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*Continuum Electromechanics*

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Problems for Chapter 11

For Section 11.2:

Prob. 11.2.1 Starting with the Lagrangian description of the particle motions afforded by Eq. 2, show that if  $\vec{E} = -\nabla\phi$  and  $\phi$  is independent of time, the sum of potential and kinetic energies,  $\frac{1}{2}m(\vec{v}\cdot\vec{v})+q\phi$ , is invariant.

For Section 11.3:

Prob. 11.3.1 A planar version of the magnetron configuration considered in this section makes use of planar electrodes that play the role of the coaxial ones in Fig. 11.3.1. The cathode, which is at zero potential, is in the plane  $x = 0$  while the anode is at  $x = a$  and has the potential  $V$ . Thus, ignoring space-charge effects, the electric potential is  $Vx/a$ . A uniform magnetic flux density,  $B_0$ , is imposed in the  $z$  direction.

- (a) Write the equations of motion for an electron in terms of its position  $\vec{r} = x\vec{i}_x + y\vec{i}_y + z\vec{i}_z$ .
- (b) Combine these expressions to obtain a single second-order differential equation for  $x(t)$ .
- (c) Under the assumption that  $V$  is constant, integrate this expression once (in a way analogous to that used in obtaining Eq. 11.3.2) to obtain a potential well that is determined by the combined effects of the applied electric and magnetic fields. Use this result to generate a potential-well picture analogous to Fig. 11.3.2 and qualitatively describe how the particle trajectories are influenced by the applied potential. (Assume that particles leave the cathode with no velocity.)
- (d) Sketch typical trajectories in the  $x$ - $y$  plane.
- (e) What is the critical potential,  $V = V_c$ ?

For Section 11.4:

Prob. 11.4.1 An electron has an initial velocity that is purely axial and moves in a region of constant potential and uniform axial magnetic field. Show that Eqs. 11.2.3-11.2.5 are satisfied by a subsequent motion in which the electron continues to move in only the  $z$  direction. How is it that Eq. 11.4.8 can predict the focusing effect of the field even though it only involves the axial component of  $\vec{B}$ ?

Prob. 11.4.2 In a magnetic lens, the potential,  $\phi$ , is constant. Use Eq. 11.4.8 to show that all magnetic lenses are converging, in the sense that  $dr/dz$  will be less as a particle leaves the lens than as it enters.

Prob. 11.4.3 If the focal length is long compared to the magnetic length of the magnetic lens, it is said to be a weak lens. In this case, most of the deflection occurs outside the field region with the lens serving to redirect the particles. Essentially,  $r$  remains constant through the lens, but its spatial derivative is altered. Use Eq. 11.4.8 to show that in this case the focal length is

$$f = \frac{8\phi m}{e} \left( \int_{z_-}^{z_+} B_z^2 dz \right)^{-1}$$

Show that this general result for a weak lens is consistent with Eq. 11.4.12.

Prob. 11.4.4 An electron beam crosses the  $z = 0$  plane from the region  $z < 0$  with the potential  $V_0$ . As it crosses, a given electron has radial position  $r_0$  and moves parallel to the  $z$  axis. In the  $z = 0$  plane, a potential  $\phi = V_0 J_0(\gamma r)$  is imposed so that for  $z > 0$ ,  $\phi = V_0 J_0(\gamma r) \exp(-\gamma r)$  (See Eq. 2.16.18 with  $jk \rightarrow \gamma$  and  $m = 0$ .) Use the paraxial ray equation, Eq. 11.4.8, to determine the electron trajectory. Assume that  $\gamma r_0 \ll 1$ .

For Section 11.5:

Prob. 11.5.1 Under what conditions does an axial magnetic field suppress transverse electron motions? To answer this question take the potential distribution as having been determined from Eq. 11.7.7 and write the transverse components of the force equation with the potential terms as "drives." Argue that transverse motions are small compared to longitudinal ones provided that the frequency is low compared to the electron-cyclotron frequency.

Prob. 11.5.2 An electron beam of annular cross section with outer radius  $a$  and inner radius  $b$  streams in the  $z$  direction with velocity  $U$ .

- (a) Show that the transfer relations are as summarized in Table 2.16.2 with  $k^2 \rightarrow \gamma^2 \equiv k^2 [1 - \omega_p^2 / (\omega - kU)^2]$  in the coefficients.
- (b) As an application of these transfer relations, consider a beam of radius  $b$  (no free-space core) with the potential constrained to be  $\hat{\phi}^c$  at the radius  $a$ . The region  $b < r < a$  is free space. Find  $\hat{D}_r^c$
- (c) Find the eigenfrequencies of the temporal modes with a perfect conductor at  $r = a$ .

