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SOLUTIONS TO CHAPTER 11

11.0 INTRODUCTION

11.0.1 The Kirchhoff voltage law gives

$$v = v_c + L\frac{di}{dt} + Ri \tag{1}$$

where

$$i = C\frac{dv_c}{dt} \tag{2}$$

Multiplying (1) by i we get the power flowing into circuit

$$vi = v_c i + \frac{d}{dt}\Big(\frac{1}{2}Li^2\Big) + Ri^2 \tag{3}$$

But

$$v_c i = C\frac{dv_c}{dt}v_c = \frac{d}{dt}\Big(\frac{1}{2}Cv_c^2\Big) \tag{4}$$

and thus we have shown

$$vi = \frac{d}{dt}w + i^2 R \tag{5}$$

where

$$w = \Big(\frac{1}{2}Cv_c^2 + \frac{1}{2}Li^2\Big) \tag{6}$$

Since w is under a total time derivative it integrates to zero, when the excitation i starts from zero and ends at zero. This indicates storage, since the energy supplied by the excitation is extracted after deexcitation. The term i^2R is positive definite and indicates power consumption.

11.1 INTEGRAL AND DIFFERENTIAL CONSERVATION STATEMENTS

11.1.1 (a) If $S = S_x\mathbf{i_x}$, then there is no power flow through surfaces with normals perpendicular to x. The surface integral

$$\oint_S \mathbf{S}\cdot d\mathbf{a}$$

gives $(x_1 > x_2)$

$$[S_x(x_1) - S_x(x_2)]A$$

because S_x is independent of y and z.

(b) Because W and P_d are also independent of y and z, the integrations transverse to the x-axis are simply multiplications by A. Hence from (11.1.1)

$$-A[S_x(x_1) - S_x(x_2)] = A\frac{d}{dt}\int W dx + A\int P_d dx$$

When $x_1 - x_2 = \Delta x$,

$$S_x(x_1) = S_x(x_2) + \frac{\partial S_x}{\partial x}\Big|_{x_2}\Delta x$$

$\int W dx = W\Delta x, \int P_d dx = P_d\Delta x$ and we get

$$-\frac{\partial S_x}{\partial x} = \frac{\partial W}{\partial t} + P_d$$

We have to use partial time derivatives, because W is also a function of x.

(c) The time rate of change of energy and the power dissipated must be equal to the net power flow, which is equal to the difference of the power flowing in and the power flowing out.

11.2 POYNTING'S THEOREM

11.2.1 (a) The power flow is

$$\mathbf{E}\times\mathbf{H} = -E_x H_z \mathbf{i}_y \tag{1}$$

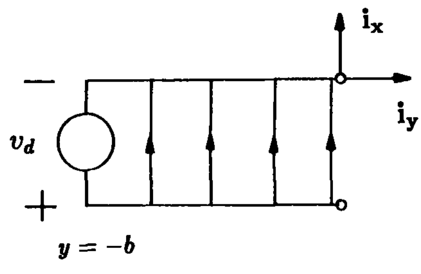

Figure S11.2.1

The EQS field is

$$E_x = \frac{V_d}{a} \tag{2}$$

$$\frac{\partial H_z}{\partial y} = \epsilon\frac{\partial E_x}{\partial t} \tag{3}$$

and thus

$$H_z = y\epsilon_o \frac{\partial E_x}{\partial t} \tag{4}$$

since $H_z = 0$ at $y = 0$. From (1), (2), and (4)

$$\mathbf{E} \times \mathbf{H} = -\mathbf{i}_y y\epsilon_o \frac{V_d}{a}\frac{d}{dt}\left(\frac{V_d}{a}\right) = -\mathbf{i}_y \frac{y\epsilon_o}{a^2} V_d \frac{dV_d}{dt} \tag{5}$$

(b) The power input is:

$$-\int \mathbf{E} \times \mathbf{H} \cdot d\mathbf{a}$$

over the cross-section at $y = -b$ where $d\mathbf{a} = -\mathbf{i}_y$ and therefore,

$$-\int \mathbf{E} \times \mathbf{H} \cdot d\mathbf{a} = \frac{b\epsilon_o}{a^2} aw V_d \frac{dV_d}{dt} = \frac{d}{dt}\left(\frac{1}{2}CV_d^2\right) \tag{6}$$

with

$$C = \frac{\epsilon_o bw}{a}$$

(c) The time rate of change of the electric energy is

$$\begin{aligned} \frac{d}{dt}\int W_e dv &= \frac{d}{dt}\int \frac{1}{2}\epsilon_o \mathbf{E}^2 dv = \frac{d}{dt}\left[\frac{1}{2}\epsilon_o\left(\frac{V_d}{a}\right)^2 abw\right] \\ &= \frac{d}{dt}\left(\frac{1}{2}\frac{\epsilon_o bw}{a}V_d^2\right) = \frac{d}{dt}\left(\frac{1}{2}CV_d^2\right) \qquad \text{QED} \end{aligned} \tag{7}$$

(d) The magnetic energy is

$$\begin{aligned} W_m &= \int \frac{1}{2}\mu_o \mathbf{H}^2 dv = \frac{1}{2}\mu_o aw \int_{-b}^{0} H_z^2 dy \\ &= \frac{1}{2}\mu_o aw \frac{b^3}{3}\left[\epsilon_o \frac{d}{dt}\frac{V_d}{a}\right]^2 \end{aligned} \tag{8}$$

Now

$$\frac{d}{dt}V_d \sim \frac{V_d}{\tau}$$

where τ is the time of interest. Therefore,

$$W_m = \frac{1}{6}\frac{\mu_o\epsilon_o b^2}{\tau^2}\epsilon_o \frac{bw}{a}V_d^2 \ll \frac{1}{2}\epsilon_o \frac{bw}{a}V_d^2$$

if

$$\frac{1}{3}\frac{\mu_o\epsilon_o b^2}{\tau^2} = \frac{1}{3}\frac{b^2}{c^2\tau^2} \ll 1$$

11.2.2 (a)

$$H_z = -\frac{I_d}{w} \tag{1}$$

From Faraday's law

$$-\frac{\partial E_x}{\partial y} = -\mu_o \frac{\partial H_z}{\partial t} \tag{2}$$

and therefore

$$E_x = -\mu_o y \frac{d}{dt}\left(\frac{I_d}{w}\right) \tag{3}$$

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = -E_x H_z \mathbf{i}_y = -\mathbf{i}_y \frac{\mu_o y}{w^2} I_d \frac{dI_d}{dt}$$

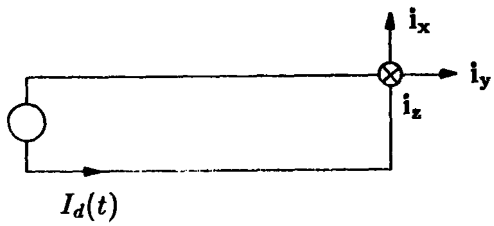

Figure S11.2.2

(b) The input power is $-\int \mathbf{S}\cdot d\mathbf{a}$, integrated over the cross-section at $y = -b$ with $d\mathbf{a} \parallel -\mathbf{i}_y$. The result is

$$-\int \mathbf{S}\cdot d\mathbf{a} = \frac{\mu_o b}{w^2} aw \frac{d}{dt}\frac{1}{2} I_d^2 = \frac{d}{dt}\frac{1}{2} L I_d^2$$

with

$$L = \frac{\mu_o ab}{w}$$

(c) The magnetic energy is

$$\int W_m dv = \int dv \frac{1}{2}\mu_o \mathbf{H}^2 = \frac{1}{2} abw\mu_o \frac{I_d^2}{w^2} = \frac{1}{2} L I_d^2$$

with the same L as defined above. Thus the magnetic energy by itself balances the conservation equation.

(d) The electric energy storage is

$$\int W_e dv = \int \frac{1}{2}\epsilon_o \mathbf{E}^2 dv = \frac{1}{2}\epsilon_o \frac{\mu_o^2}{w^2}\left(\frac{dI_d}{dt}\right)^2 \frac{b^3}{3} aw$$
$$= \frac{1}{3}\epsilon_o\mu_o b^2 \frac{1}{2}\frac{\mu_o ba}{w}\frac{I_d^2}{\tau^2} = \frac{1}{3}\frac{\epsilon_o\mu_o b^2}{\tau^2}\int W_m dv$$

where $dI_d/dt \simeq I_d/\tau$, with τ equal to the characteristic time over which I_d changes appreciably. Thus,

$$\int W_e dv \ll \int W_m dv$$

as long as

$$\frac{1}{3}\frac{\epsilon_o \mu_o b^2}{\tau^2} = \frac{1}{3}\frac{b^2}{c^2\tau^2} \ll 1$$

11.3 OHMIC CONDUCTORS WITH LINEAR POLARIZATION AND MAGNETIZATION

11.3.1 (a) The electric field of a dipole current source is

$$\mathbf{E} = \frac{i_p d}{4\pi\sigma r^3}\left[2\cos\theta \mathbf{i}_r + \sin\theta \mathbf{i}_\theta\right] \tag{1}$$

The H-field is given by Ampère's law

$$\nabla \times \mathbf{H} = \mathbf{J} = \sigma\mathbf{E} \tag{2}$$

Now, by symmetry it appears that $\mathbf{H}$ must be ϕ directed

$$\mathbf{H} = \mathbf{i}_\phi H_\phi \tag{3}$$

and thus

$$\nabla \times \mathbf{H} = \mathbf{i}_r \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(H_\phi \sin\theta) - \mathbf{i}_\theta \frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) \tag{4}$$

By inspection of the θ-component of (4), with the aid of (1) and (2), one finds

$$H_\phi = \frac{i_p d}{4\pi r^2}\sin\theta \tag{5}$$

The same result is obtained by comparing r components. Therefore,

$$\mathbf{E}\times\mathbf{H} = \left(\frac{i_p d}{4\pi}\right)^2 \frac{1}{\sigma}\frac{1}{r^5}\left[-2\cos\theta\sin\theta\mathbf{i}_\theta + \sin^2\theta\mathbf{i}_r\right] \tag{6}$$

The density of dissipated power is

$$\begin{aligned} P_d = \mathbf{E}\cdot\mathbf{J} = \sigma\mathbf{E}^2 &= \left(\frac{i_p d}{4\pi}\right)^2 \frac{1}{\sigma r^6}\left[4\cos^2\theta + \sin^2\theta\right] \\ &= \left(\frac{i_p d}{4\pi}\right)^2 \frac{1}{\sigma r^6}\left[1 + 3\cos^2\theta\right] \end{aligned} \tag{7}$$

(c) Poynting's theorem requires

$$\nabla \cdot \mathbf{S} + P_d = 0 \tag{8}$$

Now $\nabla \cdot \mathbf{S}$ in spherical coordinate is

$$\nabla \cdot \mathbf{S} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 S_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(S_\theta \sin\theta)$$

Now

$$\begin{aligned} \nabla \cdot (\mathbf{E}\times\mathbf{H}) &= \Big(\frac{i_p d}{4\pi}\Big)^2 \frac{1}{\sigma r^6}[-3\sin^2\theta - 4\cos^2\theta + 2\sin^2\theta] \\ &= -\Big(\frac{i_p d}{4\pi}\Big)^2 \frac{1}{\sigma r^6}[1 + 3\cos^2\theta] \end{aligned} \tag{9}$$

Thus, (8) is indeed satisfied according to (7) and (9).

