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SOLUTIONS TO CHAPTER 5

5.1 PARTICULAR AND HOMOGENEOUS SOLUTIONS TO POISSON'S AND LAPLACE'S EQUATIONS

5.1.1 The particular solution must satisfy Poisson's equation in the region of interest. Thus, it is the first term in the potential, associated with the charge in the upper half plane. What remains satisfies Laplace's equation everywhere in the region of interest, so it can be called the homogeneous solution. It might also be made part of the particular solution.

5.1.2 (a) The charge density follows from Poisson's equation.

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_o} \Rightarrow \rho = \rho_o \cos \beta x \quad (1)$$

(b) The first term does not satisfy Laplace's equation and indeed was responsible for the charge density, (1). Thus, it can be taken as the particular solution and the remainder as the homogeneous solution. In that case,

$$\Phi_p = \frac{\rho_o \cos \beta x}{\epsilon_o \beta^2}; \quad \Phi_h = -\frac{\rho_o \cos \beta x \cosh \beta y}{\epsilon_o \beta^2 \cosh \beta a} \quad (2)$$

and the homogeneous solution must satisfy the boundary conditions

$$\Phi_h(y = -a) = \Phi_h(y = a) = -\frac{\rho_o \cos \beta x}{\epsilon_o \beta^2} \quad (3)$$

(c) We could just have well taken the total solution as the particular solution.

$$\Phi_p = \Phi; \quad \Phi_h = 0 \quad (4)$$

in which case the homogeneous solution must be zero on the boundaries.

5.1.3 (a) Because the second derivatives with respect to y and z are zero, the Laplacian reduces to the term on the left. The right side is the negative of the charge density divided by the permittivity, as required by Poisson's equation.

(b) With C_1 and C_2 integration coefficients, two integrations of (b) give

$$\Phi = -\frac{4\rho_o}{d^2\epsilon_o} \frac{(x-d)^4}{12} + C_1x + C_2 \quad (1)$$

Evaluation of this expression at each of the boundaries then serves to determine the coefficients

$$C_1 = \frac{V}{d} - \frac{\rho_o d}{3\epsilon_o}; \quad C_2 = \frac{\rho_o d^2}{2\epsilon_o} \quad (2)$$

and hence the given potential.

- (c) From the derivation it is clear that the Laplacian of the first term accounts for all of the charge density while that of the remaining terms is zero.
- (d) On the boundaries, the homogeneous solution, which must cancel the potential of the particular solution on the boundaries, must be (d).

- 5.1.4 (a) The derivatives with respect to y and z are by definition zero, so Poisson's equation reduces to

$$\frac{d^2\Phi}{dx^2} = -\frac{\rho_o}{\epsilon_o} \sin\left(\frac{\pi x}{d}\right) \quad (1)$$

- (b) Two integrations of (1) give

$$\Phi = \frac{\rho_o d^2}{\epsilon_o \pi^2} \sin\left(\frac{\pi x}{d}\right) + C_1 x + C_2 \quad (2)$$

and evaluation at the boundaries determines the integration coefficients.

$$C_2 = 0; \quad C_1 = v/d \quad (3)$$

It follows that the required potential is

$$\Phi = \frac{\rho_o d^2}{\epsilon_o \pi^2} \sin\left(\frac{\pi x}{d}\right) + \frac{V x}{d} \quad (4)$$

- (c) From the derivation, the first term in (4) accounts for the charge density while the remaining terms have no second derivative and hence no Laplacian. Thus, the first term must be included in the particular solution while the remaining term can be defined as the homogeneous solution.

$$\Phi_p = \frac{\rho_o d^2}{\epsilon_o \pi^2} \sin\left(\frac{\pi x}{d}\right); \quad \Phi_h = \frac{V x}{d} \quad (5)$$

- (d) In the case of (c), it follows that the boundary conditions satisfied by the homogeneous solution are

$$\Phi_h(0) = -\Phi_p(0) = 0; \quad \Phi_h(d) = V - \Phi_p(d) = V \quad (6)$$

- 5.1.5 (a) There is no charge density, so the potential must satisfy Laplace's equation. $\mathbf{E} = (-v/d)\mathbf{i}_x = -\partial\Phi/\partial z$

$$\nabla^2\Phi = \frac{\partial}{\partial z}\left(\frac{\partial\Phi}{\partial z}\right) = 0 \quad (1)$$

- (b) The surface charge density on the lower surface of the upper electrode follows from applying Gauss' continuity condition to the interface between the highly

conducting metal and the free space just below. Because the field is zero in the metal,

$$\sigma_s = \epsilon_o[0 - E_x^b] = \frac{\epsilon_o v}{d} \quad (2)$$

- (c) The capacitance follows from the integration of the surface charge density over the surface of the electrode having the potential v . That amounts to multiplying (2) by the area A of the electrode.

$$q = A\sigma_s = \frac{\epsilon_o A}{d} v = CV \quad (3)$$

- (d) Enclose the upper electrode by the surface S having the volume V and the integral form of the charge conservation law is

$$\oint_S \mathbf{J} \cdot \mathbf{n} da + \frac{d}{dt} \int_V \rho dV = 0 \quad (4)$$

Contributions to the first term are confined to where the wire carrying the total current i into the volume passes through S . By definition, the second term is the total charge, q , on the electrode. Thus, (4) becomes

$$-i + \frac{dq}{dt} = 0 \quad (5)$$

Introduction of (3) into this expression then gives the current

$$i = C \frac{dv}{dt} \quad (6)$$

- 5.1.6 (a) Well away from the edges, the fields between the plates are the potential difference divided by the spacings. Thus, they are as given.
- (b) The surface charge densities on the lower surface of the upper electrode and on the upper plus lower surfaces of the middle electrode are, respectively

$$\sigma_1 = \epsilon_o(0 + E_1) = \frac{\epsilon_o v_1}{2d}; \quad \sigma_m = \epsilon_o(0 + E_m) = \epsilon_o(v_1 - v_2)/d \quad (1)$$

$$\sigma_{mm} = -\epsilon_o(v_1 - v_2)/d + \epsilon_o v_2/d \quad (2)$$

Thus, the total charge on these electrodes is these quantities multiplied by the respective plate areas

$$q_1 = w[\sigma_1(L - l) + \sigma_m l] \quad (3)$$

$$q_2 = \epsilon_o l w \sigma_m \quad (4)$$

These are the expressions summarized in matrix notation by (a).

5.2 UNIQUENESS OF SOLUTIONS OF POISSON'S EQUATION

5.3 CONTINUITY CONDITIONS

5.3.1 (a) In the plane $y = 0$, the respective potentials are

$$\Phi^a(0) = V \cos \beta x = \Phi^b(0) \quad (1)$$

and are therefore equal.

(b) The tangential fields follow from the given potentials.

$$E_x^a = \beta V \sin \beta x e^{-\beta y}; \quad E_x^b = \beta V \sin \beta x e^{\beta y} \quad (2)$$

Evaluated at $y = 0$, these are also equal. That is, if the potential is continuous in a given plan, then so also is its slope in any direction within that plane.

(c) From Gauss' continuity condition applied to the plane $y = 0$,

$$\sigma_s = -\epsilon_o \left[\frac{\partial \Phi^a}{\partial y} - \frac{\partial \Phi^b}{\partial y} \right]_{y=0} = 2\beta V \cos \beta x \quad (3)$$

and this is the given surface charge density.

5.3.2 (a) The y dependence is not given. Thus, given that $\mathbf{E} = -\nabla\Phi$, only the x and z derivatives and hence x and z components of \mathbf{E} can be found. These are the components of \mathbf{E} tangential to the surface $y = 0$. If these components are to be continuous, then to within a constant so must be the potential in the plane $y = 0$.

(b) For this particular potential,

$$E_x = -\beta V \cos \beta x \sin \beta z; \quad E_z = -\beta V \sin \beta x \cos \beta z \quad (1)$$

If these are to be the tangential components of \mathbf{E} on both sides of the interface, then the $x-z$ dependence of the potential from which they were derived must also be continuous (within a constant that must be zero if the electric field normal to the interface is to remain finite).

5.4 SOLUTIONS TO LAPLACE'S EQUATION IN CARTESIAN COORDINATES

- 5.4.1 (a) The given potential satisfies Laplace's equation. Evaluated at either $x = 0$ or $y = 0$ it is zero, as required by the boundary conditions on these boundaries. At $x = a$, it has the required potential, as it does at $y = a$ as well. Thus, it is the required potential.
- (b) The plot of equipotentials and lines of electric field intensity is obtained from Fig. 4.1.3 by cutting away that part of the plot that is outside the boundaries at $x = a, y = a, x = 0$ and $y = 0$. Note that the distance between the equipotentials along the line $y = a$ is constant, as it must be if the potential is to have a linear distribution along this surface. Also, note that except for the special point at the origin (where the field intensity is zero anyway), the lines of electric field intensity are perpendicular to the zero potential surfaces. This is as it must be because there is no component of the field tangential to an equipotential.

- 5.4.2 (a) The potentials on the four boundaries are

$$\begin{aligned}\Phi(a, y) &= V(y + a)/2a; & \Phi(-a, y) &= V(y - a)/2a \\ \Phi(x, a) &= V(x + a)/2a; & \Phi(x, -a) &= V(x - a)/2a\end{aligned}\quad (1)$$

- (b) Evaluation of the given potential on each of the four boundaries gives the conditions on the coefficients

$$\begin{aligned}\Phi(\pm a, y) &= \frac{V}{2a}y \pm \frac{V}{2} = \pm Aa + By + C + Dxy \\ \Phi(x, \pm a) &= \frac{V}{2a}x \pm \frac{V}{2} = Ax \pm Ba + C + Dxy\end{aligned}\quad (2)$$

Thus, $A = B = V/2a, C = 0$ and $D = 0$ and the equipotentials are straight lines having slope -1 .

$$\Phi = \frac{V}{2a}(x + y)\quad (3)$$

- (c) The electric field intensity follows as being uniform and having x and y components of equal magnitude.

$$\mathbf{E} = -\nabla\Phi = -\frac{V}{2a}(\mathbf{i}_x + \mathbf{i}_y)\quad (4)$$

- (d) The sketches of the potential, (3), and field intensity, (4), are as shown in Fig. S5.4.2.

