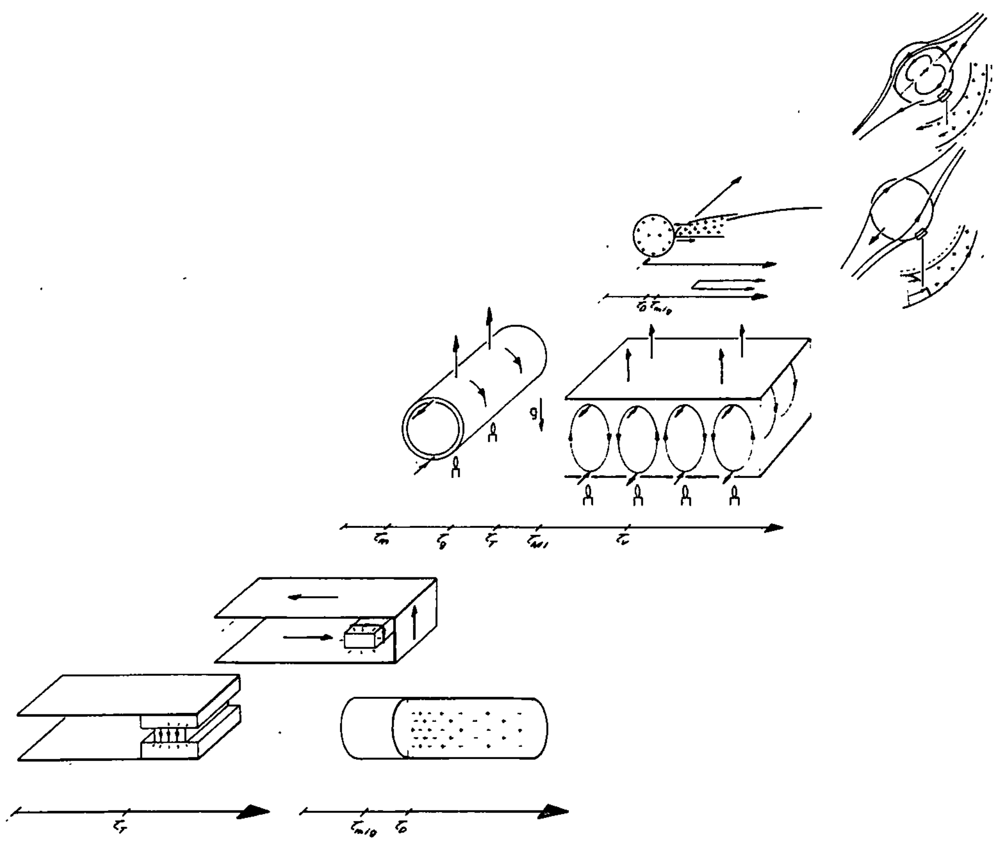
10

Electromechanics with Thermal and Molecular Diffusion



Prob. 10.2.1 (a) In one dimension, Eq. 10.2.2 is simply

$$\frac{d^2T}{dx^2} = -\frac{\phi_d}{k_T} \tag{1}$$

The motion has no effect because $\bar{v}$ is perpendicular to the heat flux. This expression is integrated twice from x=0 to an arbitrary location, x. Multiplied by $-k_T$, the constant from the first integration is the heat flux at x=0, T^{β}. The second integration has T^{β} as a constant of integration. Hence,

$$T = -\frac{1}{k_T}\int_0^x\int_0^{x'}\phi_d(x'')\,dx''dx' - \frac{T^{\beta}}{k_T}\Delta + T^{\beta} \tag{2}$$

Evaluation of this expression at x=0 where T = T^{β} gives a relation that can be solved for T^{β}. Substitution of T^{β} back into Eq. 2, gives the desired temperature distribution.

$$T = -\frac{1}{k_T}\int_0^x\int_0^{x'}\phi_d(x'')dx''dx' + T^{\beta} - \frac{x}{\Delta}(T^{\beta}-T^{\alpha}) + \frac{x}{\Delta k_T}\int_0^{\Delta}\int_0^{x'}\phi_d(x'')\,dx''dx' \tag{3}$$

(b) The heat flux is gotten from Eq. 3 by evaluating

$$T' = -k_T\frac{dT}{dx} = \int_0^x\phi_d(x')dx' + \frac{k_T}{\Delta}(T^{\beta}-T^{\alpha}) - \frac{1}{\Delta}\int_0^{\Delta}\int_0^{x'}\phi_d(x'')\,dx''\,dx' \tag{4}$$

At the respective boundaries, this expression becomes

$$T'^{\alpha} = \int_0^{\Delta}\phi_d(x')dx' + \frac{k_T}{\Delta}(T^{\beta}-T^{\alpha}) - \frac{1}{\Delta}\int_0^{\Delta}\int_0^{x'}\phi_d(x'')dx''dx' \tag{5}$$

$$T'^{\beta} = \frac{k_T}{\Delta}(T^{\beta}-T^{\alpha}) - \frac{1}{\Delta}\int_0^{x}\int_0^{x'}\phi_d(x'')\,dx''dx' \tag{6}$$