For Section 11.6:

Prob. 11.6.1 A physical situation is represented by  $m$  dependent variables  $x_j$

$$x_j = x_1, \dots, x_j, \dots, x_m$$

which satisfy  $m$  first-order partial differential equations

$$\sum_{j=1}^m [F_{ij} \frac{\partial x_j}{\partial t} + G_{ij} \frac{\partial x_j}{\partial z}] = 0$$

(For example, in Sec. 11.6,  $m = 2$ ,  $x_1 = \rho$  and  $x_2 = v$ .) The coefficients  $F_{ij}$  and  $G_{ij}$  are functions of the  $x$ 's as well as  $(z, t)$ .

- (a) Use the method of undetermined multipliers to find a determinantal equation for the first characteristic equations.
- (b) Show that the same determinantal equation results by requiring that the coefficient matrix vanish.

For Section 11.7:

Prob. 11.7.1 There is a complete analogy between shallow-water gravity-wave dynamics and the one-dimensional compressible motions of gases studied in this section. Use Eqs. 9.13.11 and 9.13.12 with  $V = 0$  and  $b = \text{constant}$  to show that if  $\gamma = 2$ , analogous quantities are

$$\rho \leftrightarrow \xi, \quad v \leftrightarrow v, \quad \frac{a^2}{\rho} \leftrightarrow g$$

The analogy is exploited in the film "Waves in Fluids," which deals with both types of wave systems. (See Reference 10, Appendix C.)

Prob. 11.7.2 The quasi-one-dimensional equations of motion for free surface flow contained by an electric field (Fig. 9.13.3) are Eqs. 9.13.4 and 9.13.9, with  $A$  and  $\rho$  given in Fig. 9.13.3. Find the associated first and second characteristic equations.

Prob. 11.7.3 An inviscid, incompressible fluid rests on a rigid flat bottom, as shown in Fig. 9.13.1. Consider the motions with  $V = 0$ .

- (a) Show that the first characteristic equations are

$$\frac{dz}{dt} = v \pm \frac{1}{2} R(\xi) \text{ on } C^\pm$$

and that the second characteristic equations are

$$v = \mp R(\xi) + c_\pm \text{ on } C^\pm$$

$$\text{where } R(\xi) = 2 \sqrt{g\xi}.$$

- (b) A simple wave propagates into a region of constant depth  $\xi_c$  and zero velocity. At  $z = 0$ ,  $\xi = \xi_s(t)$ . Find  $v$  and  $\xi$  for  $z > 0$ ,  $t > 0$ . Show that only if  $d\xi_s/dt > 0$  will a shock form.
- (c) Consider the initial value problem: when  $t = 0$ ,  $v = v_0(z, 0) = 1$  and  $\xi = \xi_0(z, 0)$ , where

Prob. 11.7.3 (continued)

$$\xi_0(z,0) = \begin{cases} 1 & \text{for } z < -3 \\ -0.3|z| + 1.9 & \text{for } -3 < z < 3 \\ 1 & \text{for } 3 < z \end{cases}$$

For computational purposes, set  $g = 1$  and use seven characteristics in each family originating at equal intervals along the initial pulse. Evaluate the Riemann invariants  $c_+$  and  $c_-$  along the seven  $C^+$  and seven  $C^-$  characteristics.

- (d) Find  $v$  and  $R(\xi)$  at each of the characteristic intersections.  
 (e) Find the characteristic slopes at each of the intersections and draw the characteristics in the  $z, t > 0$  plane.  
 (f) Plot the velocity as a function of  $z$  when  $t = 0, t = 2$  and  $t = 4$ .

Prob. 11.7.4 The region between plane parallel conducting plates having essentially infinite extent in the  $y-z$  plane is filled with material described by the constitutive laws

$$\vec{B} = \mu_0 \vec{H}; \quad \vec{D} = \epsilon \vec{E} + \delta (\vec{E} \cdot \vec{E}) \vec{E}$$

where  $\mu_0, \epsilon$  and  $\delta$  are constants. The plates form a transmission line for transverse electromagnetic plane waves propagating in the  $z$  direction.

- (a) Show that if  $\vec{E} = E(z,t) \hat{i}_x$  and  $\vec{H} = H(z,t) \hat{i}_y$ , Faraday's and Ampere's laws require that

$$\frac{\partial E}{\partial z} = -\mu_0 \frac{\partial H}{\partial t}; \quad \frac{\partial H}{\partial z} = -(3\delta E^2 + \epsilon) \frac{\partial E}{\partial t}$$

- (b) Show that the first characteristic equations are

$$\frac{dz}{dt} = \pm [\mu_0 (3\delta E^2 + \epsilon)]^{-1/2} \quad \text{on } C^\pm$$

and that the second characteristic equations are

$$H = \mp R(E) + c_\pm \quad \text{on } C^\pm$$

where  $R(E)$  is

$$R(E) = \sqrt{\frac{3\delta}{4\mu_0}} \left[ E \sqrt{E^2 + \frac{\epsilon}{2\delta}} + \frac{\epsilon}{3\delta} \ln \left( E + \sqrt{E^2 + \frac{\epsilon}{3\delta}} \right) \right]$$

