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SOLUTIONS TO CHAPTER 4

4.1 IRROTATIONAL FIELD REPRESENTED BY SCALAR POTENTIAL: THE GRADIENT OPERATOR AND GRADIENT INTEGRAL THEOREM

4.1.1 (a) For the potential

$$\Phi = \frac{V_o}{a^2}(x^2 + y^2 + z^2) \tag{1}$$

$$\text{grad}\,\Phi = \frac{2V_o}{a^2}(x\hat{i}_x + y\hat{i}_y + z\hat{i}_z) \tag{2}$$

(b) The unit normal is

$$\mathbf{n} = \frac{\nabla\Phi}{|\nabla\phi|} = \frac{x\hat{i}_x + y\hat{i}_y + z\hat{i}_z}{\sqrt{x^2 + y^2 + z^2}} = \mathbf{i_r} \tag{3}$$

4.1.2 For $\Phi = \frac{V_o}{a^2}xy$, we have

$$\mathbf{E} = -\nabla\Phi = -\frac{V_o}{a^2}(y\mathbf{i_x} + x\mathbf{i_y}) \tag{1}$$

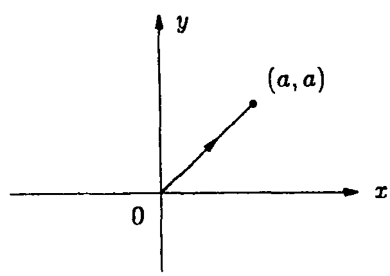

Figure S4.1.2

Integration on the path shown in Fig. S4.1.2 can be accomplished using t as a parameter, where for this curve $x = t$ and $y = d$ so that in

$$d\mathbf{s} = \mathbf{i_x}dx + \mathbf{i_y}dy \tag{2}$$

we can replace $dx = dt, dy = dt$. Thus,

$$\int_{(0,0)}^{(a,a)} \mathbf{E}\cdot d\mathbf{s} = \int_{t=0}^{a} -\frac{V_o}{a^2}(\mathbf{i_x} + \mathbf{i_y})\cdot(\mathbf{i_x} + \mathbf{i_y})dt = -V_o \tag{3}$$

Alternatively, $\Phi(0,0) = 0$ and $\Phi(a,a) = V_o$ and so $\Phi(0,0) - \Phi(a,a) = -V_o$.

4.1.3 (a) The three electric fields are respectively, $\mathbf{E} = -\nabla\Phi$,

$$\mathbf{E} = -(V_o/a)\mathbf{i_x} \tag{1}$$

$$\mathbf{E} = -(V_o/a)\mathbf{i_y} \tag{2}$$

$$\mathbf{E} = -\frac{2V_o}{a^2}(x\mathbf{i_x} - y\mathbf{i_y}) \tag{3}$$

(b) The respective equipotentials and lines of electric field intensity are sketched in the $x - y$ plane in Figs. S4.1.3a-c.

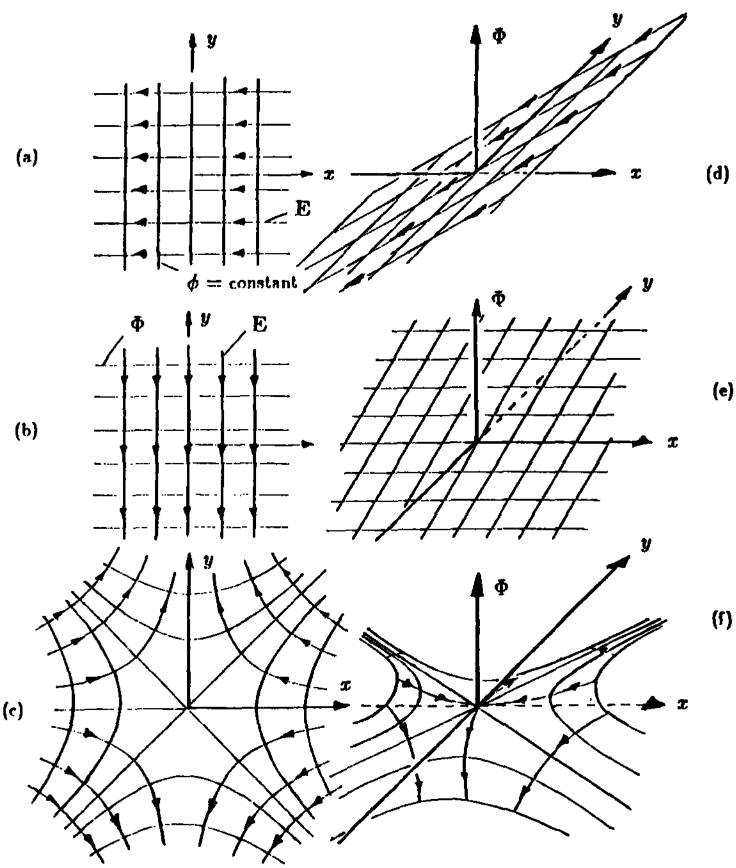

Figure S4.1.3

(c) Alternatively, the vertical axis of a three dimensional plot is used to represent the potential as shown in Figs. S4.1.3d-f.

4.1.4 (a) In Cartesian coordinates, the grad operator is given by (4.1.12). With Φ defined by (a), the desired field is

$$\begin{aligned}\mathbf{E} &= -\frac{\partial\Phi}{\partial x}\mathbf{i_x} - \frac{\partial\Phi}{\partial y}\mathbf{i_y}\\ &= \frac{-\rho_o}{\epsilon_o[(\pi/a)^2+(\pi/b)^2]}\Big[\frac{\pi}{a}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}\mathbf{i_x} + \frac{\pi}{b}\sin\frac{\pi x}{a}\cos\frac{\pi y}{b}\mathbf{i_y}\Big]\end{aligned} \quad (1)$$

(b) Evaluation of the curl gives

$$\begin{aligned}\nabla\times\mathbf{E} &= \begin{vmatrix}\mathbf{i_x} & \mathbf{i_y} & \mathbf{i_z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0\\ E_x & E_y & 0\end{vmatrix} = \mathbf{i_z}\Big[\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\Big]\\ &= \Big[\frac{\pi^2}{ab}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b} - \frac{\pi^2}{ab}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}\Big]\\ &= 0\end{aligned} \quad (2)$$

so that the field is indeed irrotational.

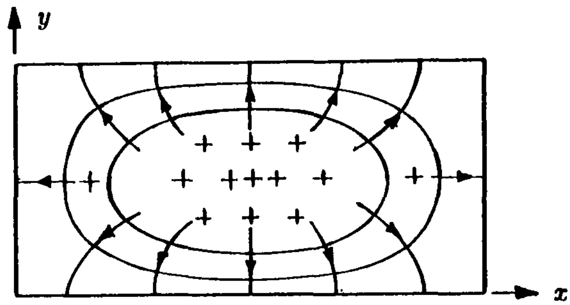

Figure S4.1.4

(c) From Gauss' law, the charge density is given by taking the divergence of (1).

$$\begin{aligned}\rho &= \nabla\cdot\epsilon_o\mathbf{E} = \epsilon_o\Big(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}\Big)\\ &= \frac{-\rho_o}{[(\pi/a)^2+(\pi/b)^2]}\Big[-(\pi/a)^2\sin\frac{\pi x}{b} - (\pi/b)^2\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\Big]\end{aligned} \quad (3)$$

(d) Evalvuation of the tantential component from (1) on each boundary gives; at

$$\begin{aligned}x = 0, E_y = 0; \quad x = a, E_y = 0\\ y = 0, E_x = 0; \quad y = a, E_x = 0\end{aligned} \quad (4)$$

(e) A sketch of the potential, the charge density and hence of $\mathbf{E}$ is shown in Fig. S4.1.5.

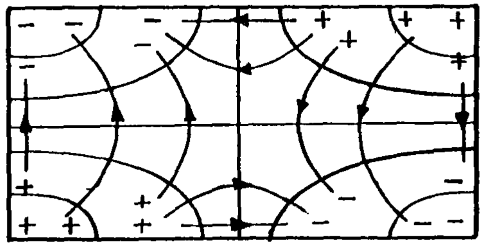

Figure S4.1.5

(f) The integration of $\mathbf{E}$ between points (a) and (b) in FIg. P4.1.5 should be the same as the difference between the potentials evaluated at these end points because of the gradient integral theorem, (16). In this particular case, let $x = t, y = (b/a)t$ so that $dx = dt$ and $dy = (b/a)dt$.

$$\begin{aligned}\int_a^b \mathbf{E}\cdot d\mathbf{s} &= \frac{-\rho_o}{\epsilon_o[(\pi/a)^2+(\pi/b)^2]}\int_{a/2}^{a}\Big[\frac{\pi}{a}\cos\frac{\pi t}{a}\sin\frac{\pi t}{a}dt \\ &+\frac{\pi}{a}\sin\frac{\pi t}{a}\cos\frac{\pi t}{a}\Big]dt \\ &= \frac{-\rho_o}{\epsilon_o[(\pi/a)^2+(\pi/b)^2]}\int_{a/2}^{a}\frac{\pi}{a}\sin\frac{2\pi t}{a}dt \\ &= \frac{\rho_o}{\epsilon_o[(\pi/a)^2+(\pi/b)^2]}\end{aligned} \tag{5}$$

The same result is obtained by taking the difference between the potentials.

$$\Phi\Big(\frac{a}{2},\frac{b}{2},t\Big)-\Phi(a,b,t)=\frac{\rho_o}{\epsilon_o[(\pi/a)^2+(\pi/b)^2]} \tag{6}$$

(g) The net charge follows by integrating the charge density given by (c) over the given volume.

$$Q=\int_V \rho dv=\int_0^d\int_0^b\int_0^a \rho_o\sin(\pi x/a)\sin(\pi y/b)dxdydz=\frac{4\rho_o abd}{\pi^2} \tag{7}$$

From Gauss' integral law, it also follows by integrating the flux density $\epsilon_o\mathbf{E}\cdot\mathbf{n}$ over the surface enclosing this volume.

$$\begin{aligned}Q=\oint_S \epsilon_o\mathbf{E}\cdot\mathbf{n}da &= \frac{-\rho_o d}{[(\pi/a)^2+(\pi/b)^2]}\Big\{\int_0^a\frac{\pi}{b}\sin(\pi x/a)\cos\pi dx \\ &-\int_0^a\frac{\pi}{b}\sin(\pi x/a)dx+\int_0^b\frac{\pi}{a}\cos\pi\sin\frac{\pi y}{b}dy \\ &-\int_0^b\frac{\pi}{a}\sin\frac{\pi y}{b}dy\Big\}=\frac{4\rho_o abd}{\pi^2}\end{aligned} \tag{8}$$

(h) The surface charge density on the electrode follows from using the normal electric field as given by (1).

$$\sigma_s = \epsilon_o E_y(y=0) = \frac{-\rho_o}{[(\pi/a)^2 + (\pi/b)^2]} \frac{\pi}{b} \sin \frac{\pi x}{a} \tag{9}$$

Thus, the net charge on this electrode is

$$q = \int_0^d \int_{a/4}^{3a/4} \frac{-\rho_o}{[(\pi/a)^2 + (\pi/b)^2]} \frac{\pi}{b} \sin \frac{\pi x}{a} dx dz = \frac{-\sqrt{2}(a/b) d\rho_o}{(\pi/a)^2 + (\pi/b)^2} \tag{10}$$

(i) The current $i(t)$ then follows from conservation of charge for a surface S that encloses the electrode.

$$\oint_S \mathbf{J} \cdot \mathbf{n} da + \frac{d}{dt} \int_V \rho dv \Rightarrow i + \frac{dq}{dt} = 0 \tag{11}$$

Thus, from (10),

$$i = \frac{\sqrt{2}(a/b)d}{(\pi/a)^2 + (\pi/b)^2} \frac{d\rho_o}{dt} \tag{12}$$

4.1.5 (a) In Cartesian coordinates, the grad operator is given by (4.1.12). With Φ defined by (a), the desired field is

$$\begin{aligned} \mathbf{E} &= -\left[\frac{\partial \Phi}{\partial x} \mathbf{i}_x + \frac{\partial \Phi}{\partial y} \mathbf{i}_y\right] \\ &= \frac{\rho_o}{\epsilon_o[(\pi/a)^2 + (\pi/b)^2]} \left[\frac{\pi}{a} \sin \frac{\pi}{a} x \cos \frac{\pi}{b} y \mathbf{i}_x + \frac{\pi}{b} \cos \frac{\pi}{a} x \sin \frac{\pi}{b} y \mathbf{i}_y\right] \end{aligned} \tag{1}$$

(b) Evaluation of the curl gives

$$\begin{aligned} \nabla \times \mathbf{E} &= \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ E_x & E_y & 0 \end{vmatrix} = \mathbf{i}_z \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) \\ &= \frac{\rho_o}{\epsilon_o[(\pi/a)^2 + (\pi/b)^2]} \left[-\frac{\pi^2}{ab} \sin \frac{\pi}{a} x \sin \frac{\pi}{b} y \right. \\ &\quad \left. + \frac{\pi^2}{ab} \sin \frac{\pi}{a} x \sin \frac{\pi}{b} y\right] = 0 \end{aligned} \tag{2}$$

so that the field is indeed irrotational.