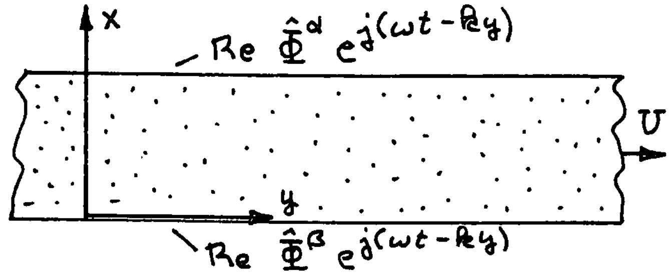
Prob. 10.3.1 In Eq. 10.3.20, the transient heat flux at the surfaces is zero, so $\hat{T}'^{\alpha} = \hat{T}'^{\beta} = 0$.

$$\begin{bmatrix} -\coth\gamma_T\Delta & \dfrac{1}{\sinh\gamma_T\Delta} \\ \dfrac{-1}{\sinh\gamma_T\Delta} & \coth\gamma_T\Delta \end{bmatrix}\begin{bmatrix}\hat{T}^{\alpha}\\ \hat{T}^{\beta}\end{bmatrix} = \sum_{i=1}^{\infty}\frac{\left(\frac{i\pi}{\Delta}\right)\hat{\Phi}_i/k_T\gamma_T}{\left[\left(\frac{i\pi}{\Delta}\right)^2 + k_z^2 + j\frac{(\omega_z - k_z U)}{K_T}\right]}\begin{bmatrix}(-1)^i\\ 1\end{bmatrix} \tag{1}$$

These expressions are inverted to find the dynamic part of the surface temperatures.

$$\begin{bmatrix}\hat{T}^{\alpha}\\ \hat{T}^{\beta}\end{bmatrix} = \sum_{i=1}^{\infty}\frac{\left(\frac{i\pi}{\Delta}\right)\hat{\Phi}_i/k_T\gamma_T}{\left[\left(\frac{i\pi}{\Delta}\right)^2 + k_z^2 + j\frac{(\omega_z - k_z U)}{K_T}\right]}\begin{bmatrix}(-1)^i\coth\gamma_T\Delta & \dfrac{-1}{\sinh\gamma_T\Delta}\\ -\coth\gamma_T\Delta & \dfrac{(-1)^i}{\sinh\gamma_T\Delta}\end{bmatrix} \tag{2}$$

<u>Prob. 10.3.2</u> (a) The EQS electrical dissipation density is



$$\phi_d=\sigma\bar{E}'\cdot\bar{E}'=\sigma\bar{E}\cdot\bar{E}$$

$$=\sigma\left[R_e\hat{\bar{E}}(x)e^{j(\omega t-ky)}\right]^2=\frac{\sigma}{4}\left[\hat{\bar{E}}e^{j(\omega t-ky)}-\hat{\bar{E}}^*e^{-j(\omega t-ky)}\right]^2$$

$$=\frac{1}{2}\sigma\left[\hat{\bar{E}}\hat{\bar{E}}^*-Re\,\hat{\bar{E}}\cdot\hat{\bar{E}}\,e^{j(\omega_2 t-k_2 y)}\right] \tag{1}$$

Thus, in Eq. 10.3.6

$$\phi_o=\frac{1}{2}\sigma\hat{\bar{E}}\cdot\hat{\bar{E}}^*\;;\;\hat{\Phi}=-\frac{1}{2}\sigma\hat{\bar{E}}^2 \tag{2}$$

The specific $\hat{\bar{E}}(x)$ follows from

$$\hat{\Phi}(x)=\hat{\Phi}^{\alpha}\frac{\sinh kx}{\sinh k\Delta}-\hat{\Phi}^{\beta}\frac{\sinh k(x-\Delta)}{\sinh k\Delta} \tag{3}$$

so that

$$\hat{\bar{E}}=-\frac{d\hat{\Phi}}{dx}\bar{i}_x+jk\hat{\Phi}\,\bar{i}_y$$

$$=\left[-k\hat{\Phi}^{\alpha}\frac{\cosh kx}{\sinh k\Delta}+k\hat{\Phi}^{\beta}\frac{\cosh k(x-\Delta)}{\sinh k\Delta}\right]\bar{i}_x \tag{4}$$

$$+jk\left[\hat{\Phi}^{\alpha}\frac{\sinh kx}{\sinh k\Delta}-\hat{\Phi}^{\beta}\frac{\sinh k(x-\Delta)}{\sinh k\Delta}\right]\bar{i}_y$$