(d)

$$\Phi = \frac{i_p d}{4\pi\sigma}\frac{\cos\theta}{r^2}$$

$$\begin{aligned} \nabla \cdot (\Phi \mathbf{J}) &= \Big(\frac{i_p d}{4\pi}\Big)^2 \nabla \cdot \frac{1}{\sigma r^5}[2\cos^2\theta \mathbf{i}_r + \sin\theta\cos\theta \mathbf{i}_\theta] \\ &= -\Big(\frac{i_p d}{4\pi}\Big)^2 \frac{1}{\sigma r^6}[6\cos^2\theta - 2\cos^2\theta + \sin^2\theta] \\ &= -\Big(\frac{i_p d}{4\pi}\Big)^2 \frac{1}{\sigma r^6}[1 + 3\cos^2\theta] = \nabla\cdot(\mathbf{E}\times\mathbf{H}) \end{aligned}$$

(e) We need not form the cross-product to obtain flow density. The power flow density is the current density weighted by local potential Φ.

11.3.2 (a) The potential is a solution of Laplace's equation

$$\Phi = -\frac{v}{ln\frac{a}{b}} ln(r/a) \tag{1}$$

$$\mathbf{E} = \frac{v}{ln(a/b)}\frac{\mathbf{i}_r}{r} \tag{2}$$

$$\nabla \times \mathbf{H} = \mathbf{J} = \sigma\mathbf{E} = \frac{\sigma v}{ln(a/b)}\frac{\mathbf{i}_r}{r} \tag{3}$$

from Ampère's law. By symmetry

$$\mathbf{H} = \mathbf{i}_\phi H_\phi \tag{4}$$

and

$$-\frac{\partial H_\phi}{\partial z} = \frac{\sigma v}{ln(a/b)}\frac{1}{r} \tag{5}$$

and thus

$$H_\phi = -\frac{\sigma v}{ln(a/b)}\frac{z}{r} \tag{6}$$

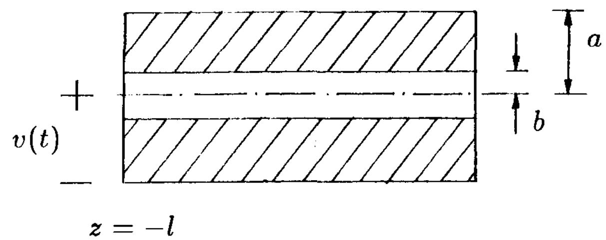

Figure S11.3.2a

(b) The Poynting vector is

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} = -\mathbf{i}_z\frac{\sigma v^2}{ln^2(a/b)}\frac{z}{r^2} \tag{7}$$

(c) The Poynting flux is

$$\begin{aligned}\oint \mathbf{S}\cdot d\mathbf{a} &= -\int_{r=b}^{r=a} S_z 2\pi r dr\Big|_{z=-l}\\ &= -\frac{2\pi\sigma v^2 l}{ln^2(a/b)}ln(a/b) = -\frac{2\pi\sigma l}{ln(a/b)}v^2\end{aligned} \tag{8}$$

(d) The dissipated power is

$$\begin{aligned}\int dv P_d = \int dv\sigma\mathbf{E}^2 &= \int_{z=-l}^{0}\int_{r=b}^{r=a}\frac{\sigma v^2}{ln^2(a/b)}\frac{2\pi r}{r^2}drdz\\ &= \frac{2\pi\sigma l}{ln(a/b)}v^2\end{aligned} \tag{9}$$

(e) The alternate form for the power flow density is

$$\mathbf{S} = \Phi\mathbf{J} = -\sigma\frac{v^2}{ln^2(a/b)}ln(r/a)\frac{\mathbf{i_r}}{r} \tag{10}$$

$$\begin{aligned}\oint \mathbf{S}\cdot d\mathbf{a} &= -[S_r(r=b) - S_r(r=a)]2\pi bl\\ &= -\frac{2\pi\sigma l}{ln(a/b)}v^2\end{aligned} \tag{11}$$

This is indeed equal to the negative of (9).

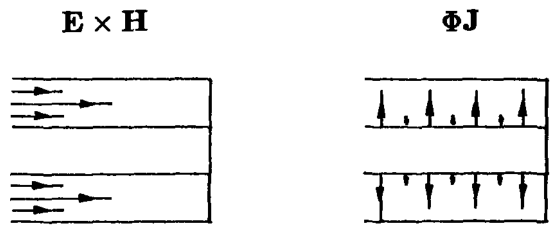

Figure S11.3.2b

(f) See Fig. S11.3.2b.

(g) At $z = -l$,

$$\oint \mathbf{H} \cdot d\mathbf{s} = \frac{2\pi\sigma l v}{ln(a/b)} = i \tag{12}$$

Thus

$$vi = \frac{2\pi\sigma l}{ln(a/b)} v^2 \quad \text{Q.E.D.} \tag{13}$$

11.3.3 (a) The electric field is

$$\mathbf{E} = \frac{v}{d}\mathbf{i}_z \tag{1}$$

From Ampère's law:

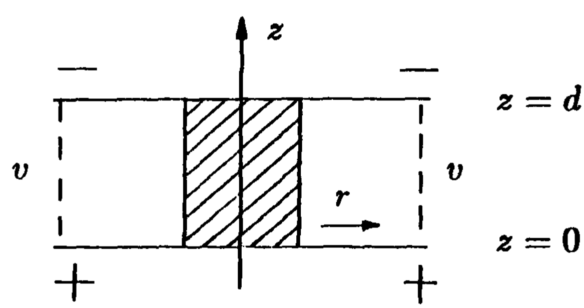

Figure S11.3.3

$$\oint \mathbf{H} \cdot d\mathbf{s} = \int \left(\mathbf{J} + \epsilon \frac{\partial \mathbf{E}}{\partial t}\right) \cdot d\mathbf{a} \tag{2}$$

$$2\pi r H_\phi = \begin{cases} \pi r^2\left[\frac{\sigma v}{d} + \epsilon\frac{d}{dt}(v/d)\right] & \text{for } r < b \\ \pi b^2\left[\sigma\frac{v}{d} + \epsilon\frac{d}{dt}(v/d)\right] + \pi(r^2 - b^2)\epsilon_o\frac{d}{dt}(v/d) & \text{for } b < r < a \end{cases} \tag{3}$$

and thus

$$H_\phi = \begin{cases} \frac{r}{2}\left[\sigma\frac{v}{d} + \epsilon\frac{d}{dt}(v/d)\right] & \text{for } r < b \\ \frac{1}{2r}\left[\frac{\sigma b^2 v}{d} + \epsilon b^2\frac{d}{dt}(v/d) + (r^2 - b^2)\epsilon_o\frac{d}{dt}(v/d)\right] & \text{for } b < r < a \end{cases} \tag{4}$$

The Poynting flux density

$$\begin{aligned} \mathbf{E}\times\mathbf{H} &= \mathbf{i}_z \times \mathbf{i}_\phi E_z H_\phi \\ &= \begin{cases} -\mathbf{i}_r\frac{r}{2}\left(\sigma\frac{v}{d} + \epsilon\frac{d}{dt}(v/d)\right)\frac{v}{d} & \text{for } r < b \\ -\mathbf{i}_r\frac{1}{2r}\left\{\frac{1}{d}\left[\epsilon b^2 + \epsilon_o(r^2 - b^2)\right]\frac{d}{dt}(v) + \frac{\sigma b^2}{d}v\right\}\frac{v}{d} & \text{for } b < r < a \end{cases} \end{aligned} \tag{5}$$

(b)

$$\begin{aligned} -\int \mathbf{E}\times\mathbf{H}\cdot d\mathbf{a} &= -\int_{z=0}^{d} \mathbf{i}_r\cdot\mathbf{E}\times\mathbf{H}\,dz 2\pi r \\ &= \begin{cases} \pi r^2\left(\sigma\frac{v}{d} + \epsilon\frac{d}{dt}(v/d)\right)v & r < b \\ \pi\left\{\frac{1}{d}\left[\epsilon b^2 + \epsilon_o(r^2 - b^2)\right]\frac{d}{dt}(v/d) + \frac{\sigma b^2}{d}v\right\}v & b < r < a \end{cases} \end{aligned} \tag{6}$$

For $r < b$,

$$\begin{aligned} \int \frac{dW}{dt}dv + \int P_d dv &= \int_{z=0}^{d}\int_{r=0}^{r}\frac{1}{2}\epsilon\frac{d}{dt}(v/d)^2 2\pi r\,dr\,dz \\ &\quad + \int_{z=0}^{d}\int_{r=0}^{r}\sigma(v/d)^2 2\pi r\,dr\,dz \\ &= \epsilon v\frac{d}{dt}(v/d)\pi r^2 + \sigma\frac{v^2}{d}\pi r^2 \end{aligned} \tag{7a}$$

For $b < r < a$:

$$\begin{aligned} \int \frac{dW}{dt}dv + \int P_d dv &= \int_{z=0}^{d}\int_{r=0}^{b}\frac{1}{2}\epsilon\frac{d}{dt}(v/d)^2 2\pi r\,dr\,dz \\ &\quad + \int_{z=0}^{d}\int_{r=b}^{r}\frac{1}{2}\epsilon_o\frac{d}{dt}(v/d)^2 2\pi r\,dr\,dz \\ &\quad + \int_{z=0}^{d}\int_{r=0}^{b}\sigma(v/d)^2 2\pi r\,dr\,dz \\ &= \pi\left\{\left[\frac{\epsilon b^2}{d}v + \epsilon_o\frac{(r^2 - b^2)}{d}v\right]\frac{d}{dt}(v) + \frac{\sigma b^2}{d}v^2\right\} \quad \text{Q.E.D.} \end{aligned} \tag{7b}$$

(c)

$$\mathbf{S} = \Phi\left(\mathbf{J} + \epsilon\frac{\partial\mathbf{E}}{\partial t}\right) \tag{8}$$

The potential Φ is given by

$$\Phi = -\frac{v}{d}(z-d)$$

and

$$\mathbf{J} + \epsilon\frac{\partial \mathbf{E}}{\partial t} = \begin{cases} \mathbf{i}_z\left(\sigma\frac{v}{d} + \epsilon\frac{d}{dt}\frac{v}{d}\right) & \text{for } r < b \\ \mathbf{i}_z\epsilon_o\frac{d}{dt}\frac{v}{d} & \text{for } b < r < a \end{cases} \tag{9}$$

Therefore,

$$\mathbf{S} = \begin{cases} -\mathbf{i}_z\left(\frac{\sigma v}{d} + \frac{\epsilon}{d}\frac{dv}{dt}\right)(z-d)\frac{v}{d} & \text{for } r < b \\ -\mathbf{i}_z\frac{\epsilon_o}{d}\frac{dv}{dt}(z-d)\frac{v}{d} & \text{for } b < r < a \end{cases} \tag{10}$$

(d) The integral is

$$-\oint \mathbf{S}\cdot d\mathbf{a} = \int_0^r 2\pi r dr[S_z(z=0) - S_z(z=d)] \tag{11}$$

For $r < b$:

$$= \int_0^r 2\pi r dr d\left(\frac{\sigma v}{d} + \frac{\epsilon}{d}\frac{dv}{dt}\right)\frac{v}{d} = \pi r^2\left(\frac{\sigma v}{d} + \frac{\epsilon}{d}\frac{dv}{dt}\right)v \tag{12a}$$

For $a < r < b$:

$$\begin{aligned} &= \int_0^b 2\pi r dr d\left(\frac{\sigma v}{d} + \frac{\epsilon}{d}\frac{dv}{dt}\right)\frac{v}{d} + \int_b^r 2\pi r dr d\frac{\epsilon_o}{d}\frac{dv}{dt}\frac{v}{d} \\ &= \pi b^2\left(\frac{\sigma v}{d} + \frac{\epsilon}{d}\frac{dv}{dt}\right)v + \pi(r^2 - b^2)\frac{\epsilon_o}{d}\frac{dv}{dt}v \end{aligned} \tag{12b}$$

Equations (12) agree with (6).