- (c) With no electric or magnetic fields between them when  $t = 0$ , the plates are driven at  $z = 0$  by a voltage source that imposes the field  $E(0,t) = E_0(t)$ . Prove that the problem is over-specified by imposition of  $H(0,t) = H_0(t)$ . Show that all  $C^+$  characteristics are straight lines. Take  $\mu_0, \epsilon$  and  $\delta$  as unity and the drive

$$E(0,t) = E_0(t)$$

where

$$E_0 = \begin{cases} 0 & \text{for } t < 0 \\ \frac{1 - \cos(t\pi/2)}{2} & \text{for } 0 < t < 2 \\ 1 & \text{for } 2 < t \end{cases}$$

and draw the  $C^+$  characteristics. (Use 7 lines originating at equal intervals in time from  $0 < t < 2$ .) Will a shock form? Use a sketch to show how the transient appears as it passes positions  $z = 0, 1/3$  and  $2/3$  and sketch to show how  $E$  depends on  $z$  when  $t = 0, 1$  and  $2$ .

Prob. 11.7.5 An axially symmetric deformation of a theta-pinch plasma is shown in the figure. The plasma column has a radius  $\xi(z,t)$  and is perfectly conducting. The total flux of magnetic field  $\Lambda_0$  trapped between the plasma and the surrounding perfectly conducting wall (radius  $a$ ) is constant.

Prob. 11.7.5 (continued)

- (a) Model the plasma as a perfectly conducting incompressible fluid and write quasi-one-dimensional equations of motion that include non-linear effects (see Sec. 9.13.)
- (b) When  $t = 0$ , the plasma is static and has the "bulge" shown in Fig. P11.7.5b. Show that the interface never subsequently has a radius greater than  $\xi_{\text{peak}}$ .

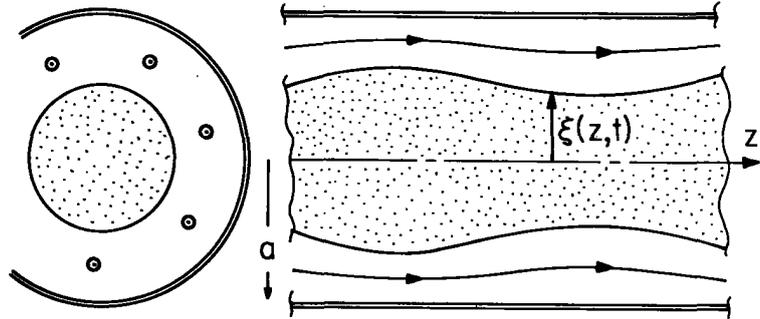


Fig. P11.7.5a

For Section 11.8:

Prob. 11.8.1 A perfectly conducting gas subject to isentropic dynamics is modeled as in Sec. 11.8. However, one-dimensional motions now considered are arbitrary in form. Consider again the one-dimensional space-time dependence of all variables; e.g.  $\vec{v} = \vec{v}(x,t)$  and  $\vec{H} = \vec{H}(x,t)$ . Determine the first characteristic equations.

For Section 11.9:

Prob. 11.9.1 Show that for small amplitudes the phase space trajectory for the configuration of Fig. 11.9.1b is represented by an ellipse.

Prob. 11.9.2 In the configuration of Fig. 11.1.1b, the driving voltage is  $V(t) = \text{Re } \hat{V} \exp j(\omega t)$ , where  $\hat{V}$  and  $\omega$  are given. Under the assumption that  $V(t)$  is small enough to give only a linear response, use the characteristic equations to find  $n(z,t)$ . Assume that the system is in the sinusoidal steady state.

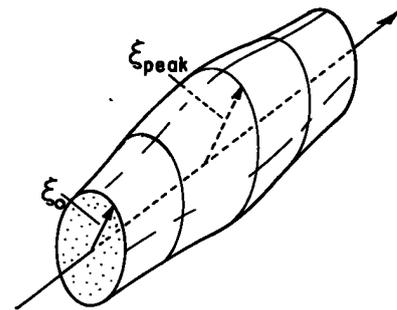


Fig. P11.7.5b

For Section 11.10:

Prob. 11.10.1 Use Eqs. 11.10.7 and 11.10.8 to derive an analytical expression for the  $P=0$  case illustrated by Fig. 11.10.3a.

Prob. 11.10.2 Consider the limit of the single-stream systems of Fig. 11.10.1 in which  $\gamma \rightarrow \infty$  and hence  $V \rightarrow 0$ . In this case, the normalization velocity should be  $U$  rather than  $V$ .

- (a) Show that the normalized equation of motion is

$$\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial z} \right)^2 \xi = \frac{P}{4} \left[ \frac{1}{(1-\xi)^2} - \frac{1}{(1+\xi)^2} \right]$$

where  $P$  is as defined after Eq. 11.10.6.