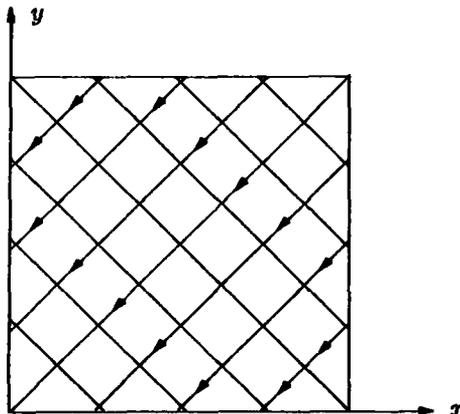


Figure S5.4.2

- (e) To make the potential zero at the origin, $C = 0$. Evaluation at $(x, y) = (0, a)$ where the potential must also be zero shows that $B = 0$. Similarly, evaluation at $(x, y) = (a, 0)$ shows that $A = 0$. Evaluation at $(x, y) = (a, a)$ gives $D = V/2a^2$ and hence the potential

$$\Phi = \frac{V}{2a^2}xy \quad (5)$$

Of course, we are not guaranteed that the postulated combination of solutions to Laplace's equation will satisfy the boundary conditions everywhere. However, evaluation of (5) on each of the boundaries shows that it does. The associated electric field intensity is

$$\mathbf{E} = -\nabla\Phi = -\frac{V}{2a^2}(yi_x + xi_y) \quad (6)$$

The equipotentials and lines of field intensity are as shown by Fig. 4.1.3 inside the boundaries $x = \pm a$ and $y = \pm a$.

- 5.4.3**
- (a) The given potential, which has the form of the first term in the second column of Table 5.4.1, satisfies Laplace's equation. It also meets the given boundary conditions on the boundaries enclosing the region of interest. Therefore, it is the required potential.
- (b) In identifying the equipotential and field lines of Fig. 5.4.1 with this configuration, note that $k = \pi/a$ and that the extent of the plot that is within the region of interest is between the zero potentials at $x = -\pi/2k$ and $x = \pi/2k$. The plot is then adapted to representing our potential distribution by multiplying each of the equipotentials by V_0 divided by the potential given on the plot at $(x, y) = (0, b)$. Note that the field lines are perpendicular to the walls at $x = \pm a/2$.

- 5.4.4 (a) Write the solution as the sum of two, each meeting zero potential conditions on three of the boundaries and the required sinusoidal distribution on the fourth.

$$\Phi = V_o \sin\left(\frac{\pi x}{a}\right) \frac{\sinh(\pi y/a)}{\sinh(\pi)} + V_o \sin\frac{\pi y}{a} \frac{\sinh\left[\frac{\pi}{a}(a-x)\right]}{\sinh(\pi)} \quad (1)$$

- (b) The associated electric field is

$$\mathbf{E} = -\frac{V_o \pi}{a \sinh(\pi)} \left\{ \left[\cos(\pi x/a) \sinh(\pi y/a) - \sin(\pi y/a) \cosh\left[\frac{\pi}{a}(a-x)\right] \right] \mathbf{i}_x + \left[\sin(\pi x/a) \cosh(\pi y/a) + \cos(\pi y/a) \sinh\left[\frac{\pi}{a}(a-x)\right] \right] \mathbf{i}_y \right\} \quad (2)$$

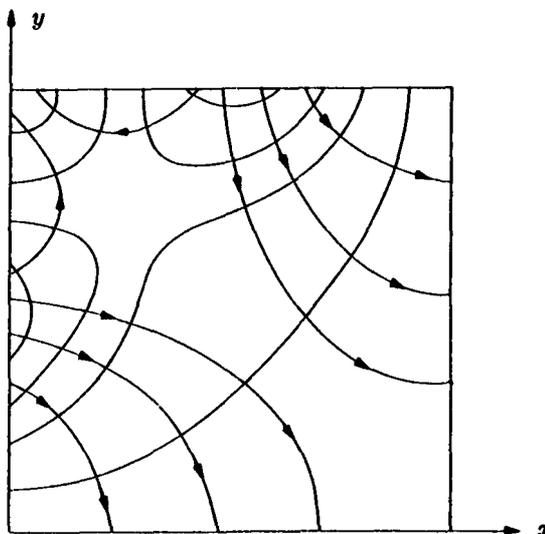


Figure S5.4.4

- (c) A sketch of the equipotentials and field lines is shown in Fig. S5.4.4.

- 5.4.5 (a) The given potential, which has the form of the second term in the second column of Table 5.4.1, satisfies Laplace's equation. The electrodes have been shaped and constrained in potential to match the potential. For example, between $y = -b$ and $y = b$, we obtain the y coordinate of the boundary $\eta(x)$ as given by (a) by setting (b) equal to the potential v of the electrode, $y = \eta$ and solving for η .
- (b) The electric field follows from (b) as $\mathbf{E} = -\nabla\Phi$.
- (c) The potential given by (b) and field given by (c) have the same (x, y) dependence as that represented by Fig. 5.4.2. To adjust the numbers given on the plot for the potentials, note that the potential at the location $(x, y) = (0, a)$ on the upper electrode is v . Thus, to make the plot fit this situation, multiply

each of the given potentials by v divided by the potential given on the plot at the location $(x, y) = (0, a)$.

- (d) The charge on the electrode is found by enclosing it by a surface S and using Gauss' integral law. To make the integration over the surface enclosing the electrode convenient, the surface is selected as enclosing the electrode in an arbitrary way in the field free region above the electrode, passing through the slits in the planes $x = \pm l$ to the y equal zero plane and closing in the $y = 0$ plane. Thus, with y_1 defined as the height of the electrode at its left and right extremities, the net charge is

$$\begin{aligned} q &= d\epsilon_o \int_{y=0}^{y_1} -E_x(x = -l) dy + d\epsilon_o \int_{y=0}^{y_1} E_x(x = l) dy \\ &+ d\epsilon_o \int_{x=-l}^l -E_y(y = 0) dx \\ &= -\frac{vd\pi\epsilon_o}{2b \sinh\left(\frac{\pi a}{2b}\right)} \left[\int_0^{y_1} -\sin\frac{\pi l}{2b} \sinh\frac{\pi y}{2b} dy \right. \\ &+ \int_0^{y_1} -\sin\frac{\pi l}{2b} \sinh\frac{\pi y}{2b} dy \\ &\left. + \int_{-l}^l -\cos\frac{\pi x}{2b} dx \right] \end{aligned} \quad (2)$$

Note that

$$\sinh ky_1 = \frac{\sinh ka}{\cos kl}; \quad -\sinh^2 ky + \cosh^2 ky = 1 \quad (3)$$

and (2) becomes the given result.

- (e) Conservation of charge for a surface enclosing the electrode through which the wire carrying the current i passes requires that $i = dq/dt$. Thus, given the result of (d) and the voltage dependence, (e) follows.

- 5.4.6** (a) Reversing the potentials on the lower electrodes turns the potential from an even to an odd function of y . Thus, the potential takes the form of the first term in the second column of Table 5.4.1.

$$\Phi = A \cosh\left(\frac{\pi y}{2b}\right) \cos\frac{\pi x}{2b} \quad (1)$$

To make the potential be v at $(x, y) = (0, a)$, the coefficient is adjusted so that

$$\Phi = v \cos kx \frac{\cosh ky}{\cosh ka}; \quad k \equiv \frac{\pi}{2b} \quad (2)$$

The shape of the upper electrode in the range between $x = -b$ and $x = b$ is then obtained by solving (2) with $\Phi = v$ and $y = \eta$ for η .

$$\eta = \frac{1}{k} \cosh^{-1} \left[\frac{\cosh ka}{\cos kx} \right] \quad (3)$$

(b) The electric field intensity follows from (2) as

$$\mathbf{E} = -\frac{vk}{\cosh ka} [-\sin(kx) \cosh(ky)\mathbf{i}_x + \cos kx \sinh ky\mathbf{i}_y] \quad (4)$$

(c) The equipotentials and field lines are as shown by Fig. 5.4.2. To adjust the given potentials, multiply each by v divided by the potential given from the plot at the location $(x, y) = (0, a)$.

(d) The charge on the electrode segment is obtained by using Gauss' integral law with a surface that encloses the electrode. This surface is arbitrary in the field free region above the electrode. For convenience, it passes through the slits to the $y = 0$ plane in the planes $x = \pm l$ and closes in the $y = 0$ plane. Note that there is no electric field perpendicular to this latter surface, so the only contributions to the surface integration come from the surfaces at $x = \pm l$.

$$\begin{aligned} q &= 2d\epsilon_o \int_0^y \left[\frac{vk}{\cosh ka} \sin(kl) \cosh(ky) \right] dy \\ &= \frac{2d\epsilon_o v}{\cosh ka} \sin kl \sinh ky_1 \end{aligned} \quad (5)$$

With the use of the identities

$$\cosh(ky_1) = \frac{\cosh ka}{\cos kl}; \quad \cosh^2 ky_1 - \sinh^2 ky_1 = 1 \quad (6)$$

(5) becomes

$$q = Cv = \frac{2d\epsilon_o v}{\cosh ka} \sin kl \sqrt{\left[\frac{\cosh(ka)}{\cos kl} \right]^2 - 1} \quad (7)$$

(e) From conservation of charge,

$$i = C \frac{dv}{dt} = -CV_o \omega \sin \omega t$$

5.5 MODAL EXPANSIONS TO SATISFY BOUNDARY CONDITIONS

5.5.1 (a) The solutions superimposed by the infinite series of (a) are chosen to be zero in the planes $x = 0$ and $x = b$ and to be the linear combination of exponentials in the y direction that are zero at $y = b$. To evaluate the coefficients, multiply both sides by $\sin(m\pi x/a)$ and integrate from $x = 0$ to $x = a$

$$\int_0^a \Phi(x, 0) \sin\left(\frac{m\pi x}{a}\right) dx = \sum_{n=1}^{\infty} A_n \sinh\left(-\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) dx \quad (1)$$

The integral on the right is zero except for $m = n$, in which case the integral of $\sin^2(n\pi x/a)$ over the interval $x = 0$ to $x = a$ gives the average value of $1/2$ multiplied by the length a , $a/2$. Thus, (1) can be solved for the coefficient A_m , to obtain (b) as given (if $m \rightarrow n$).

- (b) In the specific case where the distribution is as given, the integration of (b) gives

$$\begin{aligned} A_n &= \frac{2}{a \sinh\left(-\frac{n\pi b}{a}\right)} \int_{a/4}^{3a/4} V_1 \sin\left(\frac{n\pi x}{a}\right) dx \\ &= \frac{2V_1}{n\pi \sinh\left(\frac{n\pi b}{a}\right)} \left[\cos\left(\frac{n\pi x}{a}\right) \right]_{a/4}^{3a/4} \end{aligned} \quad (2)$$

which becomes (c) as given.

