer cylinder.



(a) The electric field at the emitter electrode is given as $E_r(r=R_i)=E_i$. If a current I flows, what are the steady-state electric field and space charge distributions?

(b) What is the dc current I in terms of the voltage under space charge limited conditions ($E_i=0$)? **Hint:**

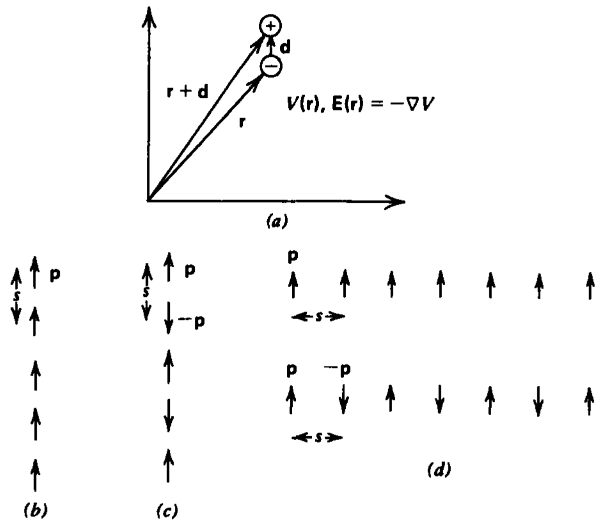
$$\int\frac{[r^2-R_i^2]^{1/2}}{r}\,dr=[r^2-R_i^2]^{1/2}-R_i\cos^{-1}\left(\frac{R_i}{r}\right)$$

(c) For what value of E_i is the electric field constant between electrodes? What is the resulting current?

(d) Repeat (a)–(b) for concentric spherical electrodes.

Section 3.8

43. (a) How much work does it take to bring a point dipole from infinity to a position where the electric field is **E**?



(b) A crystal consists of an infinitely long string of dipoles a constant distance s apart. What is the binding energy of the crystal? (**Hint:** $\sum_{n=1}^{\infty} 1/n^3 \approx 1.2$.)

(c) Repeat (b) if the dipole moments alternate in sign. (**Hint:** $\sum_{n=1}^{\infty} (-1)^n/n^3 \approx -0.90$.)

(d) Repeat (b) and (c) if the dipole moments are perpendicular to the line of dipoles for identical or alternating polarity dipoles.

44. What is the energy stored in the field of a point dipole with moment **p** outside an encircling concentric sphere with molecular radius R? **Hint:**

$$\int \cos^2\theta \sin\theta \, d\theta = -\frac{\cos^3\theta}{3}$$

$$\int \sin^3\theta \, d\theta = -\tfrac{1}{3}\cos\theta\,(\sin^2\theta + 2)$$

45. A spherical droplet of radius R carrying a total charge Q on its surface is broken up into N identical smaller droplets.

(a) What is the radius of each droplet and how much charge does it carry?

(b) Assuming the droplets are very far apart and do not interact, how much electrostatic energy is stored?

(c) Because of their surface tension the droplets also have a constant surface energy per unit area w_s. What is the total energy (electrostatic plus surface) in the system?

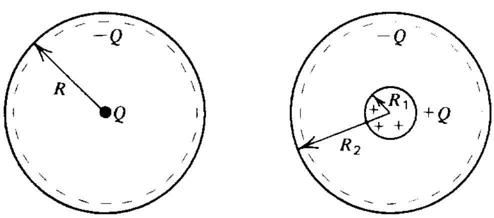
(d) How much work was required to form the droplets and to separate them to infinite spacing.

(e) What value of N minimizes this work? Evaluate for a water droplet with original radius of 1 mm and charge of 10^{-6} coul. (For water $w_s \approx 0.072$ joule/m^2.)

