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SOLUTIONS TO CHAPTER 10

10.0 INTRODUCTION

10.0.1 (a) The line integral of the electric field along C_1 is from Faraday's law:

$$\oint_{C_1} \mathbf{E} \cdot dl = 0 \tag{1}$$

because no flux is linked (see Fig. S10.0.1a). Therefore

$$-v + iR = 0$$

because the voltage drop across the resistor is iR. Hence

$$v = iR \tag{2}$$

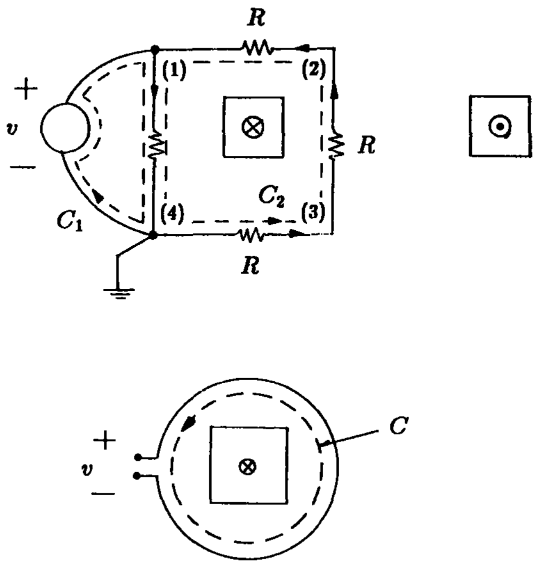


Figure S10.0.1a,b

The line integral along C_2 is

$$4iR = \frac{d\Phi_\lambda}{dt} \tag{3}$$

which leads to

$$iR = (\frac{d\Phi_\lambda}{dt})/4 \tag{4}$$

Therefore, we find for the voltage across the voltmeter

$$v = \frac{1}{4}\frac{d\Phi_\lambda}{dt} \tag{5}$$

(b) With the voltmeter connected to 2, (1) becomes

$$v = 2iR$$

Using (2),

$$v = 2\Big[\frac{1}{4}\frac{d\Phi_\lambda}{dt}\Big]$$

and similarly for the other modes

$$v(3) = 3[iR] = 3\Big[\frac{1}{4}\frac{d\Phi_\lambda}{dt}\Big]$$

$$v(4) = 4iR = 4\Big[\frac{1}{4}\frac{d\Phi_\lambda}{dt}\Big] = \frac{d\Phi_\lambda}{dt}$$

For a transformer with a one turn secondary (see Fig. S10.0.1b),

$$v = \oint_C \mathbf{E}\cdot dl = \frac{\partial}{\partial t}\int \mathbf{B}\cdot d\mathbf{a} = \frac{d}{dt}\Phi_\lambda$$

10.0.2 Given the following one-turn inductor (Figs. S10.0.2a and S10.0.2b), we want to find (a) v_2 and (b) v_1. The current per unit length (surface current) flowing along the sheet is $K = i/d$. The tangential component of the magnetic field has to have the discontinuity K. A magnetic field (the gradient of a Laplacian potential)

$$\begin{aligned} H_z &= \frac{i}{d} \quad \text{inside} \\ &= 0 \quad \text{outside} \end{aligned} \tag{1}$$

has the proper discontinuity. This is the field in a single turn "coil" of infinite width d and finite $K = i/d$. It serves here as an approximation.

(a) v_2 can be found by applying Faraday's law to the contour C_2,

$$\oint_{C_2} \mathbf{E}\cdot d\mathbf{s} = -\frac{d}{dt}\int_{S_2} \mathbf{B}\cdot d\mathbf{a}$$

Using (1), and the constitutive relation $\mathbf{B} = \mu_o\mathbf{H}$,

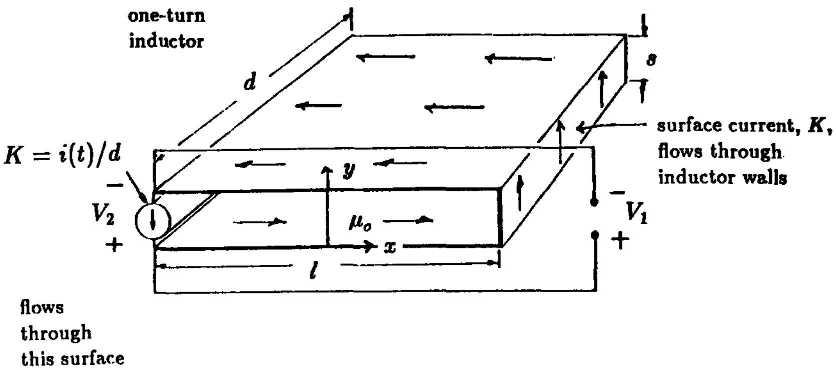
$$\int_{(A)C_2}^{(B)} \mathbf{E}\cdot d\mathbf{s} + \int_{(B)C_2}^{(A)} \mathbf{E}\cdot d\mathbf{s} = -\frac{d}{dt}\int_{S_2} \mu_o\frac{i(t)}{d}\,dx\,dy \tag{2}$$

Since the inductor walls are perfectly conducting, $\mathbf{E} = 0$ for the second integral on the left in (2). Therefore,

$$-v_2 = -\frac{d}{dt}\left(sl\mu_o \frac{i(t)}{d}\right)$$

or,

$$\Rightarrow v_2 = \frac{sl\mu_o}{d}\frac{di(t)}{dt}$$



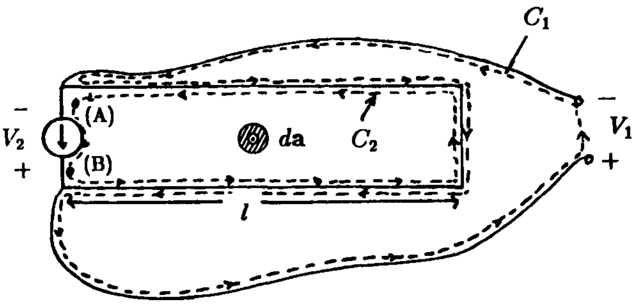


Figure S10.0.2

(b) Now, v_1 can be found by a similar method. Writing Faraday's law on C_1,

$$\oint_{C_1} \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt}\int_{S_1} \mathbf{B} \cdot d\mathbf{a} \tag{3}$$

Since C_1 does not link any flux, (3) can be written

$$-v_1 = -\frac{d}{dt}(0) = 0$$

10.1 MAGNETOQUASISTATIC ELECTRIC FIELDS IN SYSTEMS OF PERFECT CONDUCTORS

10.1.1 The magnetic field intensity from Problem 8.4.1 is

$$\mathbf{H} = \frac{i\pi R^2}{4\pi}\left[2\cos\theta\Big(\frac{1}{r^3} - \frac{1}{b^3}\Big)\mathbf{i_r} + \sin\theta\Big(\frac{1}{r^3} + \frac{2}{b^3}\Big)\mathbf{i}_\theta\right]$$

The E-field induced by Faraday's law has lines that link the dipole field and uniform field. By symmetry they are ϕ-directed. Using the integral law of Faraday's law using a spherical cap bounded by the contour r = constant, θ = constant, we have

$$\begin{aligned}\oint \mathbf{E}\cdot d\mathbf{s} = 2\pi r\sin\theta E_\phi &= -\frac{d}{dt}\int_0^\theta \mu_o \mathbf{H}_r 2\pi r\sin\theta r d\theta \\ &= -\mu_o\frac{di}{dt}\frac{\pi R^2}{4\pi}\int_0^\theta 2\pi r^2 2\sin\theta\cos\theta d\theta\Big(\frac{1}{r^3} - \frac{1}{b^3}\Big) \\ &= -\mu_o\frac{di}{dt}\frac{\pi R^2}{4\pi}\pi r^2\Big(\frac{1}{r^3} - \frac{1}{b^3}\Big)2\sin^2\theta\end{aligned}$$

Thus:

$$E_\phi = \mu_o\frac{R^2}{4b^2}\Big(\frac{r}{b} - \frac{b^2}{r^2}\Big)\sin\theta\frac{di}{dt}$$

10.1.2 (a) The H-field is similar to that of Prob. 10.0.2 with K specified. It is z-directed and uniform

$$H_z = \begin{cases} K & \text{inside} \\ 0 & \text{outside}\end{cases} \tag{1}$$

Indeed, it is the gradient of a Laplacian potential and has the proper discontinuity at the sheet.

(b) The particular solution does not need to satisfy all the boundary conditions. Suppose we look for one that satisfies the boundary conditions at $y = 0, x = 0$, and $y = a$. If we set

$$\mathbf{E}_p = \mathbf{i_x}E_{xp}(y,t) \tag{2}$$

with $E_{xp}(0,t) = 0$ we have satisfied all three boundary conditions. Now, from Faraday's law,

$$\frac{\partial E_{xp}}{\partial y} = \mu_o\frac{\partial H_z}{\partial t} = \mu_o\frac{dK}{dt} \tag{3}$$

Integration gives

$$E_{xp} = y\mu_o\frac{dK}{dt} \tag{4}$$

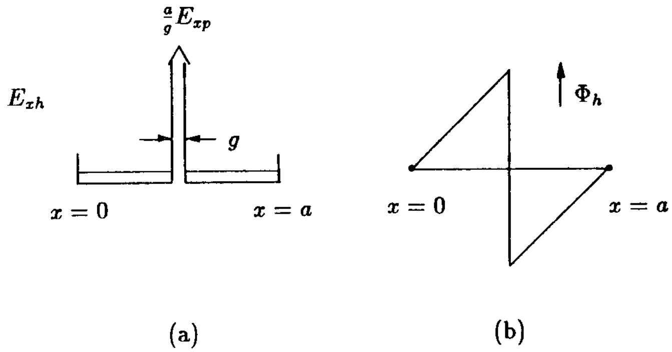


Figure S10.1.2a,b

The total field has to satisfy the boundary condition at $y = -l$. There, the field has to vanish for almost all $0 \le x \le a$, except for the short gap at the center of the interval. Thus the E_x-field must consist of a large field $\frac{a}{g}E_{xp}$, over the gap g, and zero field elsewhere. The homogeneous solution must have an E_x-field that looks as shown in Fig. S10.1.2a, or a potential that looks as shown in Fig. S10.1.2b. The homogeneous solution is derivable from a Laplacian potential Φ_h

$$\Phi_h = \sum A_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right) \tag{5}$$

which obeys all the boundary conditions, except at $y = -l$. Denote the potential Φ_h at $y = -l$ by

$$\Phi_h(y=-l) = aE_{xp}f(x) \tag{6}$$

so that the jump of $f(x)$ at $x = a/2$ is normalized to unity. Using the orthogonality properties of the sine function, we have

$$-\sinh\left(\frac{m\pi}{a}l\right)\frac{a}{2}A_m = aE_{xp}\int_{x=0}^{a} f(x)\sin\left(\frac{m\pi}{a}x\right)dx \tag{7}$$