- 5.5.2 (a) This problem illustrates how the modal approach can be applied to finding the solutions in a rectangular region for arbitrary boundary conditions on all four of the boundaries. In general, four infinite series would be used, each with zero potential on three of the walls and with coefficients to match the potential boundary condition on the fourth wall. Here, the potential is zero on two of the walls, so only two infinite series are used. The first is zero in the planes $y = 0$, $y = b$ and $x = a$ and, because the potential is constant in the plane $x = 0$, has coefficients that are as given by (5.5.8). (The roles of a and b are reversed relative to those in the section for this first term and the minus sign results because the potential is being matched at $x = 0$. Note that the argument of the sinh function is negative within the region of interest.) The coefficients of the second series are similarly determined. (This time, the roles of x and y and of a and b are as in the section discussion, but the surface where the uniform potential is imposed is at $y = 0$ rather than $y = b$.)
- (b) The surface charged density on the wall at $x = a$ is

$$\sigma_s = \epsilon_o [-E_x(x = a)] = -\epsilon_o \frac{\partial \Phi}{\partial x}(x = a) \quad (1)$$

Evaluation using (a) results in (b).

- 5.5.3 (a) For arbitrary distributions of potential in the plane $y = 0$ and $x = 0$, the potential is taken as the superposition of series that are zero on all but these planes, respectively.

$$\begin{aligned} \Phi &= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left[\frac{n\pi}{a}(y - b)\right] \\ &\quad + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi y}{b}\right) \sinh\left[\frac{n\pi}{b}(x - a)\right] \end{aligned} \quad (1)$$

The first of these series must satisfy the boundary condition in the plane $y = 0$,

$$\Phi(x = 0) = \sum_{n=1}^{\infty} A_n \sinh\left(-\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi x}{a}\right) \quad (2)$$

where

$$\Phi(x, 0) = \begin{cases} 2V_a x/a; & 0 < x < a/2 \\ 2V_a(1 - x/a); & a/2 < x < a \end{cases} \quad (3)$$

Multiplication of both sides of (2) by $\sin(m\pi x/a)$ and integration from $x = 0$ to $x = a$ gives

$$\begin{aligned} \frac{2V_a}{a} \int_0^{a/2} x \sin\left(\frac{m\pi x}{a}\right) dx + 2V_a \int_{a/2}^a \sin\left(\frac{m\pi x}{a}\right) dx \\ - \frac{2V_a}{a} \int_{a/2}^a x \sin\left(\frac{m\pi x}{a}\right) dx \\ = A_m \frac{a}{2} \sinh\left(-\frac{m\pi b}{a}\right) \end{aligned} \quad (4)$$

Integration, solution for $A_m \rightarrow A_n$ then gives $A_n = 0$, n even and for n odd

$$A_n = -\frac{8V_a \sin\left(\frac{n\pi}{2}\right)}{n^2 \pi^2 \sinh\left(\frac{n\pi b}{a}\right)} \quad (5)$$

Evaluation on the boundary at $x = 0$ leads to a similar term with the roles of V_a and a replaced by those of V_b and b , respectively. Thus, $B_n = 0$ for n even and for n odd

$$B_n = -\frac{8V_b \sin\left(\frac{n\pi}{2}\right)}{n^2 \pi^2 \sinh\left(\frac{n\pi a}{b}\right)} \quad (5)$$

(b) The surface charge density in the plane $y = b$ is

$$\begin{aligned} \sigma_s = \epsilon_0 [-E_y(y = b)] = \epsilon_0 \frac{\partial \Phi}{\partial y}(y = b) \\ = \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \left[A_n \left(\frac{n\pi}{a}\right) \sin\left(\frac{n\pi x}{a}\right) - B_n \left(\frac{n\pi}{b}\right) \sinh\left[\left(\frac{n\pi}{b}\right)(x - a)\right] \right] \end{aligned} \quad (6)$$

where A_n and B_n are given by (5) and (6).

5.5.4 (a) Far to the left, the system appears as a parallel plate capacitor. A uniform field satisfies both Laplace's equation and the boundary conditions.

$$\mathbf{E} = -\frac{V}{d} \mathbf{i}_y \Rightarrow \Phi_a = \frac{Vy}{d} \quad (1)$$

(b) Because the uniform field part of this solution, Φ_a , satisfies the conditions far to the left, the additional part must go to zero there. However, the first term produces a field tangential to the right boundary which must be cancelled by the second term. Thus, conditions on the second term are that it also satisfy Laplace's equation and the boundary conditions as given

- (c) Because of the homogeneous boundary conditions in the $y = 0$ and $y = d$ planes, the solution is selected as being sinusoidal in the y direction. Because the region extends to infinity in the $-x$ direction, exponential solutions are used in that direction, with the sign of the exponent arranged to assure decay in the $-x$ direction.

$$\Phi_b = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{d}\right) e^{n\pi x/d} \quad (2)$$

The coefficients are determined by the requirement on this part of the potential at $x = 0$.

$$-\frac{Vy}{d} = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{d}\right) \quad (3)$$

Multiplication by $\sin(m\pi y/d)$, integration from $y = 0$ to $y = d$, solution for A_m and replacement of A_m by A_n gives

$$A_n = \frac{2V}{n\pi} \cos n\pi = \frac{2V}{n\pi} (-1)^n \quad (4)$$

The sum of the potentials of (1) and (2) with the coefficient given by (4) is (e).

- (d) The equipotential lines must be those of a plane parallel capacitor, (1), far to the left where the associated field lines are y directed and uniform. Because the boundaries are either at the potential V or at zero potential to the right, these equipotential lines can only terminate in the gap at $(x, y) = (0, d)$, where the potential makes an abrupt excursion from the zero potential of the right electrode to the potential V of the top electrode. In this local, the potential lines converge and become radially symmetric. The boundaries are themselves equipotentials. The electric field, which is perpendicular to the equipotentials and directed from the upper electrode toward the bottom and right electrodes, can then be pictured as shown by Fig. 6.6.9c turned upside down.

- 5.5.5 (a) The potential far to the left is that of a plane parallel plate capacitor. It takes the form $Ax + B$, with the coefficients adjusted to meet the boundary conditions at $y = 0$ and $x = a$.

$$\Phi(y \rightarrow -\infty) \rightarrow \Phi_a = \frac{V_0}{2} \left(1 - \frac{2x}{a}\right) \quad (1)$$

- (b) With the total potential written as

$$\Phi = \Phi_a + \Phi_b \quad (2)$$

the potential Φ_b can be used to make the total potential satisfy the boundary condition at $y = 0$. Because the first part of (2) satisfies Laplace's equation and the boundary conditions far to the left, the second part must go to zero there. Thus, it is taken as a superposition of solutions to Laplace's equation

that are zero in the planes $y = 0$ and $y = a$ (so that the potential there as given by the first term is not disturbed) and that decay exponentially in the $-y$ direction.

$$\Phi_b = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) e^{n\pi y/a} \quad (3)$$

At $y = 0$, $\Phi(x, 0) = \Phi_d(x)$. Thus, $\Phi_b(x, 0) = \Phi_d(x) - \Phi_a(x)$ and evaluation of (3) at $y = 0$, multiplication by $\sin(m\pi x/a)$ and integration from $x = 0$ to $x = a$ gives

$$\int_0^a \left[\Phi_d(x) - \frac{V_o}{2} \left(1 - \frac{2x}{a}\right) \right] \sin \frac{m\pi x}{a} dx = A_m \frac{a}{2} \quad (4)$$

from which it follows that

$$A_n = \frac{2}{a} \int_0^a \Phi_d(x) \sin\left(\frac{n\pi x}{a}\right) dx - \begin{cases} \frac{2V_o}{n\pi}; & n \text{ even} \\ 0; & n \text{ odd} \end{cases} \quad (5)$$

Thus, the potential between the plates is

$$\Phi = \frac{V_o}{2} \left(1 - \frac{2x}{a}\right) + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) e^{n\pi y/a} \quad (6)$$

where A_n is given by (5).

5.5.6

The potential is taken as the sum of two, the first being zero on all but the boundary at $x = a$ where it is $V_o y/a$ and the second being zero on all but the boundary at $y = a$, where it is $V_o x/a$. The second solution is obtained from the first by interchanging the roles of x and y . For the first solution, we take

$$\Phi_1 = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \frac{\sinh\left(\frac{n\pi y}{a}\right)}{\sinh n\pi} \quad (1)$$

The coefficients follow by evaluating this expression at $x = a$, multiplying by $\sin(m\pi y/a)$ and integrating from $y = 0$ to $y = a$.

$$\int_0^a \frac{V_o x}{a} \sin\left(\frac{n\pi x}{a}\right) dx = A_n (a/2) \quad (2)$$

Thus,

$$A_n = -\frac{2V_o}{n\pi} (-1)^n \quad (3)$$

The first part of the solution is given by substituting (3) into (1). It follows that the total solution is

$$\Phi = \sum_{n=1}^{\infty} -\frac{2V_o}{n\pi} \frac{(-1)^n}{\sinh(n\pi)} \left[\sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right) + \sin\left(\frac{n\pi y}{a}\right) \sinh\left(\frac{n\pi x}{a}\right) \right] \quad (4)$$

- 5.5.7 (a) The total potential is zero at $y = 0$ and so also is the first term. Thus, Φ_1 must be zero as well at $y = 0$. The first term satisfies the boundary condition at $y = b$, so Φ_1 must be zero there as well. However, in the planes $x = 0$ and $x = a$, the first term has a potential Vy/b that must be cancelled by the second term so that the sum of the two terms is zero. Thus, Φ_1 must satisfy the conditions summarized in the problem statement.
- (b) To satisfy the conditions at $x = 0$ and $x = a$, the y dependence is taken as $\sin(n\pi y/b)$. The product form x dependence is a linear combination of exponentials having arguments $(n\pi y/b)$. Because the boundary conditions in the $x = 0$ and $x = a$ planes are even about the plane $x = a/2$, this linear combination is taken as being the cosh function displaced so that its origin is at $x = a/2$.

$$\Phi = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{b}\right) \cosh\left[\frac{n\pi}{b}\left(x - \frac{a}{2}\right)\right] \quad (1)$$

Thus, if the boundary condition is satisfied at $x = a$, it is at $x = 0$ as well. Evaluation of (1) at $x = a$, multiplication by $\sin(m\pi y/b)$ and integration from $y = 0$ to $y = b$ then gives an expression that can be solved for A_m and hence A_n

$$A_n = \frac{2V(-1)^n}{n\pi \cosh(n\pi a/2b)} \quad (2)$$

In terms of these coefficients, the desired solution is then

$$\Phi = \frac{Vy}{b} + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{b}\right) \cosh\left[\frac{n\pi}{b}\left(x - \frac{a}{2}\right)\right] \quad (3)$$

5.6 SOLUTIONS TO POISSON'S EQUATION WITH BOUNDARY CONDITIONS

- 5.6.1 The potential is the sum of two homogenous solutions that satisfy Laplace's equation and a third inhomogeneous solution that makes the potential satisfy Poisson's equation for each point in the volume. This latter solution, which follows from assuming $\Phi_p = \Phi_p(y)$ and integration of Poisson's equation, is arranged to give zero potential on each of the boundaries, so it is up to the first two to satisfy the boundary conditions. The first solution is zero at $y = 0$, has the same x dependence as the wall at $y = d$ and has a coefficient that has been adjusted so that the magnitude of the potential matches that at $y = d$. The second solution is zero at $y = d$ (the displaced sinh function is a linear combination of the sinh and cosh functions in column 2 of Table 5.4.1) and so does not disturb the potential already satisfied by the first term at that boundary. At $y = 0$, where the first term has been arranged to make no contribution, it has the same y dependence as the potential in the $y = 0$ plane and has its coefficient adjusted so that it has the correct magnitude on that boundary as well.