46. Two coaxial cylinders of radii a and b carry uniformly distributed charge either on their surfaces or throughout the volume. Find the energy stored per unit length in the z direction for each of the following charge distributions that have a total charge of zero:

(a) Surface charge on each cylinder with $\sigma_a 2\pi a = -\sigma_b 2\pi b$.
(b) Inner cylinder with volume charge ρ_a and outer cylinder with surface charge σ_b where $\sigma_b 2\pi b = -\rho_a \pi a^2$.
(c) Inner cylinder with volume charge ρ_a with the region between cylinders having volume charge ρ_b where $\rho_a \pi a^2 = -\rho_b \pi (b^2 - a^2)$.

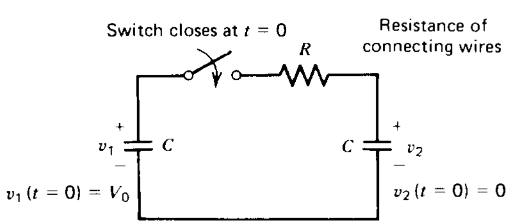
47. Find the binding energy in the following atomic models:



(a) (b)

(a) A point charge Q surrounded by a uniformly distributed surface charge $-Q$ of radius R.
(b) A uniformly distributed volume charge Q within a sphere of radius R_1 surrounded on the outside by a uniformly distributed surface charge $-Q$ at radius R_2.

48. A capacitor C is charged to a voltage V_0. At $t = 0$ another initially uncharged capacitor of equal capacitance C is



connected across the charged capacitor through some lossy wires having an Ohmic conductivity σ, cross-sectional area A, and total length l.

(a) What is the initial energy stored in the system?

(b) What is the circuit current i and voltages v_1 and v_2 across each capacitor as a function of time?

(c) What is the total energy stored in the system in the dc steady state and how does it compare with (a)?

(d) How much energy has been dissipated in the wire resistance and how does it compare with (a)?

(e) How do the answers of (b)–(d) change if the system is lossless so that $\sigma = \infty$? How is the power dissipated?

(f) If the wires are superconducting Section 3-2-5*d* showed that the current density is related to the electric field as

$$\frac{\partial J}{\partial t} = \omega_p^2 \varepsilon E$$

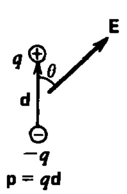
where the plasma frequency ω_p is a constant. What is the equivalent circuit of the system?

(g) What is the time dependence of the current now?

(h) How much energy is stored in each element as a function of time?

(i) At any time t what is the total circuit energy and how does it compare with (a)?

Section 3.9



49. A permanently polarized dipole with moment $\mathbf{p}$ is at an angle θ to a uniform electric field $\mathbf{E}$.

(a) What is the torque T on the dipole?

(b) How much incremental work dW is necessary to turn the dipole by a small angle $d\theta$? What is the total work required to move the dipole from $\theta = 0$ to any value of θ? (**Hint:** $dW = T\,d\theta$.)

(c) In general, thermal agitation causes the dipoles to be distributed over all angles of θ. Boltzmann statistics tell us that the number density of dipoles having energy W are

$$n = n_0\, e^{-W/kT}$$

where n_0 is a constant. If the total number of dipoles within a sphere of radius R is N, what is n_0? (**Hint:** Let $u = (pE/kT)\cos\theta$.)

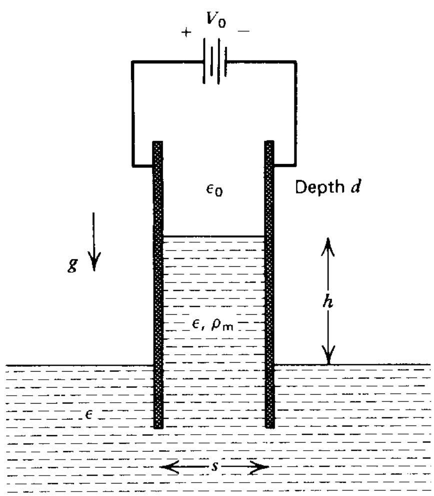
(d) Consider a shell of dipoles within the range of θ to $\theta + d\theta$. What is the magnitude and direction of the net polarization due to this shell?

(e) What is the total polarization integrated over θ? This is known as the Langevin equation. (**Hint:** $\int ue^u\,du = (u-1)e^u$.)

