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SOLUTIONS TO CHAPTER 12

12.1 ELECTRODYNAMIC FIELDS AND POTENTIALS

12.1.1 The particular part of the E-field obeys

$$\nabla \times \mathbf{E}_p = -\frac{\partial}{\partial t}\mathbf{B} \tag{1}$$

$$\nabla \cdot \epsilon_o \mathbf{E}_p = 0 \tag{2}$$

If we set

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{3}$$

then

$$\nabla \times \left(\mathbf{E}_p + \frac{\partial \mathbf{A}}{\partial t}\right) = 0 \tag{4}$$

or

$$\mathbf{E}_p = -\frac{\partial}{\partial t}\mathbf{A} - \nabla \Phi_p \tag{5}$$

Because of (2),

$$\frac{\partial}{\partial t}\nabla \cdot \mathbf{A} + \nabla^2 \Phi_p = 0 \tag{6}$$

But, because we use the Coulomb gauge,

$$\nabla \cdot \mathbf{A} = 0 \tag{7}$$

and thus

$$\nabla^2 \Phi_p = 0 \tag{8}$$

There is no source for the scalar potential of the particular solution. Further

$$\nabla^2 \mathbf{A} = -\mu_o J_u \tag{9}$$

Conversely,

$$\nabla \cdot \epsilon_o \mathbf{E}_h = \rho_u \tag{10}$$

and

$$\nabla \times \mathbf{E}_h = 0 \tag{11}$$

Therefore,

$$\mathbf{E}_h = -\nabla \Phi_h \tag{12}$$

and from (10)

$$\nabla^2 \Phi_h = -\frac{\rho_u}{\epsilon_o} \tag{13}$$

Thus (9) and (13) look like the inhomogeneous wave equation with $\partial^2/\partial t^2$ terms omitted.

12.1.2 $\frac{\partial^2}{\partial t^2}\mathbf{A}$ is of order $1/\tau^2\mathbf{A}$, $\nabla^2\mathbf{A}$ is of order $\mathbf{A}/L^2$. Thus, $\mu\epsilon\frac{\partial^2}{\partial t^2}\mathbf{A}$ is of order $\frac{\mu\epsilon}{\tau^2}L^2$ compared with $\nabla^2\mathbf{A}$. It is negligible if $\mu\epsilon L^2/\tau^2 = L^2/c^2\tau^2 \ll 1$. The same approach shows that $\mu\epsilon(\partial^2/\partial t^2)\Phi$ can be neglected compared with $\nabla^2\Phi$ if $L^2/c^2\tau^2 \ll 1$.

12.2 ELECTRODYNAMIC FIELDS OF SOURCE SINGULARITIES

12.2.1 The time dependence of $q(t)$ is the same as that of Fig. 12.2.5, except that it now extends over one full period.

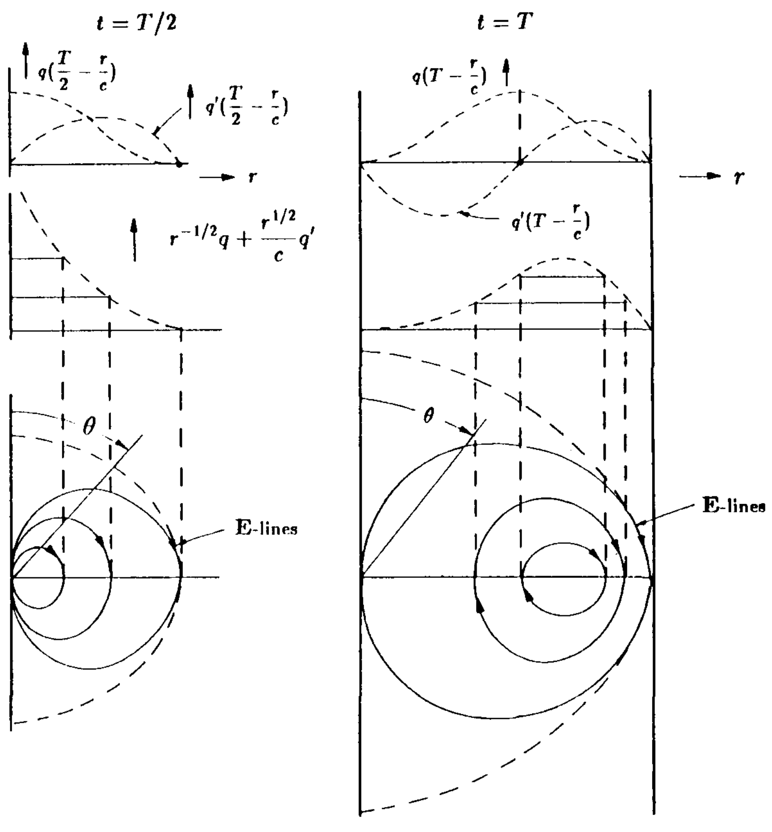

Figure S12.2.1a

Plot of Electric Dipole Field. Any set of field lines that close upon themselves may be considered to be lines of equal height of a potential. The potential does not necessarily reproduce the field intensity at every point. i.e.

$$\mathbf{E} = -\underbrace{(\mathbf{i}_\phi \times \nabla\Phi)}\cdot\overbrace{f(r,\theta)} \tag{1}$$

The "underbrace" gives the pattern. The "overbrace" is the multiplier. It does not change the direction of the field. Take

$$\mathbf{E} = \frac{d}{4\pi\epsilon}\left\{2\cos\theta\left[\frac{q}{r^3}+\frac{q'}{cr^2}\right]\mathbf{i}_r + \left[\frac{q}{r^3}+\frac{q'}{cr^2}+\frac{q''}{c^2 r}\right]\sin\theta\,\mathbf{i}_\theta\right\} \tag{2}$$

where

$$q = q(t-\frac{r}{2})$$

If one defines

$$\Phi = \left(\frac{d}{4\pi\epsilon}\right)2\sin\theta\left(qr^{-1/2}+\frac{q'}{c}r^{1/2}\right) \tag{3}$$

Then

$$\begin{aligned}\nabla\Phi = (\frac{d}{4\pi\epsilon})\Big[&2\sin\theta(-\frac{1}{2}r^{-3/2} - q'\frac{1}{c}r^{-1/2}+\frac{1}{2}q'\frac{1}{c}r^{1/2}-\frac{q''}{c^2}r^{-1/2})\mathbf{i}_r \\ &+\mathbf{i}_\theta 2\cos\theta(qr^{-3/2}+\frac{q'}{c}r^{-1/2})\Big]\end{aligned} \tag{4}$$

One constructs a vector perpendicular to $\nabla\Phi$, $\mathbf{i}_\phi \times \nabla\Phi$, by interchanging the θ and r components and reversing the sign of one of them

$$-\mathbf{i}_\phi \times \nabla\Phi = (\frac{d}{4\pi\epsilon})r^{-3/2}\{2\cos\theta(q+\frac{r}{c}q')\mathbf{i}_r + \sin\theta(q+\frac{r}{c}q'+\frac{r^2}{c^2}q'')\mathbf{i}_\theta\} \tag{5}$$

Thus if we choose $f(r,\theta) = r^{-3/2}$, we reproduce the **E**-field of the dipole by expression (1).