- 5.6.2** The particular solution is found by assuming that the particular potential is only a function of y and integration of Poisson's equation twice. With the two integration coefficients adjusted to make the potential of this particular solution zero on each of the boundaries, it is the same as the last term in (a) of Prob. 5.6.1. Thus, the homogeneous solution must be zero at $y = 0$, suggesting that it has a sinh function y dependence. The x dependence of the potential at $y = d$ then suggests the x dependence of the potential be made $\sin(kx)$. With the coefficient of this homogeneous solution adjusted so that the condition at $y = d$ is satisfied, the desired potential is

$$\Phi = \Phi_o \sinh kx \frac{\sinh ky}{\sinh kd} - \frac{\rho_o}{2\epsilon_o} y(y-d) \quad (1)$$

- 5.6.3** (a) In the volume, Poisson's equation is satisfied by a potential that is independent of y and z ,

$$\nabla^2 \Phi_p = \frac{\partial^2 \Phi_p}{\partial x^2} = -\frac{\rho_o}{\epsilon_o} \cos k(x-\delta) \quad (1)$$

Two integrations give the particular solution

$$\Phi_p = \frac{\rho_o}{\epsilon_o k^2} \cos k(x-\delta) \quad (2)$$

$$\mathbf{E}_p = \frac{\rho_o}{\epsilon_o k} \sin k(x-\delta) \mathbf{i}_x \quad (3)$$

- (b) The boundary conditions at $y = \pm d/2$ are

$$E_x = E_{px} + E_{hx} = E_o \cos kx \quad (4)$$

Because the configuration is symmetric with respect to the $x-z$ plane, use $\cosh(ky)$ as the y dependence. Thus, in view of the two x dependencies, the homogeneous potential is assumed to take the form

$$\Phi_h = [A \sin kx + B \cos k(x-\delta)] \cosh ky \quad (5)$$

The condition of (4) then requires that

$$E_{xh} = -[A \cos kx - B \sin k(x-\delta)] k \cosh ky \quad (6)$$

and it follows from the fact that at $y = d/2$ that (3) + (6) = (4)

$$A = -E_o/k \cosh(kd/2); \quad B = -\rho_o/\epsilon_o k^2 \cosh(kd/2) \quad (7)$$

so that the total potential is as given by (d) of the problem statement.

- (c) First note that because of the symmetry with respect to the z plane, there is no net force in the y direction. In integrating ρE_x over the volume, note that E_x is

$$E_x = \frac{\rho_o}{\epsilon_o k} \sin k(x - \delta) + \frac{\cosh kh}{\cosh\left(\frac{kd}{2}\right)} \left[E_o \cos kx - \frac{\rho_o}{\epsilon_o k} \sin k(x - \delta) \right] \quad (8)$$

In view of the x dependence of the charge density, only the second term in this expression makes a contribution to the integral. Also, $\rho = \rho_o \cos k(x - \delta) = \rho_o [\cos k\delta \cos kx - \sin k\delta \sin kx]$ and only the first of these two terms makes a contribution also.

$$\begin{aligned} f_x &= \int_0^{2\pi/k} \int_{-d/2}^{d/2} \rho_o \cos k\delta \cos kx \frac{\cosh ky}{\cosh\left(\frac{kd}{2}\right)} E_o \cos kx dy dx \\ &= [2\pi \rho_o E_o \cos k\delta \tanh(kd/2)] / k^2 \end{aligned} \quad (9)$$

- 5.6.4 (a) For a particular solution, guess that

$$\Phi = A \cos k(x - \delta) \quad (1)$$

Substitution into Poisson's equation then shows that $A = \rho_o / \epsilon_o k^2$ so that the particular solution is

$$\Phi_p = \frac{\rho_o}{\epsilon_o k^2} \cos k(x - \delta) \quad (2)$$

- (b) At $y = 0$

$$\Phi_h = -\Phi_p = -\frac{\rho_o}{\epsilon_o k^2} \cos k(x - \delta) \quad (3)$$

while at $y = d$,

$$\Phi_h = V_o \cos kx - \frac{\rho_o}{\epsilon_o k^2} \cos k(x - \delta) \quad (4)$$

- (c) The homogeneous solution is itself the sum of a part that satisfies the conditions

$$\Phi_1(y = d) = V_o \cos kx, \quad \Phi_1(y = 0) = 0 \quad (5)$$

and is therefore

$$\Phi_1 = V_o \cos kx \frac{\sinh ky}{\sinh kd} \quad (6)$$

and a part satisfying the conditions

$$\Phi_2(y = d) = -\frac{\rho_o}{k^2 \epsilon_o} \cos k(x - \delta); \quad \Phi_2(0) = -\frac{\rho_o}{\epsilon_o k^2} \cos k(x - \delta) \quad (7)$$

which is therefore

$$\Phi_2 = -\frac{\rho_o}{k^2 \epsilon_o} \cos k(x - \delta) \frac{\cosh k(y - \frac{d}{2})}{\cosh(kd/2)} \quad (8)$$

Thus, the total potential is the sum of (2), (6) and (8).

$$\Phi = \frac{\rho_o}{\epsilon_o k^2} \cos k(x - \delta) \left[1 - \frac{\cosh k(y - \frac{d}{2})}{\cosh(\frac{kd}{2})} \right] + V_o \cos kx \frac{\sinh ky}{\sinh kd} \quad (9)$$

(d) In view of the given charge density and (9), the force density in the x direction is

$$F_x = \frac{\rho_o}{k\epsilon_o} \sin k(x - \delta) \cos k(x - \delta) \left[1 - \frac{\cosh k(y - \frac{d}{2})}{\cosh(\frac{kd}{2})} \right] + \rho_o k V_o \sin kx \cos k(x - \delta) \frac{\sinh ky}{\sinh kd} \quad (10)$$

The first term in this expression integrates to zero while the second gives a total force of

$$f_x = \frac{\rho_o k V_o}{\sinh kd} \int_0^{2\pi/k} \int_0^d \sin kx \cos k(x - \delta) \sinh ky dy dx \quad (11)$$

With the use of $\cos k(x - \delta) = \cos kx \cos k\delta + \sin kx \sin k\delta$, this integration gives

$$f_x = \rho_o \pi V_o \frac{(\cosh kd - 1) \sin k\delta}{k \sinh kd} \quad (12)$$

5.6.5

By inspection, we know that if we look for a particular solution having only a y dependence, it will have the same y dependence as the charge distribution (the second derivative of the sin function is once again a sin function). Thus, we substitute $A \sin(\pi y/b)$ into Poisson's equation and evaluate A .

$$\Phi_p = \frac{\rho_o b^2}{\epsilon_o \pi^2} \sin\left(\frac{\pi y}{b}\right) \quad (1)$$

The homogeneous solution must therefore be zero on the boundaries at $y = b$ and $y = 0$ and must be $-\rho_o b^2 \sin(\pi y/b)/\epsilon_o \pi^2$ at $x = \pm a$. This latter condition is even in x and can be matched by the solution to Laplace's equation

$$\Phi_h = A \sin\left(\frac{\pi y}{b}\right) \frac{\cosh(\pi x/b)}{\cosh(\pi a/b)} \quad (2)$$

if the coefficient, A , is made

$$A = -\rho_o b^2 / \epsilon_o \pi^2 \quad (3)$$

Thus, the solution is the sum of (1) and (2) with A given by (3).

5.6.6 (a) The charge distribution follows from Poisson's equation.

$$-\frac{\rho}{\epsilon_0} = \nabla^2 \Phi_p \Rightarrow \rho = \epsilon_0 V \sin \beta x \sin \frac{\pi y}{b} (\beta^2 + \frac{\pi^2}{b^2}) \quad (1)$$

(b) To make the total solution satisfy the zero potential conditions, the homogeneous solution must also be zero at $y = 0$ and $y = b$. At $x = 0$ it must also be zero but at $x = a$ the homogeneous solution must be $\Phi_h = -V \sin(\pi y/b) \sin \beta a$. Thus, we select the homogeneous solution

$$\Phi_h = A \sin \frac{\pi y \sinh(\pi x/b)}{b \sinh(\pi a/b)} \quad (2)$$

make $A = -V \sin \beta a$ and obtain the potential distribution

$$\Phi = V \sin \left(\frac{\pi y}{b} \right) \left[\sin \beta x - \sin \beta a \frac{\sinh(\pi x/b)}{\sinh(\pi a/b)} \right] \quad (3)$$

5.6.7 A particular solution is found by assuming that it only depends on x and integrating Poisson's equation twice to obtain

$$\Phi_p = -\frac{\rho_0 l^2}{6\epsilon_0} \left(\frac{x}{l} - \frac{x^3}{l^3} \right) \quad (1)$$

The two integration constants have been assigned so that the potential is zero at $x = 0$ and $x = l$. The homogeneous solution must therefore satisfy the boundary conditions

$$\begin{aligned} \Phi_h(x=0) &= \Phi_h(x=l) = 0 \\ \Phi_h(y=\pm d) &= -\frac{\rho_0 l^2}{6\epsilon_0} \left(\frac{x}{l} - \frac{x^3}{l^3} \right) \end{aligned} \quad (2)$$

The first two of these are satisfied by the following solutions to Laplace's equation.

$$\Phi_h = \sum_{n=1}^{\infty} A_n \sin \left(\frac{n\pi x}{l} \right) \frac{\cosh \left(\frac{n\pi y}{d} \right)}{\cosh \left(\frac{n\pi d}{l} \right)} \quad (3)$$

This potential has an even y dependence, reflecting the fact that the boundary conditions are even in y . To determine the coefficients in (3), note that the second pair of boundary conditions require that

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} = -\frac{\rho_0 l^2}{6\epsilon_0} \left(\frac{x}{l} - \frac{x^3}{l^3} \right) \quad (4)$$

Multiplication of both sides of this expression by $\sin(m\pi x/l)$, and integration gives

$$A_m \frac{l}{2} = -\frac{\rho_0 l}{6\epsilon_0} \int_0^l x \sin \left(\frac{m\pi x}{l} \right) dx + \frac{\rho_0}{6\epsilon_0 l} \int_0^l x^3 \sin \frac{m\pi x}{l} dx \quad (5)$$

or

$$A_n = \frac{2l^3 \rho_o}{\epsilon_o (n\pi)^3} (-1)^n$$

Thus, the required potential is

$$\Phi = \frac{\rho_o l^2}{6\epsilon_o} \left(\frac{x}{l} - \frac{x^3}{l^3} \right) + \sum_{n=1}^{\infty} \frac{2}{l} \left(\frac{l}{n\pi} \right)^3 \frac{\rho_o}{\epsilon_o} (-1)^n \sin \frac{n\pi x}{l} \frac{\cosh \left(\frac{n\pi y}{l} \right)}{\cosh \left(\frac{n\pi d}{l} \right)} \quad (6)$$