(e) The power input at $r = a$ is from (12b)

$$\pi b^2\left(\frac{\sigma v}{d} + \frac{\epsilon}{d}\frac{dv}{dt}\right)v + \pi(a^2 - b^2)\frac{\epsilon_o}{d}\frac{dv}{dt}v = vi \tag{13}$$

where

$$i = \pi b^2\left[\frac{\sigma v}{d} + \epsilon\frac{d}{dt}(v/d)\right] + \pi(a^2 - b^2)\epsilon_o\frac{d}{dt}(v/d)$$

which is the sum of the displacement current and convection current between the two plates.

11.3.4 (a) From the potentials (7.5.4) and (7.5.5) we find the E-field

$$\begin{aligned} \mathbf{E} = -\nabla\Phi = \mathbf{i}_r E_o\cos\phi\left(1 + \left(\frac{R}{r}\right)^2\frac{\sigma_b - \sigma_a}{\sigma_b + \sigma_a}\right) \\ - \mathbf{i}_\phi E_o\sin\phi\left(1 - \left(\frac{R}{r}\right)^2\frac{\sigma_b - \sigma_a}{\sigma_b + \sigma_a}\right) \qquad r < R \end{aligned} \tag{1a}$$

and

$$\frac{2\sigma_a}{\sigma_b+\sigma_a}E_o(\mathbf{i_r}\cos\phi-\mathbf{i}_\phi\sin\phi) \qquad r<R \tag{1b}$$

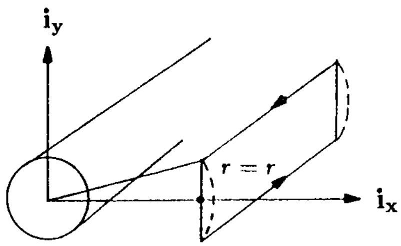

Figure S11.3.4

The **H**-field is z-directed by symmetry and can be found from Ampère's law using a contour in a $z-x$ plane, symmetrically located around the x-axis and of unit width in z-direction. If the contour is picked as shown in Fig. S11.3.4, then

$$\oint_C \mathbf{H}\cdot d\mathbf{s}=\int_S \mathbf{J}\cdot d\mathbf{a}=2H_z=2\int_0^\phi J_r r d\phi$$
$$=\begin{cases} 2r\sigma_a E_o\sin\phi\left(1+\left(\frac{R}{r}\right)^2\frac{\sigma_b-\sigma_a}{\sigma_b+\sigma_a}\right) & \text{for } r>R \\ 2r\sigma_b E_o\frac{2\sigma_a}{\sigma_b+\sigma_a}\sin\phi & \text{for } r<R\end{cases} \tag{2}$$

The Poynting vector is

$$\mathbf{E}\times\mathbf{H}=E_\phi H_z\mathbf{i_r}-E_rH_z\mathbf{i}_\phi=-\mathbf{i_r}r\sigma_aE_o^2\sin^2\phi\left[1-\left(\frac{R}{r}\right)^4\left(\frac{\sigma_b-\sigma_a}{\sigma_b+\sigma_a}\right)^2\right]$$
$$-\mathbf{i}_\phi r\sigma_aE_o^2\sin\phi\cos\phi\left[1+\left(\frac{R}{r}\right)^2\left(\frac{\sigma_b-\sigma_a}{\sigma_b+\sigma_a}\right)\right]^2 \qquad r>R$$
$$=-\mathbf{i_r}r\sigma_bE_o^2\sin^2\phi\left(\frac{2\sigma_a}{\sigma_a+\sigma_b}\right)^2$$
$$-\mathbf{i}_\phi r\sigma_bE_o^2\sin\phi\cos\phi\left(\frac{2\sigma_a}{\sigma_a+\sigma_b}\right)^2 \qquad r<R$$

(b) The alternate power flow vector $\mathbf{S}=\Phi\mathbf{J}$ follows from (7.5.4)-(7.5.5) and (1)

$$\Phi\mathbf{J}=-\mathbf{i_r}\sigma_aE_o^2r\cos^2\phi\left[1-\left(\frac{R}{4}\right)^4\left(\frac{\sigma_b-\sigma_a}{\sigma_b+\sigma_a}\right)^2\right]$$
$$+\mathbf{i}_\phi\sigma_aE_o^2r\sin\phi\cos\phi\left[1-\left(\frac{R}{r}\right)^2\frac{\sigma_b-\sigma_a}{\sigma_b+\sigma_a}\right]^2 \qquad r>R$$
$$=-\mathbf{i_r}\sigma_bE_o^2r\cos^2\phi\left(\frac{2\sigma_a}{\sigma_b+\sigma_a}\right)^2 \tag{4}$$
$$+\mathbf{i}_\phi\sigma_bE_o^2r\sin\phi\cos\phi\left(\frac{2\sigma_a}{\sigma_b+\sigma_a}\right)^2 \qquad r<R$$

(c) The power dissipation density P_d is

$$P_d = \sigma \mathbf{E}^2 = \sigma_a E_o^2 \cos^2\phi \left[1 + (\frac{R}{r})^2 \frac{\sigma_b - \sigma_a}{\sigma_b + \sigma_a}\right]^2 + \sigma_a E_o^2 \sin^2\phi \left[1 - (\frac{R}{r})^2 \frac{\sigma_b - \sigma_a}{\sigma_b + \sigma_a}\right]^2 \qquad r > R \tag{5a}$$

$$= \sigma_b E_o^2 \left(\frac{2\sigma_a}{\sigma_a + \sigma_b}\right)^2 \qquad r < R \tag{5b}$$

(d) We must now evaluate $\nabla \cdot (\mathbf{E} \times \mathbf{H})$ and $\nabla \cdot \Phi \mathbf{J}$ and show that they yield $-P_d$.

$$\begin{aligned}\nabla \cdot \mathbf{S} &= \frac{1}{r}\frac{\partial (r S_r)}{\partial r} + \frac{1}{r}\frac{\partial S_\phi}{\partial \phi} = -2\sigma_a E_o^2 \sin^2\phi \left[1 + (\frac{R}{r})^4 \left(\frac{\sigma_b - \sigma_a}{\sigma_b + \sigma_a}\right)^2\right] \\ &\quad - \sigma_a E_o^2 (\cos^2\phi - \sin^2\phi)\left[1 + (\frac{R}{r})^2 \left(\frac{\sigma_b - \sigma_a}{\sigma_b + \sigma_a}\right)\right]^2 \\ &= -\sigma_a E_o^2 \sin^2\phi \left[1 - (\frac{R}{r})^2 \left(\frac{\sigma_b - \sigma_a}{\sigma_b + \sigma_a}\right)\right]^2 \\ &\quad - \sigma_a E_o^2 \cos^2\phi \left[1 + (\frac{R}{r})^2 \left(\frac{\sigma_b - \sigma_a}{\sigma_b + \sigma_a}\right)\right]^2\end{aligned} \tag{6a}$$

for $r > R$,

$$\begin{aligned}\nabla \cdot S &= -2\sigma_b E_o^2 \sin^2\phi \left(\frac{2\sigma_a}{\sigma_a + \sigma_b}\right)^2 \\ &\quad - (\cos^2\phi - \sin^2\phi)\sigma_b E_o^2 \left(\frac{2\sigma_a}{\sigma_a + \sigma_b}\right)^2 \\ &= -\sigma_b E_o^2 \left(\frac{2\sigma_a}{\sigma_a + \sigma_b}\right)^2\end{aligned} \tag{6b}$$

for $r < R$. Comparison of (5) and (6) shows that the Poynting theorem is obeyed. Now take the other form of power flow. The analysis is simplified if we note that $\nabla \cdot \mathbf{J} = 0$. Thus

$$\begin{aligned}\nabla \cdot \Phi \mathbf{J} &= \mathbf{J} \cdot \nabla \Phi = J_r \frac{\partial}{\partial r}\Phi + J_\phi \frac{1}{r}\frac{\partial}{\partial \phi}\Phi = -\sigma \mathbf{E}^2 \\ &= -\sigma_a E_o^2 \cos^2\phi \left[1 + (\frac{R}{r})^2 \frac{\sigma_b - \sigma_a}{\sigma_b + \sigma_a})\right]^2 \\ &\quad - \sigma_a E_o^2 \sin^2\phi \left[1 - (\frac{R}{r}) \left(\frac{\sigma_b - \sigma_a}{\sigma_b + \sigma_a}\right)\right]^2 \qquad r > R\end{aligned} \tag{7a}$$

and

$$\nabla \cdot \Phi \mathbf{J} = -\sigma_b E_o^2 \left(\frac{2\sigma_a}{\sigma_a + \sigma_b}\right)^2 \qquad r < R \tag{7b}$$

Q.E.D.

11.4 ENERGY STORAGE

√

11.4.1 From (8.5.14)-(8.5.15) we find the H-fields. Integrating the energy density we find

$$
\begin{aligned}
w &= \int dv \frac{1}{2}\mu_o \mathbf{H}^2 = \frac{1}{2}\mu_o \int_0^R r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \left(\frac{Ni}{3R}\right)^2 \\
&+ \frac{1}{2}\mu_o \int_R^\infty r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \left(\frac{Ni}{6R}\right)^2 \left(\frac{R}{r}\right)^6 (4\cos^2\theta + \sin^2\theta) \\
&= \frac{1}{2}\mu_o \frac{4\pi R^3}{3}\left(\frac{Ni}{3R}\right)^2 + \frac{1}{2}\mu_o 2\pi \times 4\left(\frac{Ni}{6R}\right)^2 \times \frac{1}{3}R^3 \\
&= \frac{1}{2}\frac{2\pi N^2 \mu_o R}{9} i^2
\end{aligned}
$$

where we have used

$$
\begin{aligned}
\int_0^\pi \sin\theta d\theta (4\cos^2\theta + \sin^2\theta) &= -\int_0^\pi d(\cos\theta)(3\cos^2\theta + 1) \\
&= \int_{-1}^1 dx(3x^2+1) = (x^3+x)\Big|_{-1}^1 = 4
\end{aligned}
$$

Because

$$
w = \frac{1}{2}Li^2
$$

we find that

$$
L = \frac{2\pi N^2 \mu_o R}{9}
$$

Q.E.D.