We can sketch the function Φ for $\theta = \pi/2$.

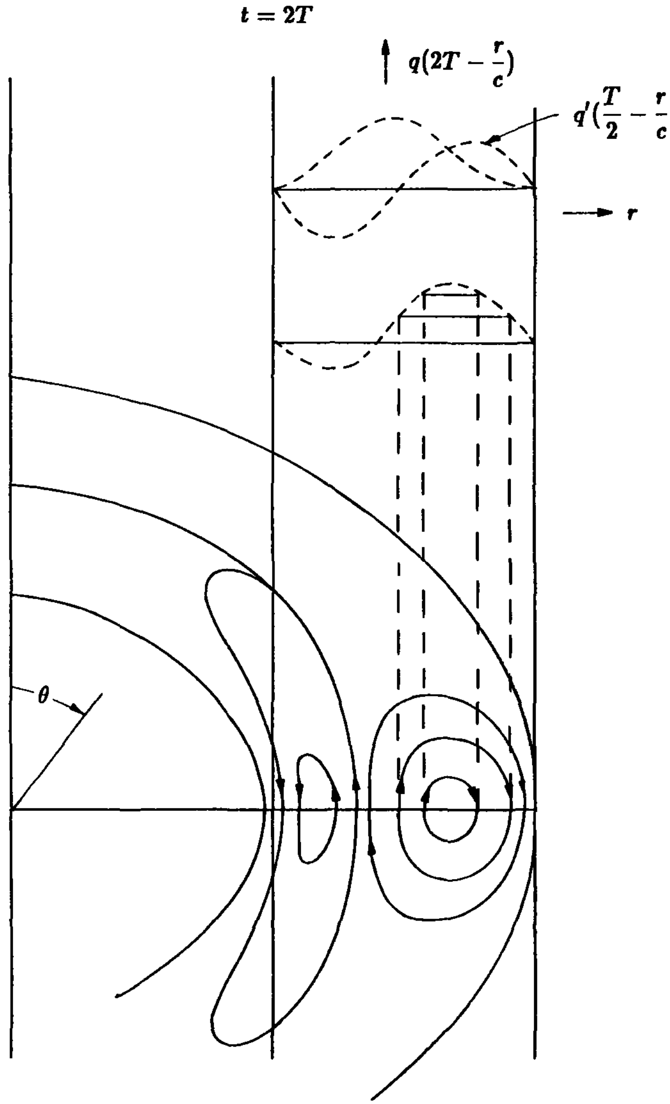

Figure S12.2.1b

12.2.2 Interchange $\mathbf{E} \rightarrow \mathbf{H}$, $\mathbf{H} \rightarrow -\mathbf{E}$ and $\mu_o \rightarrow \epsilon_o$. From (23)

$$d\hat{i} = j\omega\hat{q}d \rightarrow j\omega\hat{q}_m d = j\omega\mu_o\hat{m} \tag{1}$$

where q_m is the magnetic charge. We obtain

$$\hat{E}_\phi = -\frac{jkj\omega\mu_o\hat{m}}{4\pi}\sin\theta\frac{e^{-jkr}}{r} \tag{2}$$

and from (24)

$$\hat{H}_\theta = -\sqrt{\frac{\epsilon_o}{\mu_o}}\hat{E}_\phi = -\frac{k^2\hat{m}}{4\pi}\sin\theta\frac{e^{-jkr}}{r} \qquad \text{QED} \tag{3}$$

12.2.3 Because $\mu_o m(t) = q_m d \to qd$ in the electric dipole case, the time dependence of $q(t)d$ and $\mu_o m(t)$ correspond to each other. With $\mathbf{E} \to \mathbf{H}$ and $\mathbf{H} \to -\mathbf{E}$ we must obtain mutually corresponding field patterns.

12.2.4 We can use the field sketch of Problem 12.2.1 with proper interchange of variables.

12.3 SUPERPOSITION INTEGRAL FOR ELECTRODYNAMIC FIELDS

12.4 ANTENNAE RADIATION FIELDS IN THE SINUSOIDAL STEADY STATE

12.4.1 From (4)

$$\begin{aligned}\psi_o(\theta) &= \frac{\sin\theta}{l}\int_0^l e^{-jkz'}e^{jkz'\cos\theta}dz' \\ &= \frac{\sin\theta}{l}\frac{1}{jk(\cos\theta-1)}\{e^{-jk(1-\cos\theta)l}-1\} \\ &= \frac{\sin\theta}{l}\frac{2}{k(1-\cos\theta)}\sin\left[\frac{kl}{2}(1-\cos\theta)\right]e^{-jk(1-\cos\theta)l/2}\end{aligned} \tag{1}$$

The radiation pattern is

$$\begin{aligned}\Psi(\theta) = |\psi_o(0)|^2 &= \frac{4\sin^2\theta\sin^2\frac{kl}{2}(1-\cos\theta)}{k^2l^2(1-\cos\theta)^2} \\ &= \frac{\sin^2\theta}{k^2l^2\sin^4(\theta/2)}\sin^2\left(kl\sin^2\frac{\theta}{2}\right)\end{aligned} \tag{2}$$

With $kl = 2\pi$

$$\Psi(\theta) \equiv \frac{\sin^2\theta}{4\pi^2\sin^4(\theta/2)}\left(\sin^2 2\pi\sin^2\frac{\theta}{2}\right) \tag{3}$$

The radiation pattern peaks near $\theta = 60°$.

