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SOLUTIONS TO CHAPTER 13

13.1 INTRODUCTION TO TEM WAVES

13.1.1 (a) From (13.1.3):

$$\begin{aligned}\frac{\partial E_x}{\partial y} &= \beta \mathrm{Re}[A\cos(\beta y)\exp(j\omega t)] = \mu\frac{\partial H_z}{\partial t} \\ &= \beta|A|\cos\beta y\cos(\omega t+\phi)\end{aligned} \tag{1}$$

where ϕ is the phase angle of A. Integrating the above yields

$$H_z = \frac{\beta}{\omega\mu}|A|\cos\beta y\sin(\omega t+\phi) = -\mathrm{Re}\, j\frac{\beta}{\omega\mu}A\cos\beta y e^{j\omega t} \tag{2}$$

Introducing (2) and the expression for E_x into (13.1.2) gives

$$-\frac{\beta^2}{\omega\mu}|A|\sin\beta y\sin(\omega t+\phi) = -\omega\epsilon|A|\sin\beta y\sin(\omega t+\phi) \tag{3}$$

from which the dispersion relation follows $\beta^2 = \omega^2\mu\epsilon$.

(b) From (13.1.13)

$$H_z(-b,t) = -\mathrm{Re}\,\hat{K}_o e^{j\omega t}$$

This gives, using (2),

$$-\mathrm{Re}\, j\frac{\beta}{\omega\mu}A\cos\beta y e^{j\omega t} = -\mathrm{Re}\,\hat{K}_o e^{j\omega t} \tag{4}$$

and thus

$$A = -j\frac{\omega\mu\hat{K}_o}{\beta\cos\beta b} = -j\hat{K}_o\sqrt{\frac{\mu}{\epsilon}}\frac{1}{\cos\beta b} \tag{5}$$

Using (2) we find

$$H_z = -\mathrm{Re}\,\hat{K}_o\frac{\cos\beta y}{\cos\beta b}e^{j\omega t} \tag{6}$$

and putting the value of A from (5) into the expression for E_x gives

$$E_x = -\mathrm{Re}\, jK_o\sqrt{\frac{\mu}{\epsilon}}\frac{\sin\beta y}{\cos\beta b}e^{j\omega t} \tag{7}$$

13.1.2 (a) The standing wave

$$H_z = \text{Re}\, A \sin \beta y e^{j\omega t}$$

satisfies the boundary conditions of zero H_z at $y = 0$. From (13.1.2)

$$\frac{\partial H_z}{\partial y} = \beta \text{Re}\, A \cos \beta y e^{j\omega t} = \epsilon \frac{\partial E_x}{\partial t} \tag{1}$$

Integrating to find E_x gives

$$E_x = -\frac{\beta}{\omega\epsilon} \text{Re}\, jA \cos \beta y e^{j\omega t} \tag{2}$$

From (13.1.3) we find

$$\frac{\partial E_x}{\partial y} = \frac{\beta^2}{\omega\epsilon} \text{Re}\, jA \sin \beta y e^{j\omega t} = \mu \frac{\partial H_z}{\partial t} = \omega\mu \text{Re}\, jA \sin \beta y e^{j\omega t} \tag{3}$$

and thus

$$\beta^2 = \omega^2 \mu\epsilon \tag{4}$$

(b) Turning to the boundary conditions,

$$E_x(-b, t) = \text{Re}\, \hat{V}_d e^{j\omega t}/a \tag{5}$$

and thus from (2)

$$-\frac{\beta}{\omega\epsilon} \text{Re}\, jA \cos \beta b e^{j\omega t} = \text{Re}\, \hat{V}_d e^{j\omega t}/a \tag{6}$$

and hence

$$A = j\frac{\omega\epsilon}{\beta} \frac{\hat{V}_d}{a} \frac{1}{\cos \beta b} = j\sqrt{\frac{\epsilon}{\mu}} \frac{\hat{V}_d}{a} \frac{1}{\cos \beta b} \tag{7}$$

We find

$$H_z = \text{Re}\, j\sqrt{\frac{\epsilon}{\mu}} \frac{\hat{V}_d}{a} \frac{\sin \beta y}{\cos \beta b} e^{j\omega t}$$

$$E_x = \text{Re}\, \frac{V_d}{a} \frac{\cos \beta y}{\cos \beta b} e^{j\omega t}$$

13.1.3 Using the identity

$$\sin x = (e^{jx} - e^{-jx})/2j \tag{1}$$

one finds from (13.1.17)

$$\begin{aligned} E_x &= -\text{Re}\, j\hat{K}_o \sqrt{\frac{\mu}{\epsilon}} \frac{1}{\cos \beta b} \frac{1}{2j} (e^{j\beta y} - e^{-j\beta y}) e^{j\omega t} \\ &= -\text{Re}\, \frac{1}{2} K_o \sqrt{\frac{\mu}{\epsilon}} [e^{j(\omega t - \beta y)} - e^{-j(\omega t + \beta y)}]/\cos \beta b \end{aligned} \tag{2}$$

The exponentials in the brackets represent waves that retain constant amplitude when $dy = \pm\frac{\omega}{\beta} dt$ exhibiting the (phase) velocities $\pm\omega/\beta = \pm 1/\sqrt{\mu\epsilon}$.

13.1.4 (a) The EQS potential in a coax is a solution of Laplace's equation. The field with rotational symmetry is

$$\Phi = Aln\frac{r}{a} \tag{1}$$

satisfying $\Phi = 0$ on outer conductor of radius a. The field is z-independent with a constant potential difference. The potential difference is

$$Aln(b/a) = V \tag{2}$$

The field is

$$\mathbf{E} = -\nabla\Phi = -\mathbf{i_r}\frac{\partial}{\partial r}Aln(r/a) = -\mathbf{i_r}\frac{A}{r} = \mathbf{i_r}\frac{V}{rln(a/b)} \tag{3}$$

(b) The field has cylindrical symmetry with field-lines parallel to $\mathbf{i}_\phi$. The potential Ψ is

$$\Psi = A\phi \tag{4}$$

The H field is

$$\mathbf{H} = -\mathbf{i}_\phi\frac{1}{r}\frac{\partial}{\partial\phi}\Psi = -\mathbf{i}_\phi\frac{A}{r} \tag{5}$$

Ampère's integral law gives

$$\oint \mathbf{H}\cdot d\mathbf{s} = \int \mathbf{J}\cdot d\mathbf{a} = I \tag{6}$$

Since H is z independent, I = constant and at $z = -l$

$$-\frac{A}{r}2\pi r = -2\pi A = I \tag{7}$$

Therefore

$$\mathbf{H} = \mathbf{i_y}\frac{I}{2\pi r} \tag{8}$$

(c) The preceding analysis suggests that

$$\mathbf{E} = \mathbf{i_r}\frac{V(z,t)}{ln(a/b)r} \tag{9a}$$

and

$$\mathbf{H} = \mathbf{i}_\phi\frac{I(z,t)}{2\pi r} \tag{9b}$$

can be solutions of Maxwell's equations. To show this it is advantageous to separate the ∇ operator into

$$\nabla = \nabla_T + \mathbf{i_z}\frac{\partial}{\partial z} \tag{10}$$

where

$$\nabla_T = \mathbf{i_r}\frac{\partial}{\partial r} + \frac{1}{r}\mathbf{i}_\phi\frac{\partial}{\partial \phi} \tag{11}$$

is the transverse part of the operator. Then

$$\nabla \times \mathbf{E} = \nabla_T \times \mathbf{E} + \mathbf{i_z} \times \frac{\partial}{\partial z}\mathbf{E} \tag{12}$$