Thus,

$$\Phi_o=\frac{1}{2}\frac{\sigma k^2}{\sinh^2 k\Delta}\Big\{\Big[\hat{\Phi}^{\alpha}(\hat{\Phi}^{\alpha})^*\cosh^2 kx-\left(\hat{\Phi}^{\alpha}\hat{\Phi}^{\beta *}+\hat{\Phi}^{\alpha *}\hat{\Phi}^{\beta}\right)\cosh kx\cosh k(x-\Delta)$$
$$+\hat{\Phi}^{\beta}\hat{\Phi}^{\beta *}\cosh^2 k(x-\Delta)\Big]$$
$$+\Big[\hat{\Phi}^{\alpha}\hat{\Phi}^{\alpha *}\sinh^2 kx-\left(\hat{\Phi}^{\alpha}\hat{\Phi}^{\beta *}+\hat{\Phi}^{\alpha *}\hat{\Phi}^{\beta}\right)\sinh kx\sinh k(x-\Delta) \tag{5}$$
$$+\hat{\Phi}^{\beta}\hat{\Phi}^{\beta *}\sinh k(x-\Delta)\Big]\Big\}$$

Prob. 10.5.1 Perturbation of Eqs. 16-18 with subscript o indicating the stationary state and time dependence, exp $\underline{s}t$, gives the relations

$$\begin{bmatrix} s+(1+f) & \Omega_o & T_{yo} \\ -\Omega_o & s+(1+f) & -T_{xo} \\ -R_a & 0 & \left(\frac{s}{P_T}+1\right) \end{bmatrix} \begin{bmatrix} T_x' \\ T_y' \\ \Omega' \end{bmatrix} = 0 \tag{1}$$

Thus, the characteristic equation for the natural frequencies is

$$\frac{s^3}{P_T} + s^2\left[\frac{2(1+f)}{P_T}+1\right] + s\left[2(1+f)+\frac{(1+f)^2}{P_T}+\frac{\Omega_o^2}{P_T}+R_a T_{yo}\right] + \left[(1+f)^2+\Omega_o^2+\Omega_o T_{xo} R_a + R_a T_{yo}(1+f)\right] = 0 \tag{2}$$

To discover the conditions for incipience of overstability, note that it takes place as a root to Eq. 2 passes from the left to the right half s plane. Thus, at incipience, $\underline{s}=j\underline{\omega}$. Because the coefficients in Eq. 2 are real, it can then be divided into real and imaginary parts, each of which can be solved for the frequency. With the use of Eqs. 23, it then follows that

$$\omega^2 = P_T\left\{(1+f) + \frac{(1+f)^2}{P_T} + \left[R_a - \frac{(1+f)^2}{P_T}\right]\frac{f}{P_T}\right\}$$

$$\omega^2 = 2\left[R_a - \frac{(1+f)^2}{f}\right]f \Big/ \left[\frac{2(1+f)}{P_T}+1\right]$$

The critical R_a is found by setting these expressions equal to each other. The associated frequency of oscillation then follows by substituting that critical R_a into either Eq. 3 or 4.

Prob. 10.5.2 With heating from the left, the thermal source term enters in the x component of the thermal equation rather than the y component. Written in terms of the rotor temperature, the torque equation is unaltered. Thus, in normalized form, the model is represented by

$$\frac{dT_x}{dt} = -\Omega T_y - T_x(1+f) - f \tag{1}$$

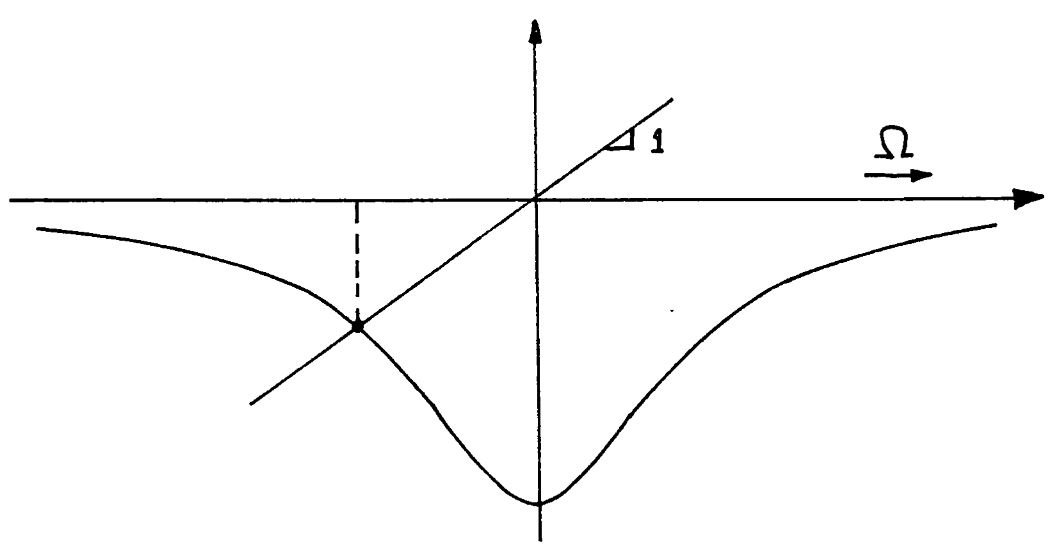
$$\frac{dT_y}{dt} = \Omega T_x - T_y(1+f) \tag{2}$$