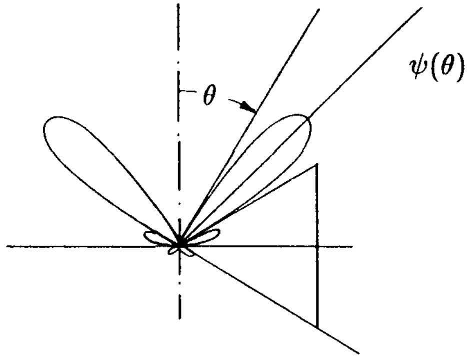

Figure S12.4.1

12.4.2 By analogy with (3) one replaces $H_\phi \to E_\phi$, $\mu \leftrightarrow \epsilon$ and $i(z')dz' = j\omega(qz)dz' \to j\omega(q_m d)dz' = j\omega\mu\mu(z')dz'$ where we interpret qd and $q_m d$ as assigned to unit length. Thus, from (2) of Prob. 12.2.2, with $\mu_o \to \mu$, $\epsilon_o \to \epsilon$,

$$E_\phi = \frac{k^2}{4\pi}\sin\theta\frac{e^{-jkr}}{r}\sqrt{\frac{\mu}{\epsilon}}\int \mathcal{M}(z')e^{jk\mathbf{r}'\cdot\mathbf{i}_r}dz'$$
$$= \frac{k^2 l}{4\pi}\sqrt{\frac{\mu}{\epsilon}}\frac{e^{-jkr}}{4}\mathcal{M}_o e^{j\alpha_o}\epsilon_o(\theta)$$

where

$$\psi_o(\theta) \equiv \frac{\sin\theta}{l}\int\frac{\mathcal{M}(r')}{\mathcal{M}_o}e^{j(k\mathbf{r}'\cdot\mathbf{i}_r-\alpha_o)}dz'$$

12.4.3

$$\psi_o(\theta) = -\frac{\sin\theta}{l}\int_0^l\frac{\sin\beta(z'-l)}{\sin\beta l}e^{jkz'\cos\theta}dz'$$
$$= -\frac{\sin\theta}{\beta l\sin\beta l}\int_0^l\frac{1}{2j}\{\left(e^{j\beta(z'-l)} - e^{-j\beta(z'-l)}\right)e^{jkz'\cos\theta}d(\beta z')$$
$$= -\frac{\sin\theta}{\beta l\sin\beta l}\frac{1}{2j}\left\{\frac{e^{j(\beta+k\cos\theta)l}-1}{j\left(1+\frac{k}{\beta}\cos\theta\right)}e^{-j\beta l} - \frac{e^{-j(\beta-k\cos\theta)l}-1}{-j\left(1-\frac{k}{\beta}\cos\theta\right)}e^{j\beta l}\right\}$$
$$= -\frac{\sin\theta}{\beta l\sin\beta l}\frac{2}{1-\frac{k^2}{\beta^2}\cos^2\theta}\left\{\cos\beta l + j\sin\beta l\frac{k}{\beta}\cos\theta - e^{jk\cos\theta l}\right\}$$

12.4.4 (a) From (12), and with $\mathbf{a}_n = n\frac{\lambda}{4}\mathbf{i}_x$,

$$\psi_a = \sum_{n=0}^{3} e^{jk\mathbf{a}_n\cdot\mathbf{i}_r}e^{j(\alpha_n-\alpha_o)} \tag{1}$$
$$= 1 + e^{j(\frac{\pi}{2}\cos\phi\sin\theta+\alpha_1-\alpha_o)} + e^{j(\pi\cos\phi\sin\theta+\alpha_2-\alpha_o)}$$

(b) Since $\psi_o = \sin\theta$, and $\alpha_i = 0$

$$|\psi_o||\psi_a| = \left|1 + 2\cos\left(\frac{\pi}{2}\cos\phi\sin\theta\right)\right|\sin\theta \tag{2}$$

(c)

$$\begin{aligned}\psi_a &= 1 + e^{j\frac{\pi}{2}(\cos\phi\sin\theta+1)} + e^{j\pi(\cos\phi\sin\theta+1)}\\ &= e^{j\frac{\pi}{2}(\cos\phi\sin\theta+1)}\left\{e^{-j\frac{\pi}{2}(\cos\phi\sin\theta+1)} + 1 + e^{j\frac{\pi}{2}(\cos\phi\sin\theta+1)}\right\}\\ &= e^{j\frac{\pi}{2}(\cos\phi\sin\theta+1)}\left[2\cos\frac{\pi}{2}(\cos\phi\sin\theta+1) + 1\right]\end{aligned} \tag{3}$$

$$|\psi_o||\psi_a| = \left|\left[1 + 2\cos\frac{\pi}{2}(\cos\phi\sin\theta+1)\right]\sin\theta\right| \tag{4}$$

12.4.5 (a)

$$\psi_a(\theta) = \sum_{n=0}^{1} e^{jk\mathbf{a}_n\cdot\mathbf{i}_r}e^{j(\alpha_n-\alpha_o)} = 1 + e^{j[\pi\cos\theta+\alpha_1-\alpha_o]} \tag{1}$$

(b)

$$|\psi_a|^2|\psi_0|^2 = 4\cos^2\left(\frac{\pi}{2}\cos\theta\right)\sin^2\theta \tag{2}$$

(c)

$$G = \frac{4\pi\cos^2\left(\frac{\pi}{2}\cos\theta\right)\sin^2\theta}{\int_0^\pi d\theta\int_0^{2\pi} d\phi\sin\theta\cos^2\left(\frac{\pi}{2}\cos\theta\right)\sin^2\theta} \tag{3}$$

Define

$$\cos\theta = u \tag{4}$$

$$\int_0^\pi d\theta\sin^3\theta\cos^2\left(\frac{\pi}{2}\cos\theta\right) = \int_{-1}^{1} du(1-u^2)\cos^2\left(\frac{\pi}{2}u\right) \tag{5}$$

Now consider integral

$$\int dx x^2\cos^2 x = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right)x^2 - \frac{x^3}{3} + \frac{2x}{8}\cos 2x - \frac{1}{8}\sin 2x \tag{6}$$

The integral is

$$\begin{aligned}\int_{-1}^{1} du(1-u^2)\cos^2\frac{\pi}{2}u &= 1 - \left(\frac{2}{\pi}\right)^3\left\{\frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right)x^2\right.\\ &\left.\quad - \frac{x^3}{3} + \frac{2x}{8}\cos 2x - \frac{1}{8}\sin 2x\right\}_{-\pi/2}^{\pi/2}\\ &= \frac{2}{3} + \frac{1}{2}\left(\frac{2}{\pi}\right)^2\end{aligned} \tag{7}$$

The gain is

$$G = \frac{4\pi \cos^2\left(\frac{\pi}{2}\cos\theta\right)\sin^2\theta}{2\pi\{\frac{2}{3}+\frac{2}{\pi^2}\}} \tag{8}$$