(c) From Gauss' law, the charge density is given by taking the divergence of (1).

$$\rho = \nabla \cdot \epsilon_o \mathbf{E} = \epsilon_o \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}\right) = \rho_o \cos \frac{\pi}{a} x \cos \frac{\pi}{b} y \tag{3}$$

(d) The electric field $\mathbf{E}$ is tangential to the boundaries only if it has no normal component there.

$$E_x(0,y)=0, \quad E_x(a,y)=0$$
$$E_y(x,0)=0, \quad E_y(x,b)=0 \tag{4}$$

(e) A sketch of the potential, the charge density and hence of $\mathbf{E}$ is shown in Fig. S4.1.4.

(f) The integration of $\mathbf{E}$ between points (a) and (b) in Fig. P4.1.4 should be the same as the difference between the potentials evaluated at these end points because of the gradient integral theorem, (16). In this particular case, where $y=(b/a)x$ on C and hence $dy=(b/a)dx$

$$\begin{aligned}\int_{(a)}^{(b)}\mathbf{E}\cdot d\mathbf{s} &= \int_{a/2}^{a}\left\{E_x\left(x,\frac{b}{a}x\right)dx+E_y\left(x,\frac{b}{a}x\right)(b/a)dx\right\}\\ &= \frac{\rho_o}{\epsilon_o[(\pi/a)^2+(\pi/b)^2]}\int_{a/2}^{a}\frac{2\pi}{a}\sin\frac{\pi}{a}x\cos\frac{\pi}{a}x\,dx\\ &= \frac{-\rho_o}{\epsilon_o[(\pi/a)^2+(\pi/b)^2]}\end{aligned}$$

The same result is obtained by taking the difference between the potentials.

$$\int_{(a)}^{(b)}\mathbf{E}\cdot d\mathbf{s}=\Phi(a)-\Phi(b)=\frac{-\rho_o}{\epsilon_o[(\pi/a)^2+(\pi/b)^2]} \tag{6}$$

(g) The net charge follows by integrating the charge density over the given volume. However, we can see from the function itself that the positive charge is balanced by the negative charge, so

$$Q=\int_V \rho dV=0 \tag{7}$$

From Gauss' integral law, the net charge also follows by integrating the flux density $\epsilon_o\mathbf{E}\cdot\mathbf{n}$ over the surface enclosing this volume. From (d) this normal flux is zero, so that the net integral is certainly also zero.

$$Q=\oint_S \epsilon_o\mathbf{E}\cdot\mathbf{n}da=0 \tag{8}$$

The surface charge density on the electrode follows from integrating $\epsilon_o\mathbf{E}\cdot\mathbf{n}$ over the "electrode" surface. Thus, the net charge on the "electrode" is

$$q=\oint_S \epsilon_o\mathbf{E}\cdot\mathbf{n}da=0 \tag{9}$$

4.1.6 (a) From (4.1.2)

$$\begin{aligned}\mathbf{E} &= -\left(\frac{\partial\Phi}{\partial x}\mathbf{i_x} + \frac{\partial\Phi}{\partial y}\mathbf{i_y}\right)\\ &= -A[m\cosh mx \sin k_y y \sin k_z z \mathbf{i_x}\\ &+ \sinh mx k_y \cos k_y y \sin k_z z \mathbf{i_y}\\ &+ k_z \sinh mx \sin k_y y \cos k_z z \mathbf{i_z}] \sin \omega t\end{aligned} \tag{1}$$

(b) Evaluation using (1) gives

$$\begin{aligned}\bar{\nabla}\times\mathbf{E} &= \begin{vmatrix}\mathbf{i_x} & \mathbf{i_y} & \mathbf{i_z}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ E_x & E_y & E_z\end{vmatrix}\\ &= \mathbf{i_x}\left[\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right] + \mathbf{i_y}\left[\frac{\partial E_x}{\partial z}\right.\\ &\left.- \frac{\partial E_z}{\partial x}\right] + \mathbf{i_z}\left[\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right]\end{aligned} \tag{2}$$

$$\begin{aligned}&= -A\sin\omega t\{\mathbf{i_x}(k_y k_z \sinh mx \cos k_y y \cos k_z z - k_y k_z \sinh mx \cos k_y y \cos k_z z)\\ &+ \mathbf{i_y}(m k_z \cosh mx \sin k_y y \cos k_z z - k_z m \cosh mx \sin k_y y \cos k_z z)\\ &+ \mathbf{i_z}(m k_y \cosh mx \cos k_y y \sin k_z z - m k_y \cosh mx \cos k_y y \sin k_z z)\\ &= 0\end{aligned} \tag{3}$$

(c) From Gauss' law, (4.0.2)

$$\rho = \nabla\cdot\epsilon_o\mathbf{E} = -\epsilon_o A(m^2 - k_y^2 - k_z^2)\sinh mx \sin k_y y \sin k_z z \sin\omega t \tag{5}$$

(d) No. The gradient of vector or divergence of scalar are not defined.

(e) For $\rho = 0$ everywhere, make the coefficient in (5) be zero.

$$m^2 = k_y^2 + k_z^2 \tag{6}$$

4.1.7 (a) The wall in the first quadrant is on the surface defined by

$$y = a - x \tag{1}$$

Substitution of this value of y into the given potential shows that on this surface, the potential is a linear function of x and hence the desired linear function of distance along the surface

$$\Phi = Aa(2x - a) \tag{2}$$

To make this potential assume the correct values at the end points, where $x = 0$ and Φ must be $-V$ and where $x = a$ and Φ must be V, make $A = V/a^2$ and hence

$$\Phi = \frac{V}{a^2}(x^2 - y^2) \tag{3}$$

On the remaining surfaces, respectively in the second, third and fourth quadrants

$$y = x + a; \quad y = -a - x; \quad y = x - a \tag{4}$$

Substitution of these functions into (3) also gives linear functions of x which respectively satisfy the conditions on the potentials at the end points.

(b) Using (4.1.12),

$$\mathbf{E} = -\left(\frac{\partial \Phi}{\partial x}\mathbf{i}_x + \frac{\partial \Phi}{\partial y}\mathbf{i}_y\right) = -\frac{V}{a^2}(2x\mathbf{i}_x - 2y\mathbf{i}_y) \tag{5}$$

From Gauss' law, (4.0.2), the charge density is

$$\rho = \epsilon_o\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}\right) = -\frac{\epsilon_o V}{a^2}(2 - 2) = 0 \tag{6}$$

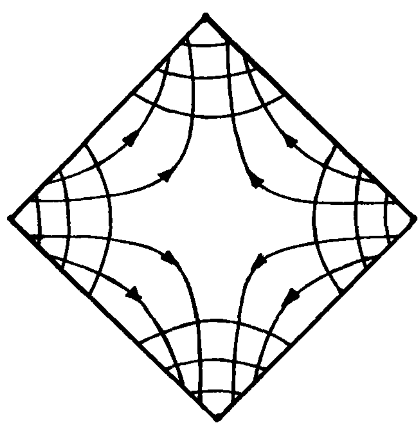

Figure S4.1.7

(c) The equipotentials and lines of $\mathbf{E}$ are shown in Fig. S4.1.7.

4.1.8 (a) For the given $\mathbf{E}$,

$$\nabla \times \mathbf{E} = \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ \partial/\partial x & \partial/\partial y & 0 \\ Cx & -Cy & 0 \end{vmatrix} = \mathbf{i}_z\left[\frac{\partial}{\partial x}(-Cy) - \frac{\partial}{\partial y}(Cx)\right] = 0 \tag{1}$$

so $\mathbf{E}$ is irrotational. To evaluate C, remember that the vector differential distance $d\mathbf{s} = \mathbf{i}_x dx + \mathbf{i}_y dy$. For ths contour, $d\mathbf{s} = \mathbf{i}_y dy$. To let the integral take account of the sign naturally, the integration is carried out from the origin to (a) (rather than the reverse) and set equal to $\Phi(0,0) - \Phi(0,h) = -V$.

$$-V = \int_0^h -Cy\,dy = -\frac{1}{2}Ch^2 \tag{2}$$

Thus, $C = 2V/h^2$.

(b) To find the potential, observe from $\mathbf{E} = -\nabla\Phi$ that

$$\frac{\partial\Phi}{\partial x} = -Cx; \qquad \frac{\partial\Phi}{\partial y} = Cy \tag{3}$$

Integration of (3a) with respect to x gives

$$\Phi = -\frac{1}{2}Cx^2 + f(y) \tag{4}$$

Differentiation of this expression with respect to y and comparison to (3b) then shows that

$$\frac{\partial\Phi}{\partial y} = \frac{df}{dy} = Cy \Rightarrow f = \frac{1}{2}y^2 + D \tag{5}$$

Because $\Phi(0,0) = 0, D = 0$ so that

$$\Phi = -\frac{1}{2}C(x^2 - y^2) \tag{6}$$

and, because $\Phi(0,h) = V$, it follows that

$$\Phi = -\frac{1}{2}C(0^2 - h^2) \tag{7}$$

so that once again, $C = 2V/h^2$.

(c) The potential and $\mathbf{E}$ are sketched in Fig. S4.1.8a.

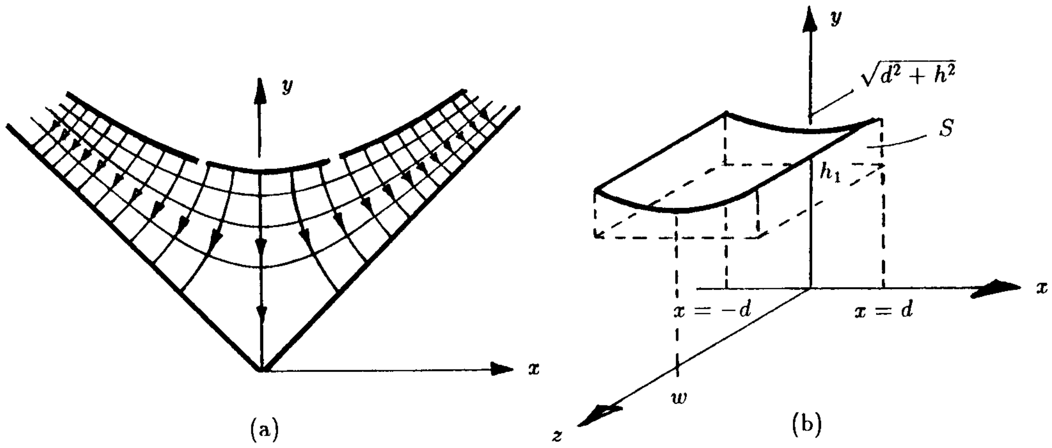

Figure S4.1.8

(d) Gauss' integral law is used to compute the charge on the electrode using the surface shown in Fig. S4.1.8b to enclose the electrode. There are six surfaces possibly contributing to the surface integration.

$$\oint_S \epsilon_o \mathbf{E} \cdot \mathbf{n} da = q \tag{8}$$

On the two having normals in the z direction, $\epsilon_o \mathbf{E} \cdot \mathbf{n} = 0$. In the region above the electrode the field is zero, so there is no contribution there either. On the two side surfaces and the bottom surface, the integrals are

$$\begin{aligned} q =& \epsilon_o \int_0^w \int_{h_1}^{\sqrt{d^2+h^2}} \mathbf{E}(d, y) \cdot \mathbf{i_x} dy dz \\ &+ \epsilon_o \int_0^w \int_{h_1}^{\sqrt{d^2+h^2}} \mathbf{E}(-d, y) \cdot (-\mathbf{i_x}) dy dz \\ &+ \epsilon_o \int_0^w \int_{-d}^{d} \mathbf{E}(x, h_1) \cdot (-\mathbf{i_y}) dx dz \end{aligned} \tag{9}$$