$$\frac{1}{P_T}\frac{d\Omega}{dt} = -\Omega + R_a T_x \tag{3}$$

In the steady state, Eq. 2 gives T_y in terms of T_x and Ω, and this substituted into Eq. 1 gives T_x as a function of Ω. Finally, $T_x(\Omega)$, substituted into the torque equation, gives

$$\Omega = \frac{-f(1+f)R_a}{(1+f)^2 + \Omega^2} \tag{4}$$

The graphical solution to this expression is shown in Fig. P10.5.2. Note that for $T_e > 0$ and $d > 0$ the negative rotation is consistent with the left half of the rotor being heated and hence rising the right half being cooled and hence falling.



Prob. 10.6.1 (a) To prove the exchange of stabilities holds, multiply Eq. 8 by $\hat{v}_x^*$ and the complex conjugate of Eq. 9 by $\hat{T}$ and add. (The objective here is to obtain terms involving quadratic functions of $\hat{v}_x$ and $\hat{T}$ that can be manipulated into positive definite integrals.) Then, integrate over the normalized cross-section.

$$\int_0^1 \left\{ \hat{v}_x^* \left[\frac{s}{P_{TM}}(D^2 - k^2) + D^2 \right] \hat{v}_x + R_{am} k^2 \hat{T} \left[s^* - (D^2 - k^2) \right] \hat{T}^* \right\} dx = 0 \tag{1}$$

The second-derivative terms in this expression are integrated by parts to obtain

$$\frac{s}{P_{TM}} \hat{v}_x^* D\hat{v}_x \Big]_0^1 - \int_0^1 |D\hat{v}_x|^2 \frac{s}{P_{TM}} dx - \frac{k^2 s}{P_{TM}} \int_0^1 |\hat{v}_x|^2 dx + \hat{v}_x^* D \hat{v}_x \Big|_0^1 - \int_0^1 |D\hat{v}_x|^2 dx$$
$$+ R_{am} k^2 \left\{ s^* \int_0^1 |\hat{T}|^2 dx - \hat{T} D\hat{T} \Big|_0^1 + \int_0^1 \left[|D\hat{T}|^2 + k^2 |\hat{T}|^2 \right] dx \right\} = 0 \tag{2}$$

Boundary conditions eliminate the terms evaluated at the surfaces. With the definition of positive definite integrals

$$I_1 \equiv \int_0^1 |D\hat{v}_x|^2 dx \quad ; \quad I_3 = \int_0^1 |\hat{T}|^2 dx$$
$$I_2 \equiv \int_0^1 \left[k^2 |\hat{v}_x|^2 + |D\hat{v}_x|^2 \right] dx \quad ; \quad I_4 \equiv \int_0^1 \left[|D\hat{T}|^2 + k^2 |\hat{T}|^2 \right] dx \tag{3}$$

The remaining terms in Eq. 2 reduce to

$$-\frac{s}{P_{TM}} I_2 - I_1 + s^* R_{am} k^2 I_3 + R_{am} k^2 I_4 = 0 \tag{4}$$

Now, let $s = \alpha + j\omega$, where α and ω are real. Then, Eq. 4 divides into real and imaginary parts. The imaginary part is

$$\frac{\omega}{P_{TM}} I_1 + \omega R_{am} k^2 I_3 = 0 \tag{5}$$

Prob. 10.6.1 (cont.)

It follows that if $R_{am} > 0$, then $\underline{\omega} = 0$. This is the desired proof. Note that if the heavy fluid is on the bottom ($R_{am} < 0$) the eigenfrequencies can be complex. This is evident from Eq. 17.

(b) Equations 8 and 9 show that with $\underline{s}=0$

$$\gamma^2(\gamma^2 - k^2) + R_{am} k^2 = 0 \tag{6}$$

which has the four roots $\pm\gamma_a$, $\pm\gamma_b$ evaluated with $\underline{s}=0$. The steps to find the eigenvalues of R_{am} are now the same as used to deduce Eq. 15, except that $\underline{s}=0$ throughout. Note that Eq. 15 is unusually simple, in that in the section it is an equation for $\underline{\omega}$. It was only because of the simple nature of the boundary conditions that it could be solved for γ_a and γ_b directly. In any case, the γ's are the same here, $jn\pi$, and Eq. 6 can be evaluated to obtain the criticality condition, Eq. 18, for each of the modes.