(d) We find for $\Psi(\theta)$ of array

$$\Psi(\theta) = \{|\psi_o(\theta)||\psi_1(\theta)||\psi_2(\theta)|\}^2 \tag{9}$$

with

$$\psi_2(\theta) = 1 - e^{jka\sin\theta\cos\phi} \tag{10}$$

In order to get maximum superposition in the direction $\phi = 0$, one needs $ka = \pi$ or $a = \lambda/2$. Thus

$$|\psi_2(\theta)| = |2\sin\left(\frac{\pi}{2}\sin\theta\cos\phi\right)|$$

12.5 COMPLEX POYNTING'S THEOREM AND RADIATION RESISTANCE

12.5.1 The radiation field Poynting vector of the antenna is from 12.4.2, 3.4.5

$$\frac{1}{2}(\hat{E}_\theta \hat{H}_\phi^*) = \frac{1}{2}\frac{(kl)^2}{(4\pi)^2}\sqrt{\frac{\mu_o}{\epsilon_o}}|I_o|^2(\psi_o(\theta))^2 \tag{1}$$

where $\psi_o(\theta)$ is from 12.4.28

$$\begin{aligned} \psi_o(\theta) &= \frac{1}{\left(\frac{3\pi}{2}\right)\sin\left(\frac{3\pi}{2}\right)}\frac{\cos\left(\frac{3\pi}{2}\right) - \cos\left(\frac{3\pi}{2}\cos\theta\right)}{\sin\theta} \\ &= \frac{2}{3\pi}\frac{\cos\left(\frac{3\pi}{2}\cos\theta\right)}{\sin\theta} \end{aligned} \tag{2}$$

The radiated power is

$$\begin{aligned} \frac{1}{2}|I_o|^2 R_{\text{rad}} &= \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \frac{1}{2}\hat{E}_\theta \hat{H}_\phi^* \\ &= \frac{1}{2}\frac{(3\pi)^2}{(4\pi)^2}\sqrt{\mu_o/\epsilon_o}|I_o|^2\left(\frac{2}{3\pi}\right)^2 2\pi \int_0^\pi \frac{\cos^2\left(\frac{3\pi}{2}\cos\theta\right)}{\sin^2\theta}\sin\theta d\theta \\ &= \frac{1}{2}|I_o|^2\frac{\sqrt{\mu_o/\epsilon_o}}{4\pi^2}(2\pi)\int_0^\pi d\theta \sin\theta \frac{\cos^2\left(\frac{3\pi}{2}\cos\theta\right)}{\sin^2\theta} \end{aligned} \tag{3}$$

Therefore

$$\begin{aligned} R_{\text{rad}} &= \frac{\sqrt{\mu_o/\epsilon_o}}{2\pi}\int_0^\pi d\theta \sin\theta \frac{\cos^2\left(\frac{3\pi}{2}\cos\theta\right)}{\sin^2\theta} \\ &= \frac{1}{2\pi}\sqrt{\mu_o/\epsilon_o}\int_{-1}^{1} dx \frac{\cos^2\left(\frac{3\pi}{2}x\right)}{1-x^2} \\ &= 104\Omega \end{aligned} \tag{4}$$

12.5.2 The scalar potential of the spherical coil is (see Eq. 8.5.17)

$$\Psi = \frac{NI_oR^2}{6r^2}\cos\theta = \frac{m}{4\pi r^2}\cos\theta \tag{1}$$

This identifies

$$\hat{m} = \frac{2\pi}{3}N\hat{I}_oR^2 \tag{2}$$

We have for the θ component of the H-field

$$\hat{H}_\theta = \frac{\hat{m}}{4\pi r^3}\sin\theta[1 + jkr + (jkr)^2] \tag{3}$$

and thus the radiation field is

$$\hat{H}_\theta \approx -\frac{k^2\hat{m}}{4\pi r}\sin\theta \tag{4}$$

The power radiated is

$$\begin{aligned} -\frac{1}{2}\int\int \hat{E}_\phi \hat{H}_\theta^* r^2 \sin\theta d\theta d\phi &= \frac{1}{2}2\pi\frac{4}{3}\sqrt{\mu_o/\epsilon_o}\left(\frac{k^2m}{4\pi}\right)^2 \\ &= \frac{1}{2}|\hat{I}_o|^2 R_{\text{rad}} \end{aligned} \tag{5}$$

Therefore,

$$R_{\text{rad}} = \frac{2\pi}{27}\sqrt{\mu_o/\epsilon_o}N^2(kR)^4 \tag{6}$$

The inductance of the coil is from (8.5.20)

$$L = \frac{2\pi}{9}\mu_oRN^2 \tag{7}$$

and therefore

$$R_{\text{rad}} = \frac{1}{3}\omega L(kR)^3 \tag{8}$$

12.6 PERIODIC SHEET-SOURCE FIELDS: UNIFORM AND NONUNIFORM PLANE WAVES

12.6.1 (a) From continuity:

$$\frac{\partial \hat{K}_x}{\partial x} + j\omega\hat{\sigma}_s = 0$$

Taking into account the x-dependence:

$$-jk_x\hat{K}_x + j\omega\hat{\sigma}_s = 0 \tag{2}$$

and therefore

$$\hat{K}_x = \frac{\omega}{k_x}\sigma_o e^{-jk_xx} \tag{3}$$

and

$$K_x = \text{Re}\left(\frac{\omega\sigma_o e^{jk_xx}}{k_x}\right)e^{j\omega t}$$

(b) The boundary condition on the tangential $\mathbf{H}$ is:

$$\mathbf{n}\times(\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K} \qquad \mathbf{n} \parallel \mathbf{i}_y$$

Since

$$\mathbf{H} \parallel \mathbf{i}_z \tag{4}$$

and thus

$$\hat{H}_z^a - \hat{H}_z^b = \hat{K}_x \tag{5}$$

H_z is antisymmetric, of opposite sign on the two sides of current sheet.

$$2\hat{H}_z^a = \hat{K}_x \tag{6}$$

and thus

$$\hat{H}^{\binom{a}{b}} = \mathbf{i}_z\text{Re}\left[\pm\frac{\omega\sigma_o}{2k_x}e^{\mp j\beta y}e^{j(\omega t - k_xx)}\right] \tag{7}$$