Completion of the integrals gives

$$q = \frac{4wd\epsilon_o \sqrt{d^2 + h^2}}{h^2} \tag{10}$$

4.1.9 By definition,

$$\Delta\Phi = \text{grad}\,(\Phi) \cdot \Delta\mathbf{r} \tag{1}$$

In cylindrical coordinates,

$$\Delta\mathbf{r} = \Delta r \mathbf{i_r} + r\Delta\phi \mathbf{i}_\phi + \Delta z \mathbf{i_z} \tag{2}$$

and

$$\begin{aligned} \Delta\phi &= \Phi(r + \Delta r, \Phi + \Delta\Phi, z + \Delta z) - \Phi(r, \phi, z) \\ &= \frac{\partial\Phi}{\partial r}\Delta r + \frac{\partial\Phi}{\partial\phi}\Delta\phi + \frac{\partial\Phi}{\partial z}\Delta z \end{aligned} \tag{3}$$

Thus,

$$\frac{\partial\Phi}{\partial r}\Delta r + \frac{\partial\Phi}{\partial\phi}\Delta\phi + \frac{\partial\Phi}{\partial z}\Delta z = \text{grad}\,\Phi \cdot (\Delta r \mathbf{i_r} + r\Delta\phi \mathbf{i}_\phi + \Delta z \mathbf{i_z}) \tag{4}$$

and it follows that the gradient operation in cylindrical coordinates is,

$$\text{grad}\,(\Phi) = \frac{\partial\Phi}{\partial r}\mathbf{i_r} + \frac{1}{r}\frac{\partial\Phi}{\partial\phi}\mathbf{i}_\phi + \frac{\partial\Phi}{\partial z}\mathbf{i_z} \tag{5}$$

4.1.10 By definition,

$$\Delta\Phi = \text{grad}\,(\Phi)\cdot\Delta\mathbf{r} \tag{1}$$

In spherical coordinates,

$$\Delta\mathbf{r} = \Delta r\mathbf{i_r} + r\Delta\theta\mathbf{i}_\theta + r\sin\theta\Delta\phi\mathbf{i}_\phi \tag{2}$$

and

$$\begin{aligned}\Delta\Phi &= \Phi(r+\Delta r, \theta+\Delta\theta, \phi+\Delta\phi) - \Phi(r,\theta,\phi)\\ &= \frac{\partial\Phi}{\partial r}\Delta r + \frac{\partial\Phi}{\partial\theta}\Delta\theta + \frac{\partial\Phi}{\partial\phi}\Delta\phi\end{aligned} \tag{3}$$

Thus,

$$\frac{\partial\Phi}{\partial r}\Delta r + \frac{\partial\Phi}{\partial\theta}\Delta\theta + \frac{\partial\Phi}{\partial\phi}\Delta\phi = \text{grad}\,(\Phi)\cdot(\Delta r\mathbf{i_r} + r\Delta\theta\mathbf{i}_\theta + r\sin\theta\Delta\phi\mathbf{i}_\phi) \tag{4}$$

and it follows that the gradient operation in spherical coordinates is,

$$\text{grad}(\Phi) = \frac{\partial\Phi}{\partial r}\mathbf{i_r} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\mathbf{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\mathbf{i}_\phi \tag{5}$$

4.2 POISSON'S EQUATION

4.2.1 In Cartesian coordinates, Poisson's equation requires that

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_o} \Rightarrow \rho = -\epsilon_o\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right) \tag{1}$$

Substitution of the potential

$$\Phi = \frac{\rho_o(t)}{\epsilon_o[(\pi/a)^2 + (\pi/b)^2]}\sin\frac{\pi}{a}x\sin\frac{\pi}{b}y \tag{2}$$

then gives the charge density

$$\begin{aligned}\rho = -&\frac{\rho_o(t)}{[(\pi/a)^2 + (\pi/b)^2]}[-(\pi/a)^2\sin\frac{\pi}{a}x\sin\frac{\pi y}{b}\\ &-(\pi/b)^2\sin\frac{\pi}{a}x\sin\frac{\pi}{b}y]\\ &= \rho_o(t)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\end{aligned} \tag{3}$$

4.2.2 In Cartesian coordinates, Poisson's equation requires that

$$\rho = -\epsilon_o\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right) \tag{1}$$

Substitution of the potential

$$\Phi = \frac{\rho_o}{\epsilon_o[(\pi/a)^2 + (\pi/b)^2]}\cos\frac{\pi}{a}x\cos\frac{\pi}{b}y \tag{2}$$

then gives the charge density

$$\rho = \rho_o\cos\frac{\pi}{a}x\cos\frac{\pi}{b}y \tag{3}$$

4.2.3 In cylindrical coordinates, the divergence and gradient are given in Table I as

$$\nabla\cdot\mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \tag{1}$$

$$\nabla u = \frac{\partial u}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial u}{\partial\phi}\mathbf{i}_\phi + \frac{\partial u}{\partial z}\mathbf{i}_z \tag{2}$$

By definition,

$$\nabla^2 u = \nabla\cdot\nabla u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial\phi}\left(\frac{1}{r}\frac{\partial u}{\partial\phi}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) \tag{3}$$

which becomes the expression also summarized in Table I.

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\phi^2} + \frac{\partial^2 u}{\partial z^2} \tag{4}$$

4.2.4 In spherical coordinates, the divergence and gradient are given in Table I as

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{1}$$

$$\nabla u = \frac{\partial u}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial u}{\partial\theta}\mathbf{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial u}{\partial\phi}\mathbf{i}_\theta \tag{2}$$

By definition,

$$\begin{aligned}\nabla^2 u = \nabla\cdot(\nabla u) &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r\sin\theta}\left(\frac{1}{r}\frac{\partial u}{\partial\theta}\sin\theta\right) \\ &+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)\end{aligned} \tag{3}$$

which becomes the expression also summarized in Table I.

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} \tag{4}$$

4.3 SUPERPOSITION PRINCIPLE

4.3.1 The circuit is shown in Fig. S4.3.1. Alternative solutions v_a and v_b must each satisfy the respective equations

$$C\frac{dv_a}{dt} + \frac{v_a}{R} = I_a(t); \quad (1)$$

$$C\frac{dv_b}{dt} + \frac{v_b}{R} = I_b(t) \quad (2)$$

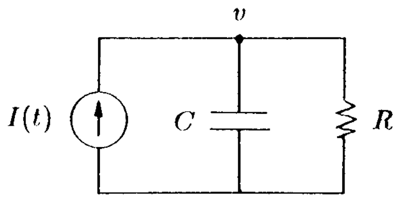

Figure S4.3.1

Addition of these two expressions gives

$$C\left[\frac{dv_a}{dt} + \frac{dv_b}{dt}\right] + \frac{1}{R}[v_a + v_b] = I_a + I_b \quad (3)$$

which, by dint of the linear nature of the derivative operator, becomes

$$C\frac{d}{dt}(v_a + v_b) + \frac{1}{R}(v_a + v_b) = I_a + I_b \quad (4)$$

Thus, if $I_a \Rightarrow v_a$ and $I_b \Rightarrow v_b$ then $I_a + I_b \Rightarrow v_a + v_b$.

4.4 FIELDS ASSOCIATED WITH CHARGE SINGULARITIES

4.4.1 (a) The electric field intensity for a line charge having linear density λ_l is

$$\mathbf{E} = \frac{\lambda_l}{2\pi\epsilon_o r}\mathbf{i_r} = -\frac{\partial\Phi}{\partial r}\mathbf{i_r} \quad (1)$$

Integration gives

$$\Phi = -\frac{\lambda_l}{2\pi\epsilon_o} ln(r/r_o) \quad (2)$$

where r_o is the position at which the potential is defined to be zero.

(b) In terms of the distances defined in Fig. S4.4.1, the potential for the pair of line charges is

$$\Phi = -\frac{\lambda_l}{2\pi\epsilon_o} ln\Big(\frac{r_+}{r_o}\Big) + \frac{\lambda_l}{2\pi\epsilon_o} ln\Big(\frac{r_-}{r_o}\Big) = \frac{\lambda_l}{2\pi\epsilon_o} ln\Big(\frac{r_-}{r_+}\Big) \tag{3}$$

where

$$r_\pm^2 = r^2 + (d/2)^2 \mp rd\cos\phi$$

Thus,

$$\Phi = \frac{\lambda}{4\pi\epsilon_o} ln\left[\frac{1 + (d/2r)^2 + \frac{d}{r}\cos\phi}{1 + (d/2r)^2 - \frac{d}{r}\cos\phi}\right] \tag{4}$$

For $d \ll r$, this is expanded in a Taylor series

$$ln\Big(\frac{1+x}{1+y}\Big) = ln(1+x) - ln(1+y) \approx x - y \tag{5}$$

to obtain the standard form of a two-dimensional dipole potential.

$$\Phi \rightarrow \frac{\lambda d}{2\pi\epsilon_o}\frac{\cos\phi}{r} \tag{6}$$

4.4.2 From the solution to Prob. 4.4.1, the potential of the pair of line charges is

$$\Phi = \frac{\lambda}{4\pi\epsilon_o} ln\left[\frac{1 + (2r/d)^2 + \frac{4r}{d}\cos\phi}{1 + (2r/d)^2 - \frac{4r}{d}\cos\phi}\right] \tag{1}$$

For a spacing that goes to infinity, $r/d \ll 1$ and it is appropriate to use the first term of a Taylor's expansion

$$ln\Big(\frac{1+x}{1+y}\Big) \simeq x - y \tag{2}$$

Thus, (1) becomes

$$\Phi = \frac{2\lambda}{\pi\epsilon_o d} r\cos\phi \tag{3}$$

In Cartesian coordinates, $x = r\cos\phi$, and (3) becomes

$$\Phi = \frac{2\lambda}{\pi\epsilon_o d} x \tag{4}$$

which is the potential of a uniform electric field.

$$\mathbf{E} = \frac{-2\lambda}{\pi\epsilon_o d}\mathbf{i_x} \tag{5}$$

4.4.3 The potential due to a line charge is

$$\Phi = \frac{\lambda}{2\pi\epsilon_o} ln \frac{r_o}{r} \tag{1}$$

where r_o is some reference. For the quadrapole,

$$\Phi = \frac{\lambda}{2\pi\epsilon_o}\left[ln\frac{r_o}{r_1} - ln\frac{r_o}{r_2} + ln\frac{r_o}{r_3} - ln\frac{r_o}{r_4}\right] = \frac{\lambda}{2\pi\epsilon_o}\left[ln\frac{r_2 r_4}{r_1 r_3}\right] \tag{2}$$

where, from Fig. P4.4.3,

$$r_1^2 = r^2[1 + (d/2r)^2 - (d/r)\sin\phi]$$

$$r_2^2 = r^2[1 + (d/2r)^2 + (d/r)\cos\phi]$$

$$r_3^2 = r^2[1 + (d/2r)^2 + (d/r)\sin\phi]$$

$$r_4^2 = r^2[1 + (d/2r)^2 - (d/r)\cos\phi]$$

With terms in $(d/2r)^2$ neglected, (2) therefore becomes

$$\Phi = \frac{\lambda}{4\pi\epsilon_o} ln\left\{\frac{1 - (d/r)^2\cos^2\phi}{1 - (d/r)^2\sin^2\phi}\right\} \tag{3}$$

for $d \ll r$.