11.4.2 The scalar potential of P9.6.3 is

$$
\Psi = \frac{N}{2}\frac{i\cos\phi}{1+\frac{\mu}{\mu_o}} \begin{cases} R/r & r > R \\ -\frac{\mu}{\mu_o}\frac{r}{R} & r < R \end{cases}
$$

The field is

$$
\mathbf{H} = \frac{N}{2R}\frac{i\cos\phi}{1+\frac{\mu}{\mu_o}} \begin{cases} (\mathbf{i}_r\cos\phi + \mathbf{i}_\phi\sin\phi)(R/r)^2; & r > R \\ \frac{\mu}{\mu_o}(\mathbf{i}_r\cos\phi - \mathbf{i}_\phi\sin\phi); & r < R \end{cases}
$$

The energy is

$$
\begin{aligned}
w_m &= l\int_0^R \frac{1}{2}\mu_o \mathbf{H}^2 r dr d\phi + l\int_R^\infty \frac{1}{2}\mu \mathbf{H}^2 r dr d\phi \\
&= \frac{1}{2}\mu_o \pi R^2 l\left(\frac{Ni}{2R}\frac{\mu/\mu_o}{1+\frac{\mu}{\mu_o}}\right)^2 \\
&\quad + \frac{1}{2}\mu l\left(\frac{Ni}{2R}\frac{1}{1+\frac{\mu}{\mu_o}}\right)^2 2\pi\int_R^\infty \left(\frac{R}{r}\right)^4 r dr \\
&= \frac{1}{2}\mu_o \pi l\frac{N^2(\mu/\mu_o)^2 i^2}{4(1+\frac{\mu}{\mu_o})^2} + \frac{1}{2}\mu\pi l\frac{N^2 i^2}{4(1+\frac{\mu}{\mu_o})^2} \\
&= \frac{1}{2}\frac{\mu\pi l N^2}{(1+\frac{\mu}{\mu_o})} i^2 = \frac{1}{2}L i^2
\end{aligned}
$$

11.4.3 The vector potential is from (8.6.32)

$$
\mathbf{A} = -\frac{\mu_o N i}{3}\left[\left(\frac{r}{a}\right)^2 - \left(\frac{r}{a}\right)\right]\sin\phi\,\mathbf{i}_z \qquad r < a \tag{1}
$$

$$
\begin{aligned}
\mu_o \mathbf{H} &= \nabla\times\mathbf{A} \\
&= -\mathbf{i}_z \times \nabla A_z = \frac{Ni}{3a}\mathbf{i}_z \times \left\{[2(r/a) - 1]\sin\phi\,\mathbf{i}_r + \left(\frac{r}{a} - 1\right)\cos\phi\,\mathbf{i}_\phi\right\} \\
&= -\frac{\mu_o N i}{3a}\left[\left(\frac{r}{a} - 1\right)\cos\phi\,\mathbf{i}_r - \left(2\frac{r}{a} - 1\right)\sin\phi\,\mathbf{i}_\phi\right]
\end{aligned}
$$

The energy is

$$
\begin{aligned}
l\int_0^a\int_0^{2\pi}\frac{1}{2}\mu_o \mathbf{H}^2 r dr d\phi &= \frac{\mu_o}{2} l\left(\frac{Ni}{3a}\right)^2 \pi\int_0^a r dr\left[\left(\frac{r}{a} - 1\right)^2 + \left(2\frac{r}{a} - 1\right)^2\right] \\
&= \frac{\mu_o}{2} l\left(\frac{Ni}{3}\right)^2 \frac{\pi}{4} = \frac{1}{2}L i^2
\end{aligned}
$$

Therefore,

$$
L = \frac{\pi}{36}\mu_o l N^2
$$

11.4.4 The energy differential is

$$
dw_m = i_1 d\lambda_1 + i_2 d\lambda_2 \tag{1}
$$

The coenergy is

$$
\begin{aligned}
dw'_m &= d(i_1\lambda_1) + d(i_2\lambda_2) - dw_m = \lambda_1 di_1 + \lambda_2 di_2 \\
&= (L_{11}i_1 + L_{12}i_2)di_1 + (L_{21}i_1 + L_{22}i_2)di_2
\end{aligned} \tag{2}
$$

with

$$L_{21} = L_{12} \tag{3}$$

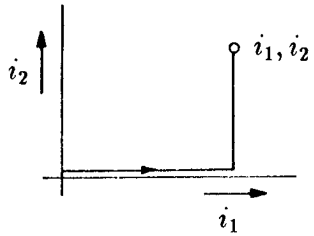

Figure S11.4.4

If we integrate this expression along a conveniently chosen path in the $i_1 - i_2$ plane as shown in Fig S11.4.4, we get

$$\begin{aligned}\int_{\substack{i_1=0\\ i_2=0}}^{i_1} L_{11} i_1 di_1 + \int_{\substack{i_2=0\\ i_1=\text{const}}}^{i_2} (L_{21} i_1 + L_{22} i_2) di_2 \\ = \frac{1}{2} L_{11} i_1^2 + L_{21} i_1 i_2 + \frac{1}{2} L_{22} i_2^2 \\ = \frac{1}{2}(L_{11} i_1^2 + L_{12} i_1 i_2 + L_{21} i_2 i_1 + L_{22} i_2^2) \\ = \frac{1}{2} L_o (N_1^2 i_1^2 + 2N_1 N_2 i_1 i_2 + N_2^2 i_2^2)\end{aligned} \tag{4}$$

when the last expression is written symmetrically, using (3).

11.4.5 If the gap is small $(a - b) \ll a$, the field is radial and can be evaluated using Ampère's law with the contour shown in Fig. S11.4.5. It is simplest to evaluate the field of stator and rotor separately and then to add. The field vanishes at $\phi = \pi/2$ and thus

$$\oint_C \mathbf{H} \cdot d\mathbf{s} = -(a - b) H_r(\phi) \tag{1}$$

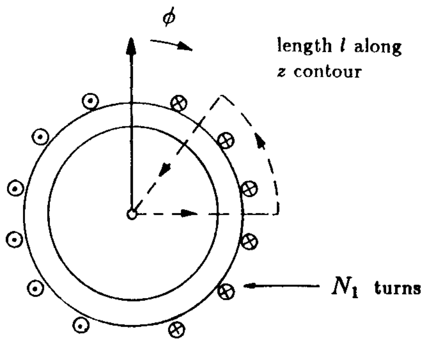

Figure S11.4.5

For the stator field, the integral of the current density is

$$\int_S \mathbf{J} \cdot d\mathbf{a} = -\int_{\phi}^{\pi/2} \frac{N_1 i_1}{2a} \sin\phi a d\phi = -\frac{N_1 i_1}{2} \cos\phi \tag{2}$$

where N_1 is the total number of terms of the stator winding. Therefore, the stator field is given by

$$\mathbf{H} \simeq \mathbf{i}_r H_r = \mathbf{i}_r \frac{N_1 i_1}{2(a-b)} \cos\phi \tag{3}$$

The rotor coil gives the field

$$H_r = \frac{N_2 i_2}{2(a-b)} \cos(\phi - \theta) \tag{4}$$

where N_2 is the total number of turns of the rotor winding. In a linear system, coenergy is equal to energy, only the independent variables have to be chosen properly, i.e. the energy expressed in terms of the currents, is coenergy. When expressed in terms of fluxes, it is energy. The coenergy density is

$$W'_m = \frac{1}{2}\mu_o H_r^2 \tag{5}$$

The coenergy is

$$\begin{aligned} w'_m &= \frac{1}{2}\mu_o(a-b)l \int_0^{2\pi} H_r^2 a d\phi \\ &= \frac{1}{2}\frac{\mu_o l a \pi}{4(a-b)}[(N_1 i_1)^2 + (N_2 i_2)^2 + 2N_1 N_2 i_1 i_2 \cos\theta] \end{aligned} \tag{6}$$

We find

$$L_{ii} = \frac{\pi\mu_o a l}{4(a-b)} N_i^2 \tag{7}$$

and

$$L_{12} = L_{21} = \frac{\pi\mu_o a l}{4(a-b)} N_1 N_2 \cos\theta$$

11.4.6

$$\mathbf{D} = \left(\frac{\alpha_1}{\sqrt{1+\alpha_2 E^2}} + \epsilon_o\right)\mathbf{E}$$

The coenergy density in the nonlinear medium is [note $\mathbf{E} \cdot d\mathbf{E} = d(\frac{1}{2}\mathbf{E}^2)$]

$$\begin{aligned} W'_e &= \int_0^{\mathbf{E}} \mathbf{D} \cdot d\mathbf{E} = \int \frac{1}{2}\left(\frac{\alpha_1}{\sqrt{1+\alpha_2 E^2}} + \epsilon_o\right) dE^2 \\ &= \frac{\alpha_1}{\alpha_2}\sqrt{1+\alpha_2 E^2} + \frac{1}{2}\epsilon_o E^2 \end{aligned}$$

In the linear material

$$w'_e = \frac{1}{2}\epsilon_o \mathbf{E}^2$$

Integrating the densities over the respective volumes one finds $(E^2 = v^2/a^2)$

$$w'_e = \left[\frac{\alpha_1}{\alpha_2}\sqrt{1+\alpha_2\frac{v^2}{a^2}} + \frac{1}{2}\epsilon_o\frac{v^2}{a^2}\right]\xi ca + \frac{1}{2}\epsilon_o\frac{v^2}{a^2}(b-\xi)ca$$

Q.E.D.

11.4.7 (a) $\mathbf{H} = \mathbf{i}_z i/w$ in both regions. Therefore,

$$\mathbf{B} = \mathbf{i}_z \mu_o i/w$$

in region (a)

$$\mathbf{B} = \mathbf{i}_z\left(\mu_o + \frac{\alpha_1}{\sqrt{1+\alpha_2 i^2/w^2}}\right)i/w$$

in region (b). The coenergy densities are

$$W'_m = \begin{cases} \frac{1}{2}\mu_o\frac{i^2}{w^2} & \text{in region (a)} \\ \frac{1}{2}\left(\mu_o\frac{i^2}{w^2} + 2\frac{\alpha_1}{\alpha_2}\sqrt{1+\alpha_2\frac{i^2}{w^2}}\frac{i^2}{w^2}\right) & \text{in region (b)} \end{cases}$$

The coenergy is

$$w'_m = wA_a\frac{1}{2}\mu_o\frac{i^2}{w^2} + wA_b\frac{1}{2}\left(\mu_o + 2\frac{\alpha_1}{\alpha_2}\sqrt{1+\alpha_2\frac{i^2}{w^2}}\right)\frac{i^2}{w^2}$$

11.5 ELECTROMAGNETIC DISSIPATION

11.5.1 From (7.9.16) we find an equation for the complex amplitude $\hat{E}_a$:

$$\hat{E}_a = \frac{j\omega\epsilon_b + \sigma_b}{(j\omega\epsilon_a+\sigma_a)b + (j\omega\epsilon_b+\sigma_b)a}\hat{v} \tag{1}$$

and since

$$a\hat{E}_a + b\hat{E}_b = \hat{v} \tag{2}$$

we find

$$\hat{E}_b = \frac{j\omega\epsilon_a + \sigma_a}{(j\omega\epsilon_a+\sigma_a)b + (j\omega\epsilon_b+\sigma_b)a}\hat{v} \tag{3}$$

(Another way of finding $\hat{E}_b$ from (1) is to note that $\hat{E}_a$ and $\hat{E}_b$ are related to each other by an interchange of a and b and of the subspcripts.) The time average power dissipation is

$$\begin{aligned}\langle p_d\rangle &= \frac{1}{2}\sigma_a|\hat{E}_a|^2 aA + \frac{1}{2}\sigma_b|\hat{E}_b|^2 bA \\ &= \frac{A}{2}\frac{a\sigma_a(\omega^2\epsilon_b^2+\sigma_b^2) + b\sigma_b(\omega^2\epsilon_a^2+\sigma_a^2)}{(b\sigma_a+a\sigma_b)^2 + \omega^2(b\epsilon_a+a\epsilon_b)^2}|\hat{v}|^2\end{aligned}$$

11.5.2 (a) The electric field follows from (7.9.36)

$$\hat{E}_b = -\nabla\hat{\Phi} = 3E_p(\cos\theta\mathbf{i}_r - \sin\theta\mathbf{i}_\theta)\frac{\sigma_a + j\omega\epsilon_a}{2\sigma_a + \sigma_b + j\omega(2\epsilon_a + \epsilon_b)}; \quad r < R \quad (1b)$$

Therefore

$$\langle P_d\rangle = \frac{1}{2}\sigma_b|\hat{E}_b|^2 = \frac{9}{2}|E_p|^2\sigma_b\frac{\sigma_a^2 + \omega^2\epsilon_a^2}{(2\sigma_a + \sigma_b)^2 + \omega^2(2\epsilon_a + \epsilon_b)^2}; \quad r < R \quad (2b)$$

The electric field in region (a) is

$$\hat{E}_a = E_p\left\{\mathbf{i}_r\cos\theta\left[1 - 2\frac{\sigma_a - \sigma_b + j\omega(\epsilon_a - \epsilon_b)}{(2\sigma_a + \sigma_b) + j\omega(2\epsilon_a + \epsilon_b)}(R/r)^3\right] - \mathbf{i}_\theta\sin\theta\left[1 + \frac{\sigma_a - \sigma_b + j\omega(\epsilon_a - \epsilon_b)}{(2\sigma_a + \sigma_b) + j\omega(2\epsilon_a + \epsilon_b)}(R/r)^3\right]\right\}$$

If we denote by

$$\hat{A} \equiv \frac{\sigma_a - \sigma_b + j\omega(\epsilon_a - \epsilon_b)}{(2\sigma_a + \sigma_b) + j\omega(2\epsilon_a + \epsilon_b)}$$

we obtain

$$\langle P_d\rangle = \frac{1}{2}\sigma_a|\hat{E}_a|^2 = |E_p|^2\{\cos^2\theta[1 - 4(R/r)^3\mathrm{Re}\,\hat{A} + 4(R/r)^6|\hat{A}|^2] + \sin^2\theta[1 + 2(R/r)^3\mathrm{Re}\,\hat{A} + (R/r)^6|\hat{A}|^2]\}$$

(b) The power dissipated is

$$\langle p_d\rangle = \frac{4\pi R^3}{3}\langle P_d\rangle \quad (3)$$

where $\langle P_d\rangle$ is taken from (2b).