From (12.6.6) and (12.6.7)

$$\mathbf{E} = \text{Re}\left[\mathbf{i_x}\left(-\frac{\beta\sigma_o}{2\epsilon_o k_x}\right) + \mathbf{i_y}\left(\pm\frac{\sigma_o}{2\epsilon_o}\right)\right]e^{\mp j\beta y}e^{j(\omega t - k_xx)} \tag{8}$$

(c) As in Problem 12.2.1, a plot of a divergence-free field can be done by defining a potential Φ and obtaining the field

$$\mathbf{E} = -\mathbf{i}_z\times\nabla\Phi f(x,y) \tag{9}$$

Now, it is clear that the potential necessary to produce (8) is

$$\Phi = \pm\frac{1}{jk_x}\left(\frac{\sigma_o}{2\epsilon_o}\right)e^{\mp j\beta y}e^{j(\omega t - k_xx)}$$

Then

$$-\mathbf{i}_z \times \nabla\Phi = \mathbf{i}_x \frac{\partial \Phi}{\partial y} - \mathbf{i}_y \frac{\partial \Phi}{\partial x}$$

and is found to be equal to (8) with $f(x, y)$ equal to unity. By visualizing the potential, one may plot E lines.

k_y imaginary:

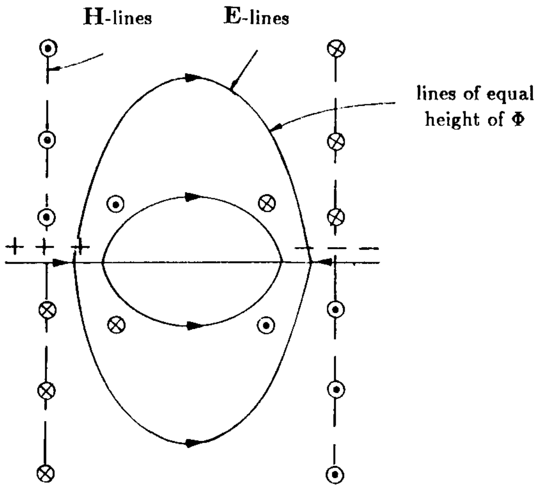

Figure S12.6.1a

At $\omega t = 0$, the potential is

$$\text{Re} - \frac{1}{jk_x}\frac{\sigma_o}{2\epsilon_o} e^{\mp j\beta y} e^{-jk_x x} = \frac{\sigma_o}{2\epsilon_o k_x} \sin(k_x x) e^{\mp|\beta| y}$$

k_y real:

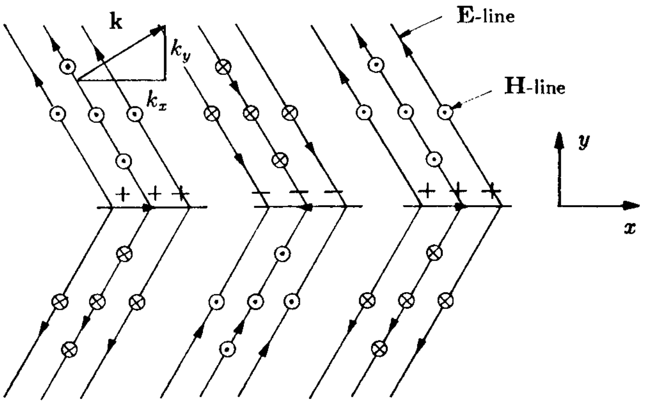

Figure S12.6.1b

At $\omega t = 0$, the potential is

$$\frac{\sigma_o}{2\epsilon_o k_x}\sin(k_x x \pm k_y y)$$

12.6.2 (a) The E-field will be z-directed, the H-field is in the $x-y$ plane

$$\hat{E}_z = A\sin(k_x x)e^{\mp jk_y y} \tag{1}$$

From (12.6.29)

$$\hat{H}_x = -\frac{1}{j\omega\mu}\frac{\partial E_z}{\partial y} = -\frac{1}{j\omega\mu}(\mp jk_y)A\sin k_x x \tag{2}$$

The discontinuity of tangential H gives:

$$\mathbf{n}\times(\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K} \tag{3}$$

in $x-z$ plane. And thus, combining (2) and (3)

$$2\frac{k_y}{\omega\mu}A\sin k_x x = -K_o \sin k_x x \tag{4}$$

and therefore

$$A = -\frac{\omega\mu K_o}{2k_y} \tag{5}$$

From (2) and (5)

$$\hat{H}_x = \mp\frac{K_o}{2}\sin(k_x x)e^{\mp jk_y y} \tag{6}$$

and from (12.6.30)

$$\hat{H}_y = j\frac{k_x}{k_y}\frac{K_o}{2}\cos(k_x x)e^{\mp jk_y y} \tag{7}$$

(b) Again we can use a potential Φ to which the H lines are lines of equal height. If we postulate

$$\Phi = \left(\frac{1}{jk_y}\right)\frac{K_o}{2}\sin k_x x e^{\mp jk_y y} \tag{8}$$

Then

$$-\mathbf{i}_z \times \nabla\Phi = \mathbf{i}_x\frac{\partial\Phi}{\partial y}\frac{K_o}{2}\frac{k_x}{jk_y}\cos k_x x \tag{9}$$

$$-\mathbf{i}_y\frac{\partial\Phi}{\partial x} = \mp\mathbf{i}_x\frac{K_o}{2}\sin k_x x - \mathbf{i}_y\frac{K_o}{2}\frac{k_x}{jk_y}\cos k_x x$$

The potential hill at $\omega t = 0$ is

$$\text{Re}[\Phi] = \mp\frac{K_o}{2}\sin k_x x \sin k_y y \tag{10}$$

$$\omega t = 0$$

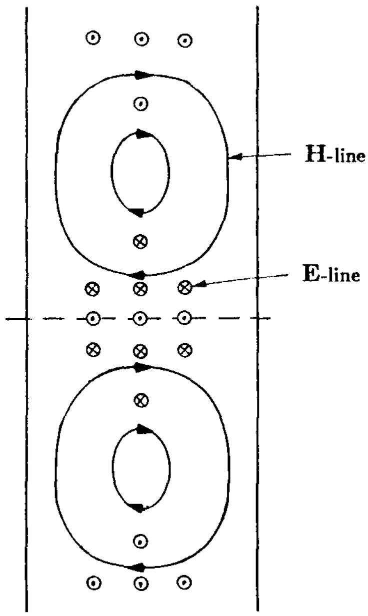

Figure S12.6.2

(c) We may write (1)

$$\hat{E}_z = -\frac{\omega\mu K_o}{2k_y}\frac{1}{2j}\left\{e^{jk_x x \mp jk_y y} - e^{-jk_x x \mp jk_y y}\right\} \tag{11}$$

and for (6) and (7)

$$\begin{aligned}\hat{\mathbf{H}} =& j\frac{K_o}{4}\Big\{ \pm \mathbf{i_x}\left(e^{jk_x x \mp jk_y y} - e^{-jk_x x \mp jk_y y}\right) \\ &+ \frac{k_x}{k_y}\mathbf{i_z}\left(e^{jk_x x \mp jk_y y} + e^{-jk_x x \mp jk_y y}\right)\Big\}\end{aligned} \tag{12}$$