Now $ln(1+x) \simeq x$ for small x so $ln[(1+x)/(1+y)] \simeq x - y$. Thus, (3) is approximately

$$\begin{aligned}\Phi &= \frac{\lambda}{4\pi\epsilon_o}\left[-(d/r)^2\cos^2\phi + (d/r)^2\sin^2\phi\right] \\ &= \frac{-\lambda d^2}{4\pi\epsilon_o r^2}[\cos^2\phi - \sin^2\phi] \\ &= \frac{-\lambda d^2}{4\pi\epsilon_o r^2}\cos 2\phi\end{aligned} \tag{4}$$

This is of the form $A\cos 2\phi/\gamma^n$ with

$$A = \frac{-\lambda d}{4\pi\epsilon_o}, \qquad n = 2 \tag{5}$$

4.4.4 (a) For $r \ll d$, we rewrite the distance functions as

$$r_1^2 = (d/2)^2\Big[\Big(\frac{2r}{d}\Big)^2 + 1 - \frac{4r}{d}\sin\phi\Big] \tag{1a}$$

$$r_2^2 = (d/2)^2\Big[\Big(\frac{2r}{d}\Big)^2 + 1 + \frac{4r}{d}\cos\phi\Big] \tag{1b}$$

$$r_3^2 = (d/2)^2\Big[\Big(\frac{2r}{d}\Big)^2 + 1 + \frac{4r}{d}\sin\phi\Big] \tag{1c}$$

$$r_4^2 = (d/2)^2\Big[\Big(\frac{2r}{d}\Big)^2 + 1 - \frac{4r}{d}\cos\phi\Big] \tag{1a}$$

With the terms $(2r/d)^2$ neglected, at follows that

$$\Phi = \frac{\lambda}{4\pi\epsilon_o} ln \left\{ \frac{1-(4r/d)^2\cos^2\phi}{1-(4r/d)^2\sin^2\phi} \right\} \tag{2}$$

Because $ln(1+x) \simeq x$ for $x \ll 1, \ln[(1+x)/(1+y)] \simeq x - y$ and (2) is approximately

$$\Phi = -\frac{\lambda}{4\pi\epsilon_o}\Big(\frac{4r}{d}\Big)^2[\cos^2\phi - \sin^2\phi] = -\frac{4\lambda r^2}{\pi\epsilon_o d^2}\cos 2\phi \tag{3}$$

This potential is seen again in Sec. 5.7. With the objective of writing it in Cartesian coordinates, (3) is written as

$$\begin{aligned}\Phi &= -\frac{4\lambda}{\pi\epsilon_o d^2}[r(\cos\phi + \sin\phi)r(\cos\phi - \sin\phi)] \\ &= \frac{-4\lambda}{\pi\epsilon_o d^2}[(x+y)(x-y)] = \frac{-4\lambda}{\pi\epsilon_o d^2}(x^2 - y^2)\end{aligned} \tag{4}$$

(b) Rotate the quadrapole by 45°.

4.5 SOLUTION OF POISSON'S EQUATION FOR SPECIFIED CHARGE DISTRIBUTIONS

4.5.1 (a) With $|\mathbf{r} - \mathbf{r}'| = \sqrt{x'^2 + y'^2 + z'^2}$, (4.5.5) becomes

$$\Phi = \int_{y'=-a}^{a}\int_{x'=-a}^{a} \frac{\sigma_s(x', y')dx'dy'}{4\pi\epsilon_o\sqrt{x'^2 + y'^2 + z^2}} \tag{1}$$

(b) For the particular charge distribution,

$$\begin{aligned}\Phi &= \frac{\sigma_o}{a^2\pi\epsilon_o}\int_{y'=0}^{a}\int_{x'=0}^{a}\frac{x'y'dx'dy'}{\sqrt{x'^2+y'^2+z^2}} \\ &= \frac{\sigma_o}{a^2\pi\epsilon_o}\int_{y'=0}^{a}\left[\sqrt{a^2+y'^2+z^2}\,y'-\sqrt{y'^2+z^2}\,y'\right]dy'\end{aligned} \tag{2}$$

To complete this second integration, let $u^2 = y'^2 + z^2, 2udu = 2y'dy'$ so that

$$\begin{aligned}\int_{y'=0}^{a} y'\sqrt{y'+z^2}dy' &= \int_{z}^{\sqrt{a^2+z^2}} u^2du = \frac{u^3}{3}\Big|_{z}^{\sqrt{a^2+z^2}} \\ &= \frac{1}{3}[(a^2+z^2)^{3/2}-z^3]\end{aligned} \tag{3}$$

Similarly,

$$\int_{y'=0}^{a} y'\sqrt{y'^2+(a^2+z^2)} = \frac{1}{3}[(2a^2+z^2)^{3/2}-(a^2+z^2)^{3/2}] \tag{4}$$

so that

$$\Phi = \frac{\sigma_o}{3a^2\pi\epsilon_o}[(2a^2+z^2)^{3/2}+z^3-2(a^2+z^2)^{3/2}] \tag{5}$$

(c) At the origin,

$$\Phi = \frac{\sigma_o}{3a^2\pi\epsilon_o}[(2a^2)^{3/2}-2a^3] = \frac{2\sigma_o a(\sqrt{2}-1)}{3\pi\epsilon_o} \tag{6}$$

(d) For $z \gg a$, (5) becomes approximately

$$\begin{aligned}\Phi &\simeq \frac{\sigma_o z^3}{3a^2\pi\epsilon_o}\left\{1+\left(\frac{2a^2}{z^2}+1\right)^{3/2}-2\left(\frac{a^2}{z^2}+1\right)^{3/2}\right\} \\ &= \frac{2\sigma_o z^3}{3a^2\pi\epsilon_o}\left\{1+\left(1+\frac{2a^2}{z^2}\right)\left(1+\frac{2a^2}{z^2}\right)^{1/2}-2\left(1+\frac{a^2}{z^2}\right)\left(1+\frac{a^2}{z^2}\right)^{1/2}\right\}\end{aligned} \tag{7}$$

For $a^2/z^2 \ll 1$, we use $(1+x)^{1/2} \simeq 1+\frac{1}{2}x$ and

$$\begin{aligned}\Phi &\simeq \frac{2\sigma_o z^3}{3a^2\pi\epsilon_o}\left\{1+\left(1+\frac{2a^2}{z^2}\right)\left(1+\frac{a^2}{z^2}\right)-2\left(1+\frac{a^2}{z^2}\right)\left(1+\frac{a^2}{2z^2}\right)\right\} \\ &= \frac{2\sigma_o z^3}{3a^2\pi\epsilon_o}\left\{1+\left(1+\frac{a^2}{z^2}\right)\left[1+\frac{2a^2}{z^2}-2-\frac{2a^2}{2z^2}\right]\right\} \\ &= \frac{2\sigma_o z^3}{3a^2\pi\epsilon_o}\left\{1+\left(1+\frac{a^2}{z^2}\right)\left(\frac{a^2}{z^2}-1\right)\right\}\end{aligned} \tag{8}$$

Thus,

$$\Phi = \frac{2\sigma_o a^2}{3\pi\epsilon_o z} \tag{9}$$

For a point charge Q at the origin, the potential along the z-axis is given by

$$\Phi = \frac{Q}{4\pi\epsilon_o z} \tag{10}$$

which is the same as the potential given by (9) if

$$Q = \frac{8\sigma_o a^2}{3} \tag{11}$$

(e) From (5),

$$\mathbf{E} = -\nabla\Phi = -\frac{\partial\Phi}{\partial z}\mathbf{i}_z = \frac{\sigma_o}{\pi a^2\epsilon_o}[z(2a^2+z^2)^{1/2} + z^2 - 2z(a^2+z^2)^{1/2}]\mathbf{i}_z \tag{12}$$

4.5.2 (a) Evaluation of (4.5.5) gives

$$\begin{aligned}\Phi &= \int_{\phi'=0}^{2\pi}\int_{\theta'=0}^{\pi}\frac{\sigma_o\cos\theta' R^2\sin\theta' d\phi' d\theta'}{4\pi\epsilon_o[R^2+z^2-4Rz\cos\theta']^{1/2}} \\ &= \frac{\sigma_o R^2}{4\epsilon_o}\int_{\theta'=0}^{\pi}\frac{\sin 2\theta' d\theta'}{\sqrt{R^2+z^2-2Rz\cos\theta'}}\end{aligned} \tag{1}$$

To integrate, let $u^2 = R^2 + z^2 - 2Rz\cos\theta'$ so that $2udu = 2Rz\sin\theta' d\theta'$ and note that $\cos\theta' = (R^2+z^2-u^2)/2Rz$. Thus, (1) becomes

$$\begin{aligned}\Phi &= \frac{\sigma_o}{4\epsilon_o z^2}\int_{z-R}^{(R+z)}(R^2+z^2-u^2)du \\ &= \frac{\sigma_o}{4\epsilon_o z^2}\Big[(R^2+z^2)(R+z) - \frac{(R+z)^3}{3} \\ &\quad - (R^2+z^2)(z-R) + \frac{(z-R)^3}{3}\Big] \\ &= \frac{\sigma_o R^3}{3\epsilon_o z^2}\end{aligned} \tag{2}$$

(b) Inside the shell, the lower limit of (2) becomes $(R-z)$. Then

$$\Phi = \frac{\sigma_o z}{3\epsilon_o} \tag{3}$$

(c) From (2) and (3)

$$\mathbf{E} = -\bar{\nabla}\Phi = -\frac{\partial\Phi}{\partial z}\mathbf{i}_z = \begin{cases} \frac{2\sigma_o R^3}{3\epsilon_o z^3}\mathbf{i}_z & z > R \\ -\frac{\sigma_o}{3\epsilon_o}\mathbf{i}_z & z < R \end{cases} \tag{4}$$

(d) Far away, the dipole potential on the z-axis would be $p/4\pi\epsilon_o z^2$ for the point charge dipole. By comparison of (2) to this expression the dipole moment is

$$p = \frac{4\pi\sigma_o R^3}{3} \tag{5}$$

4.5.3 (a) To find $\Phi(0,0,z)$ we use (4.5.4). For $\mathbf{r} = (0,0,z)$ and $\mathbf{r}' = a$ point on the cylinder of charge, $|\mathbf{r}-\mathbf{r}'| = \sqrt{(z-z')^2 + R^2}$. This distance is valid for an entire "ring" of charge. The incremental charge element is then $\sigma 2\pi R dz$ so that (4.5.4) becomes

$$\Phi(0,0,z) = \int_0^l \frac{\sigma_o 2\pi R dz'}{4\pi\epsilon_o\sqrt{(z-z')^2+R^2}} + \int_{-l}^0 \frac{-\sigma_o 2\pi R dz'}{4\pi\epsilon_o\sqrt{(z-z')^2+R^2}} \quad (1)$$

To integrate, let $q' = z - z', dq' = -dz'$ and transform the limits

$$\begin{aligned}\Phi &= \frac{\sigma_o R}{2\epsilon_o}\Big[-\int_z^{z-l}\frac{dq'}{\sqrt{q'^2+R^2}} + \int_{z+l}^{z}\frac{dq'}{\sqrt{q'^2+R^2}}\Big] \\ &= \frac{\sigma_o R}{2\epsilon_o}\left[-\ln q' + \sqrt{R^2+q'^2}\Big|_z^{z-l} + \ln|q' + \sqrt{R^2+q'^2}\Big|_{z+l}^{z}\right]\end{aligned} \quad (2)$$

Thus,

$$\begin{aligned}\Phi &= \frac{\sigma_o R}{2\epsilon_o}\ln\left[\frac{(z+\sqrt{R^2+z^2})(z+\sqrt{R^2+z^2})}{(z-l+\sqrt{R^2+(z-l)^2})(z+l+\sqrt{R^2+(z+l)^2})}\right] \\ &= \frac{\sigma_o R}{2\epsilon_o}[2\ln(z+\sqrt{R^2+z^2}) - \ln(z-l+\sqrt{R^2+(z-l)^2}) \\ &- \ln(z+l+\sqrt{R^2+(z+l)^2})]\end{aligned} \quad (3)$$

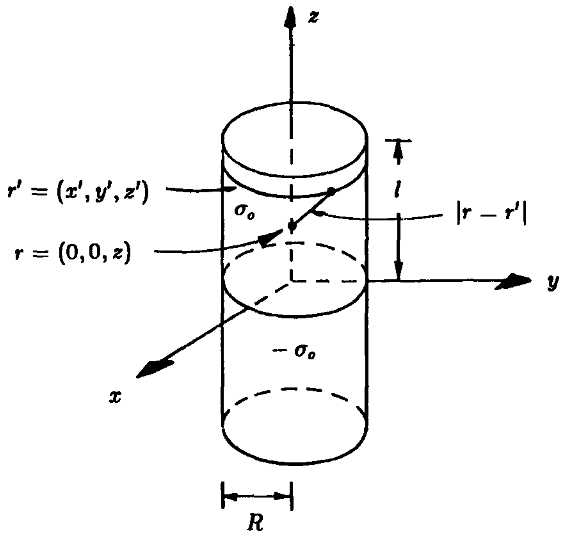

Figure S4.5.3

(b) Due to cylindrical geometry, there is no $\mathbf{i}_x$ or $\mathbf{i}_y$ field on the z axis.