Now ∇_T differentiates only r and ϕ. The EQS field, which is z independent, has $\nabla_T \times \mathbf{E} = 0$. Hence we conclude that the same holds for the "Ansatz" (9). But $\mathbf{i_z} \times \mathbf{i_r} = \mathbf{i}_\phi$ and $\mathbf{i_z} \times \mathbf{i}_\phi = -\mathbf{i_r}$. We obtain from Faraday's law

$$\frac{1}{ln(a/b)}\frac{1}{r}\frac{\partial}{\partial a}V = -\mu\frac{1}{2\pi r}\frac{\partial I}{\partial t} \tag{13}$$

The common r-dependence can be eliminated, and we find

$$\frac{\partial}{\partial z}V = -L\frac{\partial I}{\partial t} \tag{14}$$

where

$$L = \frac{\mu ln(b/a)}{2\pi} \tag{15}$$

A similar reasoning applied to $\nabla \times \mathbf{H}$ and Ampère's law yields

$$-\mathbf{i_r}\frac{1}{2\pi r}\frac{\partial I}{\partial z} = \mathbf{i_r}\frac{\epsilon}{ln(b/a)r}\frac{\partial V}{\partial t} \tag{16}$$

or

$$\frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t} \tag{17}$$

with

$$C = \frac{2\pi\epsilon}{ln(b/a)} \tag{18}$$

13.1.5 (a) With the time dependence $\exp j\omega t$, we get for the transmission line equations of (14) and (17) of Prob. 13.1.4

$$\frac{d\hat{V}}{dz} = -j\omega L\hat{I} \tag{1}$$

$$\frac{d\hat{I}}{dz} = -j\omega C\hat{V} \tag{2}$$

where

$$V = \mathrm{Re}\,\hat{V}e^{j\omega t}$$

and

$$I = \text{Re}\ \hat{I}e^{j\omega t}$$

Eliminating $\hat{V}$ from (1) and (2) one obtains

$$\frac{d^2\hat{V}}{dz^2} = -j\omega L\frac{d\hat{I}}{dz} = -\omega^2 LC\hat{V} \tag{3}$$

with the solutions

$$\hat{V} \propto \begin{cases} \cos\beta z \\ \sin\beta z \end{cases} \tag{4}$$

with

$$\beta = \omega\sqrt{LC} \tag{5}$$

We pick the solution

$$\hat{V} = A\sin\beta z \tag{6}$$

because the short forces $\hat{V}$ to be zero at $z = 0$. From (1) we find

$$\hat{I} = \frac{j}{\omega L}\frac{d\hat{V}}{dz} = \frac{j\beta}{\omega L}A\cos\beta z \tag{7}$$

and since $I = \text{Re}\ I_o e^{j\omega t}$ at $z = -l$,

$$A\cos\beta l = -j\frac{\omega L}{\beta}I_o \tag{8}$$

or

$$A = -j\sqrt{L/C}\frac{I_o}{\cos\beta l} \tag{9}$$

where we used (5). We find for the current and voltage as functions of z and t:

$$I(z,t) = \text{Re}\ \frac{I_o}{\cos\beta l}\cos\beta z e^{j\omega t} \tag{10}$$

$$\hat{V}(z,t) = -\text{Re}\ j\sqrt{L/C}I_o\frac{\sin\beta z}{\cos\beta l}e^{j\omega t} \tag{11}$$

(b) At low frequencies $\cos\beta z \simeq 1$ for all $-l < z < 0$ and $\sin\beta z \simeq \beta z = \omega\sqrt{LC}z$. Using (9) of the preceding problem,

$$\mathbf{H}(z,t) = \mathbf{i}_\phi \text{Re}\ \frac{I_o}{2\pi r}e^{j\omega t} \tag{12}$$

For the E-field we find from the preceding problem and (11) above

$$\mathbf{E} = -\mathbf{i}_r \text{Re} j\omega\frac{LzI_o e^{j\omega t}}{ln(a/b)r} = -\mathbf{i}_r\text{Re}\ j\omega\frac{\mu}{2\pi}zI_o e^{j\omega t} \tag{13}$$

This gives the voltage at $z = -l$

$$\int_b^a \mathbf{E}\cdot\mathbf{i}_r dr = \text{Re}[j\omega L l I_o e^{j\omega t}] \tag{14}$$

The inductance is Ll because L, as defined here, is the inductance per unit length. Thus we have shown that, in the limit of low frequencies, the structure behaves as a single-turn inductor.

(c) The H-field in the space between the conductors is the gradient of a potential $\Psi \propto \phi$ that is a solution of Laplace's equation. Thus,

$$\mathbf{H} = \text{Re}\,\frac{I_o}{2\pi r}\mathbf{i}_\phi e^{j\omega t} \tag{15}$$

We obtain $\mathbf{E}$ from Faraday's law

$$\nabla\times\mathbf{E} = -\frac{\mu\partial\mathbf{H}}{\partial t} = -\mu\text{Re}\,j\omega\frac{I_o}{2\pi r}\mathbf{i}_\phi e^{j\omega t} \tag{16}$$

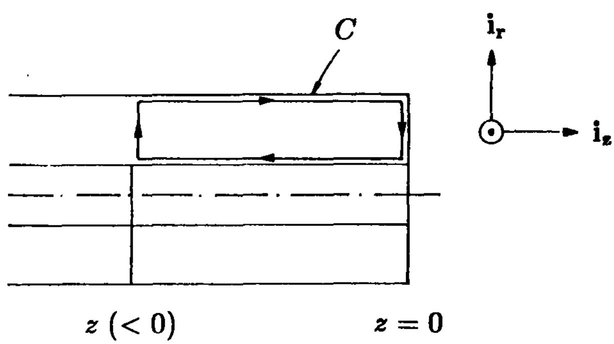

Figure S13.1.5

With the line integral along the contour C shown Fig. S13.1.5, we may find from the integral form of Faraday's law

$$\int_b^a E_r dr = \mu\text{Re}\left\{j\omega I_o e^{j\omega t}|z|\int_b^a \frac{dr}{2\pi r}\right\} \tag{17}$$

Integrals over the radial coordinate appear on both sides. Thus, comparing the integrands we find

$$E_r = -\text{Re}\,\frac{j\omega\mu I_o}{2\pi}z e^{j\omega t} \tag{18}$$

which is the same as (13).