It is clear that all odd orders integrate to zero, only even order terms remain. For an even order, except $m = 0$,

$$\begin{aligned}\int_{x=0}^{a} f(x)\sin\left(\frac{m\pi}{a}x\right) &= 2\int_{x=0}^{a/2}\frac{x}{a}\sin\left(\frac{m\pi}{a}x\right)dx\\ &= \frac{2a}{(m\pi)^2}\int_{u=0}^{m\pi/2} u\sin u\,du\\ &= \frac{2a}{(m\pi)^2}\left[-u\cos u\Big|_0^{m\pi/2} + \int_0^{m\pi/2}\cos u\,du\right]\\ &= \frac{a}{m\pi}(-1)^{\frac{m}{2}+1}\end{aligned} \tag{8}$$

Therefore

$$A_m = \begin{cases}\dfrac{2aE_{xp}}{m\pi\sinh\left(\frac{m\pi}{a}l\right)}(-1)^{m/2} & m\text{-even}\\ 0 & m\text{-odd}\end{cases} \tag{9}$$

The total field is

$$\mathbf{E} = \mu_o \frac{dK}{dt}\left\{ \mathbf{i}_x \left[y - l \sum_{\substack{m \\ \text{even}}} 2(-1)^{m/2} \frac{\sinh \frac{m\pi}{a} y}{\sinh \frac{m\pi}{a} l} \cos\left(\frac{m\pi}{a}x\right)\right] - \mathbf{i}_y l \sum_{\substack{m \\ \text{even}}} 2(-1)^{m/2} \frac{\cosh \frac{m\pi}{a} y}{\sinh \frac{m\pi}{2} l} \sin\left(\frac{m\pi}{a}x\right)\right]\right\} \tag{10}$$

10.1.3 (a) The magnetic field is uniform and z-directed

$$\mathbf{H} = \mathbf{i}_z K(t)$$

(b) The electric field is best analyzed in terms of a particular solution that satisfies the boundary conditions at $\phi = 0$ and $\phi = \alpha$, and a homogeneous solution that obeys the last boundary condition at $r = a$. The particular solution is ϕ-directed and is identical with the field encircling an axially symmetric uniform H-field

$$2\pi r E_{\phi p} = -\pi r^2 \mu_o \frac{dH_z}{dt} \tag{1}$$

and thus

$$E_{\phi p} = -\frac{r}{2}\mu_o \frac{dK}{dt} \tag{2}$$

The homogeneous solution is composed of the gradients of solutions to Laplace's equation

$$\Phi_h = \sum_n A_n (r/a)^{n\pi/\alpha} \sin\left(\frac{n\pi\phi}{\alpha}\right) \tag{3}$$

At $r = a$, these solutions must cancel the field along the boundary, except at and around $\phi = \alpha/2$. Because $\delta \ll \alpha$, we approximate the field $E_{\phi h}$ at $r = a$ as composed of a unit impulse function at $\phi = \alpha/2$ of content

$$\alpha E_{\phi p} = -\frac{a}{2}\alpha\mu_o \frac{dK}{dt} \tag{4}$$

and a constant field

$$E_{\phi h} = \frac{a}{2}\mu_o \frac{dK}{dt}$$

over the rest of the interval as shown in Fig. S10.1.3. From (3)

$$E_{\phi h}\Big|_{r=a} = -\frac{1}{a}\frac{\partial \Phi_h}{\partial \phi} = -\frac{1}{a}\sum_n (n\pi/\alpha) A_n \cos\left(\frac{n\pi\phi}{\alpha}\right) \tag{5}$$

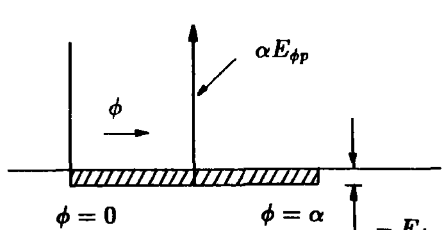


Figure S10.1.3

Here we take an alternative approach to that of 10.1.2. We do not have to worry about the part of the field over $0 < \phi < \alpha$, excluding the unit impulse function, because the line integral of $E_{\phi h}$ from $\phi = 0$ to $\phi = \alpha$ is assured to be zero (conservative field). Thus we need solely to expand the unit impulse at $\phi = \alpha/2$ in a series of $\cos\left(\frac{m\pi}{\alpha}\phi\right)$. By integrating

$$-\frac{1}{a}(m\pi/\alpha)A_m\frac{\alpha}{2} = \cos(m\pi/2)\alpha E_{\phi p} \tag{6}$$

where the right hand side is the integral through the unit impulse function. Thus,

$$A_m = \begin{cases} -\frac{2a}{m\pi}(-1)^{m/2}\alpha E_{\phi p} & \text{for } m \text{ even} \\ 0 & \text{for } m \text{ odd} \end{cases} \tag{7}$$

Therefore

$$\Phi_h = \sum_{m-\text{even}} \frac{a^2\alpha}{m\pi}(-1)^{m/2}\mu_o\frac{dK}{dt}(r/a)^{m\pi/\alpha}\sin\left(\frac{m\pi}{\alpha}\phi\right) \tag{8}$$

and

$$\mathbf{E} = -\mu_o\frac{dK}{dt}\frac{a}{2}\left\{\frac{r}{a} + \sum_{\substack{m=2\\ m-\text{even}}}^{\infty} 2(-1)^{m/2}(r/a)^{\frac{m\pi}{\alpha}-1}\left[\mathbf{i}_r\sin\left(\frac{m\pi\phi}{\alpha}\right) + \mathbf{i}_\phi\cos\left(\frac{m\pi\phi}{\alpha}\right)\right]\right\} \tag{9}$$

10.1.4 (a) The coil current produces an equivalent surface current $K = Ni/d$ and hence, because the coil is long

$$\mathbf{B} \simeq \mathbf{i}_z\mu_o\frac{Ni}{d} \tag{1}$$

(b) The (semi-) conductor is cylindrical and uniform. Thus $\mathbf{E}$ must be axisymmetric and, by symmetry, ϕ-directed. From Faraday's law applied to a circular contour of radius r inside the coil

$$2\pi r E_\phi = -\frac{dB_z}{dt}\pi r^2$$

and

$$E_\phi = -\frac{r}{2}\mu_o\frac{N}{d}\frac{di}{dt}$$

(c) The induced H-field is due to the circulating current density:

$$J_\phi = \sigma E_\phi = \omega\frac{\sigma r}{2}\mu_o\frac{N}{d}I\sin\omega t$$

where we have set

$$i(t) = I\cos\omega t$$

The H field will be axial, z- and ϕ-independent, by symmetry. (The z-"independence" follows from the fact that $d \gg b$.) From Ampère's law

$$\nabla \times \mathbf{H} = \mathbf{J}$$

we have

$$-\frac{dH_z}{dr} = J_\phi$$

and thus

$$H_{z\,\text{induced}} = -\omega\sigma\frac{r^2}{4}\mu_o\frac{N}{d}I\sin\omega t$$

For $H_{z\,\text{induced}} \ll H_{z\,\text{imposed}}$ for $r \leq b$

$$\frac{\omega\mu_o\sigma b^2}{4} \ll 1$$

10.1.5 (a) From Faraday's law

$$\nabla \times \mathbf{E}_p = -\frac{\partial}{\partial t}\mathbf{B} \tag{1}$$

and thus

$$\frac{\partial E_{yp}}{\partial x} = -\mu_o\frac{N}{d}\frac{di}{dt} \tag{2}$$

Therefore,

$$E_{yp} = -\mu_o\frac{N}{d}\left(x - \frac{b}{2}\right)\frac{di}{dt} \tag{3}$$

(b) We must maintain $\mathbf{E}\cdot\mathbf{n} = 0$ inside the material. Thus, adding the homogeneous solution, a gradient of a scalar potential Φ, we must leave $E_x = 0$ at $x = 0$ and $x = b$. Further, we must eliminate E_y at $y = 0$ and $y = a$. We need an infinite series

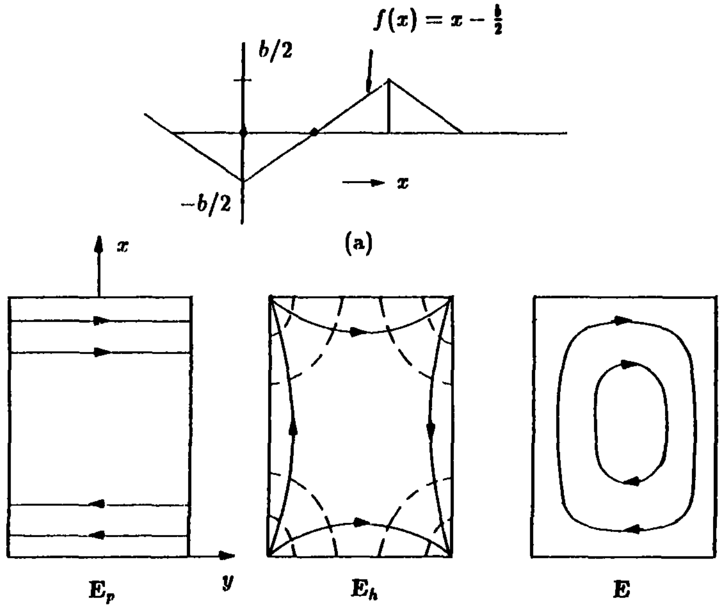
$$\Phi_h = \sum_n A_n \cos\left(\frac{n\pi}{b}x\right)\sinh\left(\frac{n\pi}{b}y\right) \tag{4}$$

with the electric field

$$\mathbf{E}_h = \sum_n A_n\left(\frac{n\pi}{b}\right)\left[\sin\left(\frac{n\pi}{b}x\right)\sinh\left(\frac{n\pi}{b}y\right)\mathbf{i}_x - \cos\left(\frac{n\pi}{b}x\right)\cosh\left(\frac{n\pi}{b}y\right)\mathbf{i}_y\right] \tag{5}$$