- 5.6.8** (a) The charge density can be found using Poisson's equation to confirm that the charge density is that given. Thus, the particular solution is indeed as given.
- (b) Continuity conditions at the interface where $y = 0$ are

$$\Phi^a = \Phi^b \quad (1)$$

$$\frac{\partial \Phi^a}{\partial y} = \frac{\partial \Phi^b}{\partial y} \quad (2)$$

To satisfy these conditions, add to the particular solution a solution to Laplace's equation in the respective regions having the same x dependence and decaying to zero far from the interface.

$$\Phi^a = A \cos \beta x e^{-\beta y} \quad (3)$$

$$\Phi^b = \frac{\rho_o}{\epsilon_o (\beta^2 - \alpha^2)} \cos \beta x e^{\alpha y} + B \cos \beta x e^{\beta y} \quad (4)$$

Substitution of these relations into (1) and (2) shows that

$$A = \frac{\rho_o}{\epsilon_o (\beta^2 - \alpha^2) 2} \left(1 - \frac{\alpha}{\beta} \right) \quad (5)$$

$$B = \frac{-\rho_o}{\epsilon_o (\beta^2 - \alpha^2) 2} \left(1 + \frac{\alpha}{\beta} \right) \quad (6)$$

and substitution of these coefficients into (3) and (4) results in the given potential distribution.

- 5.6.9** (a) The potential in each region is the sum of a part due to the wall potentials without the surface charge in the plane $y = 0$ and a part due to the surface charge and having zero potential on the walls. Each of these is continuous in the $y = 0$ plane and even in y . The x dependence of each is determined by the respective x dependencies of the wall potential and surface charge density distribution. The latter is the same as that part of its associated potential so that Gauss' continuity condition can be satisfied. Thus, with A a yet to be determined coefficient, the potential takes the form

$$\Phi = \begin{cases} V \frac{\cosh \beta y}{\cosh \beta a} \cos \beta x - A \sinh \beta (y - a) \sin \beta (x - x_o); & 0 < y < a \\ V \frac{\cosh \beta y}{\cosh \beta a} \cos \beta x - A \sinh \beta (y + a) \sin \beta (x - x_o); & -a < y < 0 \end{cases} \quad (1)$$

The coefficient is determined from Gauss' condition to be

$$-\epsilon_o \left[\frac{\partial \Phi^a}{\partial y} - \frac{\partial \Phi^b}{\partial y} \right]_{y=0} = \sigma_o \sin \beta(x - x_o) \Rightarrow A = \frac{-\sigma_o}{2\epsilon_o \beta \cosh \beta a} \quad (2)$$

(b) The force is

$$f_x = d \int_x^{x+2\pi/\beta} \sigma_o \sin \beta(x - x_o) E_x(y=0) dx \quad (3)$$

From (1),

$$E_x(y=0) = V\beta \frac{\sin \beta x}{\cosh \beta a} - \frac{\sigma_o \sinh \beta a}{2\epsilon_o \cosh \beta a} \cos \beta(x - x_o) \quad (4)$$

The integration of the second term in this expression in (3) will give no contribution. Substitution of the first term gives

$$f_x = \frac{d\sigma_o V \beta}{\cosh \beta a} \int_0^{x+2\pi/\beta} \sin \beta(x - x_o) \sin \beta x dx = \sigma_o V \beta \left(\frac{d\pi}{\beta} \right) \frac{\cos \beta x_o}{\cosh \beta a} \quad (5)$$

(d) Because the charge and wall potential are synchronous, that is $U = \omega/\beta$, the new potential distribution is just that found with x replaced by $x - Ut$. Thus, the force is that already found. The force acts on the external mechanical system (acts to accelerate the charged particles). Thus, $U f_x$ is the mechanical power output and $-U f_x$ is the mechanical power input. Because the system is loss free and the system is in the steady state so that there is no energy storage, $-U f_x$ is therefore the electrical power output.

$$\text{Electrical Power Out} = -U f_x = -U d \sigma_o V \beta \frac{\pi \cos \beta x_o}{\cosh \beta a} \quad (6)$$

(e) For (6) to be positive so that the system is a generator, $\frac{\pi}{2} < \beta x_o < \frac{3\pi}{2}$.

5.7 SOLUTIONS TO LAPLACE'S EQUATION IN POLAR COORDINATES

5.7.1 The given potentials have the correct values at $r = a$. With $m = 5$, they are solutions to Laplace's equation. Of the two possible solutions in each region having $m = 5$ and the given distribution, the one that is singular at the origin is eliminated from the inner region while the one that goes to infinity far from the origin is eliminated from the outer solution. Hence, the given solution.

- 5.7.2 (a) Of the two potentials have the same ϕ dependence as the potential at $r = R$, the one that is not singular at the origin is

$$\Phi = \frac{V}{R} r \sin \phi = \frac{V}{R} y \quad (1)$$

Note that this potential is also zero on the $y = 0$ plane, so it satisfies the potential conditions on the enclosing surface.

- (b) The surface charge density on the equipotential at $y = 0$ is

$$\sigma_s = \epsilon_o E_y = -\epsilon_o \frac{\partial \Phi}{\partial y} = -\epsilon_o \frac{V}{R} \quad (2)$$

and hence is uniform.

- 5.7.3 The solution is written as the sum of two solutions, Φ^a and Φ^b . The first of these is the linear combination of solutions matching the potential on the outside and being zero on the inside. Thus, when added to the second solution, which is zero on the outside but assumes the given potential on the inside, it does not disturb the potential on the inside boundary. Nor does the second potential disturb the potential of the first solution on the outside boundary. Note also that the correct combination of solutions, $(r/b)^3$ and $(b/r)^3$ in the first solution and (r/a) and (a/r) in the second solution can be determined by inspection by introducing r normalized to the radius at which the potential must be zero. By using the appropriate powers of r , this approach can be used for any ϕ dependence of the given potential.

- 5.7.4 From Table 5.7.1, column two, the potentials that are zero at $\phi = 0$ and $\phi = \alpha$ are

$$r^{\pm m} \sin m\phi \quad (1)$$

with $m = n\pi/\alpha, n = 1, 2, \dots$ In taking a linear combination of these that is zero at $r = a$, it is convenient to normalize the r dependence to a and write the linear combination as

$$\Phi = A(r/a)^{n\pi/\alpha} \sin\left(\frac{n\pi\phi}{\alpha}\right) + B(a/r)^{n\pi/\alpha} \sin\left(\frac{n\pi\phi}{\alpha}\right) \quad (2)$$

where A and B are to be determined. It can be seen from (2) that to make $\Phi = 0$ at $r = a$, $A = -B$ and the solution becomes

$$\Phi = A[(r/a)^{n\pi/\alpha} - (a/r)^{n\pi/\alpha}] \sin\left(\frac{n\pi\phi}{\alpha}\right) \quad (3)$$

Finally, the last coefficient and n are adjusted so that the potential meets the condition at $r = b$. Thus,

$$\Phi = V_b \frac{[(r/a)^{n\pi/\alpha} - (a/r)^{n\pi/\alpha}]}{[(b/a)^{n\pi/\alpha} - (a/b)^{n\pi/\alpha}]} \sin\left(\frac{n\pi\phi}{\alpha}\right) \quad (4)$$

5.7.5 To make the potential zero at $\phi = 0$, use the second and fourth solutions in the third column of Table 5.7.1.

$$\cos[p \ln(r)] \sinh p\phi, \quad \sin[p \ln(r)] \sinh p\phi \quad (1)$$

The linear combination of these solutions that is zero at $r = a$ is obtained by simply normalizing r to a in the second solution. This can be seen by using the double-angle formula to write that solution as

$$\begin{aligned} A \sin[p \ln(r/a)] \sinh p\phi &= A \sin[p \ln(r) - p \ln(a)] \sinh p\phi \\ &= A \{ \sin[p \ln(r)] \cos[p \ln(a)] \\ &\quad - \cos[p \ln(r)] \sin[p \ln(a)] \} \sinh p\phi \end{aligned} \quad (2)$$

This solution is made to be zero at $r = b$ by making $p = n\pi/\ln(b/a)$, where n is any integer. Finally, the last boundary condition at $\phi = 0$ is met by adjusting the coefficient A and selecting $n = 3$.

$$A = V / \sinh[3\pi\alpha/\ln(b/a)] \quad (3)$$

5.7.6 The potential is a linear combination of the first two in column one of Table 5.7.1.

$$\Phi = A\phi + B = -\frac{V}{(3\pi/2)} \left(\phi - \frac{3\pi}{2} \right) = V \left(1 - \frac{2\phi}{3\pi} \right) \quad (1)$$

This potential and the associated electric field are sketched in Fig. S5.7.6.

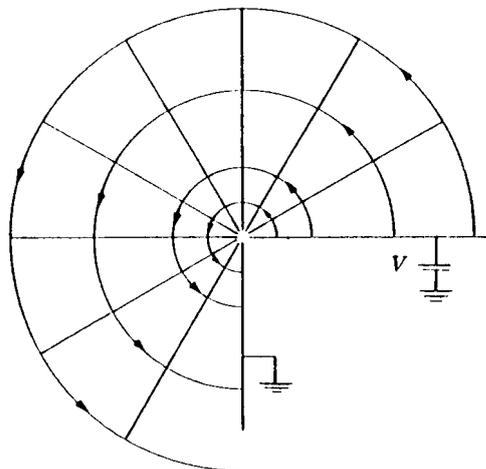


Figure S5.7.6

5.8 EXAMPLES IN POLAR COORDINATES

5.8.1 Either from (5.8.4) or from Fig. 5.8.2, it is clear that outside of the cylinder, the $z = 0$ plane is one having the same zero potential as the surface of the cylinder. Therefore, the potential and field as respectively given by (5.8.4) and (5.8.5) also describe the given situation.