13.1.6 (a) From the solutions (4) in Prob. 13.1.5 we pick the $\cos\beta z$ dependence, because the magnetic field, proportional to $\hat{I}$, is zero at $z = 0$ according to (7) of the same problem. Indeed, if $\hat{V} = A\cos\beta z$, then

$$\hat{I} = \frac{j}{\omega L}\frac{d\hat{V}}{dz} = -\frac{j\beta}{\omega L}A\sin\beta z \tag{1}$$

Since

$$\mathrm{Re}[A\cos\beta z\exp j\omega t]_{z=-l} = \mathrm{Re}[V_o\exp j\omega t] \tag{2}$$

we find

$$A = \frac{V_o}{\cos\beta l} \tag{3}$$

and

$$\hat{I} = -j\sqrt{C/L}\frac{V_o}{\cos\beta l}\sin\beta z \tag{4}$$

Therefore,

$$V(z,t) = \mathrm{Re}\left[\frac{V_o}{\cos\beta l}\cos\beta z\exp j\omega t\right] \tag{5}$$

$$I(z,t) = -\mathrm{Re}\, j\sqrt{C/L}\frac{V_o}{\cos\beta l}\sin\beta z e^{j\omega t} \tag{6}$$

$$\mathbf{E} = V(z,t)\nabla_T\frac{ln(r/a)}{ln(a/b)} = \mathbf{i}_r\frac{V(z,t)}{ln(a/b)}\frac{1}{r} \tag{7}$$

where ∇_T is the transverse gradient operator,

$$\nabla_T = \mathbf{i}_r\frac{\partial}{\partial r} + \mathbf{i}_\phi\frac{1}{r}\frac{\partial}{\partial\phi}$$

and we use the result of Prob. 13.1.4. In a similar vein

$$\mathbf{H} = \mathbf{i}_\phi\frac{I(z,t)}{2\pi r} \tag{8}$$

(b) At low frequencies, $\cos\beta z \simeq 1, \sin\beta z \simeq \beta z$ and $V(z,t) \simeq \mathrm{Re}\, V_o\exp j\omega t$. Then, assuming V_o to be real,

$$\mathbf{E} = \frac{\mathbf{i}_r}{ln(a/b)}\frac{1}{r}V_o\cos\omega t \tag{9}$$

$$\mathbf{H} = \frac{\mathbf{i}_\phi}{2\pi r}\sqrt{C/L}\beta z V_o\sin\omega t = \mathbf{i}_\phi\frac{\omega\epsilon}{r ln(a/b)}zV_o\sin\omega t \tag{10}$$

(c) At low frequencies, using EQS directly

$$\mathbf{E} = \frac{\mathbf{i}_r}{ln(a/b)}\frac{1}{r}V_o\cos\omega t \tag{11}$$

namely the gradient of a Laplacian potential $\propto ln(r/a)$. The H-field follows from

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} \tag{12}$$

with

$$\mathbf{H} = \mathbf{i}_\phi \frac{A}{r} z \tag{13}$$

introduced into (12)

$$\nabla \times \mathbf{H} = -\mathbf{i}_r \frac{\partial}{\partial z} H_\phi = -\mathbf{i}_r \frac{A}{r} = -\omega\epsilon \frac{\mathbf{i}_r}{ln(a/b)} \frac{1}{r} V_o \sin \omega t$$

and therefore

$$A = \frac{\omega}{ln(a/b)} V_o \sin \omega t \tag{14}$$

which gives the same result as (10).

13.2 TWO-DIMENSIONAL MODES BETWEEN PARALLEL-PLATES

13.2.1 We can write

$$\cos \frac{n\pi}{a} x = \frac{1}{2}\left(\exp j \frac{n\pi}{a} x + \exp -j \frac{n\pi}{a} x\right)$$

and

$$\sin \frac{n\pi}{a} x = \frac{1}{2j}\left(\exp j \frac{n\pi}{a} x - \exp -j \frac{n\pi}{a} x\right)$$

Introducing these expressions into (13.2.19)-(13.2.20) we find four terms of the form

$$\exp \mp j\beta_n y \exp \mp j \frac{n\pi}{a} x = \exp \mp j\left(\beta_n y \pm \frac{n\pi}{a} x\right) = \exp -j\mathbf{k} \cdot \mathbf{r}$$

where

$$\mathbf{k} = \pm \frac{n\pi}{a} \mathbf{i}_x \pm \beta_n \mathbf{i}_y$$

and

$$\mathbf{r} = \mathbf{i}_x x + \mathbf{i}_y y$$

This proves the assertion that the solution consists of four waves of the stated nature. These waves are phased so as to yield x-dependences of the form $\cos \frac{n\pi}{a} x$ and $\sin \frac{n\pi}{a} x$ to satisfy the boundary conditions.

13.2.2 We can start with the solutions (13.2.19) and (13.2.20) shifting x so that

$$x' = x - \frac{a}{2}$$

Considering TM modes first we note that

$$\begin{aligned} H_z \propto \cos\frac{n\pi}{a}x &= \cos\Big(\frac{n\pi x'}{a} + \frac{n\pi}{2}\Big) \\ &= \cos\frac{n\pi x'}{a}\cos\frac{n\pi}{2} - \sin\frac{n\pi x'}{a}\sin\frac{n\pi}{2} \\ &= \begin{cases} (-1)^{n/2}\cos\big(\frac{n\pi x'}{a}\big) & n \text{ even} \\ (-1)^{\frac{n-1}{2}}\sin\big(\frac{n\pi x'}{a}\big) & n \text{ odd} \end{cases} \end{aligned}$$

We see that the modes with even n are even with respect to the symmetry plane of the guide, the modes with n-odd are odd.

Next studying the TE-modes,

$$\begin{aligned} E_z \propto \sin\frac{n\pi}{a}x &= \sin\Big(\frac{n\pi x'}{a}\Big)\cos\frac{n\pi}{2} + \cos\Big(\frac{n\pi x'}{a}\Big)\sin\frac{n\pi}{2} \\ &= \begin{cases} (-1)^{n/2}\sin\frac{n\pi x'}{a} & n \text{ even} \\ (-1)^{\frac{n-1}{2}}\cos\frac{n\pi x'}{a} & n \text{ odd} \end{cases} \end{aligned}$$

We find that E_z is even for n odd, odd for n even.

(a) When $x' = \pm a/2$ and the modes are odd, $H_z = (-1)^{(n-1)/2}\sin\frac{n\pi}{2}, E_z = (-1)^{n/2}\sin\frac{n\pi}{2}$; in the first case n is odd and H_z is an extremum at $x' = \pm a/2$, and in the second case n is even and E_z is zero at both boundaries.

(b) When $x' = \pm a/2$ and the modes are even then $H_z = (-1)^{n/2}\cos(\frac{n\pi}{2})$ and $E_z = (-1)^{(n-1)/2}\cos\frac{n\pi}{2}$ we see that both boundary conditions are in both cases, because n is odd in the first case and H_z is an extrenum, n is even in the second case, and E_z is zero.

13.3 TE AND TM STANDING WAVES BETWEEN PARALLEL PLATES

13.3.1 We multiply (13.3.1) by $\hat{h}^*_{zm}$ and integrate over the interval from 0 to a.

$$\begin{aligned} \int_0^a dx\Big(\hat{h}^*_{zm}\frac{d^2\hat{h}_{zn}}{dx^2} + p_n^2\hat{h}^*_{zm}\hat{h}_{zn}\Big) &= \int_0^a dx\frac{d}{dx}\Big(\hat{h}^*_{zm}\frac{d\hat{h}_{zn}}{dx}\Big) \\ &\quad - \int_0^a dx\Big(\frac{d}{dx}\hat{h}^*_{zm}\Big)\Big(\frac{d}{dx}\hat{h}_{zn}\Big) \\ &\quad + p_n^2\int_0^a \hat{h}^*_{zm}\hat{h}_{zn}dx = 0 \end{aligned} \qquad (1)$$

where we have integrated by parts. Because $dh_{zn}/dx = 0$ at $x = 0$ and $x = a$, the integral of the integrand containing the total derivative vanishes.