At $y = \pm a/2$

$$\begin{aligned} E_{yh} &= -\sum_n A_n\left(\frac{n\pi}{b}\right)\cos\left(\frac{n\pi}{b}x\right)\cosh\left(\frac{n\pi}{b}\frac{a}{2}\right) \\ &= -E_{yp} = \mu_o\frac{N}{d}\left(x - \frac{b}{2}\right)\frac{di}{dt} \end{aligned} \tag{6}$$



Set $-\mu_o \frac{N}{d}\frac{di}{dt}$ = positive number

(b)

Figure S10.1.5

We must expand the function shown in Fig. S10.1.5a into a cosine series. Thus, multiplying (6) by $\cos \frac{m\pi}{b}x$ and integrating from $x = 0$ to $x = b$, we obtain

$$-\frac{m\pi}{b}\frac{b}{2}A_m \cosh\left(\frac{m\pi}{2b}a\right) = \mu_o \frac{N}{d}\frac{di}{dt}\int_0^b \left(x - \frac{b}{2}\right)\cos\frac{m\pi}{b}x\,dx$$
$$= \begin{cases} -\mu_o \frac{N}{d}\frac{di}{dt}2\left(\frac{b}{m\pi}\right)^2 & m \text{ - odd} \\ 0 & m \text{ - even} \end{cases} \tag{7}$$

Solving for A_m

$$A_m = \begin{cases} \frac{4b^2/(m\pi)^3}{\cosh(m\pi a/2b)}\mu_o \frac{N}{d}\frac{di}{dt} & m \text{ - even} \\ 0 & m \text{ - odd} \end{cases} \tag{8}$$

The E-field is

$$\mathbf{E} = -\mu_o \frac{N}{d}\frac{di}{dt}\left\{\left(x - \frac{b}{2}\right)\mathbf{i}_y - \sum_{n-\mathrm{odd}} \frac{4b/(m\pi)^2}{\cosh(m\pi a/2b)}\right.$$
$$\left[\sin\left(\frac{n\pi}{b}x\right)\sinh\left(\frac{n\pi}{b}y\right)\mathbf{i}_x\right. \tag{9}$$
$$\left.\left. - \cos\left(\frac{n\pi}{b}x\right)\cosh\left(\frac{n\pi}{b}y\right)\mathbf{i}_y\right]\right\}$$

(c) See Fig. S10.1.5b.

10.2 NATURE OF FIELDS INDUCED IN FINITE CONDUCTORS

10.2.1 The approximate resistance of the disk is

$$R = \frac{1}{\sigma}\frac{2\pi a}{2}\frac{1}{a\Delta}$$

where we have taken half of the circumference as the length. The flux through the disk is [compare (10.2.15)]

$$\lambda = \int \mu_o \mathbf{H} \cdot d\mathbf{a} = \mu_o \frac{i_2}{2\pi a}\pi a^2$$

$$\lambda = \frac{\mu_o i_2 a}{2}$$

This is caused by the current i_2 so the inductance of the disk L_{22} is (using $N = 1$):

$$L_{22} = \frac{\mu_o a}{2}$$

The time constant is

$$\tau_m = \frac{L_{22}}{R} = \frac{\mu_o a}{2}\frac{\sigma\Delta}{\pi} = \frac{\mu_o a\Delta\sigma}{2\pi}$$

This is roughly the same as (10.2.17).

10.2.2 Live bone is fairly "wet" and hence conducting like the surrounding flesh. Current lines have to close on themselves. Thus, if one mounts a coil with its axis perpendicular to the arm and centered with the arm as shown in Fig. S10.2.2, circulating currents are set up. If perfect symmetry prevailed and the bone were precisely at center, then no current would flow along its axis. However, such symmetry does not exist and thus longitudinal currents are set up with the bone off center.

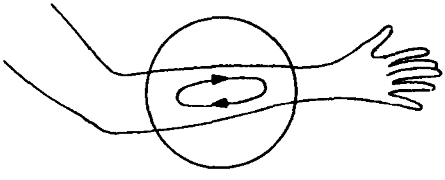


Figure S10.2.2

10.2.3 The field of coil (1) is, according to (10.2.8)

$$H_1 = \frac{N_1 i_1}{2\pi a} \tag{1}$$

The net field is

$$H_1 + H_{ind}$$

with $H_{ind} = K_\phi$ where K_ϕ is the ϕ directed current in the shell. The E-field is from Faraday's law, using symmetry

$$2\pi r E_\phi = -\mu_o \frac{d}{dt}(H_o + H_{ind})\pi r^2 \tag{2}$$

But

$$E_\phi\Big|_{r=a} = \frac{K_\phi}{\sigma\Delta} = \frac{H_{ind}}{\sigma\Delta} \tag{3}$$

and thus, for $r = a$

$$\frac{2H_{ind}}{\mu_o\sigma\Delta a} + \frac{d}{dt}H_{ind} = -\frac{d}{dt}H_o \tag{4}$$

In the sinusoidal steady state, using complex notation

$$H_o = \text{Re}\, \hat{H}_o e^{j\omega t} \quad \text{etc.} \tag{5}$$

and

$$\hat{H}_{ind} = -\frac{j\omega\tau_m}{j\omega\tau_m + 1}\hat{H}_o \tag{6}$$

where

$$\tau_m = \frac{\mu_o\sigma\Delta a}{2}$$

At small values of $\omega\tau_m$

$$|\hat{H}_{ind}| = \omega\tau_m |\hat{H}_o| \tag{7}$$

10.3 DIFFUSION OF AXIAL MAGNETIC FIELDS THROUGH THIN CONDUCTORS

10.3.1 The circulating current $K(t)$ produces an approximately uniform axial field

$$H_z = K(t) \tag{1}$$

As the field varies with time, there is an induced E-field obeying Faraday's law

$$\oint_C \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt}\int_S \mu_o \mathbf{H} \cdot d\mathbf{a} \tag{2}$$

The E-field drives the surface current

$$K = \Delta\sigma E \tag{3}$$

that must be constant along the circumference. Hence E must be constant. From (1), (2), and (3)

$$4aE = 4a\frac{K}{\Delta\sigma} = -\frac{d}{dt}\mu_o K a^2 \tag{4}$$

and thus

$$\frac{d}{dt}K + \frac{4}{\mu_o \sigma \Delta a}K = 0 \tag{5}$$

Thus

$$K(t) = K_o e^{-t/\tau_m} \tag{6}$$

with

$$\tau_m = \frac{\mu_o \sigma \Delta a}{4} \tag{7}$$

10.3.2 (a) This problem is completely analogous to 10.3.1. One has

$$H_z = K(t) \tag{1}$$

and, because $K = \Delta\sigma E$ must be constant along the surface, so that E must be constant

$$(2d + \sqrt{2}d)E = -\frac{d}{dt}\mu_o K(t)\frac{d^2}{2} \tag{2}$$

Therefore

$$(2 + \sqrt{2})\frac{K}{\Delta\sigma} = -\frac{d}{dt}(\mu_o K)\frac{d}{2} \tag{3}$$

or

$$\frac{dK}{dt} + \frac{K}{\tau_m} = 0 \tag{4}$$

with

$$\tau_m = \frac{\mu_o \sigma \Delta d}{2(2 + \sqrt{2})} \tag{5}$$

The solution for $J = K/\Delta$ is

$$J = J_o e^{-t/\tau_m} \tag{6}$$

(b) Since

$$\oint_{C_1} \mathbf{E} \cdot d\mathbf{s} = 0 \tag{7}$$

and the line integral along the surface is $\sqrt{2}dE$, we have

$$v + \sqrt{2}dE = 0 \tag{8}$$

$$v = -\sqrt{2}\frac{dK}{\Delta\sigma} = -\sqrt{2}\frac{dJ_o}{\sigma}e^{-t/\tau_m} \tag{9}$$

(c) Again from Faraday's law

$$\begin{aligned}\oint_{C_2} \mathbf{E} \cdot d\mathbf{s} = -v &= -\frac{d}{dt}\mu_o H_z \frac{d^2}{2} = -\mu_o \frac{d^2}{2}\frac{dK}{dt} \\ &= \mu_o \frac{1}{\tau_m}\frac{d^2}{2}\Delta J_o e^{-t/\tau_m} = (2+\sqrt{2})\frac{J_o d}{\sigma}e^{-t/\tau_m}\end{aligned} \tag{10}$$

10.3.3 (a) We set up the boundary conditions for the three uniform axial fields, in the regions $r < b, b < r < a, r > a$ (see Fig. S10.3.3).

$$H_o(t) - H_1(t) = -K_{\text{out}}(t) = -J_{\text{out}}\Delta = -\sigma E_{\text{out}}\Delta \tag{1}$$

$$H_1(t) - H_2(t) = -K_{\text{in}}(t) = -J_{\text{in}}\Delta = -\sigma E_{\text{in}}\Delta \tag{2}$$

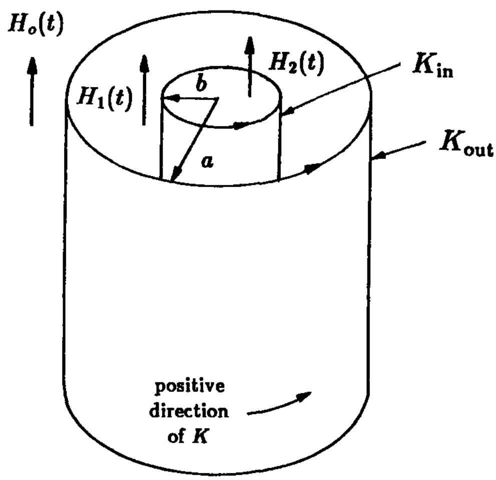


Figure S10.3.3

From the integral form of Faraday's law:

$$2\pi a E_{\text{out}} = -\mu_o \frac{d}{dt}\left[H_1(t))\pi(a^2 - b^2) + H_2(t)\pi b^2\right] \tag{3}$$

$$2\pi b E_{\text{in}} = -\mu_o \frac{d}{dt}\left[H_2(t)\pi b^2\right] \tag{4}$$

We can solve for E_{out} and E_{in} and substitute into (1) and (2)

$$H_o(t) - H_1(t) = \mu_o \frac{\sigma\Delta}{2}\left[\frac{a^2 - b^2}{a}\frac{dH_1(t)}{dt} + \frac{b^2}{a}\frac{dH_2(t)}{dt}\right] \tag{5}$$

$$H_1(t) - H_2(t) = \mu_o \frac{\sigma\Delta b}{2}\frac{dH_2(t)}{dt} \tag{6}$$

We obtain from (6)

$$\tau_m \frac{dH_2(t)}{dt} + H_2(t) - H_1(t) = 0 \tag{7}$$

where

$$\tau_m \equiv \frac{\mu_o \sigma \Delta b}{2}$$

From (5), after some rearrangement, we obtain:

$$\Rightarrow \tau_m \frac{b}{a}\frac{dH_2}{dt} + \tau_m\left(\frac{a}{b} - \frac{b}{a}\right)\frac{dH_1(t)}{dt} + H_1(t) = H_o(t) \tag{8}$$