Prob. 10.6.2 Equation 10.6.14 takes the form

$$[M_{ij}]\begin{bmatrix}\hat{T}_1\\ \cdot\\ \cdot\\ \hat{T}_4\end{bmatrix} = \begin{bmatrix}\hat{T}^\alpha\\ \hat{T}^\beta\\ \hat{v}^\alpha\\ \hat{v}^\beta\end{bmatrix} \tag{1}$$

In terms of these same coefficients $\hat{T}_1 \ldots \hat{T}_4$, it follows from Eq. 10.6.10 that the normalized heat flux is

$$\hat{T}_x = -\sum_{n=1}^{4} \gamma_n \hat{T}_n e^{\gamma_n x} \tag{2}$$

and from Eq. 11 that the normalized pressure is

$$\hat{p} = \sum_{n=1}^{4} B_n \hat{T}_n e^{\gamma_n x}$$

$$B_n \equiv \left\{\frac{R_{am} P_{TM}}{\gamma_n} j\omega\left[j\omega - (\gamma_n^2 - k^2)\right]\right\}\hat{T}_n e^{\gamma_n x} \tag{3}$$

Evaluation of these last two expressions at $\underline{x} = 1$ where $\hat{T}_x = \hat{T}_x^\alpha$ and $\hat{p} = \hat{p}^\alpha$ and at $\underline{x} = 0$ where $\hat{T}_x = \hat{T}_x^\beta$ and $\hat{p} = \hat{p}^\beta$ gives

Prob. 10.6.2 (cont.)

$$\begin{bmatrix} \hat{T}_x^{\alpha} \\ \hat{T}_x^{\beta} \\ \hat{p}^{\alpha} \\ \hat{p}^{\beta} \end{bmatrix} = \begin{bmatrix} N_{ij} \end{bmatrix} \begin{bmatrix} \hat{T}_1 \\ \hat{T}_2 \\ \hat{T}_3 \\ \hat{T}_4 \end{bmatrix} \tag{4}$$

where (note that $B_1 = B_a \Rightarrow B_2 = -B_a$; $B_3 = B_b \Rightarrow B_4 = -B_B$.)

$$N_{ij} = \begin{bmatrix} -\gamma_a e^{\gamma_a} & \gamma_a e^{-\gamma_a} & -\gamma_b & \gamma_b \\ -\gamma_a & \gamma_a & -\gamma_b & \gamma_b \\ B_a e^{\gamma_a} & -B_a e^{-\gamma_a} & B_b e^{\gamma_b} & -B_b e^{-\gamma_b} \\ B_a & -B_a & B_b & -B_b \end{bmatrix} \tag{5}$$

Thus, the required transfer relations are

$$\begin{bmatrix} \hat{T}^{\alpha} \\ \hat{T}^{\beta} \\ \hat{p}^{\alpha} \\ \hat{p}^{\beta} \end{bmatrix} = \begin{bmatrix} N_{ij} \end{bmatrix} \begin{bmatrix} M_{ij} \end{bmatrix}^{-1} \begin{bmatrix} \hat{T}^{\alpha} \\ \hat{T}^{\beta} \\ \hat{v}^{\alpha} \\ \hat{v}^{\beta} \end{bmatrix} \tag{6}$$

So

$$C_{ij} = \begin{bmatrix} N_{ij} \end{bmatrix} \begin{bmatrix} M_{ij} \end{bmatrix}^{-1} \tag{7}$$

The matrix C_{ij} is therefore determined in two steps. First, Eq. 10.6.14 is inverted to obtain

Prob. 10.6.2 (cont.)

$$M_{ij}^{-1} = \left[4(b-a)\sinh\gamma_a \sinh\gamma_b\right]^{-1} \tag{8}$$

$$\begin{bmatrix} 2b\sinh\gamma_b & -2b\sinh\gamma_b e^{-\gamma_a} & -2\sinh\gamma_b & 2\sinh\gamma_b e^{-\gamma_a} \\ -2b\sinh\gamma_b & 2b\sinh\gamma_b e^{\gamma_a} & 2\sinh\gamma_b & -2\sinh\gamma_b e^{\gamma_a} \\ -2a\sinh\gamma_a & 2a\sinh\gamma_a e^{-\gamma_b} & 2\sinh\gamma_a & -2\sinh\gamma_a e^{-\gamma_b} \\ 2a\sinh\gamma_a & -2a\sinh\gamma_a e^{\gamma_b} & -2\sinh\gamma_a & 2\sinh\gamma_a e^{\gamma_b} \end{bmatrix}$$

Finally, Eq. 7 is evaluated using Eqs. 5 and 8.