(f) Even with a large field of $E \approx 10^6$ v/m with a dipole composed of one proton and electron a distance of 10 Å (10^{-9} m) apart, show that at room temperature the quantity (pE/kT) is much less than unity and expand the results of (e). (**Hint:** It will be necessary to expand (e) up to third order in (pE/kT).

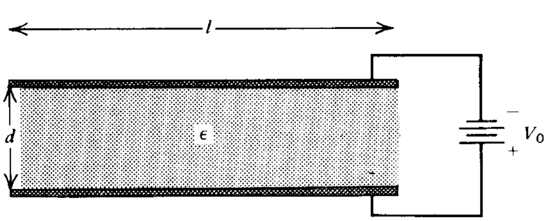
(g) In this limit what is the orientational polarizability?

50. A pair of parallel plate electrodes a distance s apart at a voltage difference V_0 is dipped into a dielectric fluid of permittivity ε. The fluid has a mass density ρ_m and gravity acts downward. How high does the liquid rise between the plates?



51. Parallel plate electrodes at voltage difference V_0 enclose an elastic dielectric with permittivity ε. The electric force of attraction between the electrodes is balanced by the elastic force of the dielectric.

(a) When the electrode spacing is d what is the free surface charge density on the upper electrode?



(b) What is the electric force per unit area that the electrode exerts on the dielectric interface?

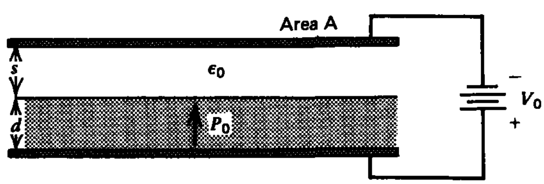
(c) The elastic restoring force per unit area is given by the relation

$$F_A = Y \ln \frac{d}{d_0}$$

where Y is the modulus of elasticity and d_0 is the unstressed ($V_0 = 0$) thickness of the dielectric. Write a transcendental expression for the equilibrium thickness of the dielectric.

(d) What is the minimum equilibrium dielectric thickness and at what voltage does it occur? If a larger voltage is applied there is no equilibrium and the dielectric fractures as the electric stress overcomes the elastic restoring force. This is called the theory of electromechanical breakdown. [See K. H. Stark and C. G. Garton, Electric Strength of Irradiated Polythene, *Nature* **176** (1955) 1225–26.]

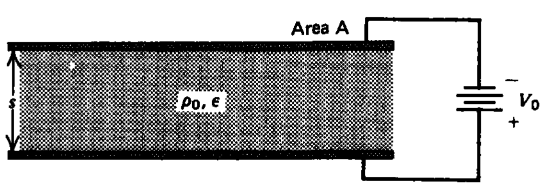
52. An electret with permanent polarization $P_0\mathbf{i}_y$ and thickness d partially fills a free space capacitor. There is no surface charge on the electret free space interface.



(a) What are the electric fields in each region?

(b) What is the force on the upper electrode?

53. A uniform distribution of free charge with density ρ_0 is between parallel plate electrodes at potential difference V_0.



(a) What is the energy stored in the system?

(b) Compare the capacitance to that when $\rho_0 = 0$.

(c) What is the total force on each electrode and on the volume charge distribution?

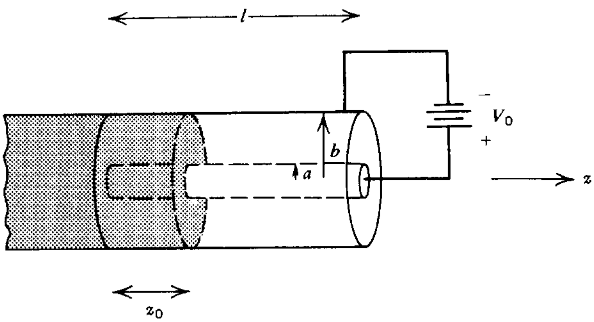
(d) What is the total force on the system?