Intuitively, we would expect the maximum electric field to be at the top of the cylinder, at $r = R, \phi = \pi/2$. From (5.8.5), the field at this point is

$$E_{\max} = 2E_o \quad (1)$$

and this maximum field is indeed independent of the cylinder radius. To be more rigorous, from (5.8.5), the magnitude of \mathbf{E} is

$$|\mathbf{E}| = E_o \mathcal{E} \quad (2)$$

where

$$\mathcal{E} \equiv \sqrt{[1 + (R/r)^2]^2 \cos^2 \theta + [1 - (R/r)^2]^2 \sin^2 \theta}$$

If this function is pictured as the vertical coordinate in a three dimensional plot where the floor coordinates are r and ϕ , its extremes are located at (r, ϕ) where the derivatives in the r and ϕ directions are zero. These are the locations where the surface represented by (2) is level and where the surface is either a maximum, a minimum or a saddle point. Thus, to locate the coordinates which are candidates for giving the maximum, note that

$$\frac{\partial \mathcal{E}}{\partial \theta} = \frac{E_o}{\mathcal{E}} \{-[1 + (R/r)^2]^2 + [1 - (R/r)^2]^2\} \cos \phi \sin \phi = 0 \quad (3)$$

and

$$\frac{\partial \mathcal{E}}{\partial r} = \frac{E_o}{\mathcal{E}} \frac{2R^2}{r^3} \{[1 + (R/r)^2]^2 \cos^2 \theta + [1 - (R/r)^2]^2 \sin^2 \theta\} = 0 \quad (4)$$

Locations where (3) is satisfied are either at

$$\phi = 0 \quad (5)$$

or at

$$\phi = \pi/2 \quad (6)$$

with r not equal to R or at

$$r = R \quad (7)$$

with ϕ not given by (5) or (6). Putting (5) into (4) shows that there is no solution for r while putting (6) into (4) shows that the associated value of r is $r = R$. Finally, putting (7) into (4) gives the same location, $r = R$ and $\phi = \pi/2$. Inspection of (5) shows that this is the location of a maximum, not a minimum.

5.8.2 Because there is no ϕ dependence of the potential on the boundaries, we use the second $m = 0$ potential from Table 5.7.1.

$$\Phi = A \ln r + B \quad (1)$$

Here, a constant potential has been added to the \ln function. The two coefficients, A and B , are determined by requiring that

$$V_b = A \ln b + B \quad (2)$$

$$V_a = A \ln a + B \quad (3)$$

Thus,

$$\begin{aligned} A &= (V_a - V_b) / \ln(a/b) \\ B &= \{V_b \ln a - V_a \ln b\} / \ln(a/b) \end{aligned} \quad (4)$$

and the required potential is

$$\begin{aligned} \Phi &= V_a \frac{\ln(r/b)}{\ln(a/b)} - V_b \frac{\ln(r/b)}{\ln(a/b)} + V_b \\ &= [V_a \ln(r/b) - V_b \ln(r/a)] / \ln(a/b) \end{aligned} \quad (5)$$

The electric field follows as being

$$\mathbf{E} = -\frac{\mathbf{i}_r (V_a - V_b) 1}{\ln(a/b) r} \quad (6)$$

and evaluation of this expression at $r = b$ shows that the field is positive on the inner cylinder, and everywhere else for that matter, if $V_a < V_b$.

5.8.3 (a) The given surface charge distribution can be represented by a Fourier series that, like the given function, is odd about $\phi = \phi_o$

$$\sigma_s = \sum_{n=1}^{\infty} \sigma_n \sin n\pi(\phi - \theta_o) \quad (1)$$

where the coefficients σ_n are determined by multiplying both sides of (1) by $\sin m\pi(\phi - \phi_o)$ and integrating over a half-wavelength.

$$\int_{\phi_o}^{\phi_o+\pi} \sigma_s(\phi) \sin m(\phi - \theta_o) d\phi = \int_{\phi_o}^{\phi_o+\pi} \sum_{n=1}^{\infty} \sigma_n \sin n(\phi - \phi_o) \sin m(\phi - \theta_o) d\phi \quad (2)$$

Thus,

$$\sigma_n = \frac{4\sigma_o}{n\pi}; \quad n \text{ odd} \quad (3)$$

and $\sigma_n = 0$, n even. The potential response to this surface charge density is written in terms of solutions to Laplace's equation that i) have the same ϕ dependence as (1), ii) go to zero far from the rotating cylinder (region a) and at the inner cylinder where $r = R$ and are continuous at $r = a$.

$$\Phi = \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \Phi_n \left\{ \begin{array}{l} [(a/R)^n - (R/a)^n](R/r)^n \\ (R/a)^n [(r/R)^n - (R/r)^n] \end{array} \right\} \sin n(\phi - \theta_o) \quad \begin{array}{l} a < r \\ R < r < a \end{array} \quad (4)$$

The coefficients Φ_n are determined by the "last" boundary condition, requiring that

$$\sigma_s(r = a) = -\epsilon_o \left[\frac{\partial \Phi^a}{\partial r} - \frac{\partial \Phi^b}{\partial r} \right]_{r=a} \quad (5)$$

Substitution of (1), (3) and (4) into (5) gives

$$\Phi_n = \frac{2\sigma_o a}{\epsilon_o \pi n^2} \quad (6)$$

(b) The surface charge density on the inner cylinder follows from using (4) to evaluate

$$\sigma_s(r = R) = -\epsilon_o \frac{\partial \Phi^b}{\partial r} \Big|_{r=R} = -\frac{\epsilon_o 2}{R} \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \Phi_n n (R/a)^n \sin n(\phi - \theta_o) \quad (7)$$

Thus, the total charge on the electrode segment in the wall of the inner cylinder is

$$q = w \int_0^\alpha \sigma_s(R) R d\phi = - \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} Q_n [\cos n\theta_o - \cos n(\alpha - \theta_o)] \quad (8)$$

where

$$Q_n \equiv \frac{4\sigma_o w a}{\pi} (R/a)^n \frac{1}{n^2}$$

(c) The output voltage is then evaluated by substituting $\theta_o \rightarrow \Omega t$ into (8) and taking the temporal derivative.

$$v_o = -R_o \frac{dq}{dt} = -\Omega R_o \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} n Q_n [\sin n\Omega t + \sin n(\alpha - \Omega t)] \quad (9)$$

5.8.4 The Fourier representation of the square-wave of surface charge density is carried out as in Prob. 5.8.3, (1) through (3), resulting in

$$\sigma_s = \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \sigma_n \sin n\pi(\phi - \theta_o) \quad (1)$$

where

$$\sigma_n = \frac{4\sigma_o}{n\pi}; \quad n \text{ odd}$$

The potential between the moving sheet at $r = R$ and the outer cylindrical wall at $r = a$, and inside the moving sheet, are respectively

$$\Phi = \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \Phi_n = \left\{ \left(\frac{a/R}{r/R} \right)^n \left[\left(\frac{r/R}{a/R} \right)^n - \left(\frac{R/r}{R/a} \right)^n \right] \right\} \sin n(\phi - \theta_o) \quad \begin{matrix} a < r < R \\ r < a \end{matrix} \quad (2)$$

where the coefficient has been adjusted so that the potential is zero at $r = R$ and continuous at the surface of the moving sheet, where $r = a$. The coefficients are determined by using Gauss' continuity condition with the surface charge density written as (1) and the potential given by (2);

$$\begin{aligned} -\epsilon_o \left(\frac{\partial \Phi^a}{\partial r} - \frac{\partial \Phi^b}{\partial r} \right)_{r=a} = \sigma_s &\Rightarrow -\epsilon_o \Phi_n (a/R)^n \left[\frac{n}{a} (a/R)^n + \frac{n}{a} (R/a)^n \right] \\ &+ \frac{n}{a} (a/R)^n [(a/R)^n - (R/a)^n] = \frac{4\sigma_o}{n\pi} \end{aligned} \quad (3)$$

which implies that

$$\Phi_n = -\frac{2\sigma_o a}{n^2 \pi \epsilon_o} \quad (4)$$

The surface charge on the detection segment is

$$\sigma_s = \epsilon_o \frac{\partial \Phi^a}{\partial r} \Big|_{r=R} = - \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{4\sigma_o}{\pi n} (a/R)^{n+1} \sin n(\phi - \theta_o) \quad (5)$$

and so the total charge on that segment is

$$q = w \int_0^\alpha \sigma_s(r=R) R d\phi = - \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} Q_n [\cos n\theta_o - \cos n(\alpha - \theta_o)] \quad (6)$$

where

$$Q_n = \frac{4\sigma_o w R}{\pi} (a/R)^{n+1} \frac{1}{n^2}$$

Finally, with $\theta_o = \Omega t$, the detected voltage is therefore

$$v_o = -R_o \frac{dq}{dt} = -\Omega R_o \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} n Q_n [\sin n\Omega t + \sin n(\alpha - \Omega t)] \quad (7)$$

5.8.5 Of the potentials in the second column of Table 5.7.1, the requirement that the potential be zero where $\phi = 0$ selects the two that vary as $\sin(m\phi)$ while the fact that the space of interest extends to the origin precludes those with negative exponents, for $m > 0$, the last two. The potential will be zero at $\phi = \alpha$ if $m = n\pi/d, n = 1, 2, \dots$. Thus, candidate potentials are

$$\Phi = \sum_{m=1}^{\infty} A_n (r/R)^{n\pi/\alpha} \sin\left(\frac{n\pi\phi}{\alpha}\right) \quad (1)$$

Evaluated at $r = R$, this potential takes the form of a Fourier series, used here to represent the uniform potential.

$$V = \sum_{m=1}^{\infty} A_n \sin\left(\frac{n\pi\phi}{\alpha}\right) \quad (2)$$

Multiplication by $\sin(q\pi\phi/\alpha)$ and integration from $\phi = 0$ to $\phi = \alpha$ gives an expression which can be solved for the coefficients in (2).

$$-\frac{V\alpha}{q\pi} \cos\left(\frac{q\pi\phi}{\alpha}\right) \Big|_0^\alpha = A_q \frac{\alpha}{2} \Rightarrow A_n = \frac{4V}{\pi} \begin{cases} 1/n; & n \text{ odd} \\ 0; & n \text{ even} \end{cases} \quad (3)$$

Thus, (1) and (3) are the given answer.