Next take the complex conjugate of (13.3.1) applied to $\hat{h}_{zm}$ multiply by $\hat{h}_{zn}$ and integrate. The result is

$$\int_0^a dx \frac{d}{dx}\hat{h}^*_{zm}\frac{d}{dx}\hat{h}_{zn} = p_m^2 \int_0^a \hat{h}^*_{zm}\hat{h}_{zn}dx \tag{2}$$

Subtraction of (1) and (2) gives

$$(p_m^2 - p_n^2)\int_0^a \hat{h}^*_{zm}\hat{h}_{zn}dx = 0$$

Thus

$$\int_0^a \hat{h}^*_{zm}\hat{h}_{zn}dx = 0$$

when $p_m^2 \neq p_n^2$ and orthogonality is proven. The steps involving $\hat{e}_{zn}$ are identical. The only difference is that

$$\int_0^a dx\frac{d}{dx}\Big(\hat{e}^*_{zm}\frac{d\hat{e}_{zn}}{dx}\Big)$$

vanishes because $\hat{e}^*_{zm}$ vanishes at $x = 0$ and $x = a$.

13.3.2 (a) The charge in the bottom plate is

$$q = \int_0^w \int_{(a-\Delta)/2}^{(a+\Delta)/2} \epsilon E_y dx dz \tag{1}$$

Using (13.3.15)

$$q = \text{Re}\Big[\sum_{\substack{n=1\\ \text{odd}}} \frac{4n\pi\epsilon}{a}\frac{\hat{v}}{\beta_n a}\frac{1}{\sin\beta_n b}e^{j\omega t}\Big]\frac{2wa}{n\pi}(-1)^{\frac{n-1}{2}}\sin\frac{n\pi\Delta}{2a} \tag{2}$$

where we have used the fact that

$$\begin{aligned}\int_0^w \int_{(a-\Delta)/2}^{(a+\Delta)/2} \sin\Big(\frac{n\pi}{a}x\Big)dxdz &= -\frac{wa}{n\pi}\Big[\cos\Big(\frac{n\pi}{a}\frac{a+\Delta}{2}\Big) - \cos\Big(\frac{n\pi}{a}\frac{a-\Delta}{2}\Big)\Big]\\ &= 2\frac{wa}{n\pi}\sin\frac{n\pi}{2}\sin\frac{n\pi\Delta}{2a}\\ &= \frac{2wa}{n\pi}(-1)^{\frac{n-1}{2}}\sin\frac{n\pi\Delta}{2a}\end{aligned} \tag{3}$$

$$\begin{aligned}v_o &= -\text{Re}\, j\omega\hat{q}e^{j\omega t}R\\ &= -\text{Re}\Big[j\omega 8\epsilon w R\hat{v}\sum_n \frac{(-1)^{\frac{n-1}{2}}}{\beta_n a}\frac{\sin\frac{n\pi\Delta}{2a}}{\sin\beta_n b}e^{j\omega t}\Big]\end{aligned} \tag{4}$$

When $\beta_n b = \pi$ we have a resonance. Now

$$\beta_n = \sqrt{\omega^2\mu\epsilon - \left(\frac{n\pi}{a}\right)^2}$$

The resonance frequency of the n-th mode occurs at

$$\beta_n \frac{b}{a} a = \sqrt{4\omega^2\mu\epsilon a^2 - (n\pi)^2} = \pi \tag{5}$$

or

$$\omega\sqrt{\mu\epsilon}a = \sqrt{(n^2+1)}\frac{\pi}{2} \tag{6}$$

(b) For $n = 1$ this is at π. The next mode resonates at $\sqrt{5/4}\pi$. Thus, in this range, two resonances occur for which the response goes to infinity. Of course, in this limit, losses have to be taken into account which will maintain the response finite. The low frequency limit is when

$$\omega\sqrt{\mu\epsilon} \ll n\frac{\pi}{a}$$

Then

$$\beta_n \sin\beta_n b \to -\frac{n\pi}{a}\sinh\frac{n\pi}{a}b$$

and

$$v_o = \mathrm{Re}\left[j\omega 8\pi\epsilon w R\hat{v}\sum_n \frac{(-1)^{\frac{n-1}{2}}\sin\frac{n\pi\Delta}{2a}}{\frac{n\pi}{a}\sinh\frac{n\pi}{a}b}e^{j\omega t}\right] \tag{7}$$

(c) From (13.3.13), when only one mode predominates,

$$H_z \simeq \mathrm{Re}\left[\frac{4j\omega\epsilon\hat{v}}{\beta_n a}\frac{\cos\beta_n y}{\sin\beta_n b}\cos\frac{n\pi}{a}\right]e^{j\omega t}$$

where $n = 1$ at $\omega\sqrt{\mu\epsilon}a = \pi$ and $n = 2$ at $\omega\sqrt{\mu\epsilon}a = \sqrt{5/4}\pi$. To get a finite answer, we need $\hat{v}/\sin\beta_n b$ to remain finite as the resonance frequency is approached.

13.3.3 (a) H_z at $x = 0$ and $x = a$ gives the surface currents in the bottom and top electrodes. Because the voltage sources push currents into the structure in opposite directions, the surface currents, and H_z, have to vanish at the symmetry plane.

(b) The x-component of the E field can be found directly from (13.3.14), replacing the $\sin\beta_n y/\sin\beta_n b$ by $\cos\beta_n y/\cos\beta_n b$ to take into account the changed symmetry of the field

$$E_x = \mathrm{Re}\left[\sum_{\substack{n=1\\ \text{odd}}}^{\infty} -\frac{4\hat{v}}{a}\frac{\cos\beta_n y}{\cos\beta_n b}\cos\frac{n\pi}{a}x\right]e^{j\omega t}$$

Because $\partial H_z/\partial y = j\omega\epsilon E_x$ we find

$$H_z = \mathrm{Re}\left[\sum_{\substack{n=1\\ n\,\text{odd}}}^{\infty}\frac{4j\omega\epsilon\hat{v}}{\beta_n a}\frac{\sin\beta_n y}{\cos\beta_n b}\cos\frac{n\pi}{a}x\right]e^{j\omega t}$$

13.3.4 (a) The flux linkage λ is

$$\lambda = \mu H_z A \tag{1}$$

and the voltage is

$$v_o = \mu A \frac{dH_z}{dt} \tag{2}$$

(b) From (13.3.13) we find that $|\hat{H}_z|$ is a maximum for $x = 0$ and $x = a$.

(c) From the detailed expression (13.3.13), using (2)

$$v_o = -\text{Re}\Big[\sum_{\substack{n=1\\ n\,\text{odd}}}^{\infty} \frac{4\omega^2\mu\epsilon\hat{v}}{\beta_n a} A \frac{1}{\sin\beta_n b} \cos\frac{n\pi}{a}X\Big]e^{j\omega t}$$

(d) The loop should lie in the $y-z$ plane. Then it links H_x that is tangential to the bottom plate.