(b) We introduce complex notation

$$H_o = H_m \cos\omega t = \text{Re}\left\{H_m e^{j\omega t}\right\} \tag{9}$$

Similarly H_1 and H_2 are replaced by $H_{1,2} = \text{Re}\left[\hat{H}_{1,2}e^{j\omega t}\right]$. We obtain two equations for the two unknowns $\hat{H}_1$ and $\hat{H}_2$:

$$-\hat{H}_1 + (1 + j\omega\tau_m)\hat{H}_2 = 0$$

$$\left[1 + j\omega\tau_m\left(\frac{a}{b} - \frac{b}{a}\right)\right]\hat{H}_1 + \frac{b}{a}j\omega\tau_m\hat{H}_2 = H_m$$

They can be solved in the usual way

$$\hat{H}_1 = \frac{\begin{vmatrix} 0 & 1 + j\omega\tau_m \\ H_m & \frac{b}{a}j\omega\tau_m \end{vmatrix}}{Det} = -\frac{(1 + j\omega\tau_m)H_m}{Det}$$

$$\hat{H}_2 = \frac{\begin{vmatrix} -1 & 0 \\ 1 + \omega\tau_m\left(\frac{a}{b} - \frac{b}{a}\right) & H_m \end{vmatrix}}{Det} = -\frac{H_m}{Det}$$

where Det is the determinant.

$$Det \equiv -\left\{\left[1 + j\omega\tau_m\left(\frac{a}{b} - \frac{b}{a}\right)\right](1 + j\omega\tau_m) + j\omega\tau_m\frac{b}{a}\right\}$$

10.3.4 (a) To the left of the sheet (see Fig. S10.3.4),

$$\mathbf{H} = K_o \mathbf{i}_z \tag{1}$$

To the right of the sheet

$$\mathbf{H} = K \mathbf{i}_z \tag{2}$$

Along the contour C_1, use Faraday's law

$$\oint_{C_1} \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{a} \tag{3}$$

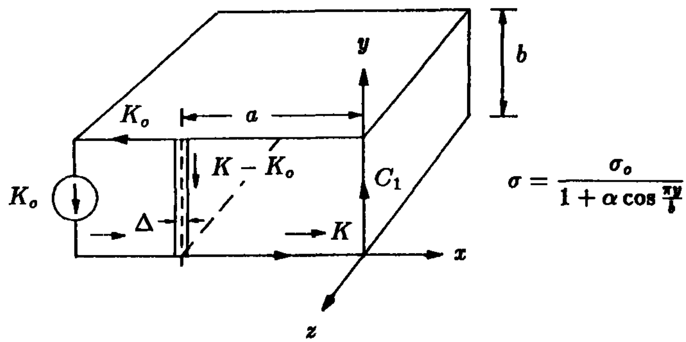


Figure S10.3.4

Along the three perfectly conducting sides of the conductor $\mathbf{E} = 0$. In the sheet the current $K - K_o$ is constant so that

$$\nabla \cdot J = 0 \Rightarrow \nabla \cdot (\sigma E) = 0 \tag{4}$$

$$\oint_{C_1} \mathbf{E} \cdot d\mathbf{s} = \int_{y=0}^{b} \left(\frac{K - K_o}{\Delta \sigma} \right) dy = -\mu_o ab \frac{dK}{dt} \tag{5}$$

$$\frac{K - K_o}{\Delta \sigma_o} \int_{y=0}^{b} \left(1 + \alpha \cos \frac{\pi y}{b} \right) dy = -\mu_o ab \frac{dK}{dt} \tag{6}$$

The integral yields b and thus

$$\frac{dK}{dt} + \frac{K}{\mu_o a \Delta \sigma_o} = \frac{K_o}{\mu_o a \Delta \sigma_o} \tag{7}$$

From (7) we can find K as a function of time for a given $K_o(t)$.

(b) The y-component of the electric field at $x = -a$ has a uniform part and a y-dependent part according to (5). The y-dependent part integrates to zero and hence is part of a conservative field. The uniform part is

$$E_{yp} b = -\frac{K - K_o}{\Delta \sigma_o} b = \mu_o ab \frac{dK}{dt} \tag{8}$$

This is the particular solution of Faraday's law

$$\frac{\partial E_{yp}}{\partial x} = -\mu_o \frac{\partial H_z}{\partial t} = -\mu_o \frac{dK}{dt} \tag{9}$$

with the integral

$$E_{yp} = -\mu_o x \frac{dK}{dt} \tag{10}$$

and indeed, at $x = -a$, we obtain (8). There remains

$$E_{yh} = -\frac{K - K_o}{\Delta\sigma_o} \alpha \cos\left(\frac{\pi y}{b}\right) \tag{11}$$

It is clear that this field can be found from the gradient of the Laplacian potential

$$\Phi = A \sin\left(\frac{\pi y}{b}\right) \sinh\left(\frac{\pi x}{b}\right) \tag{12}$$

that satisfies the boundary conditions on the perfect conductors. At $x = -a$

$$-\frac{\partial \Phi}{\partial y}\Big|_{x=-a} = \frac{\pi}{b} A \cos\frac{\pi y}{b} \sinh\left(\frac{\pi a}{b}\right) = -\frac{K - K_o}{\Delta\sigma_o} \alpha \cos\frac{\pi y}{b} \tag{13}$$

and thus

$$A = -\frac{\alpha b}{\pi \Delta\sigma_o} \frac{(K - K_o)}{\sinh(\pi a/b)} \tag{14}$$

10.4 DIFFUSION OF TRANSVERSE MAGNETIC FIELDS THROUGH THIN CONDUCTORS

10.4.1 (a) Let us consider an expanded view of the conductor (Fig. S10.4.1). At $y = \Delta$, the boundary condition on the normal component of $\mathbf{B}$ gives

$$\mathbf{i}_y \cdot [\mathbf{B}^a - \mathbf{B}^c] = 0 \tag{1}$$

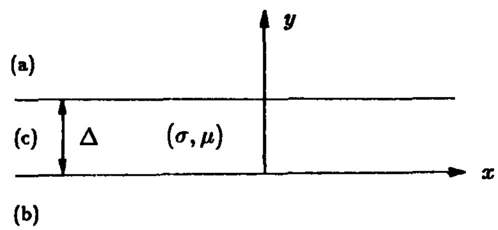


Figure S10.4.1

Therefore

$$B_y^a\Big|_{y=\Delta} - B_y^c\Big|_{y=\Delta} = 0 \tag{2}$$

At $y = 0$

$$B_y^c\Big|_{y=0} - B_y^b\Big|_{y=0} = 0 \tag{3}$$

Since the thickness, Δ, of the sheet is very small, we can assume that B is uniform across the sheet so that,

$$B_y^c\Big|_{y=\Delta} = B_y^c\Big|_{y=0} \tag{4}$$

Using (3) and (4) in (2),

$$B_y^a - B_y^b = 0 \tag{5}$$

From the continuity condition associated with Ampère's law

$$\mathbf{n} \times [\mathbf{H}^a - \mathbf{H}^b] = \mathbf{K}$$

Since

$$\mathbf{K} = K_z\mathbf{i}_z, \quad \mathbf{n} = \mathbf{i}_y,$$

$$-H_x^a + H_x^b = K_z \tag{6}$$

The current density $\mathbf{J}$ in the sheet is

$$J_z = \frac{K_z}{\Delta} \tag{7}$$

And so, from Ohm's law

$$E_z = \frac{K_z}{\Delta\sigma} \tag{8}$$

Finally from Faraday's law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{9}$$

Since only B_y matters (only time rate of change of flux normal to the sheet will induce circulating $\mathbf{E}$-fields) and $\mathbf{E}$ only has a z-component,

$$-\frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t}$$

From (8) therefore,

$$\frac{\partial}{\partial x}\left[\frac{K_z}{\Delta\sigma}\right] = \frac{\partial B_y}{\partial t}$$

and finally, from (6),

$$\frac{\partial}{\partial x}[H_x^a - H_x^b] = -\Delta\sigma\frac{\partial B_y}{\partial t} \tag{10}$$

(b) At $t = 0$ we are given $\mathbf{K} = \mathbf{i}_z K_o \sin\beta x$. Everywhere except within the current sheet, we have $\mathbf{J} = 0$

$$\Rightarrow \mathbf{H} = -\nabla\Psi$$

So from $\nabla \cdot \mu_o \mathbf{H} = 0$, we have

$$\nabla^2 \Psi = 0$$

Boundary conditions are given by (5) and (10) and by the requirement that the potential mut decay as $y \to \pm\infty$. Since H_x will match the $\sin \beta x$ dependence of the current, pick solutions with $\cos \beta x$ dependence

$$\Psi^{(a)} = A(t) \cos \beta x e^{-\beta y} \tag{11a}$$

$$\Psi^{(b)} = C(t) \cos \beta x e^{\beta y} \tag{11b}$$

$$\mathbf{H}^{(a)} = \beta A(t) \sin \beta x e^{-\beta y} \mathbf{i_x} + \beta A(t) \cos \beta x e^{-\beta y} \mathbf{i_y} \tag{12a}$$

$$\mathbf{H}^{(b)} = \beta C(t) \sin \beta x e^{\beta y} \mathbf{i_x} - \beta C(t) \cos \beta x e^{\beta y} \mathbf{i_y} \tag{12b}$$

From (5),

$$\mu_o \beta A(t) \cos \beta x e^{-\beta y} \big|_{y=0} + \mu_o \beta C(t) \cos \beta x e^{\beta y} \big|_{y=0} = 0$$

Therefore,

$$A(t) = -C(t) \tag{13}$$

From (10),

$$\frac{\partial}{\partial x} \left[\beta A(t) \sin \beta x e^{-\beta y} \big|_{y=0} - \beta C(t) \sin \beta x e^{\beta y} \big|_{y=0} \right] = -\Delta \sigma \mu_o \beta \cos \beta x e^{-\beta y} \big|_{y=0} \frac{dA(t)}{dt}$$

Using (13)

$$2\beta^2 A(t) \cos \beta x = -\Delta \sigma \mu_o \beta \cos \beta x \frac{dA(t)}{dt}$$

The cosines cancel and

$$\frac{dA(t)}{dt} + \frac{2\beta}{\Delta \sigma \mu_o} A(t) = 0 \tag{14}$$

The solution is

$$A(t) = A(0) e^{-t/\tau} \qquad \tau = \frac{\mu_o \Delta \sigma}{2\beta} \tag{15}$$

So the surface current, proportional to H_x according to (6), decays similarly as

$$\mathbf{K} = \mathbf{i_z} K_o \sin \beta x e^{-t/\tau} \qquad \tau = \frac{\mu_o \Delta \sigma}{2\beta}$$

10.4.2 (a) If the sheet acts like a perfect conductor (see Fig. S10.4.2), the component of $\mathbf{H}$ perpendicular to the sheet must be zero.