$$C_{ij} = \left[(b-a)\sinh\gamma_a \sinh\gamma_b\right]^{-1}\left[C'_{ij}\right]$$

where

$$[C'_{ij}] =$$

$$\begin{bmatrix} [a\gamma_b\sinh\gamma_a\cosh\gamma_b - b\gamma_a\sinh\gamma_b\cosh\gamma_a] & [\gamma_a b\sinh\gamma_b - \gamma_b a\sinh\gamma_a] & [\gamma_a\sinh\gamma_b\cosh\gamma_a - \gamma_b\sinh\gamma_a\cosh\gamma_b] & [\gamma_b\sinh\gamma_a - \gamma_a\sinh\gamma_b] \\ [a\gamma_b\sinh\gamma_a - b\gamma_a\sinh\gamma_b] & [b\gamma_a\sinh\gamma_b\cosh\gamma_a - a\gamma_b\sinh\gamma_a\cosh\gamma_b] & [\gamma_a\sinh\gamma_b - \gamma_b\sinh\gamma_a] & [\gamma_b\sinh\gamma_a\cosh\gamma_b - \gamma_a\sinh\gamma_b\cosh\gamma_a] \\ [bB_a\sinh\gamma_b\cosh\gamma_a - aB_b\sinh\gamma_a\cosh\gamma_b] & [-bB_a\sinh\gamma_b + aB_b\sinh\gamma_a] & [B_b\sinh\gamma_a\cosh\gamma_b - B_a\sinh\gamma_b\cosh\gamma_a] & [B_a\sinh\gamma_b - B_b\sinh\gamma_a] \\ [bB_a\sinh\gamma_b - aB_b\sinh\gamma_a] & [aB_b\sinh\gamma_a\cosh\gamma_b - bB_a\cosh\gamma_a\sinh\gamma_b] & [-B_a\sinh\gamma_b + B_b\sinh\gamma_a] & [B_a\sinh\gamma_b\cosh\gamma_a - B_b\sinh\gamma_a\cosh\gamma_b] \end{bmatrix}$$

Prob. 10.6.3 (a) To the force equation, Eq. 4, is added the viscous force density, $\eta\nabla^2\bar{v}$. Operating on this with [-curl(curl)], then adds to Eq. 7, $\eta\nabla^4\hat{v}_x$. In terms of complex amplitudes, the result is

$$\left[\eta(D^2-k^2)^2 - j\omega\rho(D^2-k^2) - \sigma(\mu_o H_o)^2 D^2\right]\hat{v}_x = -\alpha\rho_o g k^2 \hat{T} \qquad (1)$$

Normalized as suggested, this results in the first of the two given equations. The second is the thermal equation, Eq. 3, unaltered but normalized.

(b) The two equations in (v_x,T) make it possible to determine the six possible solutions exp γx.

$$\left[(\gamma^2-k^2)^2 - \frac{j\omega}{P_T}(\gamma^2-k^2) - \frac{T_m}{T_{MV}}\gamma^2\right]\left[(\gamma^2-k^2) - j\omega\right] + R_a = 0 \qquad (2)$$

The six roots comprise the solution

$$\hat{T} = \sum_{k=1}^{6} T_k e^{\gamma_k x} \qquad (3)$$

The velocity follows from the second of the given equations

$$\hat{v}_x = \sum_{k=1}^{6}\left[j\omega - (\gamma_k^2 - k^2)\right]T_k e^{\gamma_k x} \qquad (4)$$

To find the transfer relations, the pressure is gotten from the x component of the force equation, which becomes

$$D\hat{p} = \left[-j\omega + P_T(D^2-k^2)\right]\hat{v}_x + R_a P_T \hat{T} \qquad (5)$$

Thus, in terms of the six coefficients,

$$\hat{p} = \sum_{k=1}^{6}\left\{\left[-j\omega + P_T(\gamma_k^2 - k^2)\right]\left[j\omega - (\gamma^2 - k^2)\right] + R_a P_T\right\}\frac{T_k}{\gamma_k} e^{\gamma_k x} \qquad (6)$$

For two-dimensional motions where v_z=0, the continuity equation suffices to find $\hat{v}_y$ in terms of $\hat{v}_x$. Hence,

$$\hat{v}_y = \frac{1}{jk_y} D\hat{v}_x \qquad (7)$$

Prob. 10.6.3 (cont.)

From Eqs. 6 and 7, the stress components can be written as

$$\hat{S}_x = -\hat{p} + 2\eta D\hat{v}_x \tag{8}$$

$$\hat{S}_y = \eta\left(D\hat{v}_y - jk_y\hat{v}_x\right) \tag{9}$$

and the thermal flux is similarly written in terms of the amplitudes T_k.

$$\hat{T}'_x = -k_T D\hat{T} \tag{10}$$

These last three relations, respectively evaluated at the α and β surfaces provide the stresses and thermal fluxes in terms of the T_k's.

$$\begin{bmatrix} \hat{S}_x^\alpha \\ \hat{S}_x^\beta \\ \hat{S}_y^\alpha \\ \hat{S}_y^\beta \\ \hat{T}_x^\alpha \\ \hat{T}_x^\beta \end{bmatrix} = \left[A_{ij}\right]\begin{bmatrix} \hat{T}_1 \\ \hat{T}_2 \\ \hat{T}_3 \\ \hat{T}_4 \\ \hat{T}_5 \\ \hat{T}_6 \end{bmatrix} \tag{11}$$