11.5.3 (a) The magnetic field is z-directed and equal to the surface current in the sheet. In region (b)

$$\mathbf{H} = H^b\mathbf{i}_z \quad (1)$$

in region (a) it is

$$\mathbf{H} = \mathbf{i}_z K \quad (2)$$

The field at the sheet is, from Faraday's integral law

$$E_y = b\mu_o\frac{dH^b}{dt} \quad \text{at} \quad x = -b \quad (3)$$

The field at the source is

$$E_y = a\mu_o\frac{dK}{dt} + b\mu_o\frac{dH^b}{dt} \quad (4)$$

The power dissipated in the sheet is, using (3)

$$p_d = \int \sigma E_y^2 dv = \sigma \Delta w d b^2 \mu_o^2 \Big(\frac{dH^b}{dt}\Big)^2 \tag{5}$$

The stored energy is

$$\begin{aligned}\int_V W dv &= \frac{1}{2}\mu_o (H^a)^2 a d w + \frac{1}{2}\mu_o (H^b)^2 b d w \\ &= \frac{1}{2}\mu_o d w [b(H^b)^2 + aK^2]\end{aligned} \tag{6}$$

(b) The integral of the Poynting vector gives

$$\oint \mathbf{E}\times\mathbf{H}\cdot d\mathbf{a} = -E_y H_z w d = -\big(a\mu_o \frac{dK}{dt} + b\mu_o \frac{dH^b}{dt}\big) K w d \tag{7}$$

Now

$$H_b = K - E_y \sigma \Delta = K - b\mu_o \frac{dH_b}{dt}\sigma\Delta \tag{8}$$

When we introduce this into (7) we get

$$\begin{aligned}\oint \mathbf{E}\times\mathbf{H}\cdot da = &- \Big\{\frac{1}{2} a\mu_o w d \frac{dK^2}{dt} + \frac{1}{2} b\mu_o w d \frac{dH^{b2}}{dt}\Big\} \\ &- \sigma b^2 w d \mu_o^2 \Big(\frac{dH^b}{dt}\Big)^2 \sigma\Delta\end{aligned} \tag{9}$$

But the last term is p_d; and the term in wavy brackets is the time rate of change of the magnetic energy.

11.5.4 Solving (10.4.13) for $\hat{A}$, under sinusoidal, steady state conditions, gives

$$\begin{aligned}\hat{A} &= \frac{1}{(j\omega\tau_m + 1)}\left[-j\omega\tau_m + \frac{1 - \frac{\mu_o}{\mu}}{\mu_o \Delta\sigma a}\tau_m\right] a^2 H_o \\ &= \frac{1}{(j\omega\tau_m + 1)}\left[-j\omega\tau_m + \frac{\mu - \mu_o}{\mu + \mu_o}\right] a^2 H_o\end{aligned} \tag{1}$$

From (10.4.11), we obtain $\hat{C}$

$$\hat{C} = -\frac{\mu_o}{\mu}\Big(H_o + \frac{\hat{A}}{a^2}\Big) = -\frac{\frac{2\mu_o}{\mu+\mu_o}}{1 + j\omega\tau_m} H_o \tag{2}$$

The discontinuity of the tangential magnetic field gives the current flowing in the cylinder. From (10.4.10)

$$\begin{aligned}\Delta\hat{H}_\phi &= -\Big(H_o - \frac{\hat{A}}{a^2}\Big)\sin\phi - \hat{C}\sin\phi \\ &= -\Big[1 + j\omega\tau_m + j\omega\tau_m - \frac{\mu - \mu_o}{\mu + \mu_o} - \frac{2\mu_o}{\mu + \mu_o}\Big]\frac{H_o \sin\phi}{1 + j\omega\tau_m} \\ &= -2\frac{j\omega\tau_m}{1 + j\omega\tau_m}\sin\phi H_o = \hat{K}_z\end{aligned} \tag{3}$$

Note the dependence of the current upon ω: when $\omega\tau_m \gg 1$, then the current is just large enough $(-2H_o \sin\phi)$ to cancel the field internal to the cylinder. When $\omega\tau_m \to 0$, of course, the current goes to zero. The jump of H_ϕ is equal to K. The power dissipated is, per unit axial length:

$$p_d = \frac{1}{2}\int \sigma|\hat{E}|^2 dv = \frac{1}{2}\sigma\Delta a \int_0^{2\pi} |\hat{E}_z|^2 d\phi \tag{4}$$

But

$$\sigma \hat{E}_t \Delta = \hat{K}_z \tag{5}$$

and thus

$$p_d = \frac{1}{2}\int_0^{2\pi} d\phi \frac{|\hat{K}_z|^2}{\sigma^2\Delta^2}\sigma\Delta a = \frac{\pi a}{\sigma\Delta}\frac{2\omega^2\tau_m^2 a}{1+\omega^2\tau_m^2}|H_o|^2 \tag{6}$$

11.5.5 (a) The applied field is in the direction normal to the paper, and is equal to

$$H_o \cos\omega t = N i_o \cos\omega t/d \tag{1}$$

The internal field is $H_o + K$ where K is the current flowing in the cylinder. From Faraday's law in complex form

$$\oint \hat{E}\cdot d\mathbf{s} = -j\omega\mu(\hat{H}_o + \hat{K})b^2 \tag{2}$$

Because $\hat{K}$ must be a constant, $\hat{E}$ tangential to the surface of the cylindrical shell must be constant. The path length is $4b$. We have

$$\hat{K} = \sigma\Delta\hat{E} = -\frac{j\omega\mu\sigma\Delta b}{4}(\hat{H}_o + \hat{K}) \tag{3}$$

and solving for $\hat{K}$

$$\hat{K} = -\frac{j\omega\tau_m}{1+j\omega\tau_m}H_o \tag{4}$$

where

$$\tau_m = \frac{\mu\sigma\Delta b}{4} \tag{5}$$

The surface current cancels H_o in the high frequency limit $\omega\tau_m \to \infty$. In the low frequency limit, it approaches zero as $\omega\tau_m$ approaches zero. Thus

$$p_d = \frac{1}{2}\int \sigma|\hat{E}|^2 dv = \frac{1}{2}\frac{4b\Delta d\sigma}{\sigma^2\Delta^2}|\hat{K}|^2 = \frac{2b}{\sigma\Delta d}N^2 i_o^2 \frac{\omega^2\tau_m^2}{1+\omega^2\tau_m^2} \tag{6}$$

(b) The time average Poynting flux is

$$\begin{aligned}
-\text{Re}\oint \hat{\mathbf{E}}\times\hat{\mathbf{H}}\cdot d\mathbf{a} &= -\text{Re}\,\frac{1}{2}4bd\hat{E}\hat{H}^* \\
&= -\text{Re}\left\{2bdH_o^*(-j\omega\tau_m)(\hat{H}_o+\hat{K})\right\} \\
&= \text{Re}\,2bdj\omega\tau_m\hat{H}_o^*\hat{K} \\
&= \frac{2bd}{\sigma\Delta}\frac{\omega^2\tau_m^2}{1+\omega^2\tau_m^2}|H_o|^2 = \frac{2b}{\sigma\Delta d}\frac{\omega^2\tau_m^2}{1+\omega^2\tau_m^2}N^2 i_o^2
\end{aligned} \tag{7}$$

which is the same as above.

11.5.6 (a) When the volume current density is zero, then Ampère's law in the MQS limit becomes

$$\nabla \times \mathbf{H} = 0 \tag{1}$$

and Faraday's law is

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mu_o(\mathbf{H} + \mathbf{M}) \tag{2}$$

If we introduce complex notation to describe the sinusoidal steady state $\mathbf{E} = \text{Re}\,\hat{\mathbf{E}}(\mathbf{r})e^{j\omega\tau}$ etc., then we get from the above

$$\nabla \times \hat{\mathbf{H}} = 0 \tag{3}$$

$$\nabla \times \hat{\mathbf{E}} = -j\omega\mu_o(\hat{\mathbf{H}} + \hat{\mathbf{M}}) \tag{4}$$

If $\hat{M}$ is linearly related to $\hat{H}$ we may write

$$\hat{\mathbf{M}} = \hat{\chi}_m \hat{\mathbf{H}} \tag{5}$$

where $\hat{\chi}_m$ is, in general, a function of ω, we may define

$$\hat{\mu} = \mu_o(1 + \hat{\chi}_m) \tag{6}$$

and write for (4)

$$\nabla \times \hat{\mathbf{E}} = -j\omega\hat{\mathbf{B}} \tag{7}$$

with

$$\hat{\mathbf{B}} \equiv \hat{\mu}\hat{\mathbf{H}} \tag{8}$$

Because $\nabla \cdot \mu_o(\hat{\mathbf{H}} + \hat{\mathbf{M}}) = 0$, we have

$$\nabla \cdot \hat{\mathbf{B}} = 0 \tag{9}$$

(b) The magnetic dipole moment is, according to (20) of the solution to P10.4.3.

$$\hat{m} = -2\pi R^3 \hat{H}_o \frac{j\omega\tau}{1 + j\omega\tau} \tag{10}$$

with $\tau = \mu_o\sigma\Delta R/3$. As $\omega\tau_m \to \infty$, this reduces to the result (9.5.16). The susceptibility is found from (5):

$$\hat{\chi}_m = -2\pi(R/s)^3 \frac{j\omega\tau}{1 + j\omega\tau}$$

where $1/s^3$ is the density of the dipoles.