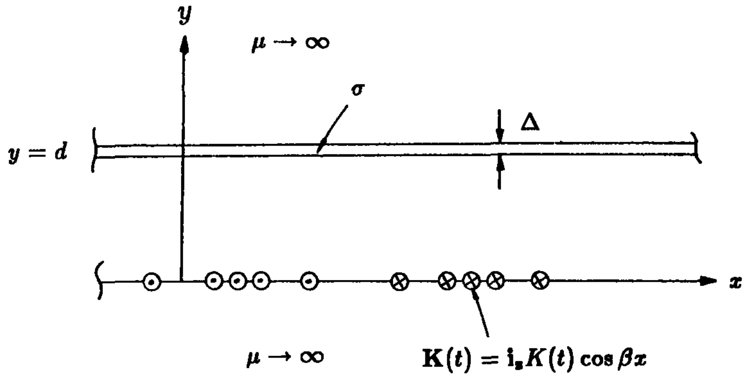


Figure S10.4.2

At $y = 0$ the magnetic field experiences a jump of the tangential component

$$\mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{K} \tag{1}$$

with $\mathbf{n} \parallel \mathbf{i_y}$ and $\mathbf{H}_2 = 0$,

$$H_x = -K(t) \cos \beta x \tag{2}$$

The field in the space $0 < y < d$ is the gradient of a Laplacian potential

$$\Psi = A \sin \beta x \cosh \beta (y - d) \tag{3}$$

The cosh is chosen so that H_y is zero at $y = d$:

$$\mathbf{H} = -A\beta [\cos \beta x \cosh \beta (y - d) \mathbf{i_x} + \sin \beta x \sinh \beta (y - d) \mathbf{i_y}] \tag{4}$$

Satisfying the boundary condition at $y = 0$

$$-A\beta \cos \beta x \cosh \beta d = -K(t) \cos \beta x \tag{5}$$

Therefore

$$A = \frac{K(t)}{\beta \cosh \beta d} \tag{6}$$

$$\Psi = \frac{K(t) \sin \beta x \cosh \beta (y - d)}{\beta \cosh \beta d} \tag{7}$$

(b) For $K(t)$ slowly varying, the magnetic field diffuses straight through so the sheet acts as if it were not there. The field "sees" $\mu \to \infty$ material and, therefore, has no tangential H

$$\Psi = A \sin \beta x \sinh \beta (y - d) \tag{8}$$

which satisfies the condition $H_x = 0$ at $y = d$. Indeed,

$$\mathbf{H} = -A\beta[\cos\beta x \sinh\beta(y-d)\mathbf{i_x} + \sin\beta x \cosh\beta(y-d)\mathbf{i_y}]$$

Matching the boundary condition at $y = 0$, we obtain

$$A = -\frac{K(t)}{\beta\sinh\beta d} \tag{9}$$

$$\Psi = -\frac{K(t)\sin\beta x \sinh\beta(y-d)}{\beta\sinh\beta d} \tag{10}$$

(c) Now solving for the general time dependence, we can use the previous results as a clue. Initially, the sheet acts like a perfect conductor and the solution (7) must apply. As $t \to \infty$, the sheet does not conduct, and the solution (10) must apply. In between, we must have a transition between these two solutions. Thus, postulate that the current $\mathbf{i_z}K_s(t)\cos\beta x$ is flowing in the top sheet. We have

$$K_s(t)\cos\beta x = \sigma\Delta E_z \tag{11}$$

Postulate the potential

$$\Psi = C(t)\frac{\sin\beta x\cosh\beta(y-d)}{\beta\cosh\beta d} - D(t)\frac{\sin\beta x\sinh\beta(y-d)}{\beta\sinh\beta d} \tag{12}$$

The boundary condition at $y = 0$ is

$$\begin{aligned} -\frac{\partial\Psi}{\partial x}\Big|_{y=0} = H_x\Big|_{y=0} &= -K(t)\cos\beta x \\ &= -C(t)\cos\beta x - D(t)\cos\beta x \end{aligned} \tag{13}$$

Therefore

$$C + D = K \tag{14}$$

At $y = d$

$$-\frac{\partial\Psi}{\partial x}\Big|_{y=d} = H_x\Big|_{y=d} = K_s(t)\cos\beta x = -C(t)\frac{\cos\beta x}{\cosh\beta d} \tag{15}$$

The current in the sheet is driven by the E-field induced by Faraday's law and is z-directed by symmetry

$$\begin{aligned} \frac{\partial E_z}{\partial y} = -\frac{\partial}{\partial t}\mu_o H_x = \mu_o&\frac{\cos\beta x\cosh\beta(y-d)}{\cosh\beta d}\frac{dC}{dt} \\ -\mu_o&\frac{\cos\beta x\sinh\beta(y-d)}{\sinh\beta d}\frac{dD}{dt} \end{aligned} \tag{16}$$

Therefore,

$$E_z = \frac{\mu_o\cos\beta x\sinh\beta(y-d)}{\beta\cosh\beta d}\frac{dC}{dt} - \mu_o\frac{\cos\beta x\cosh\beta(y-d)}{\beta\sinh\beta d}\frac{dD}{dt} \tag{17}$$

At $y = d$

$$E_z = -\mu_o \frac{1}{\beta \sinh \beta d} \cos \beta x \frac{dD}{dt} = \frac{K_s \cos \beta x}{\sigma \Delta} \tag{18}$$

Hence, combining (14), (15), and (18)

$$\cosh \beta d K_s = -C(t) = -K + D = -\frac{\mu_o \sigma \Delta}{\beta} \coth \beta d \frac{dD}{dt} \tag{19}$$

resulting in the differential equation

$$\frac{\mu_o \sigma \Delta}{\beta} \coth \beta d \frac{dD}{dt} + D = K \tag{20}$$

With K a step function

$$D = K_o \left[1 - e^{-t/\tau_m}\right] \tag{21}$$

where

$$\tau_m = \frac{\mu_o \sigma \Delta}{\beta} \coth \beta d \tag{22}$$

and

$$C = K_o e^{-t/\tau_m}$$

At $t = 0$, $D = 0$ and at $t = \infty$, $C = 0$. This checks with the previously obtained solutions.

10.4.3 (a) If the shell (Fig. S10.4.3) is thin enough it acts as a surface of discontinuity at which the usual boundary conditions are obeyed. From the continuity of the normal component of $\mathbf{B}$,

$$B_r^a - B_r^b = 0 \tag{1}$$

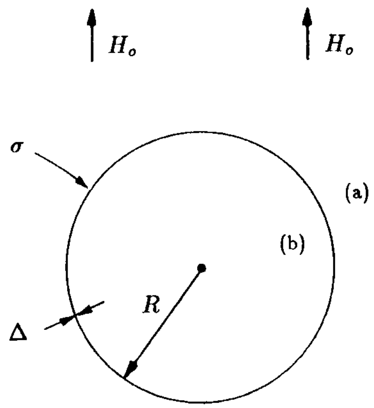


Figure S10.4.3

the continuity condition associated with Ampère's law

$$\mathbf{n} \times [\mathbf{H}^a - \mathbf{H}^b] = \mathbf{K} \tag{2}$$

use of Ohm's law

$$E = \frac{J}{\sigma} = \frac{K}{\Delta\sigma} \tag{3}$$

results in

$$H_\theta^a - H_\theta^b = K_\phi = \Delta\sigma E_\phi \tag{4}$$

The electric field obeys Faraday's law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5}$$

Only flux normal to the shell induces E in the sheet. By symmetry, $\mathbf{E}$ is ϕ-directed

$$(\nabla \times \mathbf{E})_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(E_\phi \sin\theta) = -\frac{\partial B_r}{\partial t} \tag{6}$$

And thus, at the boundary

$$\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}[\sin\theta[H_\theta^a - H_\theta^b] = -\mu_o\Delta\sigma\frac{\partial H_r}{dt} \tag{7}$$

(b) Set

$$H_o(t) = \text{Re}\left\{H_o e^{j\omega t}\right\}[\cos\theta \mathbf{i}_r - \sin\theta \mathbf{i}_\theta] \tag{8}$$

The H-field outside and inside the shell must be the gradient of a scalar potential

$$\hat{\Psi}_a = -H_o r\cos\theta + \frac{\hat{A}\cos\theta}{r^2} \tag{9}$$

$$\hat{\Psi}_b = \hat{C} r\cos\theta \tag{10}$$

$$\hat{H}_\theta^a = -H_o\sin\theta + \frac{\hat{A}}{r^3}\sin\theta \tag{11}$$

$$\hat{H}_\theta^b = \hat{C}\sin\theta \tag{12}$$

$$\hat{H}_r^a = H_o\cos\theta + \frac{2\hat{A}}{r^3}\cos\theta \tag{13}$$

$$\hat{H}_r^b = -\hat{C}\cos\theta \tag{14}$$

From (1)

$$B_r^a = B_r^b \Rightarrow H_o + \frac{2\hat{A}}{R^3} = -\hat{C} \tag{15}$$

Introducing (11), (12), and (13) into (7) we find

$$\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin^2\theta\left(-H_o + \frac{\hat{A}}{R^3} - \hat{C}\right)\right\} = -j\omega\mu_o\Delta\sigma\left\{H_o\cos\theta + \frac{2\hat{A}\cos\theta}{R^3}\right\} \tag{16}$$

from which we find A, using (15) to eliminate $\hat{C}$.

$$\hat{A} = -\frac{j\omega\mu_o\Delta\sigma R^4 H_o}{2(j\omega\mu_o\Delta\sigma R + 3)} \tag{17}$$

$\hat{A}$ provides the dipole term

$$\frac{\hat{m}}{4\pi} = \hat{A} = \frac{-j\omega\mu_o\Delta\sigma R^4 H_o}{2(j\omega\mu_o\Delta\sigma R + 3)} \tag{18}$$

and thus

$$\hat{m} = -\frac{j\omega\tau(2\pi R^3)\hat{H}_o)}{(1 + j\omega\tau)} \tag{19}$$

with

$$\tau = \frac{\mu_o\sigma\Delta R}{3}$$

(c) In the limit $\omega\tau \to \infty$, we find

$$\hat{m} \to -2\pi H_o R^3$$

as in Example 8.4.4.