By evaluating Eqs. 3, 4 and 7 at the respective surfaces, relations are obtained

$$\begin{bmatrix} \hat{v}_x^\alpha \\ \hat{v}_x^\beta \\ \hat{v}_y^\alpha \\ \hat{v}_y^\beta \\ \hat{T}^\alpha \\ \hat{T}^\beta \end{bmatrix} = \left[B_{ij}\right]\begin{bmatrix} \hat{T}_1 \\ \hat{T}_2 \\ \hat{T}_3 \\ \hat{T}_4 \\ \hat{T}_5 \\ \hat{T}_6 \end{bmatrix} \tag{12}$$

Inversion of these relations gives the amplitudes T_k in terms of the velocities and temperature. Hence,

$$\begin{bmatrix} \hat{S}_x^\alpha \\ \hat{S}_x^\beta \\ \hat{S}_y^\alpha \\ \hat{S}_y^\beta \\ \hat{T}_x^\alpha \\ \hat{T}_x^\beta \end{bmatrix} = [A][B]^{-1}\begin{bmatrix} \hat{v}_x^\alpha \\ \hat{v}_x^\beta \\ \hat{v}_y^\alpha \\ \hat{v}_y^\beta \\ \hat{T}^\alpha \\ \hat{T}^\beta \end{bmatrix} \tag{13}$$

Prob. 10.7.1 (a) The imposed electric field follows from Gauss' integral law and the requirement that the integral of $\bar{E}$ from r=R to r=a be V.

$$\bar{E} = \frac{\lambda \bar{i}_r}{2\pi \epsilon_o r} \quad ; \quad \lambda = \frac{V\, 2\pi\epsilon_o}{\ln\left(\frac{a}{R}\right)} \tag{1}$$

The voltage V can be constrained, or the cylinder allowed to charge up, in which case the cylinder potential relative to that at r=a is V. Conservation of ions in the quasi-stationary state is Eq. 10.7.4 expressed in cylindrical coordinates.

$$\frac{1}{r}\frac{d}{dr} r\left(\frac{b\lambda\rho}{2\pi\epsilon_o r} - K_+ \frac{d\rho}{dr}\right) = 0 \tag{2}$$

One integration, with the constant evaluated in terms of the current i to the cylinder, gives

$$2\pi r K_+ \frac{d\rho}{dr} - \frac{b\lambda}{\epsilon_o}\rho = i \tag{3}$$

The particular solution is $-\epsilon_o i/b\lambda$, while the homogeneous solution follows from

$$\int \frac{d\rho}{\rho} = \frac{b\lambda}{2\pi\epsilon_o K_+}\int \frac{dr}{r} \tag{4}$$

Thus, with the homogeneous solution weighted to make $\rho(a) = \rho_o$, the charge density distribution is the sum of the homogeneous and particular solutions,

$$\rho = \left(\rho_o + \frac{\epsilon_o i}{b\lambda}\right)\left(\frac{r}{a}\right)^{f\lambda} - \frac{\epsilon_o i}{b\lambda} \tag{5}$$

where f = $q/2\pi\epsilon_o kT$.

(b) The current follows from requiring that at the surface of the cylinder, r=R, the charge density vanish.

$$i = \frac{\rho_o b}{\epsilon_o}\lambda \frac{1}{\left[\left(\frac{a}{R}\right)^{f\lambda} - 1\right]} \tag{6}$$

With the voltage imposed, this expression is completed by using Eq. 1b.

Prob. 10.7.1 (cont.)

(c) With the cylinder free to charge up, the charging rate is determined by

$$i = \frac{d\lambda}{dt} \tag{7}$$

This expression can be integrated by writing it in the form

$$\int_0^t \frac{\rho_o b}{\epsilon_o} dt = \int_0^\lambda \frac{\left[\left(\frac{a}{R}\right)^{f\lambda} - 1\right]}{\lambda} d\lambda \tag{8}$$

By defining $g \equiv \ln(a/R)^f$ this becomes

$$t\left(\frac{\rho_o b}{\epsilon_o}\right) = \int_0^{g\lambda} \frac{[e^{g\lambda} - 1]}{g\lambda} dg\lambda = \frac{g\lambda}{1!} + \frac{g^2\lambda^2}{2\cdot 2!} + \frac{g^3\lambda^3}{3\cdot 3!} + \cdots \tag{9}$$