12.6.3 (a) At first it is best to find the field E_z due to a single current sheet at $y = 0$. We have

$$\hat{E}_z = Ae^{-jk_x x}e^{\mp j\beta y} \tag{1}$$

From (12.6.29)

$$\hat{H}_x = -\frac{1}{j\omega\mu}\frac{\partial \hat{E}_z}{\partial y} = -\frac{1}{j\omega\mu}(\mp j\beta)Ae^{-jk_x x}e^{\mp j\beta y} \tag{2}$$

From the boundary condition

$$\mathbf{n} \times (\mathbf{H}^{a} - \mathbf{H}^{(b)}) = \mathbf{K} \tag{3}$$

we get

$$2\frac{\beta}{\omega\mu} A e^{-jk_z x} = -K e^{-jk_z x}$$

and thus

$$A = -\frac{\omega\mu K}{2\beta} \tag{4}$$

Now we can add the fields due to each source

$$\begin{aligned} \hat{E}_z = &-\frac{\omega\mu}{2\beta}[\hat{K}^a e^{-jk_z x} \begin{Bmatrix} e^{-j\beta(y-\frac{d}{2})} \\ e^{j\beta(y-\frac{d}{2})} \\ e^{j\beta(y-\frac{d}{2})} \end{Bmatrix} \\ &+ \hat{K}^b e^{-jk_z x} \begin{Bmatrix} e^{-j\beta(y+\frac{d}{2})} \\ e^{-j\beta(y+\frac{d}{2})} \\ e^{j\beta(y+\frac{d}{2})} \end{Bmatrix} \end{aligned} \tag{5}$$

(b) When

$$\hat{K}_a e^{-j\beta\frac{d}{2}} + \hat{K}_b e^{j\beta\frac{d}{2}} = 0 \tag{6}$$

Then

$$\hat{K}_b = -\hat{K}_a e^{-j\beta d} \tag{7}$$

there is cancellation at $y < -d/2$

(c)

$$\begin{aligned} \hat{E}_z &= -\frac{\omega\mu}{2\beta}\left[\hat{K}_a e^{-jk_z x} e^{-j\beta(y-\frac{d}{2})} - \hat{K}_a e^{-jk_z x} e^{-j\beta(y+\frac{3d}{2})}\right] \\ &= -j\frac{\omega\mu\hat{K}_a}{\beta} e^{-jk_z x} e^{-j\beta(y-\frac{d}{2})} \sin\beta d e^{-j\beta d} \end{aligned} \tag{8}$$

(d) In order to produce maximum radiation we want the endfire array situation of $\beta d = \pi/2$. (Indeed, $\sin\beta d = 1$ in this case.) Because

$$\beta = \sqrt{\omega^2\mu\epsilon - k_x^2} \tag{9}$$

we have

$$\omega = \frac{1}{\sqrt{\mu\epsilon}}\left[k_x^2 - \left(\frac{\pi}{2d}\right)^2\right]^{1/2} \tag{10}$$

The direction is

$$\mathbf{k} = k_x \mathbf{i_x} + \beta \mathbf{i_y} = k_x \mathbf{i_x} + \frac{\pi}{2d}\mathbf{i_y}$$

12.6.4 (a) If we want cancellations, we again want (compare P12.6.3)

$$\hat{\sigma}_b = -\hat{\sigma}_a e^{-jk_y d} \tag{1}$$

(b) A single sheet at $y = d/2$ gives

$$H_z = \pm A e^{-jk_x x} e^{\mp jk_y (y - \frac{d}{2})} \tag{2}$$

Now,

$$\frac{\partial \hat{K}_x}{\partial x} + j\omega\hat{\sigma} = 0 \tag{3}$$

gives

$$\hat{K}_x = \frac{\omega}{k_x}\hat{\sigma}_a \tag{4}$$

and

$$2\hat{H}_z^a\Big|_{y=0_+} = \frac{\omega}{k_x}\hat{\sigma}_a \tag{5}$$

Therefore

$$A = \frac{\omega}{2k_x}\hat{\sigma}_a \tag{6}$$

and the field of both sheets is

$$H_z = j\frac{\omega}{k_x}\hat{\sigma}_a e^{-jk_x x} e^{-jk_y\left(y + \frac{d}{2}\right)} \sin k_x d \tag{7}$$

(c) $k_y d = \pi/2$. Therefore, as in P12.6.3,

$$\omega = \frac{1}{\sqrt{\mu\epsilon}}\left[k_x^2 - \left(\frac{\pi}{2d}\right)^2\right]^{1/2} \tag{8}$$

12.7 ELECTRODYNAMIC FIELDS IN THE PRESENCE OF PERFECT CONDUCTORS

12.7.1 The field of the antenna is that of a current distribution $|\cos kz|$. We may treat it in terms of an array factor of three antennae spaced $\lambda/2$ apart along the z-axis. From 12.4.12

$$\begin{aligned} |\psi_a(\theta)| = \Big|\sum_{i=0}^{3} e^{ik\frac{\lambda}{2}\cos\theta}\Big| &= |1 + e^{j\pi\cos\theta} + e^{2j\pi\cos\theta}| \\ &= |e^{-j\pi\cos\theta} + 1 + e^{j\pi\cos\theta}| \\ &= 1 + 2\cos(\pi\cos\theta) \end{aligned} \tag{1}$$

The function $\psi_o(\theta)$ follows from 12.4.8 with $kl = \pi$

$$\psi_o(\theta) = \frac{2}{\pi}\cos\left(\frac{\pi}{2}\cos\theta\right)/\sin\theta \tag{2}$$

Combining (1) and (2) we complete the proof.

12.7.2 The current distribution, with image, is proportional to $|\sin kz|$. The point at which the current is fed into the antenna calls for zero current. Since the radiated power is finite, R_{rad} is infinite. In practice, because of the finite losses, it is not infinite but much larger than $\sqrt{\mu_o/\epsilon_o}$.