(c) The magnetic field at $x = -l$ is

$$\hat{\mathbf{H}} = \mathbf{i}_z \hat{K} \tag{14}$$

The electric field follows from Faraday's law: applied to a contour along the perfect conductor and current generator

$$-a\hat{E}_y(-l) = -j\omega\hat{\mu}\hat{H}_z al \tag{15}$$

and thus

$$\hat{E}_y = j\omega\hat{\mu}l\hat{H}_z \tag{16}$$

The power dissipated is

$$\begin{aligned} p_d &= -\frac{1}{2}\text{Re}\oint \hat{\mathbf{E}}\times\hat{\mathbf{H}}^*\cdot d\mathbf{a} \\ &= \frac{1}{2}\text{Re}\ \hat{E}_y\hat{H}_z^*\big|_{x=-l}ad \\ &= \frac{1}{2}\text{Re}\ j\omega\hat{\mu}|\hat{K}|^2 adl \end{aligned} \tag{17}$$

Introducing (12) and (13) we find

$$p_d = \pi(R/s)^3\mu_o\frac{\omega^3\tau}{1+\omega^2\tau^2}|\hat{K}|^2 adl \tag{18}$$

11.5.7 From (10.7.15) we find

$$\hat{H}_z = \hat{K}_s \exp -(1+j)\Big(\frac{x+b}{\delta}\Big) \tag{1}$$

so that $H_z = K_s$ at the surface at $x = -b$. The current density is

$$\hat{\mathbf{J}} \simeq \nabla\times\hat{\mathbf{H}} = -\mathbf{i}_y\frac{\partial H_z}{\partial x} = \mathbf{i}_y\frac{(1+j)}{\delta}K_s \exp -(1+j)\Big(\frac{x+b}{\delta}\Big) \tag{2}$$

The power dissipation density is

$$P_d = \frac{1}{2}\frac{|\hat{J}_y|^2}{\sigma} \tag{3}$$

and thus the power dissipated per unit area is

$$\int_{x=-b}^{x=0} P_d dx \simeq \frac{|\hat{K}_s|^2}{\sigma}\int_{x=-b}^{\infty}\exp -\frac{2(x+b)}{\delta}dx = \frac{|\hat{K}_s|^2}{2\sigma\delta} \quad \text{watts/m}^2$$

11.5.8 (a) From (10.7.10) we find $\hat{H}_z$ everywhere. The current density is

$$\hat{J} = (\nabla \times \hat{\mathbf{H}})_y = -\frac{\partial H_z}{\partial x} = \frac{(1+j)}{\delta}\hat{K}_s \frac{e^{-(1+j)\frac{x}{\delta}} + e^{(1+j)\frac{x}{\delta}}}{e^{(1+j)\frac{b}{\delta}} - e^{-(1+j)\frac{b}{\delta}}} \tag{1}$$

The density of dissipated power is:

$$\begin{aligned} P_d = \frac{1}{2}\frac{|\hat{J}|^2}{\sigma} &= \frac{1}{\sigma\delta^2}|\hat{K}_s|^2 \frac{e^{-2x/\delta} + 2\cos\frac{2x}{\delta} + e^{2x/\delta}}{e^{2b/\delta} - 2\cos\frac{2b}{\delta} + e^{-2b/\delta}} \\ &= \frac{1}{\sigma\delta^2}|\hat{K}_s|^2 \frac{\cosh\frac{2x}{\delta} + \cos\frac{2x}{\delta}}{\cosh\frac{2b}{\delta} - \cos\frac{2b}{\delta}} \end{aligned} \tag{2}$$

The total dissipated power is

$$\begin{aligned} p_d = ad\int_{x=-b}^{0} P_d dx &= ad\frac{1}{\sigma\delta^2}|\hat{K}_s|^2 \frac{\delta}{2}\frac{\sinh\frac{2x}{\delta} + \sin\frac{2x}{\delta}}{\cosh\frac{2b}{\delta} - \cos\frac{2b}{\delta}}\Bigg|_{-b}^{0} \\ &= ad\frac{|\hat{K}_s|^2}{2\sigma\delta}\frac{\sinh\frac{2b}{\delta} + \sin\frac{2b}{\delta}}{\cosh\frac{2b}{\delta} - \cos\frac{2b}{\delta}} \end{aligned} \tag{3}$$

(b) Take the limit $\delta \ll b$. Then $\sinh\frac{2b}{\delta} \simeq \cosh\frac{2b}{\delta} = \frac{1}{2}e^{2b/\delta}$ and the sines and cosines are negligible.

$$p_d = \frac{ad}{2\sigma\delta}|\hat{K}_s|^2 \tag{4}$$

which is consistent with P11.5.7. When $2b/\delta \ll 1$, then

$$\cosh\left(\frac{2b}{\delta}\right) - \cos\left(\frac{2b}{\delta}\right) \approx 1 + \frac{1}{2}\left(\frac{2b}{\delta}\right)^2 - \left(1 - \frac{1}{2}\left(\frac{2b}{\delta}\right)^2\right) = \left(\frac{2b}{\delta}\right)^2 \tag{5}$$

$$\sinh\left(\frac{2b}{\delta}\right) + \sin\left(\frac{2b}{\delta}\right) \simeq \frac{4b}{\delta} \tag{6}$$

and thus

$$p_d = ad\frac{1}{2\sigma\delta}|\hat{K}_s|^2\frac{\delta}{b} = \frac{ad|\hat{K}_s|^2}{2\sigma b} \tag{7}$$

The total current is

$$\hat{i} = \hat{K}_s d \tag{8}$$

The resistance is

$$R = \frac{a}{\sigma b d} \tag{9}$$

and

$$\frac{1}{2}|\hat{i}|^2 R = ad\frac{|\hat{K}_s|^2}{2\sigma b} \tag{10}$$

Q.E.D.

11.5.9 The constitutive law

$$\frac{\partial \mathbf{M}}{\partial t} = \gamma \mathbf{H} \tag{1}$$

gives for complex vector amplitudes

$$j\omega \hat{\mathbf{M}} = \gamma \hat{\mathbf{H}} \tag{2}$$

and thus

$$\hat{\chi}_m = \frac{\gamma}{j\omega} \tag{3}$$

and

$$\hat{\mu} = \mu_o(1 + \hat{\chi}_m) = \mu_o\left(1 + \frac{\gamma}{j\omega}\right) \tag{4}$$

The flux is

$$\mathbf{B} = \hat{\mu}\hat{\mathbf{H}} = \mu_o\left(1 + \frac{\gamma}{j\omega}\right)\hat{\mathbf{H}} \tag{5}$$

The induced voltage is

$$v = \frac{d\lambda}{dt} \Rightarrow \hat{v} = j\omega\hat{\lambda} \tag{6}$$

and

$$\hat{\lambda} = N_1 \frac{\pi w^2}{4} \hat{B}_\phi \tag{7}$$

But

$$\hat{H}_\phi = \frac{N_1 \hat{i}}{2\pi R} \tag{8}$$

and thus

$$\hat{\lambda} = \hat{\mu} \frac{N_1^2 w^2}{8R} \hat{i} \tag{9}$$

and thus

$$\hat{v} = j\omega\hat{\lambda} = j\omega\mu_o \frac{N_1^2 w^2}{8R}\hat{i} + \mu_o \frac{\gamma N_1^2 w^2 \hat{i}}{8R} = (j\omega L + R_m)\hat{i} \tag{10}$$

Thus

$$L = \mu_o \frac{N_1^2 w^2}{8R} \qquad R_m = \frac{\mu_o \gamma N_1^2 w^2}{8R} \tag{11}$$

11.5.10 (a) The peak H field is

$$H_{\text{peak}} = \frac{N_1 i_{\text{peak}}}{2\pi R} = \frac{N_1}{2\pi R}\frac{2H_c 2\pi R}{N_1} = 2H_c \tag{1}$$

Thus (see Fig. S11.5.10a).

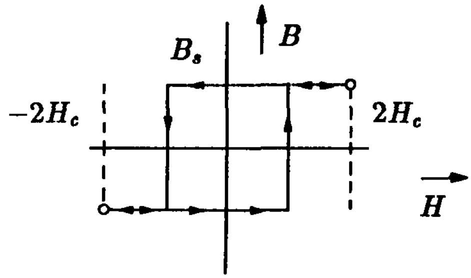

Figure S11.5.10a

(b) The terminal voltage is

$$v = \frac{d}{dt} N_1 \frac{\pi w^2}{4} B \propto \frac{dB}{dt} \tag{2}$$

The B field jumps suddenly, when $H = H_c$. This is shown in Fig. S11.5.10b. The voltage is impulse like with content equal to the flux discontinuity: $2N_1 \frac{\pi w^2}{4} B_s$.

(c) The time average power input is $\int vidt$ integrated over one period. Contributions come only at impulses of voltage and are equal to

$$\int vidt = 2 \times 2N_1 \frac{\pi w^2}{4} B_s \cdot i(t_o) \tag{3}$$

But

$$\frac{N_1 i(t_o)}{2\pi R} = H_c \tag{4}$$

and thus

$$\int vidt = 4N_1 \frac{\pi w^2}{4} B_s H_c \frac{2\pi R}{N_1} = \left(2\pi R \frac{\pi w^2}{4}\right) 4 B_s H_c \tag{5}$$

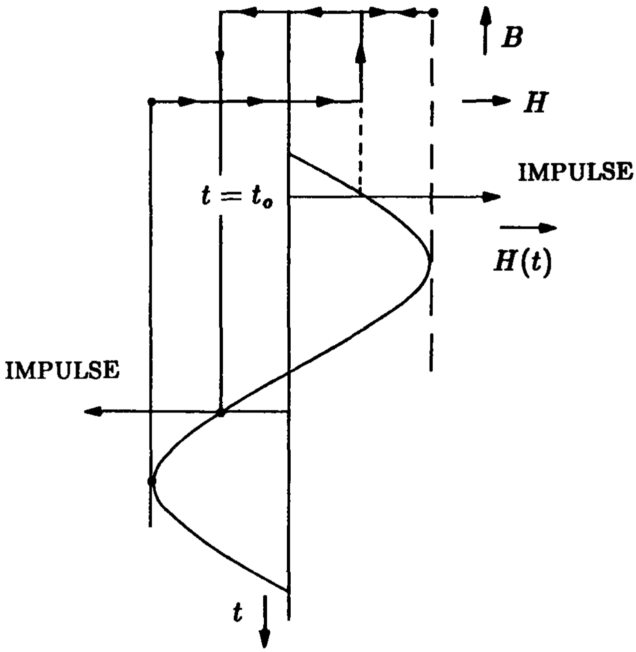

Figure S11.5.10b

(d) The energy fed into the magnetizable material per unit volume within time dt is

$$dt\mathbf{H} \cdot \frac{\partial}{\partial t}\mu_o(\mathbf{H} + \mathbf{M}) = dt\mathbf{H} \cdot \frac{\partial}{\partial t}\mathbf{B} = \mathbf{H} \cdot d\mathbf{B} \tag{6}$$

As one goes through a full cycle,

$$\oint \mathbf{H} \cdot d\mathbf{B} = \text{area of hysteresis loop} \tag{7}$$

This is $4H_cB_s$. Thus the total energy fed into the material in one cycle is

$$\text{volume} \oint \mathbf{H} \cdot d\mathbf{B} = \left(2\pi R\frac{\pi w^2}{4}\right)4B_sH_c \tag{8}$$

11.6 ELECTRICAL FORCES ON MACROSCOPIC MEDIA

11.6.1 The capacitance of the system is

$$C = \frac{\epsilon_o(b - \xi)d}{a}$$

The force is

$$f_e = \frac{1}{2}v^2\frac{dC}{d\xi} = -v^2\frac{\epsilon_o d}{2a}$$

11.6.2 The capacitance per unit length is from (4.6.27)

$$C = \frac{\pi\epsilon_o}{ln(\frac{l}{R} + \sqrt{(l/R)^2 - 1})} \tag{1}$$

where the distance between the two cylinders is $2l$. Thus replacing l by $\xi/2$, we can find the force per unit length on one cylinder by the other from

$$\begin{aligned} f_e = \frac{1}{2}v^2\frac{dC}{d\xi} &= \frac{1}{2}v^2\frac{d}{d\xi}\left[\frac{\pi\epsilon_o}{ln[\frac{\xi}{2R} + \sqrt{(\xi/2R)^2 - 1}]}\right] \\ &= -\frac{1}{2}v^2\frac{\pi\epsilon_o}{ln^2[(\xi/2R) + \sqrt{(\xi/2R)^2 - 1}]}\frac{\frac{1}{2R} + \frac{\xi}{(2R)^2}\frac{1}{\sqrt{(\xi/2R)^2-1}}}{\frac{\xi}{2R} + \sqrt{(\xi/2R)^2 - 1}} \end{aligned} \tag{2}$$

This expression can be written in a form, in which it is more recognizable. Using the fact that $\lambda_l = Cv$ we may write

$$f_o = -\frac{\lambda_l^2}{4\pi\epsilon_o R}\frac{1 + (\xi/2R)/\sqrt{(\xi/2R)^2 - 1}}{\frac{\xi}{2R} + \sqrt{(\xi/2R)^2 - 1}} \tag{3}$$

When $\xi/2R \gg 1$, and the cylinder radii are much smaller than their separation, the above becomes

$$f_e = -\frac{\lambda_l^2}{2\pi\epsilon_o 2\xi} \tag{4}$$

This is the force on a line charge λ_l in the field $\lambda_l/(2\pi\epsilon_o 2\xi)$.