- (b) In this limit the situation is similar to that of Sec. 11.9, in that the characteristics degenerate into the same lines. Show that with  $\underline{v} \equiv \partial \xi / \partial t + \partial \xi / \partial z$  the normalized characteristic equations are

$$\frac{d}{dt} \left( \frac{1}{2} v^2 + E \right) = 0; \quad E \equiv -\frac{P}{4} \left( \frac{1}{1-\xi} + \frac{1}{1+\xi} \right) \quad \text{on } \frac{dz}{dt} = 1$$

- (c) In the phase-plane  $(v, \xi)$  a given element of the system starts with  $(v, \xi) = (v_0, \xi_0)$ . Sketch typical phase-plane trajectories for  $P > 0$  and  $P < 0$  and relate to the physical situations of Fig. 11.10.1.

Prob. 11.10.3 Starting with Eqs 11.10.19 to 11.10.22, use the determinant approach to deduce Eqs. 11.10.23 to 11.10.26. To deduce the solution at  $E$  in Fig. 11.10.5 from that at points  $A$  through  $D$ , the procedure is similar to that for the single stream situations (illustrated by Eqs. 11.10.11 to 11.10.14.) For example, because the solution at  $E$  must be the same whether obtained along the  $C_1^+$  or  $C_1^-$  characteristics,  $v_{1E} = v_{1A} + \Delta v_{1A}^+ = v_{1B} + \Delta v_{1B}^+$ . Write eight equations, linear in  $(\Delta v_{1A}^+, \Delta v_{1B}^+)$ ,

Prob. 11.10.3 (continued)

$\Delta v_{2C}^+$ ,  $\Delta v_{2D}^+$ ,  $\Delta e_{1A}^+$ ,  $\Delta e_{1B}^+$ ,  $\Delta e_{2C}^+$ ,  $\Delta e_{2D}^+$ ), that can be used to find these quantities in terms of the initial data ( $v_{1A}$ ,  $v_{1B}$ ,  $v_{2C}$ ,  $v_{2D}$ ,  $e_{1A}$ ,  $e_{1B}$ ,  $e_{2C}$ ,  $e_{2D}$ ,  $f_{1A}$ ,  $f_{1B}$ ,  $f_{2C}$ ,  $f_{2D}$ ). Check Eqs. 11.10.27 and 11.10.28, and the analogous expressions for ( $\Delta v_{2C}^+$ ,  $\Delta e_{2C}^+$ ).

Prob. 11.11.1 For the magnetic configuration of Fig. 11.11.1, follow steps paralleling Eqs. 11.11.1 and 11.11.2 to show that the long-wave linearized equation of motion is Eq. 11.11.3 with parameters defined by Eqs. 11.11.4b and 11.11.5b.

Prob. 11.11.2 A wire having tension  $T$  and mass per unit length  $m$  is stretched along the  $z$  axis. It suffers transverse displacements  $\xi(z,t)$ , in the  $x$ - $z$  plane and carries a current  $I$ , as shown in Fig. P11.11.2. External coils are used to impose a magnetic flux density  $\vec{B} = (B_0/d)(y\hat{i}_x + x\hat{i}_y)$ . ( $B_0$  and  $d$  are given constants) in the neighborhood of the  $z$  axis. This is the configuration for the experiment shown in Fig. 11.11.3. Show that the equation of motion takes the form of Eq. 11.11.3 with  $M = 0$  and  $f = 0$ . What is  $P$ ?

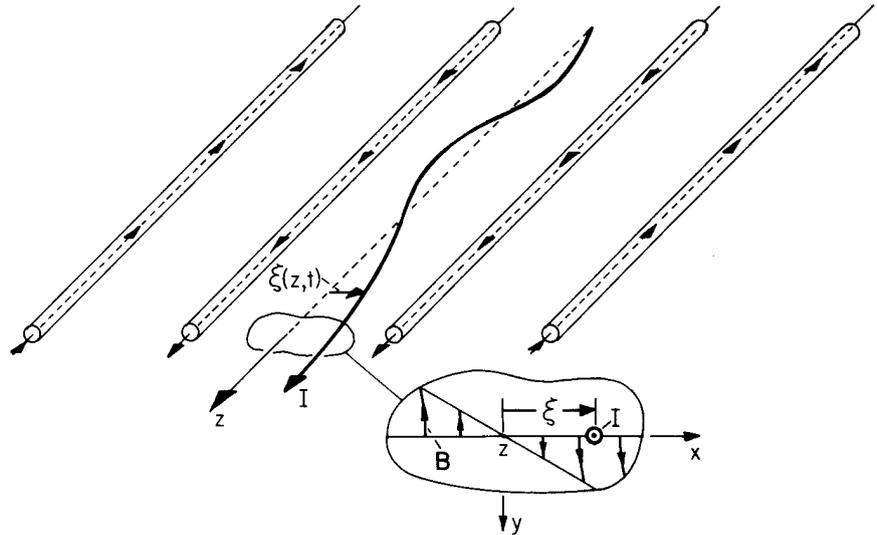


Fig. P11.11.2

Prob. 11.11.3 Derive Eqs. 11.11.13 and 11.11.14 starting with the roots of the dispersion equation, Eq. 11.11.11. Sketch the response for  $\omega_0$  less than and greater than the cut-off frequency, indicating the physical significance of  $\eta$  and  $\gamma$ .