10.4.4 (a) The field is that of a dipole of dipole moment $m = ia$

$$\Psi = \frac{ia}{4\pi r^2}\cos\theta \tag{1}$$

(b) The normal component has to vanish on the shell. We add a uniform field

$$\Psi = Ar\cos\theta + \frac{ia}{4\pi r^2}\cos\theta \tag{2}$$

The normal component of H at $r = R$ is

$$-\frac{\partial\Psi}{\partial r}\Big|_{r=R} = 0 = -\big(A - 2\frac{ia}{4\pi R^3}\big)\cos\theta$$

and thus

$$A = \frac{2ia}{4\pi R^3} \tag{3}$$

and

$$\Psi = \frac{ia}{4\pi R^2}\cos\theta\big(2\frac{r}{R} + \frac{R^2}{r^2}\big)$$

(see Fig. S10.4.4).

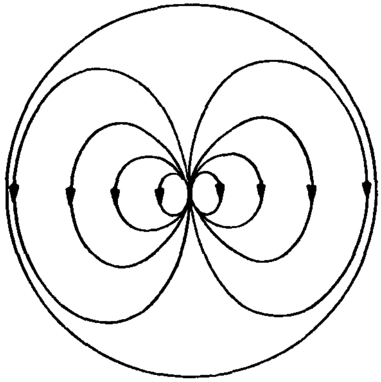


Figure S10.4.4

(c) There is now also an outside field. For $r < R$

$$\Psi = \frac{ia}{4\pi r^2}\cos\theta + A(t) r\cos\theta \tag{5}$$

For $r > R$,

$$\Psi = C(t)\frac{\cos\theta}{r^2} \tag{6}$$

The θ-components of $\mathbf{H}$ are

$$H_\theta = \frac{ia}{4\pi r^3}\sin\theta + A\sin\theta; \quad r < R \tag{7a}$$

and

$$H_\theta = \frac{C}{r^3}\sin\theta; \quad r > R \tag{7b}$$

The normal component at $r = R$ is

$$H_r = \left(\frac{2ia}{4\pi R^3} - A\right)\cos\theta \tag{8a}$$

and

$$H_r = \frac{2C}{R^3}\cos\theta \tag{8b}$$

With the boundary condition (7) of Prob. 10.4.3, we have

$$\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left[\sin^2\theta\left(\frac{C}{R^3} - \frac{ia}{4\pi R^3} - A\right)\right] = -\frac{2\mu_o\Delta\sigma}{R^3}\cos\theta\frac{dC}{dt} \tag{9}$$

From the continuity of the normal component of B, we find

$$\frac{2ia}{4\pi R^3} - A = 2\frac{C}{R^3} \tag{10}$$

The equation for C becomes

$$\frac{1}{R^4 \sin\theta}\frac{\partial}{\partial\theta}\left[\sin^2\theta\left(C - \frac{ia}{4\pi} + 2C - \frac{2ia}{4\pi}\right)\right] = -\frac{2\mu_o\Delta\sigma}{R^3}\cos\theta\frac{dC}{dt} \tag{11}$$

or

$$\tau_m\frac{dC}{dt} + C = \frac{ia}{4\pi} \tag{12}$$

with $\tau_m = \mu_o\sigma\Delta R/3$. If we consider the steady state, then

$$C = \text{Re}\left[\hat{C}e^{j\omega t}\right] \tag{13}$$

$$\hat{C} = \frac{1}{(1 + j\omega\tau_m)}\frac{ia}{4\pi} \tag{14}$$

$$\hat{A} = \frac{2ia}{4\pi R^3} - \frac{2C}{R^3} = \frac{2ia}{4\pi R^3}\frac{j\omega\tau_m}{1 + j\omega\tau_m} \tag{15}$$

Jointly with (5) and (6), this determines Ψ.

(d) When $\omega\tau_m \to \infty$, we have $\hat{C} \to 0$, no outside field and $\hat{A} = 2ia/4\pi R^3$ which checks with (3). When $\omega\tau_m \to 0$, we have no shield and $\hat{A} \to 0$. The shell behaves as if it were infinitely conducting in the limit $\omega\tau_m \to \infty$.

10.4.5 (a) If the current density varies so rapidly that the sheet is a perfect conductor, then it imposes the boundary condition (see Fig. S10.4.5),

$$\mathbf{n} \cdot \mu_o\mathbf{H} = 0 \quad \text{at} \quad r = b$$

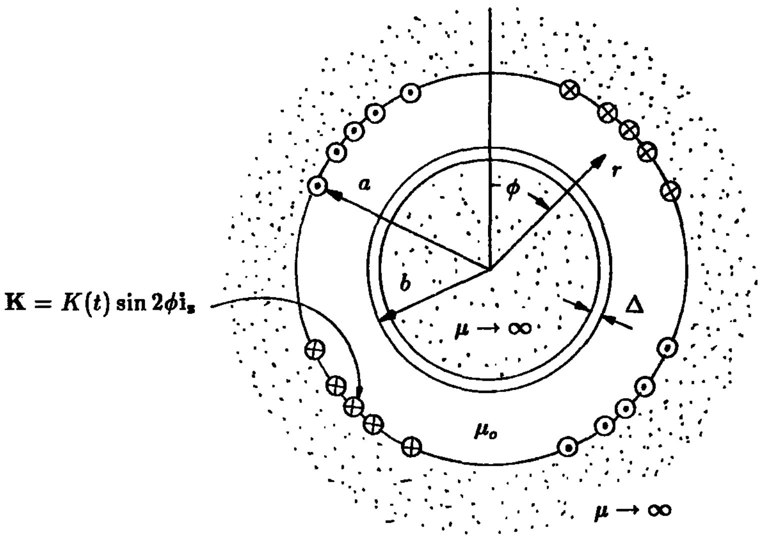


Figure S10.4.5

Inside the high μ material $\mathbf{H} = 0$ to keep $\mathbf{B}$ finite. So at $r = a$,

$$\mathbf{n} \times \mathbf{H} = \mathbf{K}$$

Therefore

$$-\mathbf{i}_z H_\phi = K(t) \sin 2\phi \mathbf{i}_z$$

Thus, the potential has to obey the boundary conditions

$$\frac{\partial \Psi}{\partial r} = 0 \quad \text{at} \quad r = b \tag{1}$$

$$-\frac{1}{r}\frac{\partial \Psi}{\partial \phi} = -K(t) \sin 2\phi \quad \text{at} \quad r = a \tag{2}$$

In order to satisfy (2), we must pick a $\cos 2\phi$ dependence for Ψ. To satisfy (1), one picks a $[(r/b)^2 + (b/r)^2] \cos 2\phi$ type solution. Guess

$$\Psi = A[(r/b)^2 + (b/r)^2] \cos 2\phi$$

Indeed,

$$\frac{\partial \Psi}{\partial r} = A\left[\frac{2r}{b^2} - \frac{2b^2}{r^3}\right] \cos 2\phi = 0 \quad \text{at} \quad r = b$$

$$\frac{\partial \Psi}{\partial \phi} = -A[(r/b)^2 + (b/r)^2] 2 \sin 2\phi$$

From (2),

$$\frac{A}{a}[(a/b)^2 + (b/a)^2] 2 \sin 2\phi = -K(t) \sin 2\phi$$

Therefore,

$$\Psi = -\frac{K(t)a}{2}\frac{[(r/b)^2 + (b/r)^2]}{[(a/b)^2 + (b/a)^2]} \cos 2\phi \tag{3}$$

(b) Now the current induced in the sheet is negligible, so all the field diffuses straight through. The sheet behaves as if it were not there at all. But at $r = b$ we have $\mu \to \infty$ material, so $H = 0$ inside. Also, since now there is no $\mathbf{K}$ at $r = b$, we must have

$$H_\phi = 0 \quad \text{at} \quad r = b$$

It is clear that the following potential obeys the boundary condition at $r = b$

$$\Psi = A[(r/b)^2 - (b/r)^2] \cos 2\phi$$

$$H_\phi = -\frac{1}{r}\frac{\partial \Psi}{\partial \phi} = \frac{A}{r}[(r/b)^2 - (b/r)^2] 2 \sin 2\phi = 0 \quad \text{at} \quad r = b$$

Again, applying (2)

$$\frac{A}{a}[(a/b)^2 - (b/a)^2] 2 \sin 2\phi = -K(t) \sin 2\phi$$

Thus,

$$\Psi = -\frac{K(t)a}{2}\frac{[(r/b)^2-(b/r)^2]}{[(a/b)^2-(b/a)^2]}\cos 2\phi \tag{4}$$

(c) At the sheet, the normal B is continuous assuming that Δ is small. Also, from Faraday's law,

$$\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt} \tag{5}$$

Since only a time varying field normal to the sheet will induce currents, we are only interested in $(\nabla \times \mathbf{E})_r$

$$\left(\frac{1}{r}\frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z}\right) = -\frac{dB_r}{dt}$$

By symmetry there is only a z-component of E

$$\frac{1}{r}\frac{\partial}{\partial \phi}E_z = -\frac{\partial B_r}{\partial t} \tag{6}$$

One should note, however, that there are some subtleties involve in the determination of the E-field. We do not attempt to match the boundary conditions on the coil surface. Such matching would require the addition of the gradient of a solution of Laplace's equation to $\mathbf{E}_p = \mathbf{i}_z E_z$. Such a field would induce surface charges in the conducting sheet, but otherwise not affect its current distribution. Remember that in MQS $\epsilon_o \frac{\partial \mathbf{E}}{\partial t}$ is ignored which means that the charging currents responsible for the build-up of charge are negligible compared to the MQS currents flowing in the systems.