54. Coaxial cylindrical electrodes at voltage difference V_0 are partially filled with a polarized material. Find the force on this

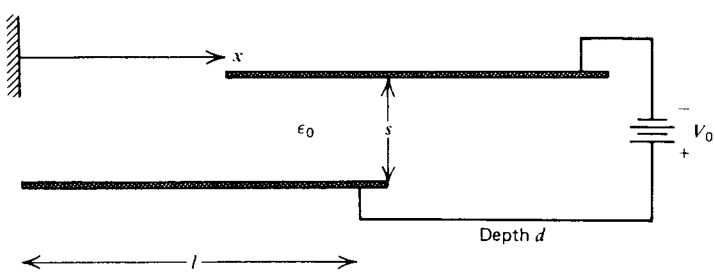
material if it is

(a) permanently polarized as $P_0\mathbf{i}_r$;

(b) linearly polarized with permittivity ε.



55. The upper electrode of a pair at constant potential difference V_0 is free to slide in the x direction. What is the x component of the force on the upper electrode?



56. A capacitor has a moveable part that can rotate through the angle θ so that the capacitance $C(\theta)$ depends on θ.

(a) What is the torque on the moveable part?

(b) An electrostatic voltmeter consists of $N+1$ fixed pie-shaped electrodes at the same potential interspersed with N plates mounted on a shaft that is free to rotate for $-\theta_0<\theta<\theta_0$. What is the capacitance as a function of θ?

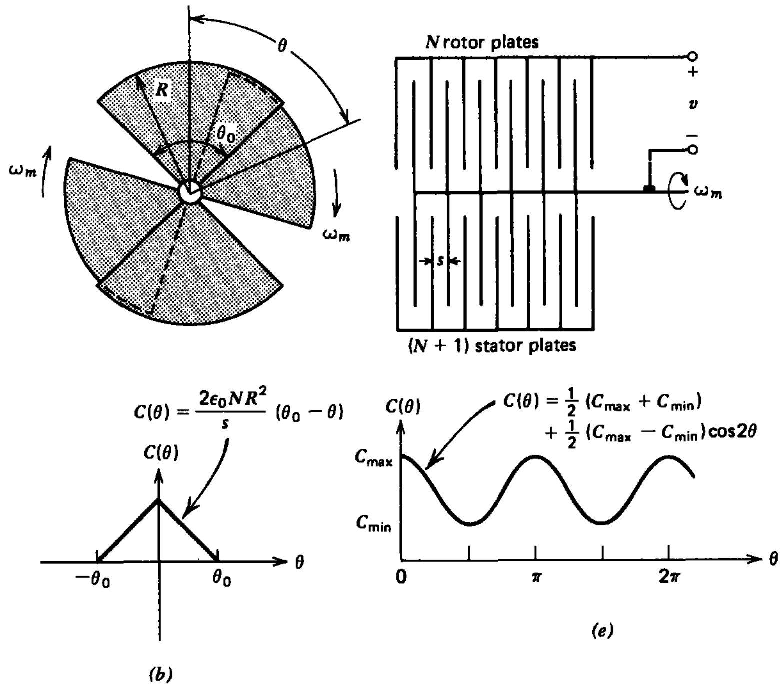
(c) A voltage v is applied. What is the electric torque on the shaft?

(d) A torsional spring exerts a restoring torque on the shaft

$$T_s = -K(\theta-\theta_s)$$

where K is the spring constant and θ_s is the equilibrium position of the shaft at zero voltage. What is the equilibrium position of the shaft when the voltage v is applied? If a sinusoidal voltage is applied, what is the time average angular deflection $<\theta>$?

(e) The torsional spring is removed so that the shaft is free to continuously rotate. Fringing field effects cause the



capacitance to vary smoothly between minimum and maximum values of a dc value plus a single sinusoidal spatial term

$$C(\theta)=\tfrac{1}{2}[C_{\max}+C_{\min}]+\tfrac{1}{2}[C_{\max}-C_{\min}]\cos 2\theta$$

A sinusoidal voltage $V_0 \cos \omega t$ is applied. What is the instantaneous torque on the shaft?