Prob. 11.11.4 Derive Eqs. 11.11.15 and 11.11.16 and sketch the response for  $\omega_0^2$  greater than and less than  $P(M^2-1)$ . Indicate the physical significance of  $\eta$  and  $\gamma$ .

Prob. 11.11.5 A liquid jet having equilibrium radius  $R$  streams in the  $z$  direction with velocity  $U$ . Coaxial with the jet is a circular cylindrical electrode having inner radius  $a$  and constant potential,  $-V$ , relative to the jet. The stream, perhaps tap water, can be regarded as perfectly conducting. Except for the coaxial electrode, the configuration is the same as described in Prob. 8.13.1.

- Determine the dispersion equation for perturbations having  $m = 1$  (kinking motions).
- Take the long-wave limit of this relation ( $ka \ll 1$  and hence  $kR \ll 1$ ) to obtain a quadratic in  $\omega$  and  $k$  representing the dominant modes of the system. Normalize this expression so that it takes the same form as Eq. 11.11.10 and define  $M$  and  $P$  accordingly.

Prob. 11.12.1 The equation for the deflection  $\xi(z,t)$  of a "string" is the forced wave equation

$$\frac{\partial^2 \xi}{\partial t^2} = v^2 \frac{\partial^2 \xi}{\partial z^2} + \frac{f(z,t)}{m}$$

where  $v^2$  is the tension divided by the mass per unit length,  $m$ . The force per unit length,  $f(z,t)$ , is an impulse in space and a sinusoidal function of time turned on when  $t=0$ .

$$f(z,t) = [(\Delta z)f_0]u_0(z) \cos \omega_0 t u_1(t)$$

- Determine the Fourier-Laplace transform of the deflection,  $\hat{\xi}(k,\omega)$ . Assume that when  $t = 0$  the deflection is zero.
- Invert the Fourier transform and use the residue theorem to find  $\hat{\xi}(z,\omega)$ .
- Invert the Laplace transform by using the residue theorem and causality to determine  $\xi(z,t)$ . Check the result to see that it satisfies initial and boundary conditions and the equation of motion.

Prob. 11.12.2 In Sec. 5.17, the spatial transient of charge induced on a moving semi-insulating sheet is considered. It is assumed that in the sinusoidal steady state considered the complex waves decay away from the region of excitation. Show that all spatial modes are indeed evanescent, thus justifying the identification of spatial modes used in Sec. 5.17. Consider in particular the two

Prob. 11.12.2 (continued)

dominant modes represented by the long-wave dispersion equation, Eq. 5.17.11, and show the loci of complex  $k$  as  $\omega$  is varied ( $\omega = \omega_r - j\sigma$ ,  $\sigma$  increasing from  $\infty \rightarrow 0$ ).

For Section 11.13:

Prob. 11.13.1 For the subelectric second order system ( $|M| < 1$ ,  $P > 0$ ), evaluate the asymptotic response by integrating Eq. 11.13.1 on the contour shown in Fig. 11.13.4.

Prob. 11.13.2 A flexible tube is the conduit for liquid having an average velocity  $U$ . The tube is immersed in a viscous liquid which dampens transverse (kinking) motions. Its walls are elastic and (perhaps due to the viscous shear stresses from the liquid flowing within) under tension. A model for transverse motions pictures the tube as a string under tension. Because most of the inertia is due to the moving fluid within, the inertial term in the equation of motion is a convective derivative. But, because damping of transverse motions is largely due to the stationary external fluid, the damping force per unit length is proportional to the rate of change of the deflection at a given location,  $z$ . Thus, the equation of motion is

$$\left[ \frac{\partial}{\partial t} + U \frac{\partial}{\partial z} \right]^2 \xi = V^2 \frac{\partial^2 \xi}{\partial z^2} - \nu \frac{\partial \xi}{\partial t}$$

where  $V$  is the wave-velocity associated with the tension and mass per unit length and  $\nu$  is the damping coefficient per unit length divided by the effective mass per unit length.

- Show that, if  $|U| > V$ , the tube is subject to kinking motions that are unstable.
- Show that if  $|U| > V$ , this instability is convective.
- Argue the result of part (b) using the method of characteristics.