$$\mathbf{E} = -\frac{\partial\Phi}{\partial z}\mathbf{i}_z = \frac{\mathbf{i}_z\sigma_o R}{2\epsilon_o}\Bigg[\frac{-2\left(1+\frac{z}{\sqrt{R^2+z^2}}\right)}{z+\sqrt{R^2+z^2}} + \left(\frac{1+\frac{(z-l)}{\sqrt{R^2+(z-l)^2}}}{z-l+\sqrt{R^2+(z-l)^2}}\right) + \left(\frac{1+\frac{(z+l)}{\sqrt{R^2+(z+l)^2}}}{z+l+\sqrt{R^2+(z+l)^2}}\right)\Bigg]$$
$$= \mathbf{i}_z\frac{\sigma_o R}{2\epsilon_o}\left[\frac{-2}{\sqrt{R^2+z^2}} + \frac{1}{\sqrt{R^2+(z-l)^2}} + \frac{1}{R^2+(z+l)^2}\right] \tag{4}$$

(c) First normalize all terms in Φ to z

$$\Phi = \frac{\sigma_o R}{2\epsilon_o}ln\left[\frac{\left(1+\sqrt{1+\frac{R^2}{z^2}}\right)\left(1+\sqrt{1+\frac{R^2}{z^2}}\right)}{\left(1-\frac{l}{z}+\sqrt{(R/z)^2+\left(1-\frac{l}{z}\right)^2}\right)\left(1+\frac{l}{z}+\sqrt{(R/z)^2+\left(1+\frac{l}{z}\right)^2}\right)}\right] \tag{5}$$

Then, for $z \gg l$ and $z \gg R$,

$$\approx \frac{\sigma_o R}{2\epsilon_o}ln\left[\frac{(1+1)(1+1)}{\left(1-\frac{l}{z}+1-\frac{l}{z}\right)\left(1+\frac{l}{z}+1+\frac{l}{z}\right)}\right]$$
$$= \frac{\sigma_o R}{2\epsilon_o}ln\left[\frac{4}{r(1-(l/z)^2)}\right]$$
$$= \frac{\sigma_o R}{2\epsilon_o}ln\left[\frac{1}{1-(l/z)^2}\right] \approx \frac{\sigma_o R}{2\epsilon_o}ln\left[1+(l/z)^2\right] \tag{6}$$
$$\approx \frac{\sigma_o R}{2\epsilon_o}\frac{l^2}{z^2}$$

The potential of a dipole with dipole moment p is

$$\Phi_{\text{dipole}} = \frac{p}{4\pi\epsilon_o}\frac{\cos\theta}{r^2} \tag{7}$$

In our case, $\cos\theta/r^2 = 1/z^2$, so $p = 2\pi R l^2$ (note the $p = qd, q = 2\pi R l\sigma_o, d_{eff} = l$).

4.5.4 From (4.5.12),

$$\Phi(x,y,z) = \int_{y'=-d/2}^{d/2} \frac{\lambda dy'}{4\pi\epsilon_o\sqrt{(x-a)^2+(y-y')^2+z^2}} \tag{1}$$

To integrate, let $u = y' - y$ so that (1) becomes

$$\Phi = \frac{\lambda}{4\pi\epsilon_o}\int_{-y-d/2}^{-y+d/2}\frac{du}{\sqrt{u^2+(x-a)^2+z^2}}$$
$$= \frac{\lambda}{4\pi\epsilon_o}ln\left[u+\sqrt{u^2+(x-a)^2+z^2}\right]_{-y-d/2}^{-y+d/2} \tag{2}$$

which is the given expression.

4.5.5 From (4.5.12),

$$\Phi(0,0,z) = \frac{\lambda_o}{4\pi\epsilon_o l}\left\{\int_{x'=0}^{l} \frac{x'dx'}{\sqrt{x'^2+(a-z)^2}} - \frac{x'dx'}{\sqrt{x'^2+(a+z)^2}}\right\}$$
$$= \frac{\lambda_o}{4\pi\epsilon_o l}\{2z + \sqrt{l^2+(a-z)^2} - \sqrt{l^2+(a^1+z^2)}\}$$

4.5.6 From (4.5.12),

$$\begin{aligned}\Phi(0,0,z) &= \int_{z'=-a}^{a} \frac{\lambda_o z' dz'}{4\pi\epsilon_o a(z-z')} = \frac{\lambda_o}{4\pi\epsilon_o a}\int_{z'=-a}^{a}\left(-1+\frac{z}{z-z'}\right)dz' \\ &= \frac{\lambda_o}{4\pi\epsilon_o a}[-a - z ln(z-a) - z + z ln(z+a)]\end{aligned} \tag{1}$$

Thus,

$$\Phi(0,0,z) = \frac{-\lambda_o}{4\pi\epsilon_o}[2a + z ln(\frac{z-a}{z+a})] \tag{2}$$

Because of the symmetry about the z axis, the only component of $\mathbf{E}$ is in the z direction

$$\begin{aligned}\mathbf{E} &= -\frac{\partial\Phi}{\partial z}\mathbf{i}_z = \frac{\lambda_o}{4\pi\epsilon_o}\left[ln(\frac{z-a}{z+a}) + z\{\frac{1}{z-a} - \frac{1}{z+a}\}\right]\mathbf{i}_z \\ &= \frac{\lambda_o}{4\pi\epsilon_o}\left[ln(\frac{z-a}{z+a}) + \frac{2az}{z^2-a^2}\right]\mathbf{i}_z\end{aligned} \tag{3}$$

4.5.7 Using (4.5.20)

$$\begin{aligned}\Phi &= -\int_{y'=0}^{\Delta}\int_{x'=-b}^{b} \frac{\sigma_o(d-b)}{\Delta 2\pi\epsilon_o(d-x')} ln|d-x'|dx'dy' \\ &= -\frac{\sigma_o(d-b)}{2\pi\epsilon_o}\int_{x'=-b}^{b}\frac{ln(d-x')}{(d-x')}dx' \\ &= -\frac{\sigma_o(d-b)}{2\pi\epsilon_o}\{-\frac{1}{2}[ln(d-x')]^2|_{-b}^{b}\} \\ &= \frac{\sigma_o(d-b)}{4\pi\epsilon_o}\{[ln(d-b)]^2 - [ln(d+b)]^2\}\end{aligned}$$

4.5.8 From (4.5.20),

$$\Phi(d,0) = -\int_{x'=0}^{2d} \frac{\sigma_o ln|d-x'|}{2\pi\epsilon_o}dx' + \int_{x'=-2d}^{0} \frac{\sigma_o ln|d-x'|dx'}{2\pi\epsilon_o} \tag{1}$$

To integrate let $u = d - x'$ and $du = -dx'$.

$$\begin{aligned}\Phi(d,0) &= \int_{d}^{-d} \frac{\sigma_o lnudu}{2\pi\epsilon_o} - \int_{3d}^{d} \frac{\sigma_o lnudu}{2\pi\epsilon_o} \\ &= \frac{\sigma_o}{2\pi\epsilon_o}\{u(ln|u|-1)|_{d}^{-d} - u(ln|u|-1)|_{3d}^{d}\} \\ &= \frac{\sigma_o}{2\pi\epsilon_o} 3dln3\end{aligned} \tag{2}$$

Thus, setting $\Phi(d,0) = V$ gives

$$\sigma_o = \frac{2\pi\epsilon_o V}{3dln3} \tag{3}$$

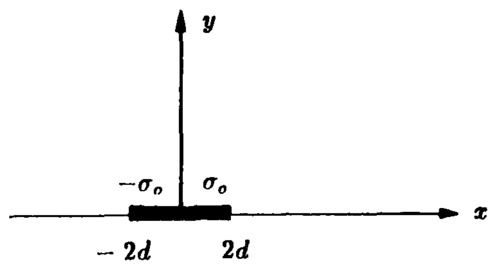

Figure S4.5.8

4.5.9 (a) (This problem might best be given while covering Sec. 8.2, where a stick model is developed for MQS systems.) At the lower end of the charge, ξ_c is the projection of **c** on **a**. This is given by

$$\xi_c = \frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|} \tag{1}$$

Similarly,

$$\xi_b = \frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|} \tag{2}$$

(b) From (4.5.20),

$$\Phi = \int_{\xi'=\xi_c}^{\xi_b} \frac{\lambda d\xi'}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|} = \int_{\frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|}}^{\frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|}} \frac{\lambda d\xi}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|} \tag{3}$$

where

$$|\mathbf{r}-\mathbf{r}'|^2 = \xi^2 + d^2$$

With θ defined as the angle between **a** and **b**,

$$|\mathbf{d}| = |\mathbf{b}|\sin\theta \tag{4}$$

But in terms of **a** and **b**,

$$\sin\theta = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|} \tag{5}$$

so that

$$d = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|} \tag{6}$$

and

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{\xi^2 + \frac{|\mathbf{a}\times\mathbf{b}|^2}{|\mathbf{a}|^2}} \tag{7}$$

(c) Integration of (3) using (6) and (7) gives

$$\Phi = \frac{\lambda}{4\pi\epsilon_o} ln\left\{\xi + \sqrt{\xi^2 + \frac{|\mathbf{a}\times\mathbf{b}|^2}{|\mathbf{a}|^2}}\right|_{\frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|}}^{\frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|}} \tag{8}$$

and hence the given result.

(d) For a line charge λ_o between $(x,y,z) = (0,0,d)$ and $(x,y,z) = (d,d,d)$,

$$\begin{aligned}
\mathbf{a} &= d\mathbf{i_x} + d\mathbf{i_y} \\
\mathbf{b} &= (d-x)\mathbf{i_x} + (d-y)\mathbf{i_y} + (d-z)\mathbf{i_z} \\
\mathbf{c} &= -x\mathbf{i_x} - y\mathbf{i_y} + (d-z)\mathbf{i_z} \\
\mathbf{b}\cdot\mathbf{a} &= d(d-x) + d(d-y) \\
\mathbf{c}\cdot\mathbf{a} &= -xd - yd \\
\mathbf{a}\times\mathbf{b} &= \begin{vmatrix} \mathbf{i_x} & \mathbf{i_y} & \mathbf{i_z} \\ d & d & 0 \\ d-x & d-y & d-z \end{vmatrix} \\
&= d(d-z)\mathbf{i_x} - d(d-z)\mathbf{i_y} + d(x-y)\mathbf{i_z} \\
|\mathbf{a}\times\mathbf{b}|^2 &= d^2[2(d-z)^2 + (x-y)^2] \\
(\mathbf{b}\cdot\mathbf{a})^2 &= d^2[(d-x)+(d-y)]^2 \\
(\mathbf{c}\cdot\mathbf{a})^2 &= d^2(x+y)^2
\end{aligned} \tag{9}$$

and evaluation of (c) of the problem statement gives (d).

4.5.10 This problem could be given in connection with covering Sec. 8.2. It illustrates the steps followed between (8.2.1) and (8.2.7), where the distinction between source and observer coordinates is also essential. Given that the potential has been found using the superposition integral, the required electric field is found by taking the gradient with respect to the observer coordinates, $\mathbf{r}$, not $\mathbf{r}'$. Thus, the gradient operator can be taken inside the integral, where it operates as though $\mathbf{r}'$ is a constant.

$$\mathbf{E} = -\nabla\Phi = -\int_V \nabla\Big[\frac{\rho(\mathbf{r}')}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|}\Big]dv' = -\int_{V'} \frac{\rho(\mathbf{r}')}{4\pi\epsilon_o}\nabla\Big[\frac{1}{|\mathbf{r}-\mathbf{r}'|}\Big]dv' \quad (1)$$

The arguments leading to (8.2.6) apply equally well here

$$\nabla\Big[\frac{1}{|\mathbf{r}-\mathbf{r}'|}\Big] = -\frac{1}{|\mathbf{r}-\mathbf{r}'|^2}\mathbf{i}_{r'r} \quad (2)$$

The result given with the problem statement follows. Note that we could just as well have derived this result by superimposing the electric fields due to point charges $\rho(\mathbf{r}')dv'$. Especially if coordinates other than Cartesian are used, care must be taken to recognize how the unit vector $\mathbf{i}_{r'r}$ takes into account the vector addition.

4.5.11 (a) Substitution of the given charge density into Poisson's equation results in the given expression for the potential.