13.3.5 The E_x field is derivable from a potential that is a square wave as shown in Fig. S13.3.5. We have

$$\Phi(x) = \sum_m A_m \sin\frac{m\pi}{a}x \tag{1}$$

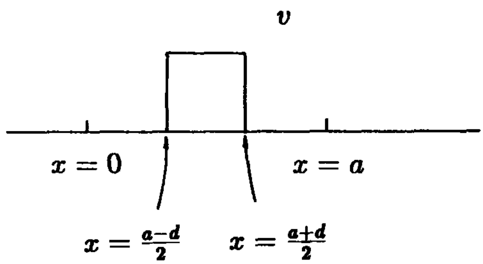

Figure S13.3.5

and using orthogonality, multiplication of both sides by $\sin\frac{n\pi}{a}x$ and integration gies

$$\frac{a}{2}A_n = v\int_{\frac{a-d}{2}}^{\frac{a+d}{2}} \sin\frac{n\pi}{a}x dx = -\frac{av}{n\pi}\Big[\cos\Big(\frac{n\pi}{a}\frac{a+d}{2}\Big) - \cos\Big(\frac{n\pi}{a}\frac{a-d}{2}\Big)\Big]$$
$$= 2\frac{av}{n\pi}\sin\Big(\frac{n\pi}{2}\Big)\sin\Big(\frac{n\pi d}{2a}\Big)$$

We find

$$A_n = \frac{4v}{n\pi}\sin\Big(\frac{n\pi}{2}\Big)\sin\Big(\frac{n\pi d}{2a}\Big)$$

We may adapt (13.3.13)-(13.3.15) for this case by replacing $4v/n\pi$ by

$$4v/n\pi \sin\Big(\frac{n\pi}{2}\Big)\sin\frac{n\pi d}{2a}$$

$$H_z = \text{Re}\left[\sum_{\substack{n=1\\ \text{odd}}}^{\infty} \frac{4j\omega\epsilon\hat{v}}{\beta_n a} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{2a}d\right) \frac{\cos\beta_n y}{\sin\beta_n b} \cos\frac{n\pi}{a}x\right] e^{j\omega t}$$

$$E_x = \text{Re}\left[\sum_{\substack{n=1\\ \text{odd}}}^{\infty} -\frac{4v}{a} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi d}{2a}\right) \frac{\sin\beta_n y}{\sin\beta_n b} \cos\frac{n\pi}{a}x\right] e^{j\omega t}$$

$$E_y = \text{Re}\left[\sum_{\substack{n=1\\ \text{odd}}}^{\infty} \frac{4n\pi}{a} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi d}{2a}\right) \frac{\cos\beta_n y}{\sin\beta_n b} \sin\frac{n\pi}{a}x\right] e^{j\omega t}$$

13.3.6 In (13.3.5) we recognized that E_x at $y = b$ must be the derivative of a potential that is a square wave. This, of course, is equivalent to the statement that E_x possesses two impulse functions. In a similar manner, H_y can be considered the derivative of a flux function $\int_0^x \mu H_y dx$. Note the analogy between (13.3.19) and (13.3.14). We may, therefore, adapt the expansion of P13.3.5 to this problem, because the flux function of Example 13.3.2 is the same as the potential of example 13.3.1. From (13.3.17)-(13.3.19):

$$E_z = \text{Re}\left[\sum_{\substack{m=1\\ \text{odd}}}^{\infty} -\frac{4j\hat{\Lambda}\omega}{m\pi} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi d}{2a}\right) \frac{\sin\beta_m y}{\sin\beta_m b} \sin\frac{m\pi}{a}x\right] e^{j\omega t}$$

$$H_x = \text{Re}\left[\sum_{\substack{m=1\\ \text{odd}}}^{\infty} \frac{4\beta_m\hat{\Lambda}}{\mu m\pi} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{2a}d\right) \frac{\cos\beta_m y}{\sin\beta_m b} \sin\frac{m\pi}{a}x\right] e^{j\omega t}$$

$$H_y = \text{Re}\left[\sum_{\substack{m=1\\ \text{odd}}}^{\infty} -\frac{4\hat{\Lambda}}{\mu a} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{2a}d\right) \frac{\sin\beta_m y}{\sin\beta_m b} \cos\frac{m\pi}{a}x\right] e^{j\omega t}$$

13.4 RECTANGULAR WAVEGUIDE MODES

13.4.1 The loop in the $y - z$ plane produces H-field lines along the x-direction. If placed in the center of the waveguide, at $x = a/2$, these field lines have the same symmetry as those of the TE_{10} mode and thus excite this mode. The detection loop links these fields lines as well. Of course, the position of the exciting loop must be displaced along y by one quarter wavelength compared to the capacitive probe for maximum excitation.

13.4.2 The cutoff frequencies are given by

$$\frac{\omega_c}{c} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{w}\right)^2}$$

The dominant mode has $n = 0$ and thus it has the (lowest) cutoff frequency

$$\frac{\omega_c}{c}\Big|_{m=1,n=0} = \left(\frac{\pi}{a}\right)$$

The higher order modes have cutoff frequencies

$$\frac{\omega_c}{c}\Big|_{mn} = \left(\frac{\pi}{a}\right)\sqrt{m^2 + n^2(a/w)^2} = \frac{\pi}{a}\sqrt{m^2 + n^2(4/3)^2}$$

The cuttoff frequencies are in the ratio to that of the dominant mode:

TE_{01}	1.33
TE_{11} and TM_{11}	1.66
TE_{20}	2.0
TE_{21} and TM_{21}	2.4

13.4.3 (a) TM-modes have all three E-field components. They approach the quasistatic fields of Ex. 5.10.1 which imposes the same boundary conditions as this example. Hence the modes are TM. From (9) we find that $\hat{e}_x \propto -jk_y \frac{\partial e_y}{\partial x} = \frac{\partial^2 e_y}{\partial y \partial x}$ and $\hat{e}_z \propto -jk_y \frac{\partial e_y}{\partial z} = \frac{\partial^2 e_y}{\partial y \partial z}$. Since $\hat{e}_x$ and $\hat{e}_z$ must vanish at $y = 0$, $\hat{e}_y$ must behave as a cosine function of y, so that $\hat{e}_x$ and $\hat{e}_z$ are sine functions of y. Therefore,

$$\begin{aligned} E_y &= \mathrm{Re} \sum_m \sum_n (A^+_{mn} e^{-j\beta_{mn} y} + A^-_{mn} e^{j\beta_{mn} y}) \sin \frac{m\pi}{a} x \sin \frac{n\pi}{w} z e^{j\omega t} \\ &= \mathrm{Re} \sum_m \sum_n 2A^+_{mn} \cos \beta_{mn} y \sin \frac{m\pi}{a} x \sin \frac{n\pi}{a} x \sin \frac{n\pi}{w} z e^{j\omega t} \end{aligned} \quad (1)$$

where

$$\beta_{mn} = \sqrt{\omega^2 \mu\epsilon - (m\pi/a)^2 - (n\pi/w)^2} \quad (2)$$

From (13.4.9):

$$\begin{aligned} E_x &= \mathrm{Re} \sum_m \sum_n \frac{-j\beta_{mn}\left(\frac{m\pi}{a}\right)}{\omega^2\mu\epsilon - \beta^2_{mn}} (A^+_{mn} e^{-j\beta_{mn} y} - A^-_{mn} e^{j\beta_{mn} y}) \\ &\quad \cos \frac{m\pi}{a} x \sin \frac{n\pi}{w} z e^{j\omega t} \\ &= \mathrm{Re} \sum_m \sum_n \frac{\beta_{mn} \frac{m\pi}{a}}{\omega^2\mu\epsilon - \beta^2_{mn}} 2A^+_{mn} \sin \beta_{mn} y \cos \frac{m\pi}{a} x \sin \frac{n\pi}{w} z e^{j\omega t} \end{aligned} \quad (3)$$

and similarly,

$$E_z = \text{Re} \sum_m \sum_n \frac{\beta_{mn} \frac{n\pi}{w}}{\omega^2 \mu\epsilon - \beta_{mn}^2} 2A_{mn}^+ \sin \beta_{mn} y \sin \frac{m\pi}{a} x \cos \frac{n\pi}{w} z e^{j\omega t} \tag{4}$$

(b) At $y = b$, E_x as a function of x must possess two equal and opposite unit impulse functions of content $v(t)/\Delta$ to give the proper voltage drop at the edges. The integral of E_x, $-\int_0^x E_x dx$ must be a square wave function of amplitude v. The same holds with regard to the integral of E_z with respect to z. In summary, E_x and E_z at $y = b$ must be derivable from a potential that is a two-dimensional square wave with the Fourier expansion (5.10.15) (compare 5.10.11):

$$\Phi(x, y) = \text{Re} \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{16\hat{v}}{mn\pi^2} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{w} z e^{j\omega t} \tag{5}$$