From Ohm's law, $\mathbf{J} = \sigma \mathbf{E}$. But, $\mathbf{J} = \mathbf{K}/\Delta$.

$$\frac{1}{r}\frac{\partial}{\partial \phi}\frac{K_z}{\Delta\sigma} = -\frac{\partial B_r}{\partial t} \tag{7}$$

Applying the boundary conditions from Ampère's law,

$$\mathbf{n} \times [\mathbf{H}_{\text{gap}}|_{r=b} - \mathbf{H}_{\mu\to\infty}] = K_z \mathbf{i}_z$$

$$H_\phi|_{r=b} = K_z$$

So at $r = b$

$$\frac{1}{b}\frac{\partial}{\partial \phi}\frac{H_\phi}{\Delta\sigma} = -\mu_o \frac{\partial H_r}{\partial t} \tag{8}$$

Now guess a solution for Ψ in the gap. Since we have two current sources (the windings at $r = a$ and the sheet at $r = b$) and we do not necessarily know that they are in phase, we need to use superposition. This involves setting up the field due to each of the two sources individually

$$\Psi = \left\{A(t)\big[\underbrace{(r/a)^2 - (a/r)^2}\big] + C(t)\big[\underbrace{(r/b)^2 - (b/r)^2}\big]\right\}\cos 2\phi \tag{9}$$

Here, A represents the field due to the current at $r = b$, and C is produced by the current at $r = a$. Apply the boundary condition (2), at $r = a$. We find from the tangential H-field

$$\frac{2C(t)}{a}[(a/b)^2 - (b/a)^2] = -K(t)$$

Thus,

$$C(t) = \frac{-aK(t)}{2[(a/b)^2 - (b/a)^2]} \tag{10}$$

The normal and tangential components of $\mathbf{H}$ at $r = b$ are

$$H_r = -\{A(t)[\frac{2b}{a^2} + \frac{2a^2}{b^3}] + C(t)\frac{4}{b}\}\cos 2\phi \tag{11}$$

$$H_\phi = \{\frac{A(t)}{b}[(b/a)^2 - (a/b)^2]\}2\sin 2\phi \tag{12}$$

From (8)

$$\frac{1}{\mu_o \Delta\sigma b}\frac{\partial}{\partial\phi}[\frac{A(t)}{b}[(b/a)^2 - (a/b)^2]2\sin 2\phi] = \left\{(\frac{2b}{a^2} + \frac{2a^2}{b^3})\frac{dA(t)}{dt} + \frac{4}{b}\frac{dC}{dt}\right\}\cos 2\phi$$

Using (10),

$$\frac{dA(t)}{dt} + A(t)\frac{2}{\mu_o b\Delta\sigma}\frac{[(a/b)^2 - (b/a)^2]}{[(a/b)^2 + (b/a)^2]} = \frac{a}{[(a/b)^2 + (b/a)^2][(a/b)^2 - (b/a)^2]}\frac{dK(t)}{dt}$$

Simplifying,

$$\frac{dA(t)}{dt} + \frac{A(t)}{\tau} = D\frac{dK(t)}{dt} \tag{13}$$

$$\tau = \frac{\mu_o b\Delta\sigma}{2}\frac{[(a/b)^2 + (b/a)^2]}{[(a/b)^2 - (b/a)^2]} \tag{14}$$

$$D = \frac{a}{[(a/b)^2 + (b/a)^2][(a/b)^2 - (b/a)^2]} \tag{15}$$

dK/dt is a unit impulse function in time. The homogeneous solution for A is

$$A(t) \propto e^{-t/\tau} \tag{16}$$

and the solution that has the proper discontinuity at $t = 0$ is

$$A = DK_o \tag{17}$$

Using (10) and (17) in (9), we obtain,

$$\Psi = \frac{aK_o}{[(a/b)^2 - (b/a)^2]}\left\{\frac{[(r/a)^2 - (a/r)^2]}{[(a/b)^2 + (b/a)^2]}e^{-t/\tau} - \frac{(r/b)^2 - (b/r)^2}{2}\right\}\cos 2\phi$$

First consider the early time $t \to 0^+$

$$\Psi = \frac{aK_o}{[(a/b)^2 - (b/a)^2]}\left\{\frac{[(r/a)^2 - (a/r)^2]}{[(a/b)^2 + (b/a)^2]} - \frac{(r/b)^2 - (b/r)^2}{2}\right\}\cos 2\phi$$

Therefore

$$\Psi = \frac{-aK_o}{2}\left[\frac{(r/b)^2 + (b/r)^2}{(a/b)^2 + (b/a)^2}\right]\cos 2\phi$$

It is the same as if the surface currents spontaneously arose to buck out the field. At $t \to \infty$, $e^{-t/\tau} \to 0$

$$\Psi = \frac{-aK_o}{2}\left[\frac{(r/b)^2 - (b/r)^2}{(a/b)^2 - (b/a)^2}\right]\cos 2\phi$$

This is when the field has enough time to diffuse through the shell so it is as if no surface currents were present.

10.4.6 (a) When ω is very high, the sheet behaves as a perfect conductor, and (see Fig. S10.4.6)

$$\Psi = bK\frac{[(r/a) + (a/r)]}{[\frac{b}{a} + \frac{a}{b}]}\cos\phi \qquad (1)$$

Then, indeed, $\partial\Psi/\partial r = 0$ at $r = a$, and $-\frac{1}{b}\frac{\partial\Psi}{\partial\phi}$ accounts for the surface current K.

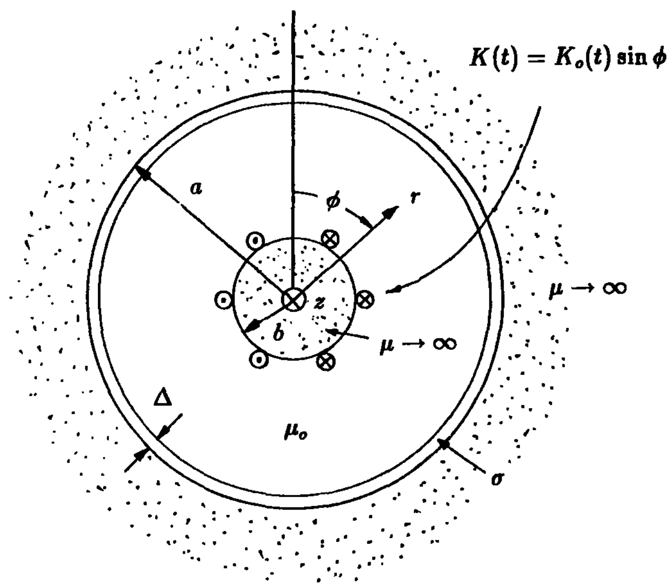


Figure S10.4.6

(b) When ω is very low, then $\partial\Psi/\partial\phi = 0$ at $r = a$ and

$$\Psi = bK\frac{[(r/a) - (a/r)]}{\left[\frac{b}{a} - \frac{a}{b}\right]}\cos\phi \tag{2}$$

(c) As before in Prob. 10.4.5, we superimpose the field caused by the two current distributions

$$\Psi = \left\{A(t)\left[\frac{r}{a} - \frac{a}{r}\right] + C(t)\left[\frac{r}{b} - \frac{b}{r}\right]\right\}\cos\phi \tag{3}$$

The r- and ϕ-components of the field are:

$$H_r = -\left\{A(t)\left[\frac{1}{a} + \frac{a}{r^2}\right] + C(t)\left[\frac{1}{b} + \frac{b}{r^2}\right]\right\}\cos\phi \tag{4}$$

$$H_\phi = \left\{\frac{A(t)}{r}\left[\frac{r}{a} - \frac{a}{r}\right] + \frac{C(t)}{r}\left[\frac{r}{b} - \frac{b}{r}\right]\right\}\sin\phi \tag{5}$$

At $r = b$,

$$H_\phi = K_o(t)\sin\phi \tag{6}$$

and thus

$$A(t) = \frac{K_o(t)b}{\frac{b}{a} - \frac{a}{b}} \tag{7}$$

At $r = a$,

$$-H_\phi\Big|_{r=a} = K_s$$

where K_s is the current in the sheet. From (7) of the preceding problem solution, we have at $r = a$

$$-\frac{1}{a}\frac{\partial}{\partial\phi}\frac{H_\phi}{\Delta\sigma} = -\mu_o\frac{\partial H_r}{\partial t} \tag{8}$$

Thus, using (4) and (5) in (8):

$$\frac{C(t)}{a}\left[\frac{a}{b} - \frac{b}{a}\right] = -\mu_o\Delta\sigma a\left\{\frac{2}{a}\frac{dA(t)}{dt} + \frac{dC(t)}{dt}\left[\frac{1}{b} + \frac{b}{a^2}\right]\right\} \tag{9}$$

Replacing A through (7) we obtain

$$\frac{dC}{dt} + \frac{\left[\frac{a}{b} - \frac{b}{a}\right]C(t)}{\mu_o\Delta\sigma a\left[\frac{a}{b} + \frac{b}{a}\right]} = \frac{2b}{(a/b)^2 - (b/a)^2}\frac{dK_o(t)}{dt} \tag{10}$$

Thus

$$\frac{dC}{dt} + \frac{C(t)}{\tau} = D\frac{dK}{dt} \tag{11}$$

with

$$\tau = \mu_o a\Delta\sigma\frac{\left[\frac{a}{b} + \frac{b}{a}\right]}{\left[\frac{a}{b} - \frac{b}{a}\right]} \tag{12}$$

$$D = \frac{2b}{(a/b)^2 - (b/a)^2}$$

The solution for a step of $K_o(t)$ is

$$C = DK_o e^{-t/\tau} \tag{13}$$

$$C(t) = DK_o e^{-t/\tau} = \frac{2bK_o}{(a/b)^2 - (b/a)^2} e^{-t/\tau}$$

Combining all the expressions gives the final answer:

$$\Psi = \frac{K_o b}{\frac{b}{a} - \frac{a}{b}} \left\{ \left[\frac{r}{a} - \frac{a}{r}\right] - 2\frac{\left[\frac{r}{b} - \frac{b}{r}\right]}{\left[\frac{a}{b} + \frac{b}{a}\right]} e^{-t/\tau} \right\} \cos\phi$$

For very short times $t/\tau \ll 1$, one has

$$\Psi = \frac{K_o b}{\frac{b}{a} - \frac{a}{b}} \left[\frac{r}{a} - \frac{a}{r} - 2\frac{\frac{r}{b} - \frac{b}{r}}{\frac{a}{b} + \frac{b}{a}}\right] \cos\phi = \frac{K_o b}{\frac{b}{a} + \frac{a}{b}} \left(\frac{r}{a} + \frac{a}{r}\right) \cos\phi$$

which is the same as (1). For very long times $\exp -t/\tau = 0$ and one obtains (2).