11.6.3 The capacitance is made up of two capacitors connected in parallel.

$$C = \frac{2\pi\epsilon_o(l-\xi)}{\ln(a/b)} + \frac{2\pi\epsilon\xi}{ln(a/b)}$$

(a) The force is

$$f_e = \frac{1}{2}v^2\frac{dC}{d\xi} = v^2\frac{\pi(\epsilon - \epsilon_o)}{ln(a/b)}$$

(b) The electric circuit is shown in Fig. S11.6.3. Since R is very small, the output voltage is

$$v_o = iR$$

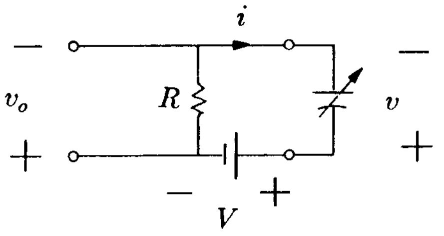

Figure S11.6.3

From Kirchoff's voltage law

$$iR + V = v$$

Now

$$q = Cv$$

and

$$i = \frac{dq}{dt} = \frac{d}{dt}(Cv) = \frac{dC}{dt}v + C\frac{dv}{dt}$$

If R is small, then v is still almost equal to V and dv/dt is much smaller than $(vdC/dt)/C$. Then

$$-i \approx V\frac{dC}{dt}$$

and

$$v_o = Ri = -2\pi RV(\epsilon - \epsilon_o)\frac{d\xi}{dt}/ln(a/b)$$

11.6.4 The capacitance is determined by the region containing the electric field

$$C = \frac{2\pi\epsilon_o(l-\xi)}{ln(a/b)}$$

(a) The force is

$$f_e = \frac{1}{2}V^2\frac{dC}{d\xi} = -\frac{\pi\epsilon_o}{ln(a/b)}V^2$$

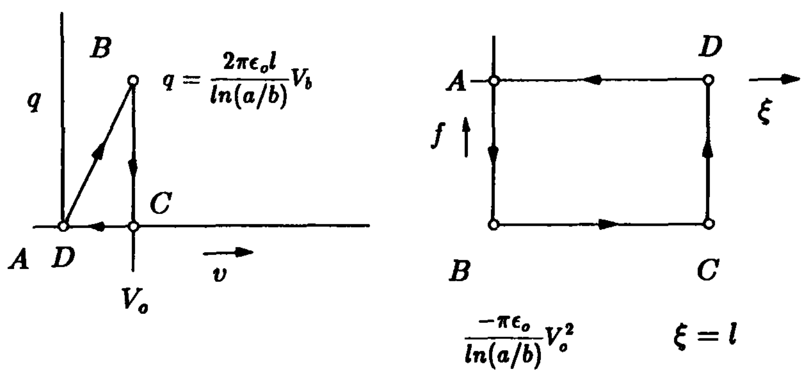

Figure S11.6.4

(b) See Fig. S11.6.4. When $\xi = 0$, then the value of capacitance is maximum. Going from A to B in the $f - \xi$ plane changes the force from 0 to a finite negative value by application of a voltage. Travel from B to C maintains the force while ξ is increasing. Thus ξ increases at constant voltage. The motion from C to D is done at constant ξ by decreasing to voltage from a finite value to zero. Finally as one returns from D to A the inner cylinder is pushed back in. In the $q-v$ plane, the point A is one of zero voltage and maximum capacitance. As the voltage is increased to V_o, the charge increases to

$$q = CV_o = \frac{2\pi\epsilon_o l}{ln(a/b)}V_o$$

The trajectory from B to C keeps the voltage fixed while increasing ξ, decreasing the capacitance. Thus the charge decreases. As one moves from C to D at constant ξ decreasing the voltage to zero, one moves back to the origin. Changing ξ to zero at zero voltage does not change the charge so that D and A coincide in the $q-v$ plane.

(c) The energy input is evaluated as the areas in the $q-v$ plane and the $\xi - f$ plane. The area in the $\xi - f$ plane is

$$\frac{\pi\epsilon_o l}{ln(a/b)}V_o^2$$

and the area in the $v-q$ plane is

$$\frac{1}{2}\frac{2\pi\epsilon_o l}{ln(a/b)}V_o^2$$

which is the same.

11.6.5 Using the coenergy value obtained in P11.4.6, we find the force is

$$f_e = \frac{\partial w_e'}{\partial \xi}\Big|_v = \left[\frac{\alpha_1}{\alpha_2}\left(\sqrt{1+\frac{\alpha^2 v^2}{\alpha^2}}-1\right)+\frac{1}{2}\epsilon_o\frac{v^2}{\alpha^2}\right]ca - \frac{1}{2}\frac{\epsilon_o v^2}{a}c$$

11.7 MACROSCOPIC MAGNETIC FORCES

11.7.1 The magnetic coenergy is

$$w_m' = \frac{1}{2}(L_{11}i_1^2 + 2L_{12}i_1i_2 + L_{22}i_2^2)$$

The force is

$$\begin{aligned} f_m = \frac{\partial w_m'}{\partial x}\Big|_{i_1,i_2} &= \frac{1}{2}\left(\frac{dL_{11}}{dx}i_1^2 + 2\frac{dL_{12}}{dx}i_1i_2 + \frac{dL_{22}}{dx}i_2^2\right) \\ &= \frac{1}{2}\left(N_1^2\frac{dL_o}{dx}i_1^2 + 2N_1N_2\frac{dL_o}{dx}i_1i_2 + N_2^2\frac{dL_o}{dx}i_2^2\right) \end{aligned}$$

Since

$$L_o = \frac{aw\mu_o}{x(1+\frac{a}{b})}$$

we have

$$f_m = -\frac{1}{2}(N_1^2i_1^2 + 2N_1N_2i_1i_2 + N_2^2i_2^2)\frac{aw\mu_o}{x^2(1+\frac{a}{b})}$$

11.7.2 The inductance of the coil is, according to the solution to (9.7.6)

$$f_m = \frac{1}{2}i^2\frac{dL}{dx} = -\frac{1}{2}i^2\frac{\mu_o N^2}{\left[\frac{x}{\pi a^2} + \frac{g}{2\pi a d}\right]^2}\frac{1}{\pi a^2}$$

√

11.7.3 We first compute the inductance of the circuit. The two gaps are in series so that Ampère's law for the electric field gives

$$y(H_1 + H_2) = ni \tag{1}$$

where H_1 is the field on the left, H_2 is the field on the right. Flux conservation gives

$$H_1(a-x)d = H_2 x d \tag{2}$$

Thus

$$H_1 = \frac{ni}{y}\frac{x}{a}$$

The flux is

$$\Phi_\lambda = \frac{\mu_o ni}{y}\left(\frac{a-x}{a}\right)xd$$

The inductance is

$$L = n\Phi_\lambda = \frac{\mu_o n^2}{y}\frac{xd(a-x)}{a}$$

The force is

$$\mathbf{f}_m = \frac{1}{2}i^2\left(\frac{\partial L}{\partial x}\mathbf{i}_x + \frac{\partial L}{\partial y}\mathbf{i}_y\right) = \frac{1}{2}i^2\frac{\mu_o n^2 d}{a}\left\{\frac{(a-2x)}{y}\mathbf{i}_x - \frac{x(a-x)}{y^2}\mathbf{i}_y\right\}$$

11.7.4 Ampère's law applied to the fields H_o and H at the inner radius in the media μ_o and μ, respectively, gives

$$H_o\int_b^a \frac{b}{r}dr = H\int_b^a \frac{b}{r}dr = Ni \tag{1}$$

and thus

$$H_o = H = \frac{Ni}{b\,ln\frac{a}{b}} \tag{2}$$

The flux is composed of the two individual fluxes

$$\Phi_\lambda = 2\pi\frac{Ni}{ln\frac{a}{b}}[\mu_o(l-\xi) + \mu\xi] \tag{3}$$

The inductance is

$$L = N\Phi_\lambda/i = \frac{2\pi}{ln(a/b)}N^2\{\mu\xi + \mu_o(l-\xi)\} \tag{4}$$

The force is

$$f(i,\xi) = \frac{1}{2}i^2\frac{dL}{d\xi} = \frac{\pi(\mu-\mu_o)}{ln(a/b)}N^2 i^2 \tag{5}$$

11.7.5 The H-field in the two gaps follows from Ampère's integral law

$$2H\Delta = 2Ni \tag{1}$$

The flux is

$$\Phi_\lambda = \mu_o H d(2\alpha - \theta)R = \mu_o N i d(2\alpha - \theta)R/\Delta \tag{2}$$

and the inductance

$$L = \frac{2N\Phi_\lambda}{i} = 2N^2\mu_o \frac{dR(2\alpha - \theta)}{\Delta} \tag{3}$$

The torque is

$$\tau = \frac{1}{2}i^2\frac{dL}{d\theta} = -\mu_o dRN^2 i^2/\Delta \tag{4}$$

11.7.6 The coenergy is

$$\begin{aligned} w'_m &= \int [\lambda_a di_a + \lambda_b di_b + \lambda_r di_r] \\ &= \frac{1}{2}L_s i_a^2 + \frac{1}{2}L_s i_b^2 + \frac{1}{2}L_r i_r^2 \\ &\quad + M\cos\theta i_a i_r + M\sin\theta i_r i_b \end{aligned} \tag{1}$$

where we have taken advantage of the fact that the integral is independent of path. We went from $i_a = i_b = i_r = 0$ first to i_a, then raised i_b to its final value and then i_r to its final value.

(b) The torque is

$$\tau = \frac{\partial w'_m}{\partial\theta} = i_r(-M\sin\theta i_a + M\cos\theta i_b)$$

(c) The two coil currents i_a and i_b produce effective z-directed surface currents with the spatial distributions $\sin\phi$ and $\sin(\phi - \frac{\pi}{2}) = -\cos\phi$ respectively. If they are phased as indicated, the effective surface current is proportional to

$$\cos(\omega t)\sin\phi - \sin\omega t\cos\phi = \sin(\phi - \omega t)$$

Thus the rate of change of the maximum of the current density is $d\phi/dt = \omega$.