5.8.6 Far from $r = R$, the field becomes that of a pair of electrodes extending from the origin to infinity in the planes $\phi = 0$ (with zero potential) and $\phi = \alpha$ (with potential V). The associated electric field is ϕ directed and simply the voltage V divided by the distance αr between the electrodes, following lines of constant r .

$$\Phi(r \rightarrow \infty) = V \frac{\phi}{\alpha} \Rightarrow \mathbf{E}(r \rightarrow \infty) = \frac{V}{\alpha r} \mathbf{i}_\phi \quad (1)$$

Although this potential satisfies the boundary conditions on the "wedge" electrodes, it does not satisfy the boundary conditions over the surface at $r = R$. On that surface, the potential should be the constant V . To satisfy this boundary condition, we add to (1) a potential that is zero on the surfaces $\phi = 0$ and $\phi = \alpha$ where (1) already satisfies the boundary conditions and that goes to zero at $r \rightarrow \infty$, where (1) is also the correct potential.

$$\Phi = V \frac{\phi}{\alpha} + \sum_{n=1}^{\infty} A_n (r/R)^{-(n\pi/\alpha)} \sin\left(\frac{n\pi\phi}{\alpha}\right) \quad (2)$$

The coefficients A_n are determined from evaluating (2) on the electrode at $r = R$, where

$$V = \frac{V\phi}{\alpha} + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi\phi}{\alpha}\right) \quad (3)$$

The first term on the right in (3) is transferred to the left, both sides of the expression multiplied by $\sin(m\pi\phi/\alpha)$ and both sides integrated from $\phi = 0$ to $\phi = \alpha$ to obtain

$$V \left\{ -\frac{\alpha}{m\pi} \cos\left(\frac{m\pi\phi}{\alpha}\right) - \frac{\alpha}{(m\pi)^2} \left[\sin\frac{m\pi\phi}{\alpha} - \frac{m\pi\phi}{\alpha} \cos\left(\frac{m\pi\phi}{\alpha}\right) \right] \right\}_0^\alpha = \frac{A_m\alpha}{2} \quad (4)$$

This expression can be solved for the coefficient, which (with $m \rightarrow n$) is

$$A_n = \frac{2V}{n\pi} \quad (5)$$

Evaluated using this coefficient, (2) is the desired potential.

- 5.8.7 (a) From the four equations in the second column of Table 5.7.1, the \sin functions satisfy the boundary conditions that $\Phi = 0$ at $\phi = 0$ and $\phi = 2\pi$ if $m = n/2, n = 1, 2, \dots$. With the understanding that n is positive, the solutions with exponents $-m$ are excluded so that the potential is finite as $r \rightarrow 0$. Thus, the remaining potential is the superposition of the modes

$$\Phi = \sum_{n=1}^{\infty} A_n (r/R)^{n/2} \sin\left(\frac{n}{2}\phi\right) \quad (1)$$

- (b) The boundary condition at $r = R$ requires that

$$V_o = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n}{2}\phi\right) \quad (2)$$

Multiplication of both sides of this expression by $\sin(p\phi/2)$ and integration gives

$$\int_0^{2\pi} V_o \sin\left(\frac{m}{2}\phi\right) d\phi = \sum_{n=1}^{\infty} \int_0^{2\pi} A_n \sin\left(\frac{n}{2}\phi\right) \sin\left(\frac{m}{2}\phi\right) d\phi \quad (3)$$

or

$$-\frac{2}{m} V_o [\cos(m\pi) - 1] = \pi A_m \quad (4)$$

so that it follows that $A_n = 0, n$ even and for n odd

$$A_n = \frac{4V_o}{n\pi} \quad (5)$$

Substitution of this coefficient into (1) then gives the desired potential.

$$\Phi = \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{4V_o}{n\pi} (r/R)^{n/2} \sin\left(\frac{n}{2}\phi\right) \quad (6)$$

(c) The associated electric field follows from this expression as

$$\mathbf{E} = -\frac{4V_o}{\pi} \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{1}{n} \left[i_r \frac{n}{2} \frac{r^{\frac{n}{2}-1}}{R^{n/2}} \sin\left(\frac{n}{2}\phi\right) + i_\phi \frac{n}{2} \frac{r^{\frac{n}{2}-1}}{R^{n/2}} \cos\left(\frac{n}{2}\phi\right) \right] \quad (7)$$

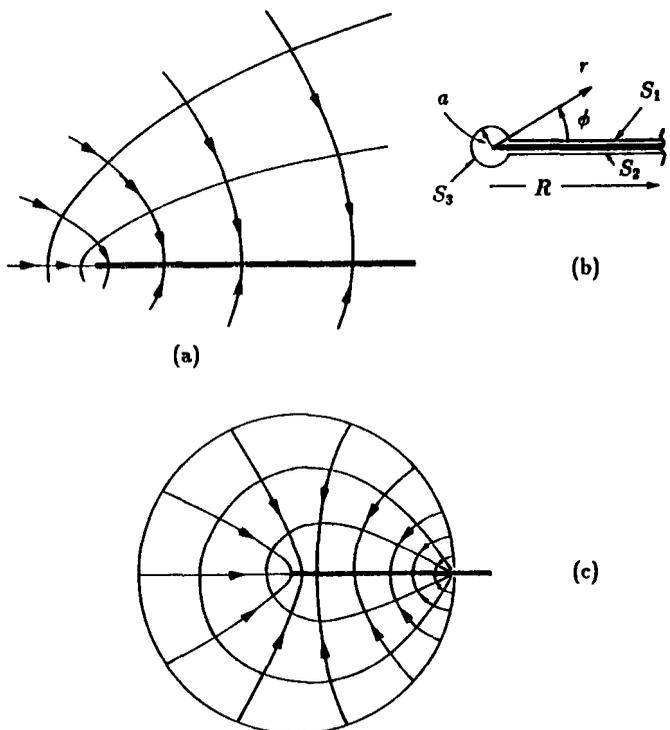


Figure S5.8.7

A sketch of the lead term in (6) and (7) is shown in Fig. S5.8.7a. The potential is finite at the tip of the fin but the electric field intensity varies as $1/\sqrt{r}$ at the tip. On the surface S_1 shown in Fig. S5.8.7b, the surface charge density follows from (7) as

$$\epsilon_o E_\phi(r, \phi = 0) = -\frac{4\epsilon_o V_o}{\pi} \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{1}{2} \frac{r^{\frac{n}{2}-1}}{R^{n/2}} \quad (8)$$

On the circular cylindrical surface S_2 at radius a , also shown in Fig. S5.8.7b,

$$\epsilon_o E_r(r = a, \phi) = -\frac{4\epsilon_o V_o}{\pi} \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{1}{2} \frac{a^{\frac{n}{2}-1}}{R^{n/2}} \sin\left(\frac{n}{2}\phi\right) \quad (9)$$

while on surface S_3 ,

$$-\epsilon_o E_\phi = -\frac{4\epsilon_o V_o}{\pi} \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{1}{2} \frac{r^{\frac{n}{2}-1}}{R^{n/2}} \quad (10)$$

The total charge represented by the first mode in the series is therefore

$$\frac{2\epsilon_0 V_o}{\pi\sqrt{R}} \left[-\int_a^R r^{-1/2} dr - \int_0^{2\pi} a^{-1/2} \sin(\phi/2) a d\phi - \int_a^R r^{-1/2} dr \right] = \frac{8\epsilon_0 V_o}{\pi} \quad (11)$$

(d) The potential and field distribution is sketched in Fig. S5.8.7b.

5.8.8 The potential takes the form of (5.8.15) with azimuthal coordinate displaced so that $\phi \rightarrow \phi_o - \phi$.

$$\Phi = \sum_{n=1}^{\infty} A_n \sin \left[n\pi \frac{\ln(r/b)}{\ln(a/b)} \right] \sinh \left[\frac{n\pi}{\ln(a/b)} (\phi_o - \phi) \right] \quad (1)$$

Evaluated at $\phi = 0$, this expression is then the same as (5.8.15) evaluated at $\phi = \phi_o$. Thus, the coefficients are the same as given by (5.8.17). For n even, $A_n = 0$ and for n odd

$$A_n = 4v \ln(a/b) / n\pi \sinh \left[\frac{n\pi}{\ln(a/b)} \phi_o \right] \quad (2)$$

5.8.9 The radial distribution $R_n(r)$ is governed by (5.7.5).

$$r \frac{d}{dr} \left(r \frac{dR_n}{dr} \right) + p_n^2 R_n = 0 \quad (1)$$

Multiplication of this expression by another of the eigenfunctions and the weighting factor $1/r$ and integration results in the expression

$$\int_a^b \left[\frac{R_m}{r} r \frac{d}{dr} \left(r \frac{dR_n}{dr} \right) + p_n^2 \frac{1}{r} R_n R_m \right] dr = 0 \quad (2)$$

With the identification $u dv = d(uv) - v du$ where

$$du = d \left(r \frac{dR_n}{dr} \right), \quad v = R_m \quad (3)$$

Eq. (2) can be integrated by parts

$$r \left. \frac{dR_n}{dr} R_m \right|_b^a - \int_b^a \left(r \frac{dR_n}{dr} \frac{dR_m}{dr} \right) dr + p_n^2 \int_b^a \frac{1}{r} R_n R_m dr = 0 \quad (4)$$

This same procedure can be repeated with the roles of n and m reversed. Subtraction of the resulting expression from (4) gives

$$r \left[\frac{dR_n}{dr} R_m - R_n \frac{dR_m}{dr} \right]_b^a + (p_n^2 - p_m^2) \int_b^a \frac{1}{r} R_n R_m dr = 0 \quad (5)$$

If boundary conditions require that the first term is zero, or in particular that $R_n(a) = 0$ and $R_m(b) = 0$, then the orthogonality condition follows.

$$(p_n^2 - p_m^2) \int_b^a \frac{1}{r} R_n R_m dr = 0 \quad (6)$$

5.9 THREE SOLUTIONS TO LAPLACE'S EQUATION IN SPHERICAL COORDINATES

- 5.9.1 (a) The given surface potential has the same θ dependence as for the uniform field potential of (5.9.4) and the dipole field potential of (5.9.3). With the coefficients of these potentials adjusted to match the given potential at $r = a$,

$$\Phi = \begin{cases} V(r/a) \cos \theta; & r < a \\ V(a/r)^2 \cos \theta; & a < r \end{cases} \quad (1)$$

- (b) A sketch of Φ and \mathbf{E} is shown in Fig. 6.3.1.

- 5.9.2 (a) The surface charge density has the same θ dependence at $r = a$ as the discontinuity in the normal derivative of the potential. This suggests representing the potentials inside and outside the sphere with the same θ dependence as the given surface charge distribution. In addition, these potentials must be finite at the origin and at infinity. The natural choices are the uniform field potential given by (5.9.4) inside the sphere and the dipole potential of (5.9.3) outside the sphere.

$$\Phi = \begin{cases} A(a/r)^2 \cos \theta; & a < r \\ A(r/a) \cos \theta; & r < a \end{cases} \quad (1)$$

The coefficients have already been adjusted so that the potential is continuous at $r = a$. Gauss' continuity condition then requires that

$$-\epsilon_o \left(\frac{\partial \Phi^a}{\partial r} - \frac{\partial \Phi^b}{\partial r} \right)_{r=a} = \sigma_o \cos \theta \Rightarrow -\epsilon_o \left[\frac{1}{a} + \frac{2}{a} \right] A = \sigma_o \quad (2)$$

so that $A = \sigma_o a / 3\epsilon_o$ and the potential is as given with the problem.

- (b) In Example 6.3.1, the potentials inside and outside the sphere take the same form as in (1) [(6.3.9) and (6.3.8)] and satisfy boundary conditions which take the same form as used here [(6.3.6) and (6.3.7)]. Indeed, we will see in Sec. 6.3 that with the polarization density given the polarization charge density is specified and the determination of the associated potential and field is much the same as in this chapter when the charge is specified. Hence, Fig. 6.3.1 portrays the potential and field.