(f) If the shaft is rotating at constant angular speed ω_m so that

$$\theta=\omega_m t+\delta$$

where δ is the angle of the shaft at $t=0$, under what conditions is the torque in (e) a constant? **Hint:**

$$\begin{aligned}\sin 2\theta \cos^2 \omega t &= \tfrac{1}{2}\sin 2\theta(1+\cos 2\omega t)\\ &= \tfrac{1}{2}\sin 2\theta+\tfrac{1}{4}[\sin(2(\omega t+\theta))-\sin(2(\omega t-\theta))]\end{aligned}$$

(g) A time average torque T_0 is required of the shaft. What is the torque angle δ?

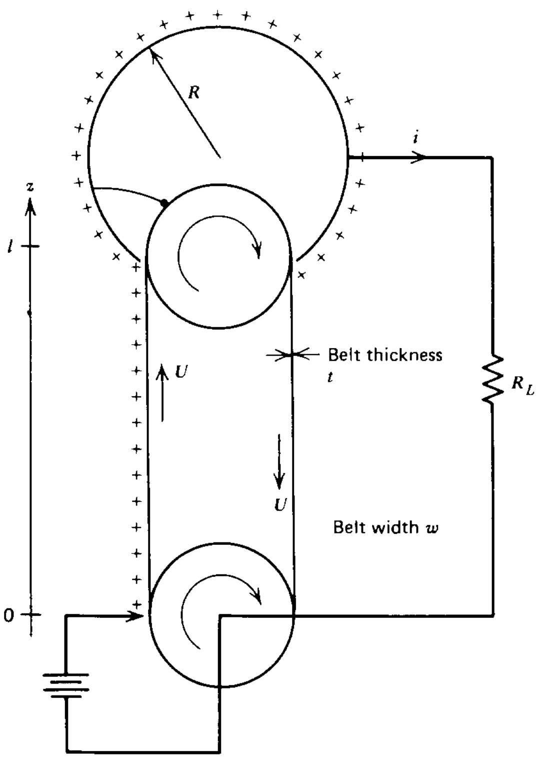
(h) What is the maximum torque that can be delivered? This is called the pull-out torque. At what angle δ does this occur?

Section 3-10

57. The belt of a Van de Graaff generator has width w and moves with speed U carrying a surface charge σ_f up to the spherical dome of radius R.

(a) What is the time dependence of the dome voltage?

(b) Assuming that the electric potential varies linearly between the charging point and the dome, how much power as a function of time is required for the motor to rotate the belt?



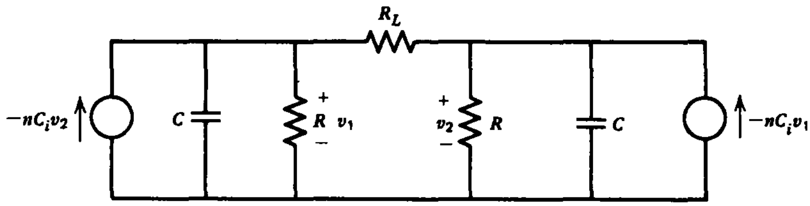
58. A Van de Graaff generator has a lossy belt with Ohmic conductivity σ traveling at constant speed U. The charging point at $z = 0$ maintains a constant volume charge density ρ_0 on the belt at $z = 0$. The dome is loaded by a resistor R_L to ground.

(a) Assuming only one-dimensional variations with z, what are the steady-state volume charge, electric field, and current density distributions on the belt?

(b) What is the steady-state dome voltage?

59. A pair of coupled electrostatic induction machines have their inducer electrodes connected through a load resistor R_L. In addition, each electrode has a leakage resistance R to ground.

(a) For what values of n, the number of conductors per second passing the collector, will the machine self-excite?



(b) If $n = 10$, $C_i = 2$ pf, and $C = 10$ pf with $R_L = R$, what is the minimum value of R for self-excitation?

(c) If we have three such coupled machines, what is the condition for self-excitation and what are the oscillation frequencies if $R_L = \infty$?

(d) Repeat (c) for N such coupled machines with $R_L = \infty$. The last machine is connected to the first.