12.7.3 (a) We have a surface current $\hat{K}_x$

$$\frac{\partial \hat{K}_x}{\partial x} + j\omega\hat{\sigma}_s = 0 \tag{1}$$

Therefore

$$\hat{K}_x = -\frac{j\omega}{\pi/a}\sigma_o \sin\left(\frac{\pi x}{a}\right) \tag{2}$$

The **H**-field is z-directed and antisymmetric with respect to y.

$$\hat{H}_z = \pm A \sin\left(\frac{\pi x}{a}\right)e^{\mp jk_y y} \tag{3}$$

From the boundary condition

$$\mathbf{n} \times (\hat{\mathbf{H}}^a - \hat{\mathbf{H}}^b) = \hat{\mathbf{K}} \tag{4}$$

with $\mathbf{n} \parallel \mathbf{i}_y$

$$2A\sin\left(\frac{\pi x}{a}\right) = -\frac{j\omega}{\pi/a}\sigma_o \sin\left(\frac{\pi x}{a}\right) \tag{5}$$

$$A = -\frac{j\omega}{2\pi/a}\sigma_o \tag{6}$$

The E-field is from (12.6.6)

$$\begin{aligned}\hat{E}_x &= \frac{1}{j\omega\epsilon_o}\frac{\partial \hat{H}_z}{\partial y} = \pm\frac{1}{j\omega\epsilon_o}\left(-\frac{j\omega}{2\pi/a}\sigma_o\right)(\mp jk_y)\sin\left(\frac{\pi x}{a}\right)e^{\mp jk_y y}\\ &= \frac{jk_y\sigma_o}{\epsilon_o(2\pi/a)}\sin\left(\frac{\pi x}{a}\right)e^{\mp jk_y y}\end{aligned} \tag{7}$$

and from 12.6.7

$$\begin{aligned}\hat{E}_y &= -\frac{1}{j\omega\epsilon_o}\frac{\partial \hat{H}_z}{\partial x} = \left(-\frac{1}{j\omega\epsilon_o}\right)\left(\mp\frac{j\omega\sigma_o}{2(\pi/a)}\right)\frac{\pi}{a}\cos\left(\frac{\pi}{a}x\right)e^{\mp jk_y y}\\ &= \pm\frac{\sigma_o}{2\epsilon_o}\cos\left(\frac{\pi}{a}x\right)e^{\mp jk_y y}\end{aligned} \tag{8}$$

(b) On the plate at $x = -a/2$

$$\hat{\sigma}_s = \epsilon_o\hat{E}_x|_{x=-a/2} = -\frac{jk_y\sigma_o}{2\pi/a}e^{\mp jk_y y} \tag{9}$$

At $x = a/2$ it is of opposite sign. The surface current is

$$\hat{K}_y = -\hat{H}_z|_{x=-a/2} = \pm \frac{j\omega\sigma_o}{2\pi/a} e^{\mp jk_y y} \tag{10}$$

and is the negative of that at $x = a/2$.

(c)

$$k_x^2 + k_y^2 = \omega^2 \mu_o \epsilon_o \tag{11}$$

and thus

$$k_y = \sqrt{\omega^2 \mu_o \epsilon_o - \left(\frac{\pi}{a}\right)^2} \tag{12}$$

Again we may identify a potential whose lines of equal height give $\mathbf{E}$. Indeed,

$$\Phi = \mp \frac{\sigma_o}{\epsilon_o(2\pi/a)} \sin\left(\frac{\pi x}{a}\right) e^{\mp jk_y y} \tag{13}$$

gives

$$\begin{aligned} -\mathbf{i}_z \times \nabla\Phi &= \mathbf{i}_x \frac{\partial \Phi}{\partial y} - \mathbf{i}_y \frac{\partial \Phi}{\partial x} \\ &= \mp \frac{\sigma_o}{\epsilon_o(2\pi/a)} e^{\mp jk_y y} \left[\mp jk_y \mathbf{i}_x \sin\left(\frac{\pi x}{a}\right) - \frac{\pi}{a} \mathbf{i}_y \cos\left(\frac{\pi x}{a}\right)\right] \end{aligned} \tag{14}$$

(d) For k_y imaginary and $\omega t = 0$

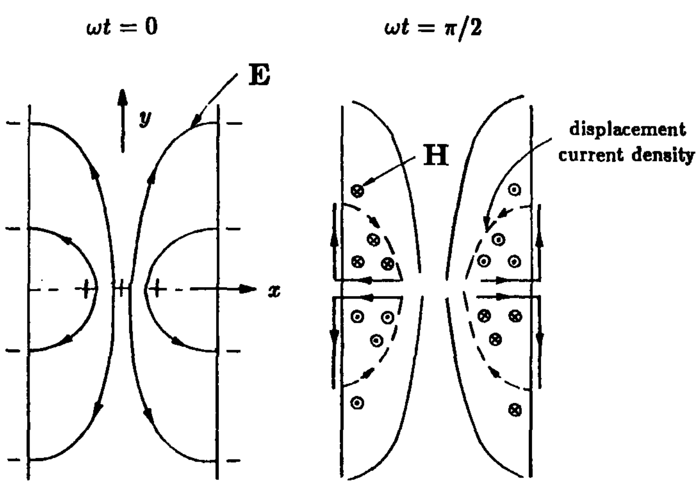

Figure S12.7.3a

$$\mathrm{Re}[\Phi] = \mp \frac{\sigma_o}{\epsilon_o(2\pi/a)} \sin\left(\frac{\pi x}{a}\right) e^{\mp |k_y| y}$$

For k_y real, $\omega t = 0$

$$\text{Re}[\Phi] = \mp \frac{\sigma_o}{\epsilon_o(2\pi/a)} \sin\left(\frac{\pi x}{a}\right) \cos k_y y$$