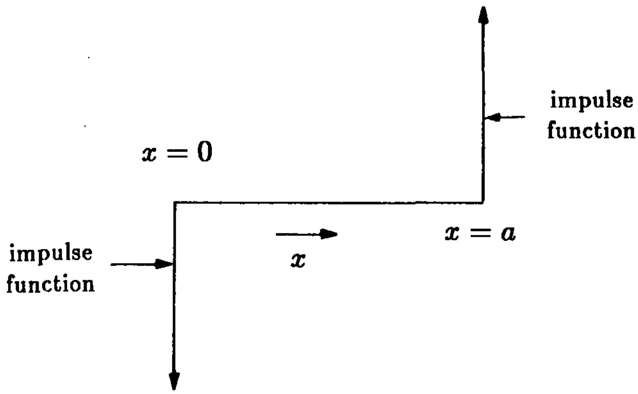

Figure S13.4.3

Thus, at $y = b$

$$E_x(y = b) = -\text{Re} \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \sum_{\substack{n=1 \\ n \text{ odd}}} \frac{16\hat{v}}{mn\pi^2} \left(\frac{m\pi}{a}\right) \cos \frac{m\pi}{a} x \sin \frac{n\pi}{w} z e^{j\omega t} \tag{6}$$

Comparison with (3) gives

$$2A_{mn} \beta_{mn} \frac{m\pi}{a} / (\omega^2 \mu\epsilon - \beta_{mn}^2) \sin \beta_{mn} b = \frac{16\hat{v}}{mn\pi^2} \frac{m\pi}{a} \tag{7}$$

for m and n odd. This gives the quoted result. An analogous relation may be obtained for E_z which yields the same result.

(c) The amplitudes go to infinity when $\sin \beta_{mn} b = 0$ or

$$\sqrt{\omega^2 \mu\epsilon - (m\pi/a)^2 - (n\pi/w)^2} b = p\pi$$

or

$$\omega\sqrt{\mu\epsilon}a = \pi\sqrt{m^2 + \left(\frac{a}{w}n\right)^2 + \left(\frac{a}{b}p\right)^2}$$

(d) We have already used the fact that the distribution of E_x and E_z in the $y = b$ plane is the same as in the quasistatic case. The only difference lies in the y-dependence which, for low frequencies gives the propagation constant

$$\beta_{mn} \simeq j\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{w}\right)^2}$$

and is pure imaginary. The EQS solution according to (5.10.11) and (5.10.15) is

$$\Phi = \text{Re} \sum_{\substack{m=1\\ \text{odd}}}^{\infty} \sum_{\substack{n=1\\ \text{odd}}}^{\infty} \frac{16\hat{v}}{mn\pi^2} \frac{\sinh k_{mn}y}{\sinh k_{mn}b} \sin \frac{m\pi}{a}x \sin \frac{n\pi}{w}z e^{j\omega t}$$

and gives for E_x:

$$E_x = -\text{Re} \sum_{\substack{m=1\\ m\,\text{odd}}}^{\infty} \sum_{\substack{n=1\\ n\,\text{odd}}}^{\infty} \frac{16\hat{v}}{mn\pi^2}\left(\frac{m\pi}{a}\right) \frac{\sinh k_{mn}y}{\sinh k_{mn}b} \cos \frac{m\pi}{a}x \sin \frac{n\pi}{w}z e^{j\omega t}$$

This is the same expression as the EQS result.

13.4.4

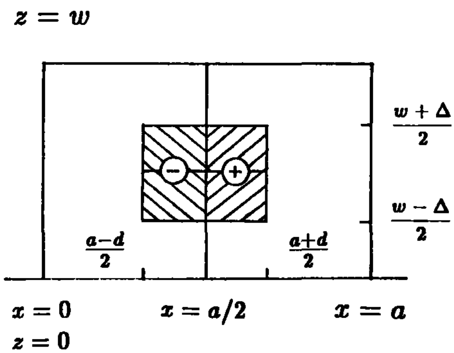

Figure S13.4.4a

The excitation produces a H_y. It looks like TE-modes are going to satisfy all the boundary conditions. H_y must be zero at $y = 0$ and thus from (25) of text

$$\begin{aligned} H_y &= \text{Re} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (C_{mn}^{+} e^{-j\beta_{mn}y} + C_{mn}^{-} e^{j\beta_{mn}y}) \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{w}z\right) e^{j\omega t} \\ &= -\text{Re} \sum_{m=0} \sum_{n=0} 2jC_{mn}^{+} \sin \beta_{mn} y \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{w}z\right) e^{j\omega t} \end{aligned} \quad (1)$$

At $y = b$ we must represent the two dimensional square-wave in the x-direction and in Fig. S13.4.4b in the z-direction as shown in Fig S13.4.4c.

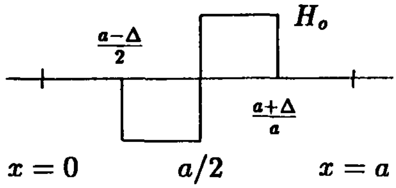

Figure S13.4.4b

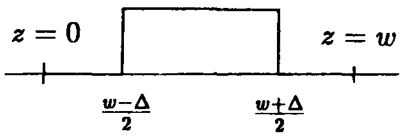

Figure S13.4.4c

We have, setting

$$-2jC_{mn}^{+}\sin\beta_{mn}b \equiv A_{mn} \tag{2}$$

$$\int_0^w\int_0^a\sum\sum A_{mn}\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{w}z\right)\cos\left(\frac{p\pi}{a}x\right)\cos\left(\frac{q\pi}{w}z\right)dx\,dz=-\frac{1}{4}A_{pq}(aw)$$

$$=-H_o\int_{\frac{w-\Delta}{2}}^{\frac{w+\Delta}{2}}dz\int_{\frac{a-d}{2}}^{\frac{a}{2}}dx\cos\left(\frac{p\pi}{a}x\right)\cos\frac{q\pi}{w}z$$

$$+H_o\int_{\frac{w-\Delta}{2}}^{\frac{w+\Delta}{2}}dz\int_{\frac{a}{2}}^{\frac{a+d}{2}}dx\cos\left(\frac{p\pi}{a}x\right)\cos\frac{q\pi}{w}z$$

$$=-\frac{H_o}{\left(\frac{p\pi}{a}\right)\left(\frac{q\pi}{w}\right)}\left[\sin\left(\frac{p\pi}{a}\frac{a}{2}\right)-\sin\left(\frac{p\pi}{a}\frac{a-d}{2}\right)\right]$$

$$\left[\sin\left(\frac{q\pi}{w}\frac{w+\Delta}{2}\right)-\sin\left(\frac{q\pi}{w}\frac{w-\Delta}{2}\right)\right]$$

$$+\frac{H_o}{\left(\frac{p\pi}{a}\right)\left(\frac{q\pi}{w}\right)}\left[\sin\frac{p\pi}{a}\frac{a+d}{2}-\sin\frac{p\pi}{a}\frac{a}{2}\right]$$

$$\left[\sin\left(\frac{q\pi}{w}\frac{w+\Delta}{2}\right)-\sin\left(\frac{q\pi}{w}\frac{w-\Delta}{2}\right)\right.$$