(b) If the given solution is indeed the response to a singular source at the origin, it must (i) satisfy the differential equation, (a), at every point except the origin and (ii) it must satisfy (c). With the objective of showing that (i) is true, note that in spherical coordinates with no θ or ϕ dependence, (b) becomes

$$\frac{1}{r^2}\frac{d}{dr}\Big(r^2\frac{d\Phi}{dr}\Big) - \kappa^2\Phi = s(r) \quad (1)$$

Substitution of (e) into this expression gives zero for the left hand side at every point, r, except the origin. The algebra is as follows. First,

$$\frac{d}{dr}\Big(\frac{A}{r}e^{-\kappa r}\Big) = -\frac{A\kappa}{r}e^{-\kappa r} - \frac{Ae^{-\kappa r}}{r^2} \quad (2)$$

Then,

$$-\frac{1}{r^2}\frac{d}{dr}\Big(\frac{A\kappa}{r}e^{-\kappa r} + \frac{e^{-\kappa r}}{r^2}\Big) - \kappa^2\frac{Ae^{-\kappa r}}{r} = \frac{Ak^2}{r^2}e^{-\kappa r} + \frac{Ak^2}{r}e^{-\kappa r}$$
$$= 0; \quad r \neq 0 \quad (3)$$

To establish the coefficient, A, integrate Poisson's equation over a spherical volume having radius r centered on the origin. By virtue of its being singular there, what is being integrated has value only at the origin. Thus, we take the limit where the radius of the volume goes to zero.

$$\lim_{r\to 0}\left\{\int_V \nabla\cdot\nabla\Phi dv - \kappa^2\int_V \Phi dv\right\} = \lim_{r\to 0}\left\{-\frac{1}{\epsilon_o}\int_V s dv\right\} \quad (4)$$

Gauss' theorem shows that the first integral can be converted to a surface integral. Thus,

$$\lim_{r\to 0}\left\{\oint_S \nabla\Phi\cdot d\mathbf{a} - \kappa^2\int_V \Phi dv\right\} = \lim_{r\to 0}\left\{-\frac{1}{\epsilon_o}\int_V s dv\right\} \quad (5)$$

If the potential does indeed have the r dependence of (e), then it follows that

$$\lim_{r\to 0}\int_V \Phi dv = \lim_{r\to 0}\int_0^r \Phi 4\pi r^2 dr = 0 \quad (6)$$

so that in the limit, the second integral on the left in (5) makes no contribution and (5) reduces to

$$\lim_{r\to 0}\left(-\frac{A\kappa}{r}e^{-\kappa r} - \frac{Ae^{-\kappa r}}{r^2}\right)4\pi r^2 = -4\pi A = -\frac{Q}{\epsilon_o} \quad (7)$$

and it follows that $A = Q/4\pi\epsilon_o$.

(c) We have found that a point source, Q, at the origin gives rise to the potential

$$Q \Rightarrow \Phi = \frac{Q}{4\pi\epsilon_o}\frac{e^{-\kappa r}}{r} \quad (8)$$

Arguments similar to those given in Sec. 4.3 show that (b) is linear. Thus, given that we have shown that the response to a point source $\rho(\mathbf{r}')dv$ at $\mathbf{r} = \mathbf{r}'$ is

$$\rho(\mathbf{r}')dv \Rightarrow \Phi = \frac{\rho(\mathbf{r}')dv}{4\pi\epsilon_o}\frac{e^{-\kappa|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \quad (9)$$

it follows by superposition that the response to an arbitrary source distribution is

$$\Phi(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')e^{-\kappa|\mathbf{r}-\mathbf{r}'|}}{4\pi\epsilon_o(\mathbf{r}-\mathbf{r}'|}dv \quad (10)$$

4.5.12 (a) A cross-section of the dipole layer is shown in Fig. S4.5.12a. Because the field inside the layer is much more intense than that outside and because the layer is very thin compared to distances over which the surface charge density varies with position in the plane of the layer, the fields inside are as though the surface charge density resided on the surfaces of plane parallel planes. Thus, Gauss' continuity condition applied to either of the surface charge densities shows that the field inside has the given magnitude and the direction must be that of the normal vector.

$$\mathbf{E}_{int} = -\sigma_s \mathbf{N}/\epsilon_o \tag{1}$$

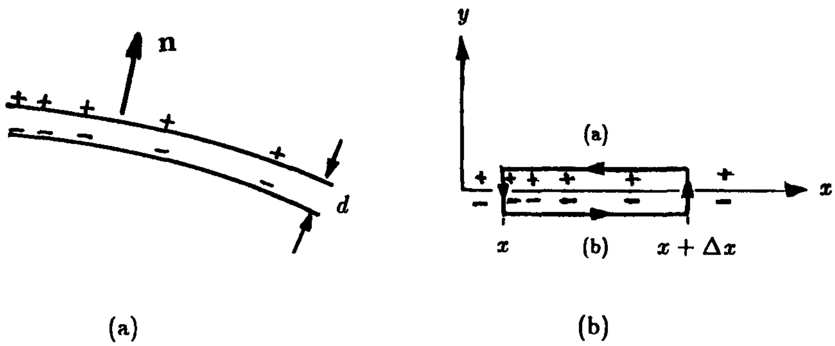

Figure S4.5.12

(b) It follows from (4.1.1) and the contour shown in Fig. S4.5.12b having incremental length Δx in the x direction that

$$-E_x^a \Delta x + E_x^b \Delta x + E_y(x + \Delta x)d - E_y(x)d = 0 \tag{2}$$

Divided by Δx, this expression becomes

$$-E_x^a + E_x^b + d\frac{\partial E_y}{\partial x} = 0 \tag{3}$$

The given expression then follows by using (1) to replace E_y with $-\sigma/\epsilon_o$ and recognizing that $\pi_s \equiv \sigma_s d$.

4.6 ELECTROQUASISTATIC FIELDS IN THE PRESENCE OF PERFECT CONDUCTORS

4.6.1 In view of (4.5.12),

$$\Phi(0,0,a) = \int_c^b \frac{\lambda\left(\frac{a-z'}{a-c}\right)}{4\pi\epsilon_o(a - z')} dz' \tag{1}$$

The z dependence of the integrand cancels out so that the integration amounts to a multiplication.

$$\Phi(0,0,a) = \frac{\lambda_o}{4\pi\epsilon_o(a-c)}(b-c) \tag{2}$$

The net charge is

$$Q = \frac{1}{2}\left[\lambda_o\left(\frac{a-b}{a-c}\right) + \lambda_o\right](b-c) \tag{3}$$

Provied that the equipotential surface passing through $(0,0,a)$ encloses all of the segment, the capacitance of an electrode having the shape of this surface is then given by

$$C = \frac{Q}{\Phi(0,0,a)} = 2\pi\epsilon_o(2a - b - c) \tag{4}$$

4.6.2 (a) The potential is the sum of the potentials due to the charge producing the uniform field and the point charges. With $r_\pm$ defined as shown in Fig. S4.6.2a,

$$\Phi = -E_o z + \frac{q}{4\pi\epsilon_o r_+} - \frac{q}{4\pi\epsilon_o r_-} \tag{1}$$

where

$$z = r\cos\theta$$

$$r_\pm = \sqrt{r^2 + (d/2)^2 \mp 2r\frac{d}{2}\cos\theta}$$

To write (1) in terms of the normalized variables, divide by $E_o d$ and multiply and divide $r_\pm$ by d. The given expression, (b), then follows.

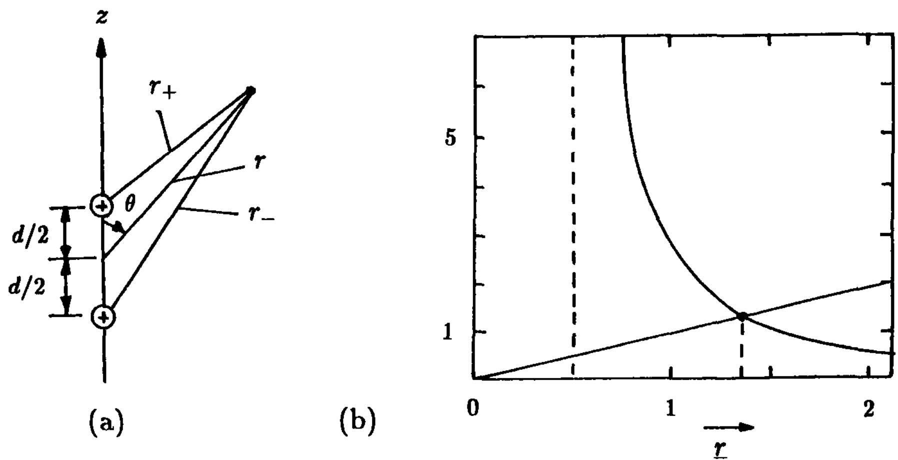

Figure S4.6.2

(b) An implicit expression for the intersection point $d/2 < r$ on the z axis is given by evaluating (b) with $\Phi = 0$ and $\theta = 0$.

$$\underline{r} = \frac{\underline{q}}{(\underline{r} - \frac{1}{2})} - \frac{\underline{q}}{(\underline{r} + \frac{1}{2})} \tag{2}$$

The graphical solution of this expression for $d/2 < r(1/2 < \underline{r})$ is shown in Fig. S4.6.2b. The required intersection point is $\underline{r} = 1.33$. Because the right hand side of (2) has an asymptote at $\underline{r} = 0.5$, there must be an intersection between the straight line representing the left side in the range $0.5 < \underline{r}$.

(c) The plot of the $\Phi = 0$ surface for $0 < \theta < \pi/2$ is shown in Fig. S4.6.2c.

(c)

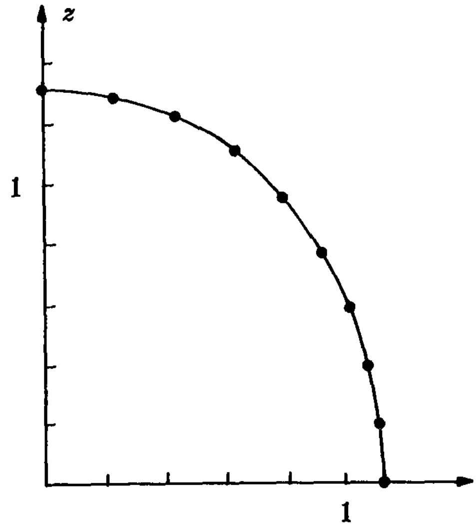

Figure S4.6.2

(d) At the north pole of the object, the electric field is z-directed. It therefore follows from (b) as $(0.5 < \underline{r})$

$$
\begin{aligned}
E_z &= -\frac{\partial \Phi}{\partial r} = -E_o \frac{\partial \underline{\Phi}}{\partial \underline{r}} = -E_o \frac{\partial}{\partial \underline{r}}\left(-\underline{r} + \frac{\underline{q}}{\underline{r} - \frac{1}{2}} - \frac{\underline{q}}{\underline{r} + \frac{1}{2}} \right) \\
&= E_o \left[1 + \frac{\underline{q}}{\left(\underline{r} - \frac{1}{2}\right)^2} = \frac{\underline{q}}{\left(\underline{r} + \frac{1}{2}\right)^2} \right]
\end{aligned}
\tag{3}
$$

Evauation of this expression at $\underline{r} = 1.33$ and $\underline{q} = 2$ gives $E_z = 3.33 E_o$.