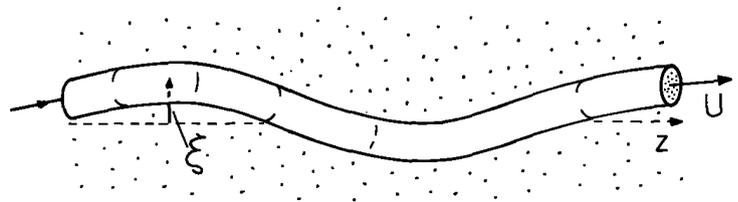


Fig. P11.13.2. Mechanical system exhibiting resistive wall instability.

For Section 11.14:

Prob. 11.14.1 In the configuration of Fig. 11.14.1, the lower fluid has a charge relaxation time that is short compared to times of interest, while the upper one is a good insulator, having uniform permittivity  $\epsilon$ . The channel walls are metal and have a potential difference,  $V$ . Thus, with the interface flat, there is an  $x$ -directed uniform electric field in the upper fluid,  $E_0 = V/a$ .

- Show that a stationary equilibrium exists, much as for the mechanical configuration. What is the stationary pressure difference across the interface under equilibrium conditions?
- For a perturbation having a given real wavenumber,  $k$ , what streaming velocity is required to produce instability? Discuss the conditions for instability in the short-wave and long-wave limits.
- Is this electromechanical Kelvin-Helmholtz type instability absolute or convective?

Prob. 11.14.2 In the configuration of Fig. 11.14.1, the lower fluid is perfectly conducting in the MQS sense while the upper fluid is a perfect insulator. The channel walls are also perfectly conducting. In a state of stationary equilibrium, much as for the mechanical system considered in Sec. 11.14, the region occupied by the upper fluid is also filled by an initially uniform  $z$ -directed magnetic field intensity  $H_0$ .

- What are the static pressures in each fluid under the stationary equilibrium conditions?
- For a given perturbation wavenumber,  $k$ , what velocity is required to produce instability? Does the magnetic field tend to stabilize the interface against Kelvin-Helmholtz instability?
- Are instabilities convective or absolute?

Prob. 11.14.3 A  $z$ -theta pinch has the configuration shown in Fig. 8.12.1. However, rather than having a static equilibrium, the perfectly conducting column now has a stationary equilibrium in which it streams in the  $z$  direction with a uniform velocity,  $U$ . Also, it is now surrounded by an insulating fluid having a mass density,  $\rho_v$ , that is appreciable. In the stationary equilibrium, the column has a uniform radius,  $R$ , and the surrounding fluid is static. For temporal modes, having given real wavenumber and mode number  $(k, m)$ , what velocity  $U$  is required to produce instability?

Prob. 11.14.4 The configuration shown in Fig. 8.14.2 is revised by having the upper uniformly charged fluid stream in a direction parallel to the equilibrium interface with a velocity  $U$ .

- Determine the dispersion equation for perturbations of the interface.
- Determine the velocity required to induce instability of a perturbation having a given wavenumber,  $k$ .

For Section 11.15:

Prob. 11.15.1 Streams described by Eqs. 11.15.1 and 11.15.2 have equal and opposite velocities. That is,  $M_1 = -M_2 = M$ .

- Determine the dispersion equation and show that it is biquadratic in both  $\omega$  and  $k$ .
- To describe temporal modes, make plots of complex  $\omega$  for real  $k$  in the electric and magnetic cases.
- Are these instabilities absolute or convective?

For Section 11.16:

Prob. 11.16.1 Figure 11.16.4 shows the eigenfunctions implied by the first three longitudinal modes. Given the eigenfrequencies, describe how these functions would be obtained.

Prob. 11.16.2 Counterstreaming continua described by Eqs. 11.15.1 and 11.15.2 have equal and opposite velocities, and hence a dispersion equation as derived in Prob. 11.15.1.

- For  $M < 1$ , argue that boundary conditions consistent with causality are  $\xi_1(0,t) = \xi_2(0,t) = \xi_1(\ell,t) = \xi_2(\ell,t) = 0$ . What is the eigenfrequency equation? In the limit  $M \rightarrow 0$ , what are the solutions to this expression?
- For  $M > 1$ , argue that boundary conditions consistent with causality are  $\xi_1 = 0$  and  $\partial \xi_1 / \partial z = 0$  at  $z = 0$  and  $\xi_2 = 0$  and  $\partial \xi_2 / \partial z = 0$  at  $z = \ell$ . What is the eigenfrequency equation?