$$=-\frac{H_o}{\left(\frac{p\pi}{a}\right)\left(\frac{q\pi}{w}\right)}\left[\sin\left(\frac{p\pi}{a}\frac{a}{2}\right)\right]$$

$$\left.-\sin\left(\frac{p\pi}{a}\frac{a-d}{2}\right)-\sin\left(\frac{p\pi}{a}\frac{a+d}{2}\right)+\sin\left(\frac{p\pi}{a}\frac{a}{2}\right)\right]$$

$$\left[\sin\frac{q\pi}{w}\frac{w+\Delta}{2}-\sin\frac{q\pi}{w}\frac{w-\Delta}{2}\right]$$

$$=-\frac{H_o}{\left(p\frac{\pi}{a}\right)\left(\frac{q\pi}{w}\right)}\left[2\sin\left(\frac{p\pi}{2}\right)-2\sin\left(\frac{p\pi}{2}\right)\cos\left(\frac{p\pi}{2a}\Delta\right)\right]2\cos\frac{q\pi}{2}\sin\frac{q\pi\Delta}{2w}$$

$$=-\frac{4H_o}{\left(p\frac{\pi}{a}\right)\left(\frac{q\pi}{w}\right)}\sin\left(\frac{p\pi}{2}\right)\cos\left(\frac{q\pi}{2}\right)\sin\frac{q\pi\Delta}{2w}\left[1-\cos\frac{p\pi}{2a}\Delta\right] \tag{3}$$

We find that p must be odd and q must be even for a finite amplitude to result.

$$A_{pq}=\frac{H_o}{pq\pi^2}(-1)^{p-1}(-1)^{\frac{q}{2}-1}\left[1-\cos\frac{p\pi}{2a}\Delta\right] \tag{4}$$

The case $q = 0$ must be handled separately.

$$\begin{aligned}\frac{1}{2}A_{po}(aw) &= -\frac{H_o}{(p\frac{\pi}{a})}\Big[\sin\big(\frac{p\pi}{a}\frac{a}{2}\big) - \sin\big(\frac{p\pi}{a}\frac{a-d}{2}\big)\Big]w \\ &+ \frac{H_o}{(\frac{p\pi}{a})}\big[\sin\frac{p\pi}{a}\big(\frac{a+d}{2}\big) - \sin\big(\frac{p\pi}{a}\frac{a}{2}\big)\big]w \\ &= -\frac{2wH_o}{(p\frac{\pi}{a})}\Big[1 - \cos\frac{p\pi}{2a}\Big]\sin\frac{p\pi}{2}\end{aligned} \tag{5}$$

and thus

$$A_{po} = \frac{H_o}{p\pi}[1 - \cos\frac{p\pi}{2a}\Delta](-1)^{p-1}$$

From (13.4.7) and (13.4.8), one finds

$$H_x = \text{Re}\sum_m\sum_n \frac{2jC^+_{mn}\beta_{mn}\big(\frac{m\pi}{a}\big)}{\omega^2\mu\epsilon - \beta^2_{mn}}\cos\beta_{mn}y\sin\frac{m\pi}{a}x\cos\frac{n\pi}{w}ze^{j\omega t} \tag{7}$$

$$H_z = \text{Re}\sum_m\sum_n \frac{2jC^+_{mn}\beta_{mn}\big(\frac{n\pi}{w}\big)}{\omega^2\mu\epsilon - \beta^2_{mn}}\cos\beta_{mn}y\cos\frac{m\pi}{a}x\sin\frac{n\pi}{w}ze^{j\omega t} \tag{8}$$

with C^+_{mn} expressed in terms of the A_{mn}'s by (2)

13.5 DIELECTRIC WAVEGUIDES: OPTICAL FIBERS

13.5.1 (a) To get an odd function of x for $\hat{e}_z$ one uses the Ansatz

$$\hat{e}_z = \begin{cases} Ae^{-\alpha_x(x-d)} & d < x < \infty \\ A\frac{\sin k_x x}{\sin k_x d} & -d < x < d \\ -Ae^{\alpha_x(x+d)} & -\infty < x < -d \end{cases} \tag{1}$$

which has been adjusted so that $\hat{e}_z$ is continuous at $x = \pm d$. Since

$$\frac{\partial \hat{e}_z}{\partial x} = j\omega\mu\hat{h}_y \tag{2}$$

and thus

$$\hat{h}_y = \frac{1}{j\omega\mu}\begin{cases} -\alpha_x Ae^{-\alpha_x(x-d)} \\ k_x A\frac{\cos k_x x}{\sin k_x d} \\ -\alpha_x Ae^{\alpha_x(x+d)} \end{cases} \tag{3}$$

$\hat{h}_y$ and $\hat{e}_z$ are continuous at $x = d$. The continuity of $\hat{e}_z$ has already been established. From the continuity of h_y:

$$\alpha_x = -k_x\cot k_x d \tag{4}$$

The cutoffs are at $k_x d = (2n-1)\frac{\pi}{2}$ (see Fig. S13.5.1a).

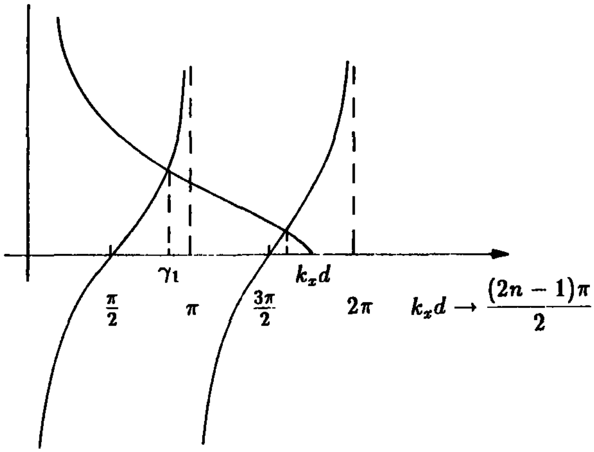

Figure S13.5.1a

(c) When according to 13.5.3

$$k_x d = \sqrt{\omega^2 \mu \epsilon_i - k_y^2} = (2n-1)\frac{\pi}{2}$$

and ω goes to infinity, then k_y must approach $\omega\sqrt{\mu\epsilon_i}$ asymptotically.

(d) See Fig. S13.5.1b (Fig. 6.4 from **Waves and Fields in Optoelectronics**, H. A. Haus, Prentice-Hall, 1984).

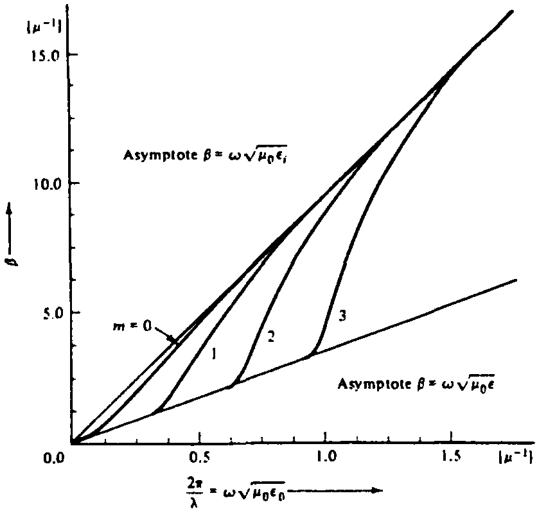

Figure S13.5.1b

13.5.2 The antisymmetric mode comes in when

$$\frac{\alpha_x}{k_x} = 0 \qquad \text{and} \qquad k_x d = \frac{\pi}{2}$$

and from (13.5.8)