- 5.9.3 Because the given charge density does not depend on ϕ , the potential is also independent of ϕ . In that case, Poisson's equation in spherical coordinates reduces to

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Phi}{\partial \theta} \right) = -\frac{\rho_o \cos \theta}{\epsilon_o} \quad (1)$$

First, given the dependence of the charge density on θ , look for a particular solution having the form $\Phi_p = Ar^p \cos \theta$. Substitution into (1) then shows that $p = 2$ and $A = -\rho_o / 4\epsilon_o$ so that a particular solution is

$$\Phi_p = -\frac{\rho_o}{4\epsilon_o} r^2 \cos \theta \quad (2)$$

The sum of this potential and a solution to Laplace's equation must satisfy the condition that the potential be zero at $r = a$. Again, for the θ dependence of the particular solution, it is natural to take a uniform field as the homogeneous solution. Thus, with B an adjustable coefficient,

$$\Phi = -\frac{\rho_o}{4\epsilon_o} r^2 \cos \theta + Br \cos \theta \quad (3)$$

and by requiring that the total potential be zero at $r = a$, it follows that $B = \rho_o a / 4\epsilon_o$ so that the potential is as given with the problem statement.

5.9.4 Because the given charge density does not depend on ϕ , the potential is also independent of ϕ . In that case, Poisson's equation in spherical coordinates reduces to

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Phi}{\partial \theta} \right) = -\frac{\rho_o}{\epsilon_o} (r/a)^m \cos \theta \quad (1)$$

First, given the dependence of the charge density on θ , look for a particular solution having the form $(r/a) \cos \theta$. Substitution into (1) then shows that $p = m + 2$ and $A = -\rho_o a^2 / \epsilon_o (m + 1)(m + 4)$ so that a particular solution is

$$\Phi_p = -\frac{\rho_o a^2}{\epsilon_o (m + 1)(m + 4)} (r/a)^{m+2} \cos \theta \quad (2)$$

The sum of this potential and a solution to Laplace's equation must satisfy the condition that the potential be zero at $r = a$. Again, for the θ dependence of the particular solution, it is natural to take a uniform field as the homogeneous solution. Thus, with B an adjustable coefficient,

$$\Phi = \Phi_p + B(r/a) \cos \theta \quad (3)$$

and by requiring that the total potential be zero at $r = a$, it follows that the required potential is

$$\Phi = \frac{-\rho_o a^2}{\epsilon_o (m + 1)(m + 4)} (r/a) [(r/a)^{m+1} - 1] \cos \theta \quad (4)$$

5.10 THREE-DIMENSIONAL SOLUTIONS TO LAPLACE'S EQUATION

5.10.1 Given the zero potential surfaces at $y = 0$ and $y = b$ and at $z = 0$ and $z = w$, it is natural to construct the solution from product solutions having the form

$$\Phi = X(x) \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{w} \quad (1)$$

where, to satisfy Laplace's equation

$$X(x) = \begin{cases} \sinh k_{mn} x \\ \cosh k_{mn} x \end{cases}$$

and

$$k_{mn} = \sqrt{(m\pi/b)^2 + (n\pi/w)^2}$$

The boundary conditions on the surfaces at $x = 0$ and $x = a$ are the same. Thus, if $X(x)$ is chosen to be even about an origin at $x = a/2$, the potential that satisfies the condition of being v at $x = 0$ will also be v at $x = a$. Thus, $X(x)$ is made a linear combination of the solutions given with (1) which is the cosh function displaced so that its argument is zero where $x = a/2$.

$$X(x) = A_{mn} \cosh k_{mn} \left(x - \frac{a}{2}\right) \quad (2)$$

The solution therefore takes the form of (a) given with the problem. At $x = 0$, the condition at $x = 0$ requires that

$$v = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cosh \left(\frac{k_{mn} a}{2}\right) \sin \left(\frac{m\pi y}{b}\right) \sin \left(\frac{n\pi z}{w}\right) \quad (3)$$

Note that this expression is the same as (11) if the $\sinh(k_{mn} b)$ is replaced by $\cosh(k_{mn} a/2)$ and $x/a \rightarrow y/b$. The evaluation of the coefficient using the orthogonality of the product solutions is therefore essentially the same as given by (5.10.11)-(5.10.15), resulting in (b) as given with the problem.

5.10.2 Given the x and z dependence of the surface charge density, which is the same as that of the components of \mathbf{E} in the z direction on either side of the surface $y = a/2$, look for solutions of the form

$$\Phi = Y(y) \sin \left(\frac{\pi x}{a}\right) \sin \left(\frac{\pi z}{w}\right) \quad (1)$$

where

$$Y(y) = \begin{cases} \sinh k_{11} y \\ \cosh k_{11} y \end{cases}$$

and

$$k_{11} = \sqrt{(\pi/a)^2 + (\pi/w)^2}$$

To satisfy the continuity conditions at $y = b/2$, the potential function is given a piece-wise representation. The function in the upper region must be zero at $y = b$, so $Y(y)$ is chosen as a sinh with its argument displaced to $y = b$. In the lower region, the sinh function with its origin at $y = 0$ does the job. Thus,

$$\Phi = \begin{cases} A \sinh k_{11}(y - b) \\ B \sinh k_{11} y \end{cases} \sin \left(\frac{\pi x}{a}\right) \sin \left(\frac{\pi z}{w}\right) \quad (2)$$

At $y = b/2$, the potential must be continuous and Gauss' continuity condition must be satisfied.

$$-A \sinh(k_{11} b/2) = B \sinh(k_{11} b/2) \quad (3)$$

$$-\epsilon_o k_{11} (A - B) \cosh(k_{11} b/2) = \sigma_o \quad (4)$$

It follows that the coefficients in (2) are

$$A = -B = -\sigma_o / 2\epsilon_o k_{11} \cosh(k_{11} b/2) \quad (5)$$

5.10.3 In each case, the solution can be regarded as the superposition of a particular solution to Poisson's equation and a homogeneous solution to satisfy the boundary conditions. The determination of representation begins with the selection of the former.

As a first solution, select a particular solution that is only x dependent. Then, Poisson's equation reduces to

$$\frac{d^2\Phi}{dx^2} = -\frac{\rho_o}{\epsilon_o} \quad (1)$$

and the particular solution that (for convenience) is also zero at $x = 0$ and $x = a$ is

$$\Phi_p = -\frac{\rho_o}{2\epsilon_o}x^2 + Ax + B = -\frac{\rho_o}{2\epsilon_o}x(x-a) \quad (2)$$

With this potential satisfying the boundary conditions on two of the surfaces, the homogeneous solution must assure satisfying the conditions on the remaining four surfaces. This is done by adding to (2) solutions designed to satisfy the conditions at $y = 0$ and $y = b$ while being zero at all the other surfaces and therefore neither disturbing the already satisfied conditions at $x = 0$ and $x = a$ nor those to be satisfied by the next homogeneous solution. To satisfy both the conditions at $y = 0$ and $y = b$, the y dependence is taken as even about $y = b/2$. A second homogeneous solution is then added to this one to assure satisfaction of the conditions at $z = 0$ and $z = w/2$ while not disturbing the potential at the other four surfaces. Thus, the potential takes the form

$$\begin{aligned} \Phi = \frac{-\rho_o}{2\epsilon_o}x(x-a) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \cosh k_{mn}(y - \frac{b}{2}) \sin(\frac{m\pi}{a}x) \sin(\frac{n\pi}{w}z) \\ + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \cosh k_{mn}(z - \frac{w}{2}) \sin(\frac{m\pi}{a}x) \sin(\frac{n\pi}{b}y) \end{aligned} \quad (3)$$

The coefficients B_{mn} and C_{mn} are determined by requiring that the potential indeed be zero on the surfaces $y = 0$ and $z = 0$ (and hence also at $y = b$ and $z = w$).

$$\frac{\rho_o}{2\epsilon_o}x(x-a) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \cosh(\frac{k_{mn}b}{2}) \sin(\frac{m\pi}{a}x) \sin(\frac{n\pi}{w}z) \quad (4)$$

$$\frac{\rho_o}{2\epsilon_o}x(x-a) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \cosh(\frac{k_{mn}w}{2}) \sin(\frac{m\pi}{a}x) \sin(\frac{n\pi}{b}y) \quad (5)$$

The coefficients therefore follow from the same procedure as illustrated by (5.10.11) through (5.10.15). For m or n even the coefficients are zero. For m and n odd,

$$\begin{aligned} B_{mn} &= \frac{\rho_o}{2\epsilon_o \cosh(\frac{k_{mn}b}{2})} (4/n\pi) \int_0^a x(x-a) \sin(\frac{m\pi}{a}x) dx \\ &= \frac{-\rho_o}{2\epsilon_o \cosh(\frac{k_{mn}b}{2})} (4/n\pi) \frac{8a^2}{(m\pi)^3} \end{aligned} \quad (6)$$

$$\begin{aligned}
 C_{mn} &= \frac{\rho_o}{2\epsilon_o \cosh\left(\frac{k_{mn}w}{2}\right)} (4/n\pi) \int_0^a x(x-a) \sin\left(\frac{m\pi}{a}x\right) dx \\
 &= \frac{-\rho_o}{2\epsilon_o \cosh\left(\frac{k_{mn}w}{2}\right)} (4/n\pi) \frac{8a^2}{(m\pi)^3}
 \end{aligned} \tag{7}$$

Two more solutions are obtained by replacing the role of x with that of y and of z . As a fourth solution, expand the charge distribution in a three dimensional Fourier series

$$\rho_o = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{q=1}^{\infty} R_{mnq} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{q\pi z}{w}\right) \tag{8}$$

The coefficients R_{mnq} follow by multiplying by

$$\sin\left(\frac{r\pi x}{a}\right) \sin\left(\frac{s\pi y}{b}\right) \sin\left(\frac{u\pi z}{w}\right)$$

integrating over the volume and solving for R_{rsu} . Then, with $rsu \rightarrow mnq$,

$$R_{mnq} = \frac{16\rho_o}{mnq\pi^3} \tag{9}$$

for m and n and q odd and zero for m or n or q even. Given this (x, y, z) dependence and given that the second derivative of each of the sinusoids results in the same sinusoidal function, we are motivated to look for a particular solution having the same form.

$$\Phi = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{q=1}^{\infty} \Phi_{mnq} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{q\pi z}{w}\right) \tag{10}$$

Substitution of this expression into Poisson's equation shows that term by term it is not only a solution to Poisson's equation (and therefore a particular solution) if

$$\Phi_{mnq} = \frac{R_{mnq}}{\epsilon_o \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{q\pi}{w}\right)^2 \right]} \tag{11}$$

but satisfies the boundary conditions as well.