(e) Gauss' integral law, applied to a surface comprised of the equipotential and the plane $z = 0$, shows that the net charge on the northern half of the object is q. For the given equipotential, $\underline{q} = 2$. It follows from the definition of $\underline{q}$ that

$$
\underline{q} = 2 = \frac{q}{4\pi\epsilon_o E_o d^2} \Rightarrow Q = q = 8\pi\epsilon_o E_o d^2 \tag{4}
$$

4.6.3 For the disk of charge in Fig. 4.5.3, the potential is given by (4.5.7)

$$
\Phi = \frac{\sigma_o}{2\epsilon_o}\left(\sqrt{R^2 + z^2} - |z|\right) \tag{1}
$$

At $(0, 0, d)$,

$$
\Phi(0, 0, d) = \frac{\sigma_o}{2\epsilon_o}\left(\sqrt{R^2 + d^2} - d\right) \tag{2}
$$

and

$$q = \sigma_o \pi R^2 \tag{3}$$

Thus

$$C \equiv \frac{q}{\Phi(0,0,d)} = \frac{2\epsilon_o \pi R^2}{\sqrt{R^2+d^2}-d} \tag{4}$$

4.6.4 (a) Due to the top sphere,

$$\Phi_+ = \frac{Q}{4\pi\epsilon_o r_1} \tag{1}$$

and similarly,

$$\Phi_- = \frac{-Q}{4\pi\epsilon_o r_2} \tag{2}$$

At the bottom of the top sphere

$$\Phi_+ = \frac{Q}{4\pi\epsilon_o R} - \frac{Q}{4\pi\epsilon_o (h-R)} \tag{3}$$

while at the top of the bottom sphere

$$\Phi_-\Big|_{r=R} = \frac{-Q}{4\pi\epsilon_o R} + \frac{Q}{4\pi\epsilon_o (h-R)} \tag{4}$$

The potential difference between the two spherical conductors is therefore

$$V = \frac{2Q}{4\pi\epsilon_o R} - \frac{2Q}{4\pi\epsilon_o (h-R)} = \frac{Q}{2\pi\epsilon_o R}(1 - \frac{R/h}{1-R/h}) \tag{3}$$

The maximum field occurs at $z = 0$ on the axis of symmetry where the magnitude is the sum of that due to point charges.

$$\mathbf{E}_{\max} = \frac{-2Q\mathbf{i}_z}{4\pi\epsilon_o (h/2)^2} = \frac{-2Q}{\pi\epsilon_o h^2}\mathbf{i}_z \tag{4}$$

(b) Replace point charge Q at $z = h/2$ by $Q_1 = Q\frac{R}{h}$ at $z = \frac{h}{2} - \frac{R^2}{h}$ and $Q_o = Q[1 - \frac{R}{h}]$ at $z = h/2$. The potential on the surface of the top sphere is now

$$\Phi_{\text{top}} = \frac{Q_o}{4\pi\epsilon_o R} + \frac{Q_1}{4\pi\epsilon_o (R - \frac{R^2}{h})} - \frac{Q}{4\pi\epsilon_o (h-R)} \tag{5}$$

The potential on the surface of the bottom sphere is

$$\Phi_{\text{bottom}} = \frac{Q_o}{4\pi\epsilon_o (h-R)} + \frac{Q_1}{4\pi\epsilon_o (h - R - \frac{R^2}{h})} - \frac{Q}{4\pi\epsilon_o R} \tag{6}$$

The potential difference is then,

$$V = \frac{Q_o}{4\pi\epsilon_o}[(\frac{1}{R} - \frac{1}{h-R})] + \frac{Q_1}{4\pi\epsilon_o}\left[\frac{1}{R-\frac{R^2}{h}} - \frac{1}{h-R-\frac{R^2}{h}}\right] - \frac{Q}{4\pi\epsilon_o}(\frac{1}{h-R} - \frac{1}{R})$$

For four charges $Q_1 = QR/h$ at $z = h/2 - R^2/h$; $Q_o = Q(1-\frac{R}{h})$ at $z = h/2$; $Q_2 = -QR/h$ at $z = -h/2 + R^2/h$; $Q_3 = -Q(1-\frac{R}{h})$ at $z = -h/2$ and

$$\Phi_{\text{top}} = \frac{Q_o}{4\pi\epsilon_o R} + \frac{Q_1}{4\pi\epsilon_o R(1-\frac{R}{h})} + \frac{Q_2}{4\pi\epsilon_o(h-R-\frac{R^2}{h})} + \frac{Q_3}{4\pi\epsilon_o(h-R)} \tag{7}$$

which becomes

$$\Phi_{\text{top}} = \frac{Q(1-\frac{R}{h})}{4\pi\epsilon_o R} + \frac{Q\frac{R}{h}}{4\pi\epsilon_o R(1-\frac{R}{h})} - \frac{Q(R^2/h^2)}{4\pi\epsilon_o(1-\frac{R}{h}-\frac{R^2}{h^2})} - \frac{Q(R/h)}{4\pi\epsilon_o R} \tag{8}$$

Similarly,

$$\Phi_{\text{bottom}} = \frac{Q(R/h)}{4\pi\epsilon_o R} + \frac{Q(R^2/h^2)}{4\pi\epsilon_o R(1-\frac{R}{h}-\frac{R^2}{h^2})} - \frac{QR/h}{4\pi\epsilon_o R(1-\frac{R}{h})} - \frac{Q(1-R/h)}{4\pi\epsilon_o} \tag{9}$$

so that

$$V = \frac{2Q}{4\pi\epsilon_o R}\left\{1 - \frac{R}{h} + \frac{R/h}{1-\frac{R}{h}} - \frac{R^2/h^2}{1-\frac{R}{h}-\frac{R^2}{h^2}} - \frac{R}{h}\right\} \tag{10}$$

$$V = \frac{Q}{2\pi\epsilon_o R}\left\{1 - \frac{2R}{h} + \frac{R/h}{1-R/h} - \frac{(R/h)^2}{1-\frac{R}{h}-(R/h)^2}\right\} \tag{11}$$

$$C = \frac{Q}{V} = \frac{2\pi\epsilon_o R}{1 - \frac{2R}{h} + \frac{(R/h)}{1-(R/h)} - \frac{(R/h)^2}{1-(R/h)-(R/h)^2}} \tag{12}$$

4.6.5 (a) The potential is the sum of that given by (a) in Prob. 4.5.4 and a potential due to a similarly distributed negative line charge on the line at $x = -a$ between $y = -d/2$ and $y = d/2$.

$$\Phi = \frac{\lambda_l}{4\pi\epsilon_o} ln\left\{ \left[\frac{d}{2} - y + \sqrt{(x-a)^2 + (\frac{d}{2} - y)^2 + z^2}\right] \left[-\frac{d}{2} - y + \sqrt{(x+a)^2 + (\frac{d}{2} + y)^2 + z^2}\right] / \left[-\frac{d}{2} - y + \sqrt{(x-a)^2 + (\frac{d}{2} + y)^2 + z^2}\right] \left[\frac{d}{2} - y + \sqrt{(x+a)^2 + (\frac{d}{2} - y)^2 + z^2}\right] \right\} \tag{1}$$

(b) The equipotential passing through $(x, y, z) = (a/2, 0, 0)$ is given by evaluating (1) at that point

$$V = \frac{\lambda_l}{4\pi\epsilon_o} ln\left\{ \frac{[\frac{d}{2} + \sqrt{\frac{a^2}{4} + \frac{d^2}{4}}][-\frac{d}{2} + \sqrt{\frac{9}{4}a^2 + \frac{d^2}{4}}]}{[-\frac{d}{2} + \sqrt{\frac{a^2}{4} + \frac{d^2}{4}}][\frac{d}{2} + \sqrt{\frac{9}{4}a^2 + \frac{d^2}{4}}]} \right\} \tag{2}$$

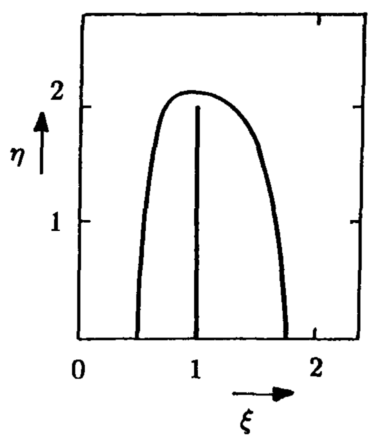

Figure S4.6.5

(c) In normalized form, (2) becomes

$$\underline{\phi} = \frac{ln\left\{ \frac{\left(2-\eta+\sqrt{(\xi-1)^2+(2-\eta)^2+z^2}\right)\left(-2-\eta+\sqrt{(\xi+1)^2+(2+\eta)^2+z^2}\right)}{\left(-2-\eta+\sqrt{(\eta-1)^2+(2+\eta)^2+z^2}\right)\left(2-\eta+\sqrt{(2-\eta)^2+(2-\eta)^2+z^2}\right)} \right\}}{ln\left\{ \frac{(4+\sqrt{1+16})(-4+\sqrt{9+16})}{(-4+\sqrt{1+16})(4+\sqrt{9+16})} \right\}} \tag{3}$$

where $\underline{\Phi} = \Phi/\Phi(\frac{a}{2},0,0), \xi = x/a, \eta = y/a$ and $d = 4a$. Thus, $\underline{\Phi} = 1$ for the equipotential passing through $(\frac{a}{2},0,0)$. This equipotential can be found by writing it in the form $f(\xi,\eta) = 0$, setting η and having a programmable calculator determine ξ. In the first quadrant, the result is as shown in Fig. S4.6.5.

(d) The lines of electric field intensity are sketched in Fig. S4.6.5.

(e) The charge on the surface of the electrode is the same as the charge enclosed by the equipotential in part (c), $Q = \lambda_l d$. Thus,

$$C = \frac{\lambda_l d}{V} = 4\pi\epsilon_o d/ln\left\{\frac{[d+\sqrt{a^2+d^2}][-d+\sqrt{9a^2+d^2}]}{[-d+\sqrt{a^2+d^2}][d+\sqrt{9a^2+d^2}]}\right\} \tag{4}$$

4.7 METHOD OF IMAGES

4.7.1 (a) The potential is due to Q and its image, $-Q$, located at $z = -d$ on the z axis.

(b) The equipotential having potential V and passing through the point $z = a < d, x = 0, y = 0$ is given by evaluating this expression and taking care in taking the square root to recognize that $d > a$.

$$V = \frac{Q}{4\pi\epsilon_o}\left(\frac{1}{d-a} - \frac{1}{d+a}\right) = \frac{Q}{4\pi\epsilon_o}\left(\frac{2a}{d^2-a^2}\right) \tag{1}$$

In general, the equipotential surface having potential V is

$$V = \frac{Q}{4\pi\epsilon_o}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right] \tag{2}$$

The given expression results from equating these last two expressions.

(c) The potential is infinite at the point charge and goes to zero at infinity and in the plane $z = 0$. Thus, there must be an equipotential contour that encloses the point charge. The charge on the electrode having the shape given by (2) must be equal to Q so the capacitance follows from (1) as

$$C = \frac{Q}{V} = 2\pi\epsilon_o\frac{(d^2-a^2)}{a} \tag{3}$$

4.7.2 (a) The line charge and associated square boundaries are shown at the center of Fig. S4.7.2. In the absence of image charges, the equipotentials would be circular. However, with images that alternate in sign to infinity in each direction, as shown, a grid of square equipotentials is established and hence the boundary conditions on the central square are met. At each point on the boundary, there is an equal distance to both a positive and a negative line charge. Hence, the potential on the boundary is zero.

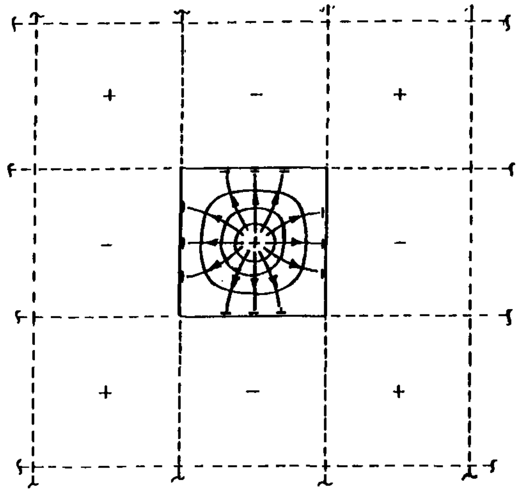

Figure S4.7.2

(b) The equipotentials close to the line charge are circular. As the other boundary is approached, they approach the square shape of the boundary. The lines of electric field intensity are as shown, terminating on negative surface charges on the surface of the boundary.

4.7.3 (a) The bird acquires the same potential as the line, hence has charges induced on it and conserves charge when it flies away.

(b) The fields are those of a charge Q at $y = h, x = Ut$ and an image at $y = -h$ and $x = Ut$.