$$\sqrt{\frac{\omega^2 \mu \epsilon_i d^2}{(k_x d)^2}\left(1 - \frac{\epsilon}{\epsilon_i}\right) - 1} = 0$$

or

$$\omega\sqrt{\mu\epsilon_i}\, d = \frac{\pi/2}{\left(1 - \frac{\epsilon}{\epsilon_i}\right)}$$

or

$$\omega = \frac{1}{\sqrt{\mu\epsilon_i}\, d}\frac{\pi/2}{\sqrt{\left(1 - \frac{\epsilon}{\epsilon_i}\right)}} = \sqrt{\frac{\epsilon_o}{\epsilon}}\frac{c}{d}\sqrt{\frac{\epsilon}{\epsilon_i}}\frac{\pi/2}{\sqrt{1 - \epsilon/\epsilon_i}}$$

$$= \frac{3 \times 10^8}{10^{-2}}\frac{1}{\sqrt{2.5}}\frac{\pi/2}{\sqrt{1 - \frac{1}{2.5}}} = 3.85 \times 10^{10}$$

$$f = \frac{\omega}{2\pi} = 6.1 \times 10^9 \text{ Hz}$$

13.5.3 (a) For TE modes

$$e_z = \begin{cases} Ae^{-\alpha_x(x-d)} & x > d \\ A\frac{\cos k_x x}{\cos k_x d} \quad \text{or} \quad A\frac{\sin k_x x}{\sin k_x d} & -d < x < d \\ Ae^{\alpha_x(x+d)} \quad \text{or} \quad -Ae^{\alpha_x(x+d)} & x < -d \end{cases}$$

where we have allowed for symmetric and antisymmetric modes. Continuity of $\hat{e}_z$ has been assured on both boundaries. The magnetic field follows from

$$\hat{h}_y = \frac{1}{j\omega\mu}\frac{de_z}{dx} \tag{2}$$

and thus

$$\hat{h}_y = \frac{1}{j\omega}\begin{cases} -\frac{\alpha_x}{\mu}Ae^{-\alpha_x(x-d)} & x > d \\ -\frac{k_x}{\mu_i}A\frac{\sin k_x x}{\cos k_x d} \quad \text{or} \quad \frac{k_x}{\mu_i}\frac{\cos k_x x}{\sin k_x d} & -d < x < d \\ \frac{\alpha_x}{\mu}Ae^{\alpha_x(x+d)} \quad \text{or} \quad -\frac{\alpha_x}{\mu}Ae^{\alpha_x(x+d)} & x < -d \end{cases} \tag{3}$$

Continuity of $\hat{h}_y$ gives

$$\frac{\alpha_x}{\mu} = \frac{k_x}{\mu_i}\tan k_x d \tag{4a}$$

for even modes, and

$$\frac{\alpha_x}{\mu} = -\frac{k_x}{\mu_i}\cot k_x d \tag{4b}$$

for odd modes. Here

$$\alpha_x = \sqrt{k_y^2 - \omega^2\mu\epsilon} \tag{5}$$

$$k_x = \sqrt{\omega^2\mu_i\epsilon_i - k_y^2} \tag{6}$$

and thus, eliminating k_y,

$$k_x^2 = \omega^2(\mu_i\epsilon_i - \mu\epsilon) - \alpha_x^2$$

and

$$\frac{\alpha_x}{k_x} = \sqrt{\frac{\omega^2\mu_i\epsilon_i d^2}{k_x^2 d^2}\left[1 - \frac{\mu\epsilon}{\mu_i\epsilon_i}\right] - 1} \tag{7}$$

(b) Cutoff occurs when $\alpha_x/k_x = 0$ and $k_x d$ is fixed. We find that when μ_i is increased above μ, ω must be lowered.

(c) The constitutive law (a) for symmetric modes has the graphic solution of Fig. 13.5.2. The only change is the expression for α_x/k_x but its $k_x d$ dependence is qualitatively the same; α_x/k_x increases when μ_i/μ increases at constant ω. This means that the intersection point moves to greater $k_x d$ values. k_y^2 increases directly with increasing μ_i/μ according to (6) and decreases with increasing k_x. The intersection point of $k_x d$ does not change as fast, in particular, at high frequencies it does not move at all. Hence, the direct dependence on μ_i predominates, k_y goes up and λ decreases.

13.5.4 (a) The fields are now

$$\hat{h}_z = \begin{cases} Ae^{-\alpha_x(x-d)} & x > d \\ A\frac{\cos k_x x}{\cos k_x d} \quad \text{or} \quad A\frac{\sin k_x x}{\sin k_x d} & -d < x < d \\ Ae^{\alpha_x(x+d)} \quad \text{or} \quad -Ae^{\alpha_x(x+d)} & x < -d \end{cases} \tag{1}$$

where we have allowed for both symmetric and antisymmetric solutions. Cointinuity of $\hat{h}_z$ has been asured on both boundaries. Further,

$$\alpha_x = \sqrt{k_y^2 - \omega^2\mu\epsilon} \tag{2}$$

$$k_x = \sqrt{\omega^2\mu_i\epsilon_i - k_y^2} \tag{3}$$

Since

$$\hat{e}_y = -\frac{1}{j\omega\epsilon}\frac{d\hat{h}_z}{dx} \tag{4}$$

$$\hat{e}_y = -\frac{1}{j\omega}\begin{cases} -\frac{\alpha_x}{\epsilon}Ae^{-\alpha_x(x-d)} \\ -\frac{k_x}{\epsilon_i}A\frac{\sin k_x x}{\cos k_x d} \quad \text{or} \quad \frac{k_x}{\epsilon_i}\frac{\cos k_x x}{\sin k_x d} \\ \frac{\alpha_x}{\epsilon}Ae^{\alpha_x(x+d)} \quad \text{or} \quad -\frac{\alpha_x}{\epsilon}Ae^{\alpha_x(x+d)} \end{cases} \tag{5}$$

Continuity of $\hat{e}_y$ at $x = \pm d$ gives

$$\frac{\alpha_x}{\epsilon} = \frac{k_x}{\epsilon_i}\tan k_x d \tag{6a}$$

for even modes, and

$$\frac{\alpha_x}{\epsilon} = -\frac{k_x}{\epsilon_i}\cot k_x d \tag{6b}$$

for odd modes. Further,

$$k_y^2 = \alpha_x^2 + \omega^2\mu\epsilon = -k_x^2 + \omega^2\mu_i\epsilon_i \tag{7}$$

Thus

$$\alpha_x^2 = \omega^2(\mu_i\epsilon_i - \mu\epsilon) - k_x^2 \tag{8}$$

and

$$\frac{\alpha_x}{k_x} = \sqrt{\frac{\omega^2\mu_i\epsilon_i d^2}{k_x^2 d^2}\left[1 - \frac{\mu\epsilon}{\mu_i\epsilon_i}\right] - 1} \tag{9}$$

(b) The cutoff frequencies are determined by $k_x d = m\frac{\pi}{2}$ and $\alpha_x = 0$. From (9)

$$\frac{\omega_c^2\mu_i\epsilon_i d^2}{\left(m\frac{\pi}{2}\right)^2}\left[1 - \frac{\mu\epsilon}{\mu_i\epsilon_i}\right] - 1 = 0$$

or

$$\omega_c = \frac{m\frac{\pi}{2}}{\sqrt{\mu_i\epsilon_i}d}\frac{1}{\sqrt{1 - \frac{\mu\epsilon}{\mu_i\epsilon_i}}}$$