For Section 11.17:

Prob. 11.17.1 Figure P11.17.1 shows distributed series of coils, each having  $n$  turns, lying in the  $y$ - $z$  plane. Each coil is of dimension  $\Delta y$  in the  $y$  direction;  $\Delta y$  is small compared to other dimensions of interest shown. The system of coils is connected to a transmission line. Insofar as distances in the  $x$  direction are concerned, the coils comprise a thin sheet and are connected to the line outside the volume of interest. Hence, their effect on an MQS system is to be represented by the boundary conditions. Represent the circuit by an inductance and a capacitance per unit length,  $L$  and  $C$  respectively, and show that (with the understanding that  $B_x$  is continuous through the sheet) the coils are represented by the boundary conditions

$$-\frac{\partial v}{\partial y} = L \frac{\partial i}{\partial t} - n w \frac{\partial B_x}{\partial t}$$

$$-\frac{\partial i}{\partial y} = C \frac{\partial v}{\partial t}$$

$$[[H_y]] = -n \frac{\partial i}{\partial y}$$

Prob. 11.17.2 In what might be termed a magnetoquasistatic resistive wall interaction, the resistive wall moves and the continuum of "oscillators" remains fixed. Thus, the thin conducting sheet of Sec. 6.3 is backed by a perfectly permeable material and moves in the  $y$  direction with a velocity  $U$ . In a plane parallel to that of the sheet is the system of distributed coils of Prob. 11.17.1, connected to the transmission line as shown in Fig. P11.17.1. These coils comprise the "stator" structure, and are backed by an infinitely permeable material. There

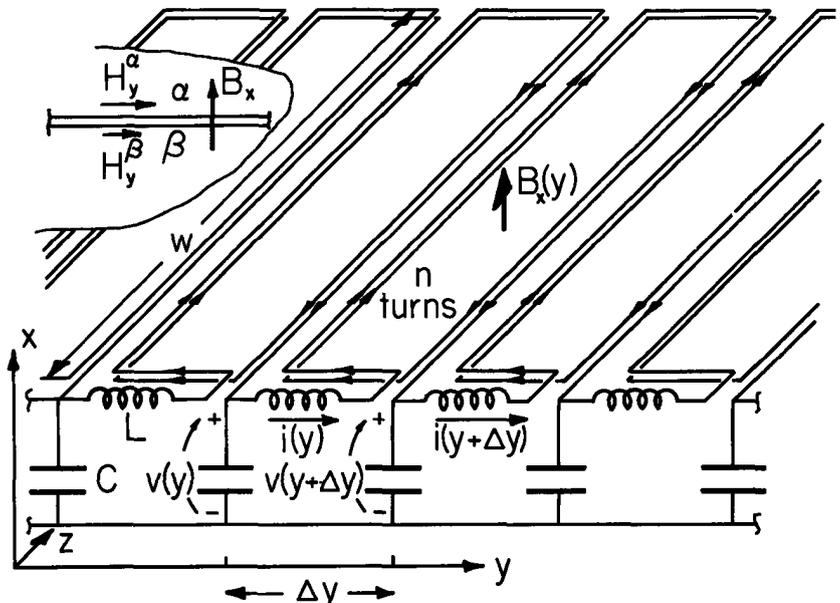


Fig. P11.17.1

Prob. 11.17.2 (continued)

is an air gap of thickness  $a$  between the moving sheet and coils.

- Find the dispersion equation describing the coupled transmission-line waves and convective magnetic diffusion modes.
- Show that the equation is cubic in  $\omega$ , and discuss the temporal modes. Instability implies that, in a rotating configuration, the system could be used as a generator. Indeed, the foregoing is a distributed form of a self-excited induction generator. More conventionally, such generators are made by tuning the stator windings of a machine like that of Sec. 6.3.
- In a system that is infinite in the  $y$  direction, is the instability absolute or convective?

Prob. 11.17.3 Electrodes lying in the  $y$ - $z$  plane, as shown in Fig. P11.17.3, have a length  $w$  in the  $y$  direction and are segmented in the  $z$  direction. Connected to the ends of the segments is a transmission line, having capacitance and inductance per unit length  $C$  and  $L$  respectively. The width of the segments in the  $z$  direction is short compared to lengths of interest in that direction. For an EQS system in which these electrodes appear as a plane having upper and lower surfaces denoted by  $\alpha$  and  $\beta$  respectively, and hence where  $\phi^\alpha = \phi^\beta \equiv v$ , show that the jump conditions relating variables on the two sides are

$$-\frac{\partial v}{\partial z} = L \frac{\partial i}{\partial t}$$

$$-\frac{\partial i}{\partial z} = C \frac{\partial v}{\partial t} - w \frac{\partial \sigma_f}{\partial t}$$