(c) The potential is the sum of that due to Q and its image $-Q$.

$$\Phi = \frac{Q}{4\pi\epsilon_o}\left[\frac{1}{\sqrt{(x-Ut)^2+(y-h)^2+z^2}} - \frac{1}{\sqrt{(x-Ut)^2+(y+h)^2+z^2}}\right] \quad (1)$$

(d) From this potential

$$E_y = -\frac{\partial\Phi}{\partial y} = \frac{Q}{4\pi\epsilon_o}\left\{\frac{y-h}{(x-Ut)^2+(y-h)^2+z^2]^{3/2}} - \frac{y+h}{[(x-Ut)^2+(y+h)^2+z^2]^{3/2}}\right\} \quad (2)$$

Thus, the surface charge density is

$$\sigma_s = \epsilon_o E_y\Big|_{y=0} = \frac{Q\epsilon_o}{4\pi\epsilon_o}\left[\frac{-h}{[(x-Ut)^2+h^2+z^2]^{3/2}} - \frac{h}{[(x-Ut)^2+h^2+z^2]^{3/2}}\right] = \frac{-Qh}{2\pi[(x-Ut)^2+h^2+z^2]^{3/2}} \tag{3}$$

(e) The net charge q on electrode at any given instant is

$$q = \int_{z=0}^{w}\int_{x=0}^{l} \frac{-Qh\,dx\,dz}{2\pi[(x-Ut)^2+h^2+z^2]^{3/2}} \tag{4}$$

If $w \ll h$,

$$q = \int_{x=0}^{l} \frac{-Qhw\,dx}{2\pi[(x-Ut)^2+h^2]^{3/2}} \tag{5}$$

For the remaining integration, $x' = (x - Ut)$, $dx' = dx$ and

$$q = \int_{-Ut}^{l-Ut} \frac{-Qhw\,dx'}{2\pi[x'^2+h^2]^{3/2}} \tag{6}$$

Thus

$$q = -\frac{Qw}{2\pi h}\left[\frac{l-Ut}{\sqrt{(l-Ut)^2+h^2}} + \frac{Ut}{\sqrt{(Ut)^2+h^2}}\right] \tag{7}$$

(f) The dahsed curves (1) and (2) in Fig. S4.7.3 are the first and second terms in (7), respectively. They sum to give (3)

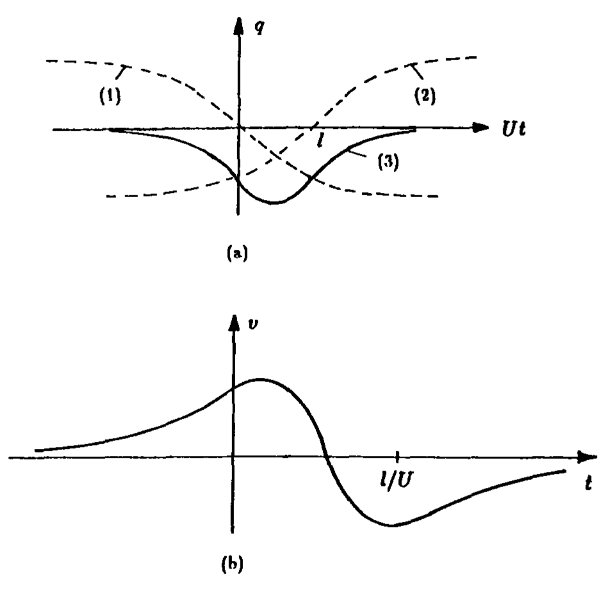

Figure S4.7.3

(g) The current follows from (7) as

$$i = \frac{dq}{dt} = -\frac{Qw}{2\pi h}\left[\frac{-Uh^2}{[(l-Ut)^2+h^2]^{3/2}} + \frac{Uh^2}{[(Ut)^2+h^2]^{3/2}}\right] \tag{8}$$

and the voltage is then $v = -iR = -Rdq/dt$. A sketch is shown in Fig. S4.7.3b.

4.7.4 For no normal **E**, we want image charges of the same sign; $+\lambda$ at $(-a, 0)$ and $-\lambda$ at $(-b, 0)$. The potential in the $x = 0$ plane is then,

$$\begin{aligned}\Phi &= -\frac{2\lambda}{2\pi\epsilon_o} ln(a^2+y^2)^{1/2} + \frac{2\lambda}{2\pi\epsilon_o} ln(b^2+y^2)^{1/2} \\ &= -\frac{\lambda}{2\pi\epsilon_o} ln\left(\frac{a^2+y^2}{b^2+y^2}\right)\end{aligned} \tag{1}$$

4.7.5 (a) The image to make the $x = 0$ plane an equipotential is a line charge $-\lambda$ at $(x, y) = (-d, d)$. The image of these two line charges that makes the plane $y = 0$ an equipotential is a pair of line charges, $+\lambda$ at $(-d, -d)$ and $-\lambda$ at $(d, -d)$. Thus

$$\begin{aligned}\Phi &= -\frac{\lambda}{4\pi\epsilon_o} ln[(x-d)^2+(y-d)^2] - \frac{\lambda}{4\pi\epsilon_o} ln[(x+d)^2+(y+d)^2] \\ &+ \frac{\lambda}{4\pi\epsilon_o} ln[(x-d)^2+(y+d)^2] + \frac{\lambda}{4\pi\epsilon_o} ln[(x+d)^2+(y-d)^2] \\ &= \frac{\lambda}{4\pi\epsilon_o} ln\left\{\frac{[(x-d)^2+(y+d)^2][(x+d)^2+(y-d)^2]}{[(x-d)^2+(y-d)^2][(x+d)^2+(y+d)^2]}\right\}\end{aligned} \tag{1}$$

(b) The surface of the electrode has the potential

$$\Phi(a,a) = \frac{\lambda}{4\pi\epsilon_o} ln\left\{\frac{[(a-d)^2+(a+d)^2][(a+d)^2+(a-d)^2]}{[(a-d)^2+(a-d)^2][(a+d)^2+(a+d)^2]}\right\} = V \tag{2}$$

Then

$$\frac{C}{\text{length}} = \frac{\lambda}{V} = \frac{4\pi\epsilon_o}{\ln\left\{\frac{[(a+d)^2+(a-d)^2]^2}{2[a-d]^2 2[a+d]^2}\right\}} = \frac{2\pi\epsilon_o}{ln\left[\frac{a^2+d^2}{a^2-d^2}\right]} \tag{3}$$

4.7.6 (a) The potential of a disk at $z = s$ is given by 4.5.7 with $z \rightarrow z - s$

$$\Phi(z > s) = \frac{\sigma_o}{2\epsilon_o}\left[\sqrt{R^2+(z-s)^2} - |z-s|\right] \tag{1}$$

The ground plane is represented by an image disk at $z = -s$; (4.5.7) with $z \rightarrow z + s$. Thus, the total potential is

$$\Phi = \frac{\sigma_o}{2\epsilon_o}\left[\sqrt{R^2+(z-s)^2} - |z-s| - \sqrt{R^2+(z+d)^2} + |z+s|\right] \tag{2}$$

(b) The potential at $z = d < s$ is

$$\begin{aligned}\Phi(z = d < s) &= \frac{\sigma_o}{2\epsilon_o}[\sqrt{R^2 + (d-s)^2} - |d-s| - \sqrt{R^2 + (d+s)^2} + |d+s|] \\ &= \frac{\sigma_o}{2\epsilon_o}[\sqrt{R^2 + (d-s)^2} - (s-d) - \sqrt{R^2 + (d+s)^2} + s + d] \\ &= \frac{\sigma_o}{2\epsilon_o}[\sqrt{R^2 + (d-s)^2} + 2d - \sqrt{R^2 + (d+s)^2}] = V\end{aligned} \tag{3}$$

Thus,

$$C = \frac{Q}{V} = \frac{2\epsilon_o \pi R^2}{\sqrt{R^2 + (d-s)^2} - \sqrt{R^2 + (d+s)^2} + 2d}$$

4.7.7 From (4.5.4),

$$\begin{aligned}\Phi(0,0,a) &= \int_{\phi=0}^{2\pi}\int_{r=0}^{R} \frac{\frac{\sigma_o}{R} r dr d\phi}{4\pi\epsilon_o\sqrt{r^2 + (h-a)^2}} + \int_{\phi=0}^{2\pi}\int_{r=0}^{R} \frac{-\frac{\sigma_o r}{R} r dr d\phi}{4\pi\epsilon_o\sqrt{r^2 + (h+a)^2}} \\ &= \frac{\sigma_o}{2\epsilon_o R}\left[\int_{r=0}^{R} \frac{r^2 dr}{\sqrt{r^2 + (h-a)^2}} - \int_{r=0}^{R} \frac{r^2 dr}{\sqrt{r^2 + (h+a)^2}}\right] \\ &= \frac{\sigma_o}{4\epsilon_o R}\left[\frac{R}{2}(\sqrt{R^2 + (h-a)^2}\right. \\ &\quad - \sqrt{R^2 + (h+a)^2}) + (h-a)^2 ln(\frac{h-a}{\sqrt{R^2 + (h-a)^2}}) \\ &\quad \left. + (h+a)^2 ln(\frac{R + \sqrt{R^2 + (h+a)^2}}{h+a})\right]\end{aligned} \tag{1}$$

The total charge in the disk is

$$Q = \int_{\phi=0}^{2\pi}\int_{r=0}^{R} \frac{\sigma_o r}{R} r dr d\phi = \frac{2\pi}{3} R^2 \sigma_o$$

Thus,

$$\begin{aligned}C = \frac{Q}{V} = &\left\{2\pi R^3 \epsilon_o\right\} / \left\{\frac{R}{2}[\sqrt{R^2 + (h-a)^2}\right. \\ &- \sqrt{R^2 + (h+a)^2}] \\ &+ (h-a)^2 ln(\frac{h-a}{\sqrt{R^2 + (h-a)^2}}) \\ &\left. + (h+a)^2 ln(\frac{R^2 + \sqrt{R^2 + (h+a)^2}}{(h+a)})\right\}\end{aligned}$$

4.7.8 Because there is perfectly conducting material at $z = 0$ there is the given line charge and an image from $(0,0,-d)$ to $(d,d,-d)$. Thus, for these respective line charges

$$
\begin{gathered}
\mathbf{a} = d\mathbf{i_x} + d\mathbf{i_y} \\
\mathbf{f} = (d-x)\mathbf{i_x} + (d-y)\mathbf{i_y} + (\pm d - z)\mathbf{i_z} \\
\mathbf{c} = -x\mathbf{i_x} - y\mathbf{i_y} + (\pm d - z)\mathbf{i_z} \qquad (1) \\
\mathbf{b}\cdot\mathbf{a} = d[(d-x)+(d-y)] \\
\mathbf{c}\cdot\mathbf{a} = -xd - yd \\
\mathbf{a}\times\mathbf{b} = d(\pm d - z)\mathbf{i_x} - \mathbf{i_y} d(\pm d - z) + \mathbf{i_z} d[(d-y)-(d-x)] \\
|\mathbf{a}\times\mathbf{b}| = d^2(\pm d - z)^2 + d^2(\pm d - z)^2 + d^2[(d-y)-(d-x)]^2
\end{gathered}
$$

The potential due to the line charge and its image then follows (c) of Prob. 4.5.9.

$$
\begin{aligned}
\Phi = \frac{\lambda}{4\pi\epsilon_o} ln \Bigg\{ & \frac{2d - x - y + \sqrt{2[(d-x)^2 + (d-y)^2 + (d-z)^2]}}{-x - y + \sqrt{2[x^2 + y^2 + (d-z)^2]}} \\
& \cdot \frac{-x - y + \sqrt{2[x^2 + y^2 + (d+z)^2]}}{2d - x - y + \sqrt{2[(d-x)^2 + (d-y)^2 + (d+z)^2]}} \Bigg\}
\end{aligned} \qquad (2)
$$

4.8 CHARGE SIMULATION APPROACH TO BOUNDARY VALUE PROBLEMS

4.8.1 For the six-segment system, the first two of (4.8.5) are

$$S_{11}\sigma_1 + S_{12}\sigma_2 + S_{13}\sigma_3 + S_{14}\sigma_4 + S_{15}\sigma_5 + S_{16}\sigma_6 = \frac{V}{2} \qquad (1)$$

$$S_{21}\sigma_1 + S_{22}\sigma_2 + S_{23}\sigma_3 + S_{24}\sigma_4 + S_{25}\sigma_5 + S_{26}\sigma_6 = \frac{V}{2} \qquad (2)$$

Because of the symmetry,

$$\sigma_1 = \sigma_3 = -\sigma_4 = -\sigma_6, \quad \sigma_2 = -\sigma_3 \qquad (3)$$

and so these two expressions reduce to two equations in two unknowns. (The other four expressions are identical to (4).)

$$\begin{bmatrix} (S_{11} + S_{13} - S_{14} - S_{16}) & (S_{12} - S_{15}) \\ (S_{21} + S_{23} - S_{24} - S_{26}) & (S_{22} - S_{25}) \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} V/2 \\ V/2 \end{bmatrix} \qquad (4)$$

Thus,

$$\sigma_1 = \frac{V}{2D}[(S_{22} - S_{25}) - (S_{12} - S_{15})] \tag{5}$$

$$\sigma_2 = \frac{V}{2D}[(S_{11} + S_{13} - S_{14} - S_{16}) - (S_{21} + S_{23} - S_{24} - S_{26})] \tag{6}$$

where

$$D = (S_{11} + S_{13} - S_{14} - S_{16})(S_{22} - S_{25}) - (S_{21} + S_{23} - S_{24} - S_{26})(S_{12} - S_{15})$$

and from (4.8.3)

$$C = \frac{1}{V}(b/3)[2\sigma_1 + \sigma_2] \tag{7}$$