(d) The torque is

$$\begin{aligned} \tau &= I_r[-M\sin(\Omega t - \gamma)I\cos\omega t + M\cos(\Omega t - \gamma)I\sin\omega t] \\ &= I_r I(-M\sin(\Omega t - \gamma - \omega t) \end{aligned}$$

But if $\Omega = \omega$, then

$$\tau = I_r I M \sin\gamma$$

11.8 FORCES ON MACROSCOPIC ELECTRIC AND MAGNETIC DIPOLES

11.8.1 (a) The potential obeys Laplace's equation and must vanish for $y \to \infty$. Thus the solution is of the form $e^{-\beta y}\cos\beta x$. The voltage distribution of $y = 0$ picks the amplitude as V_o. The E field is

$$\mathbf{E} = \beta V_o(\sin\beta x\mathbf{i_x} + \cos\beta x\mathbf{i_y})e^{-\beta y}$$

(b) The force on a dipole is

$$\mathbf{f} = \mathbf{p}\cdot\nabla\mathbf{E} = 4\pi\epsilon_o R^3(\mathbf{E}\cdot\nabla)\mathbf{E}$$

It behooves us to compute $(\mathbf{E}\cdot\nabla)\mathbf{E}$. We first construct the operator

$$\mathbf{E}\cdot\nabla = \beta V_o e^{-\beta y}(\sin\beta_x\frac{\partial}{\partial x} + \cos\beta x\frac{\partial}{\partial y})$$

Thus

$$\begin{aligned}\mathbf{E}\cdot\nabla\mathbf{E} &= \beta V_o e^{-\beta y}\Big\{\sin\beta x\frac{\partial}{\partial x}\big[\beta V_o(\sin\beta_x\mathbf{i_x} + \cos\beta x\mathbf{i_y})e^{-\beta y}\big] \\ &\quad + \cos\beta x\frac{\partial}{\partial y}\big[\beta V_o(\sin\beta x\mathbf{i_x} + \cos\beta x\mathbf{i_y})e^{-\beta y}\big] \\ &= \beta^2 V_o^2\beta\big[(\sin\beta x\cos\beta x\mathbf{i_x} - \sin^2\beta x\mathbf{i_y})e^{-\beta y} \\ &\quad - (\cos\beta x\sin\beta x\mathbf{i_x} + \cos^2\beta x\mathbf{i_y})e^{-\beta y}\big] \\ &= -\beta^2 V_o^2\beta\mathbf{i_y}e^{-\beta y}\end{aligned}$$

and thus

$$\mathbf{f} = -4\pi\epsilon_o R^3(\beta V_o)^2\beta\mathbf{i_y}e^{-\beta y}$$

11.8.2 Again we compute, as in P11.8.1,

$$(\mathbf{E}\cdot\nabla)\mathbf{E}$$

in spherical coordinates

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_o r^2}\mathbf{i_r} \tag{1}$$

and the gradient operator is

$$\nabla = \mathbf{i_r}\frac{\partial}{\partial r} + \mathbf{i}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{i}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \qquad)2)$$

Thus,

$$\mathbf{E}\cdot\nabla = \frac{Q}{4\pi\epsilon_o r^2}\frac{\partial}{\partial r} \tag{3}$$

and

$$\mathbf{E}\cdot\nabla\mathbf{E} = -\frac{Q}{4\pi\epsilon_o r^2}\frac{2Q}{4\pi\epsilon_o r^3} = -\frac{2Q^2}{(4\pi\epsilon_o)^2 r^5} \tag{4}$$

and the force is

$$\mathbf{f} = \mathbf{p}\cdot\nabla\mathbf{E} = -4\pi\epsilon_o R^3\frac{2Q^2}{(4\pi\epsilon_o)^2 r^5} = -\frac{2Q^2R^3}{4\pi\epsilon_o r^5} \tag{5}$$

Note that the computation was simple, because $(\partial/\partial r)\mathbf{i}_r = 0$. In general, derivatives of the unit vectors in spherical coordinates are not zero.

11.8.3 The magnetic potential Ψ is of the form

$$\Psi = \begin{cases} A\cos\beta x e^{-\beta y} & y > 0 \\ A\cos\beta x e^{\beta y} & y < 0 \end{cases}$$

At $y = 0$, the potential has to be continuous and the normal component of $\mu_o\mathbf{H}$ has to be discontinuous to account for the magnetic surface charge density

$$\rho_m = \nabla\cdot\mu_o\mathbf{M} \Rightarrow \mu_o M_o\cos\beta x$$

Thus

$$\Psi = \frac{M_o}{2\beta}\cos\beta x e^{-\beta y}$$

This is of the same form as Φ of P11.8.1 with the correspondence

$$V_o \leftrightarrow M_o/2\beta$$

The infinitely permeable particle must have $H = 0$ inside. Thus, in a uniform field $H_o\mathbf{i}_z$, the potential around the particle is (We use, temporarily, the conventional orientation of the spherical coordinate, $\theta = 0$ axis as along z. Later we shall identify it with the orientation of the dipole moment.)

$$\Psi = -H_oR\cos\theta\left[\frac{r}{R} - (R/r)^2\right]$$

The particle produces a dipole field

$$\frac{H_oR^3}{r^3}(2\cos\theta\mathbf{i}_r + \sin\theta\mathbf{i}_\theta) = \frac{m}{4\pi r^3}(2\cos\theta\mathbf{i}_r + \sin\theta\mathbf{i}_\theta)$$

Thus the magnetic dipole is

$$\mu_o m = 4\pi\mu_o H_o R^3$$

This is analogous to the electric dipole with the correspondence

$$\mathbf{p} \leftrightarrow \mu_o\mathbf{m}$$
$$H_o \leftrightarrow E_o$$
$$\mu_o \leftrightarrow \epsilon_o$$

Since the force is

$$\mathbf{f} = \mu_o\mathbf{m}\cdot\nabla\mathbf{H}$$

we find perfect correspondence.

11.8.4 The field of a magnetic dipole $\mu_o \mathbf{m} \parallel \mathbf{i}_z$ is

$$\mathbf{H} = \frac{\mu_o m}{4\pi\mu_o r^3}(2\cos\theta \mathbf{i}_r + \sin\theta \mathbf{i}_\theta)$$

The image dipole is at distance $-Z$ below the plane and has the same orientation. According to P11.8.3, we must compute

$$\mathbf{f} = \mu_o \mathbf{m}\cdot\nabla\mathbf{H} = \mu_o\mathbf{m}\cdot\nabla\frac{\mu_o m}{4\pi\mu_o r^3}(2\cos\theta\mathbf{i}_r + \sin\theta\mathbf{i}_\theta)$$

where we identify

$$r = 2Z$$

after the differentiation. Now

$$\mu_o\mathbf{m}\cdot\nabla = \mu_o m\frac{\partial}{\partial r}$$

$\mathbf{i}_r$ and $\mathbf{i}_\theta$ are independent of r and thus

$$\mathbf{f} = -\mu_o m\frac{4\mu_o m}{4\pi\mu_o r^4}\mathbf{i}_r$$

since $\theta = 0$. But

$$\mu_o m = 4\pi\mu_o R^3 H_o$$

and thus

$$\mathbf{f} = -\frac{\pi\mu_o R^6}{Z^4}H_o^2\mathbf{i}_r$$

11.9 MACROSCOPIC FORCE DENSITIES

11.9.1 Starting with (11.9.14) we note that $J = 0$ and thus

$$\mathbf{f} = \int \mathbf{F}dv = -\int\frac{1}{2}H^2\nabla\mu dv \tag{1}$$

The gradient of μ of the plunger is directed to the right, is singular (unit impulse-like) and of content $\mu - \mu_o$. The only contribution is from the flat end of the plunger (of radius a). We take advantage of the fact that μH is constant as it passes from the outside into the inside of the plunger. Denote the position just outside by x_-, that just inside by x_+.

$$\begin{aligned} -\int\frac{1}{2}H^2\nabla\mu dv &= -\mathbf{i}_x\pi a^2\int_{x_-}^{x_+}H^2\frac{d\mu}{dx}dx \\ &\simeq -\mathbf{i}_x\frac{\pi a^2}{2}\left[\mu H^2\Big|_{x_-}^{x_+} - \int\mu\frac{d}{dx}H^2dx\right] \end{aligned} \tag{2}$$

where we have integrated by parts. The integrand in the second term can be written

$$\mu \frac{d}{dx} H^2 = 2\mu H \frac{dH}{dx} \tag{3}$$

and the integral is

$$\int_{x_-}^{x_+} \mu H \frac{dH}{dx} = \mu H H\Big|_{x_-}^{x_+} = -\mu_o H^2\Big|_{x_-} \tag{4}$$

where we have taken into account that μH is x-independent and that $H(x_+) = 0$. Combining (2), (3), and (4), we find

$$\mathbf{f} = -\mathbf{i_x} \frac{\pi a^2}{2} \mu_o H^2 \tag{5}$$

Using the H-field of Prob. 9.7.6, we find

$$\mathbf{f} = -\mathbf{i_x} \frac{\pi a^2}{2} \frac{\mu_o N^2 i^2}{\left(x + \frac{g\pi a^2}{2\pi a d}\right)^2} \tag{6}$$

This is the same as found in Prob. 11.7.2.

11.9.2 (a) From (11.9.14) we have

$$\mathbf{F} = \mathbf{J} \times \mathbf{B} \tag{1}$$

Now B varies from $\mu_o H_o$ to $\mu_o H_i$ in a linear way, whereas $\mathbf{J}$ is constant

$$\begin{aligned} \mathbf{i_r} T_r &= \int_a^{a+\Delta} \mathbf{J} \times \mathbf{B} dr = \int_a^{a+\Delta} dr J \mu_o H (\mathbf{i}_\phi \times \mathbf{i_z}) \\ &= \mathbf{i_r} \mu_o K \left(\frac{H_o + H_i}{2}\right) \end{aligned} \tag{2}$$

where

$$\int_a^{a+\Delta} dr J \equiv K \tag{3}$$

Now, both J and H_i are functions of time. We have from (10.3.11)-(10.3.12)

$$T_r = -\mathbf{i_r} \frac{1}{2} \mu_o H_o e^{-t/\tau_m} [H_o + H_o(1 - e^{t/\tau_m})] = -\mathbf{i_r} \frac{1}{2} \mu_o H_o^2 (2 - e^{-t/\tau_m}) e^{-t/\tau_m}$$

11.9.3 (a) Here the first step is analogous to the first three equations of P11.9.2. Because **J** is constant and H varies linearly

$$\mathbf{i_r}T_r = \mu_o K \frac{(H_o + H_i)}{2}(\mathbf{i_z} \times \mathbf{i_\phi}) \tag{1}$$

(b) If we introduce the time dependence of A from (10.4.16), with $\mu = \mu_o$,

$$A = -H_m a^2 e^{-t/\tau_m} \tag{2}$$

and of K_z from (19)

$$K_z = H^o_\phi - H^i_\phi = 2\frac{A}{a^2}\sin\phi = -2H_m \sin\phi e^{-t/\tau_m} \tag{3}$$

Further note that $H^i_\phi = 0$ at $t = 0$. Therefore from (3) and (2)

$$H^o_\phi = -2H_m \sin\phi \quad \text{at} \quad t = 0 \tag{4}$$

At $t = \infty$

$$H^o_\phi = -H_m \sin\phi \tag{5}$$

because the field has fully penetrated. Thus

$$H^o_\phi = -H_m \sin\phi[1 + e^{-t/\tau_m}] \tag{6}$$

From (6) and (3) we find

$$H^i_\phi = -H_m \sin\phi[1 - e^{-t/\tau_m}] \tag{7}$$

Thus we find from (1), (3), (6), and (7)

$$\begin{aligned}\mathbf{i_r}T_r &= -\mathbf{i_r}\frac{\mu_o}{2}[(H^o_\phi)^2 - (H^i_\phi)^2] \\ &= -\mathbf{i_r}\frac{\mu_o}{2}H_m^2 \sin^2\phi[(1 + e^{-t/\tau_m})^2 - (1 - e^{-t/\tau_m})^2] \\ &= -\mathbf{i_r}2\mu_o H_m^2 \sin^2\phi e^{-t/\tau_m}\end{aligned}$$

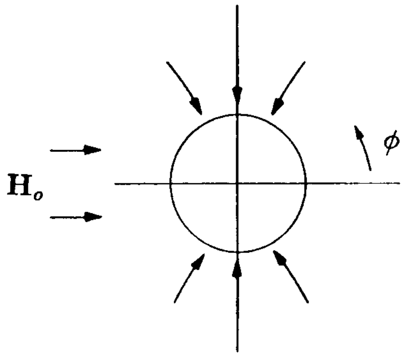

Figure S11.9.2

The force is inward, peaks at $t = 0$ and then decays. This shows that the cylinder will get crushed when a magnetic field is applied suddenly (Fig. S11.9.2).