$$\sigma_f = \llbracket D_x \rrbracket$$

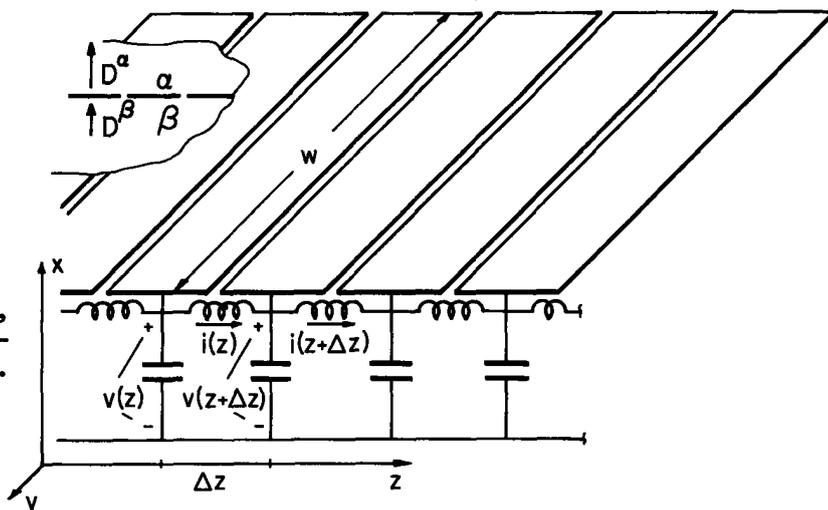


Fig. P11.17.3

Prob. 11.17.4 As a model for a traveling-wave electron-beam amplifier, consider the planar electron beam as shown in cross-section by Fig. 11.5.1. Two traveling-wave structures, described in Prob. 11.17.3, are coupled to this beam by placing the segmented electrodes in the planes (c) and (i). Respectively above and below these planes are parallel perfectly conducting sheets having spacings  $d$  from the planes of the segmented electrodes.

- For the even motions described in Sec. 11.5, determine the dispersion equation for the interaction.
- Take the long-wave limit of this expression to obtain a polynomial in  $\omega$  and  $k$ .
- Show that there can be unstable temporal modes.
- Are these modes absolutely or convectively unstable?

Prob. 11.17.5 The cross-section of a magnetoquasistatic candidate for a resistive-wall amplifier is shown in Fig. P11.17.5. In a state of stationary equilibrium, a perfectly conducting inviscid incompressible fluid, having a thickness  $2b$  and mass density  $\rho$ , streams to the right. The adjacent gaps are filled by an insulating fluid of negligible mass density. At distances  $a$  from the jet surfaces are walls composed of thin conducting sheets having surface conductivity  $\sigma_s$ , backed by highly permeable materials. Uniform surface currents flow in these sheets and on the free surfaces so as to give rise to uniform magnetic fields,  $H_0 \hat{y}$ , in the gaps. There is no equilibrium magnetic field in the jet or in the highly permeable backing materials. Effects of gravity are absent, but those of surface tension  $\gamma$  are included.

- Show that the stationary equilibrium is possible.
- Now consider kinking motions of the stream, in which the transverse displacement of both interfaces are upward and of equal magnitude. Determine the dispersion equation.
- Take the longwave limit of this result and plot complex  $\omega$  for real  $k$ . Assume that  $U > V$  where  $V \equiv \sqrt{2\gamma/\Delta\rho}$ .
- If  $U > V$ , can the system be unstable and, if so, is the instability convective or absolute?

**Prob. 11.17.6** Electron beams having equilibrium velocities  $U_1$  and  $U_2$  in the  $z$  direction share the same space. They have equilibrium number densities  $n_{o1}$  and  $n_{o2}$  respectively, and only interact through the macroscopic electric field  $\vec{E} = -\nabla\Phi$ .

- (a) Use the EQS equations of motion for the two species suggested by Eqs. 11.10.1 - 11.10.4 to show that the dispersion equation for one dimensional motions is

$$1 = \frac{\omega_{p1}^2}{(\omega - kU_1)^2} + \frac{\omega_{p2}^2}{(\omega - kU_2)^2}$$

where

$$\omega_{p1} = \sqrt{n_{o1}e^2/\epsilon_0 m} \quad , \quad \omega_{p2} = \sqrt{n_{o2}e^2/\epsilon_0 m}$$

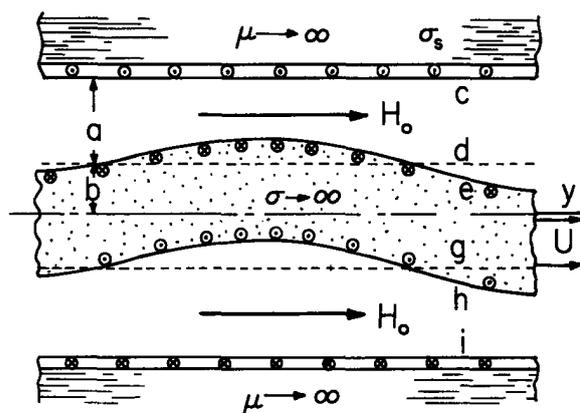


Fig. P11.17.5

- (b) In the limit  $U_2 = 0$ , this is the dispersion equation for an electron beam interacting with a stationary cold plasma. Show that the system is unstable and that the instability is convective.
- (c) With  $U_1 = -U_2$ , what is described is the interaction of counterstreaming beams. Show that in this case the system is absolutely unstable.