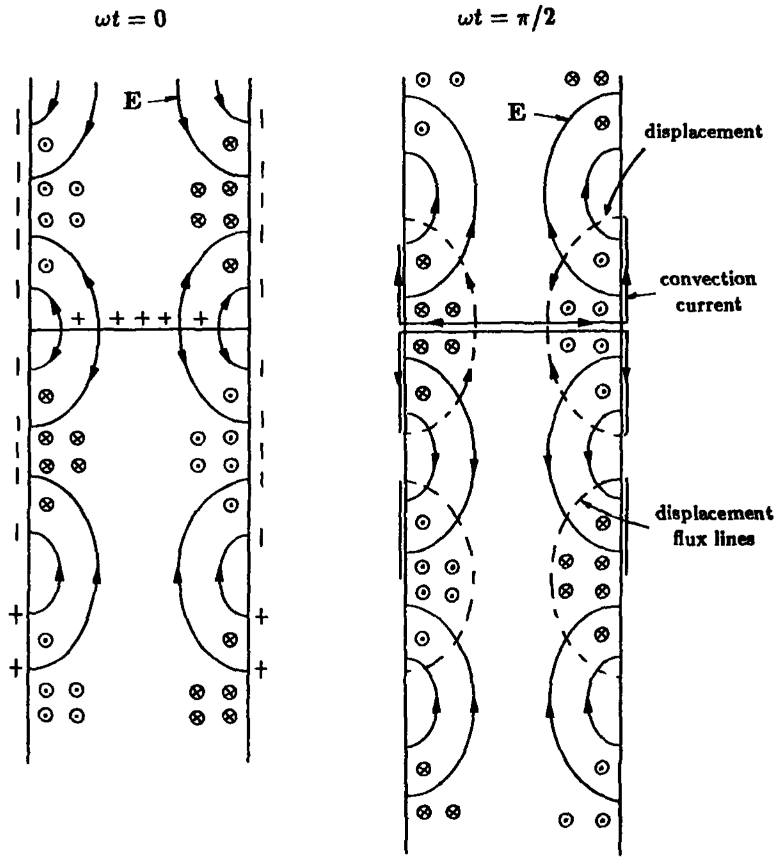

Figure S12.7.3b

12.7.4 (a) We now have a TE field with

$$\hat{E}_z = A \cos\left(\frac{\pi x}{a}\right) e^{\mp j k_y y} \tag{1}$$

From (12.6.29)

$$\begin{aligned} \hat{H}_x &= -\frac{1}{j\omega\mu} \frac{\partial \hat{E}_z}{\partial y} = -\frac{1}{j\omega\mu}(\mp j k_y) A \cos\left(\frac{\pi x}{a}\right) e^{\mp j k_y y} \\ &= \pm \frac{k_y}{\omega\mu} A \cos\left(\frac{\pi x}{a}\right) e^{\mp j k_y y} \end{aligned} \tag{2}$$

and the boundary condition

$$\mathbf{n} \times (\hat{\mathbf{H}}^a - \hat{\mathbf{H}}^b) = \hat{\mathbf{K}} \tag{3}$$

we obtain relation for A:

$$-2\frac{k_y}{\omega\mu} A \cos\frac{\pi x}{a} = K_o \cos\frac{\pi x}{a} \tag{4}$$

or

$$A = -\frac{\omega\mu K_o}{2k_y} \tag{5}$$

and thus

$$\hat{H}_x = \mp\frac{K_o}{2}\cos\left(\frac{\pi x}{2}\right)e^{\mp jk_y y} \tag{6}$$

From (12.6.60)

$$\begin{aligned}\hat{H}_y = \frac{1}{j\omega\mu}\frac{\partial \hat{E}_z}{\partial x} &= -\frac{\pi/a}{j\omega\mu}A\sin\left(\frac{\pi x}{a}\right)e^{\mp jk_y y} \\ &= -j\frac{\pi/a}{2k_y}K_o\sin\left(\frac{\pi x}{a}\right)e^{\mp jk_y y}\end{aligned} \tag{7}$$

(b) Since the E-field is z-directed, it vanishes at the walls and there is no surface charge density. On wall at $x = -a/2$

$$\hat{K}_z = \hat{H}_y \tag{8}$$

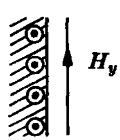

Figure S12.7.4a

and thus

$$\hat{K}_z = j\frac{\pi/a}{2k_y}K_o e^{\mp jk_y y} \tag{9}$$

On the other wall, the current is opposite.

(c)

$$k_y = \sqrt{\omega^2\mu_o\epsilon_o - (\pi/a)^2} \tag{10}$$

since $k_x = \pi/a$. Again we have a potential Φ, the lines of equal height of which give $\mathbf{H}$.

$$\Phi = \frac{1}{jk_y}\frac{K_o}{2}\cos\left(\frac{\pi x}{a}\right)e^{\mp jk_y y} \tag{11}$$

(d) For k_y imaginary:

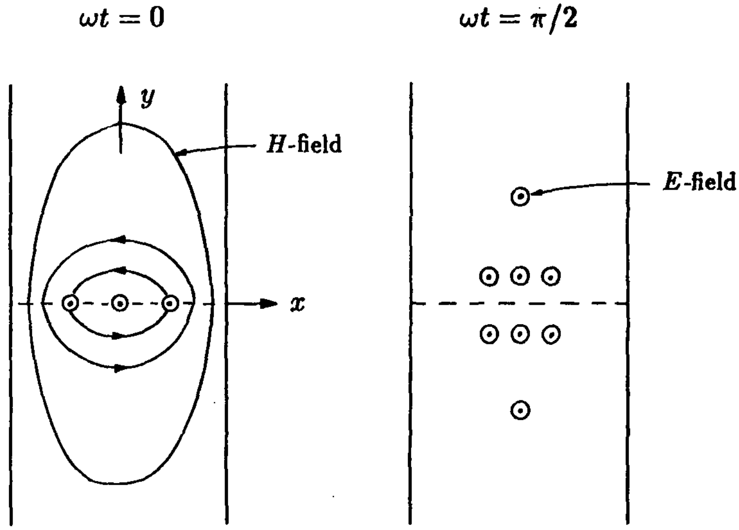

Figure S12.7.4b

for k_y real:

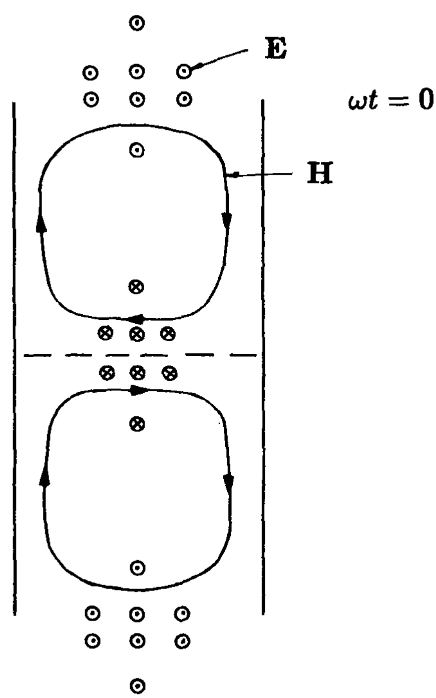

Figure S12.7.4c