10.5 MAGNETIC DIFFUSION LAWS

10.5.1 (a) We first list the five equations (10.5.1)-(10.5.5)

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{10.5.1}$$

$$\mathbf{J} = \sigma \mathbf{E} \tag{10.5.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \mu \mathbf{H} \tag{10.5.3}$$

$$\nabla \cdot \mu \mathbf{H} = 0 \tag{10.5.4}$$

$$\nabla \cdot \mathbf{J} = 0 \tag{10.5.5}$$

Take the curl of (10.5.3) and use the identity

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla\nabla \cdot \mathbf{F} - \nabla^2 \mathbf{F} \tag{1}$$

also note that

$$\nabla \cdot \mathbf{J} = \nabla \cdot \sigma \mathbf{E} = \sigma \nabla \cdot \mathbf{E} = 0 \tag{2}$$

because σ is uniform. Therefore,

$$-\nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} \nabla \times \mu \mathbf{H} \tag{3}$$

or

$$-\nabla^2(\mathbf{J}/\sigma) = -\mu\frac{\partial}{\partial t}\mathbf{J} \tag{4}$$

(b) Since $\mathbf{J} = \mathbf{i}_z J_z$, equation (b) follows immediately from (4). We now use (10.5.3)

$$\nabla\times(\mathbf{J}/\sigma) = -\frac{\partial}{\partial t}\mu\mathbf{H}$$

But

$$\nabla\times(\mathbf{J}/\sigma) = \frac{1}{\sigma}\nabla\times(\mathbf{i}_z J_z(x,y)) = \frac{1}{\sigma}\big(\mathbf{i}_x\frac{\partial}{\partial y}J_z - \mathbf{i}_y\frac{\partial}{\partial x}J_z\big)$$

and thus

$$\frac{\partial\mathbf{H}}{\partial t} = -\frac{\partial}{\partial y}\big(\frac{J_z}{\sigma\mu}\big)\mathbf{i}_x + \frac{\partial}{\partial x}\big(\frac{J_z}{\sigma\mu}\big)\mathbf{i}_y$$

10.6 MAGNETIC DIFFUSION TRANSIENT RESPONSE

10.6.1 The expressions for H_z and J_y obey the diffusion equation, no matter what signs are assigned to the coefficients. The summations cancel the field $-K_p x/b$ and current density K_p/b respectively, at $t = 0$ and eventually decay. If one turns off a drive from a steady state, the current density is initially uniform, equal to K_p/b and the field is equal to $-K_p x/b$ and then decays. But, the symmations with reversed signs have precisely that behavior.

10.6.2 (a) The magnetic field is

$$\mathbf{H} = \mathbf{i}_z H_z = K_p \tag{1}$$

and there is no E-field, nor J within the block.

(b) When the current-source is suddenly turned off, the H-field cannot disappear instantaneously; the current returns through the conducting block, but still circulates in the perfect conductor around the block. For this boundary value problem we must change the eigenfunctions. At $x = 0$, the field remains finite, because there is a circulation current terminating it. Thus we have, instead of (10.6.15),

$$H_z = \sum_{n-\text{odd}}^{\infty} C_n\cos\big(\frac{n\pi}{2b}x\big)e^{-t/\tau_n} \tag{2}$$

with the decay times

$$\tau_n = \frac{4\mu\sigma b^2}{(n\pi)^2} \tag{3}$$

Initially, H_z is uniform, and thus, using orthogonality

$$\int_{-b}^{0} H_z\cos\frac{m\pi}{2b}x\,dx = K_p\frac{2b}{m\pi}\sin\frac{m\pi}{2} = \frac{b}{2}C_m \tag{4}$$

and thus

$$C_m = (-1)^{\frac{m-1}{2}}\left(\frac{4}{m\pi}\right)K_p; \quad m \text{ odd} \tag{5}$$

$$H_z = \sum_{n-\text{odd}} (-1)^{\frac{n-1}{2}} \frac{4}{n\pi} K_p \cos\left(\frac{n\pi}{2b}x\right)e^{-t/\tau_n}$$

The current density is

$$J_y = -\frac{\partial H_z}{\partial x} = \frac{2}{b}\Sigma(-1)^{\frac{n-1}{2}} K_p \sin\left(\frac{n\pi}{2b}x\right)e^{-t/\tau_n}$$

If we pick a new origin at $x' = x + b$, then

$$\begin{aligned}\sin\left(\frac{n\pi}{2b}x\right) &= \sin\left(\frac{n\pi}{2b}x' - \frac{n\pi}{2}\right) = -\cos\frac{n\pi}{2b}x' \sin\left(\frac{n\pi}{2}\right)\\ &= -(-1)^{\frac{n-1}{2}}\cos\left(\frac{n\pi}{2b}x'\right) \quad \text{for } n \text{ odd}\end{aligned}$$

Interestingly, we find

$$J_y = -\frac{2K_p}{b}\sum_{n-\text{odd}} \cos\left(\frac{n\pi}{2b}x'\right)e^{-t/\tau_n}$$

At $t = 0$ this is the expansion of a unit impulse function at $x' = 0$ of content $-2K_p$. All the current now flows through a thin sheet at the end of the block. The factor of 2 comes in because the problem has been solved as a symmetric problem at $x' = 0$, and thus half of the current "flows" in the "imagined" other half.

10.7 SKIN EFFECT

10.7.1 (a) In order to find the impedance, we need to know the voltage v, the complex current being $\hat{K}_s$. The voltage is (see Fig. 10.7.2)

$$v = aE_y \tag{1}$$

and, from Faraday's law

$$\frac{\partial \hat{E}_y}{\partial x} = -j\omega\mu \hat{H}_z \tag{2}$$

From (2) and (10.7.10)

$$E_y = \frac{j\omega\mu\delta}{(1+j)} \frac{\left(e^{(1+j)\frac{x}{\delta}} + e^{-(1+j)\frac{x}{\delta}}\right)}{\left(e^{(1+j)\frac{b}{\delta}} - e^{-(1+j)\frac{b}{\delta}}\right)}\hat{K}_s \tag{3}$$

and thus the impedance is at $x = -b$

$$Z = \frac{a\hat{E}_y}{d\hat{K}_s} = \frac{ja\omega\mu\delta}{d(1+j)} \frac{e^{(1+j)\frac{b}{\delta}} + e^{-(1+j)\frac{b}{\delta}}}{e^{(1+j)\frac{b}{\delta}} - e^{-(1+j)\frac{b}{\delta}}} \tag{4}$$

But the factor in front is

$$\frac{ja\omega\mu\delta}{d(1+j)} = \frac{a(1+j)}{d\sigma\delta} \tag{5}$$

(b) When $b \ll \delta$, we can expand the exponentials and obtain

$$\begin{aligned} Z &= \frac{a(1+j)}{d\sigma\delta} \frac{1+(1+j)\frac{b}{\delta} + 1 - (1+j)\frac{b}{\delta}}{1+(1+j)\frac{b}{\delta} - 1 + (1+j)\frac{b}{\delta}} \\ &= \frac{a(1+j)}{d\sigma\delta} \frac{1}{(1+j)\frac{b}{\delta}} = \frac{a}{d\sigma b} \end{aligned} \tag{6}$$

(c) When $b \gg \delta$, then we need retain only the exponential $\exp[(1+j)b/\delta]$ with the result:

$$Z = \frac{a(1+j)}{d\sigma\delta} \tag{7}$$

so that

$$\mathrm{Re}(Z) = \frac{a}{d\sigma\delta}$$

This looks like (6) with b replaced by δ.

10.7.2 (a) When the block is shorted, we have to add the two solutions $\exp \pm(1+j)\frac{x}{\delta}$ so that they add at the termination. Indeed, if we set

$$\hat{H}_z = A[e^{-(1+j)\frac{x}{\delta}} + e^{(1+j)\frac{x}{\delta}}] \tag{1}$$

then the E-field is, from

$$\frac{\partial \hat{E}_y}{\partial x} = -j\omega\mu\hat{H}_z \tag{2}$$

and thus through integration

$$\hat{E}_y = \frac{j\omega\mu\delta}{(1+j)}\hat{A}[e^{-(1+j)\frac{x}{\delta}} - e^{(1+j)\frac{x}{\delta}}] \tag{3}$$

and is indeed zero at $x = 0$. In order to obtain $\hat{H}_z = \hat{K}_s$ at $x = -b$ we adjust $\hat{A}$ so that

$$\hat{H}_z = \hat{K}_s \frac{e^{(1+j)\frac{x}{\delta}} + e^{-(1+j)\frac{x}{\delta}}}{e^{(1+j)\frac{b}{\delta}} + e^{-(1+j)\frac{b}{\delta}}} \tag{4}$$

(b) The high frequency distribution is governed by the $\exp -(1+j)\frac{x}{\delta}(x<0)$ and thus

$$\hat{H}_z \simeq \hat{K}_s \frac{e^{-(1+j)\frac{x}{\delta}}}{e^{(1+j)\frac{b}{\delta}}} = \hat{K}_s e^{-(1+j)\frac{x-b}{\delta}} \tag{5}$$

This is the same expression as the one obtained from (10.7.10) by neglecting $\exp -(1+j)\frac{x}{\delta}$ and $\exp(1+j)b/\delta$.

(c) The impedance is obtained from (3) and (4)

$$\frac{a\hat{E}_y}{dK_s}\Big|_{x=-b} = \frac{a(1+j)}{d\sigma\delta}\frac{e^{(1+j)b/\delta} - e^{-(1+j)b/\delta}}{e^{(1+j)b/\delta} + e^{-(1+j)b/\delta}}$$