By defining $\lambda_D \equiv \frac{1}{g} = (q/2\pi\epsilon_o kT)/\ln(a/R)$ this takes the normalized form

$$\underline{t} = \frac{\underline{\lambda}}{1!} + \frac{\underline{\lambda}^2}{2\cdot 2!} + \frac{\underline{\lambda}^3}{3\cdot 3!} + \cdots \tag{10}$$

where

$$\underline{t} = t/\tau_e \quad ; \quad \tau_e \equiv \epsilon_o/\rho_o b$$

$$\underline{\lambda} = \lambda/\lambda_D$$

Prob. 10.7.2 Because there is no equilibrium current in the x direction,

$$\rho b E - K_+ \frac{d\rho}{dx} = 0 \tag{1}$$

For the unipolar charge distribution, Gauss' law requires that

$$\frac{d\epsilon E}{dx} = \rho \tag{2}$$

Substitution for ρ using Eq. 2 in Eq. 1 gives an expression that can be integrated once by writing it in the form

$$\frac{d}{dx}\left(\frac{1}{2} b E^2 - K_+ \frac{dE}{dx}\right) = 0 \tag{3}$$

As $x \to \infty$, $E \to 0$ and there is no charge density, so $dE/dx \to 0$. Thus, the quantity in brackets in Eq. 3 is zero, and a further integration can be performed

$$\int_E^{E_o} \frac{dE}{E^2} = \frac{1}{2}\frac{b}{K_+}\int_x^0 dx \tag{4}$$

Prob. 10.7.2 (cont.)

It follows that the desired electric field distribution is

$$E = E_o/(1-\frac{x}{\ell_d}) \tag{5}$$

where $\ell_d \equiv 2K_+/bE_o$.

The charge distribution follows from Eq. 2

$$\rho = -\frac{\epsilon E_o}{\ell_d}/(1-\frac{x}{\ell_d})^2 \tag{6}$$

The Einstein relation shows that $\ell_d = 2(kT/q)/E_o \simeq 2(25\times10^{-3})/10^4 = 5\mu m$

Prob. 10.8.1 (a) The appropriate solution to Eq. 8 is simply

$$\Phi = -S\frac{\cosh(x-\frac{\Delta}{2})}{\cosh(\Delta/2)} \tag{1}$$

Evaluated at the midplane, this gives

$$\Phi_c = -S/\cosh(\Delta/2) \tag{2}$$

(b) Symmetry demands that the slope of the potential vanish at the midplane. If the potential there is called Φ_c, evaluation of the term in brackets from Eq. 9 at the midplane gives $-\cosh\Phi_c$, and it follows that

$$\frac{1}{2}\left(\frac{d\Phi}{dx}\right)^2 - \cosh\Phi = -\cosh\Phi_c \tag{3}$$

so that instead of Eq. 10, the expression for the potential is that given in the problem.

(c) Evaluation of the integral expression at the midplane gives

$$\frac{\Delta}{2} = \int_{-S}^{\Phi_c}\frac{d\Phi}{\sqrt{2(\cosh\Phi - \cosh\Phi_c)}} \tag{4}$$

In principal, an iterative evaluation of this integral can be used to determine Φ_c and hence the potential distribution. However, the integrand is singular at the end point of the integration, so the integration in the vicinity of this end point is carried out analytically. In the neighborhood of Φ_c, $\cosh\Phi \simeq \cosh\Phi_c + \sinh\Phi_c(\Phi-\Phi_c)$ and the integrand of Eq. 4 is approximated by

$$\frac{1}{\sqrt{2}}(\cosh\Phi - \cosh\Phi_c)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}\left[\sinh\Phi_c(\Phi-\Phi_c)\right]^{-\frac{1}{2}}$$

With the numerical integration ending at $\Phi_c + \Delta\Phi$, short of Φ_c , the remainder of the integral is taken analytically.

Prob. 10.8.1 (cont.)

$$\frac{1}{\sqrt{2}}\int_{\Phi_c}^{\Phi_c+\Delta\Phi}\left[\sinh\Phi_c(\Phi-\Phi_c)\right]^{-\frac{1}{2}}d\Phi=\frac{2}{\sqrt{2}}\left(\frac{\Phi-\Phi_c}{\sinh\Phi_c}\right)^{1/2}\Bigg|_{\Phi_c}^{\Phi_c+\Delta\Phi}=\sqrt{2}\left(\frac{\Delta\Phi}{\sinh\Phi_c}\right) \qquad (6)$$

Thus, the expression to be evaluated numerically is

$$\frac{\Delta}{2}=\int_{-\zeta}^{\Phi_c+\Delta\Phi}\frac{d\Phi}{\sqrt{2(\cosh\Phi-\cosh\Phi_c)}}=\sqrt{2}\left(\frac{\Delta\Phi}{\sinh\Phi_c}\right)^{1/2} \qquad (7)$$

where Φ_c and $\Delta\Phi$ are negative quantities and ζ is a positive number. At least to obtain a rough approximation, Eq. 7 can be repeatedly evaluated with Φ_c altered to make Δ the prescribed value. For $\Delta/2=1$, $\zeta=-3$ the distribution is shown in Fig. P10.8.1 and $\Phi_c \simeq 1$.

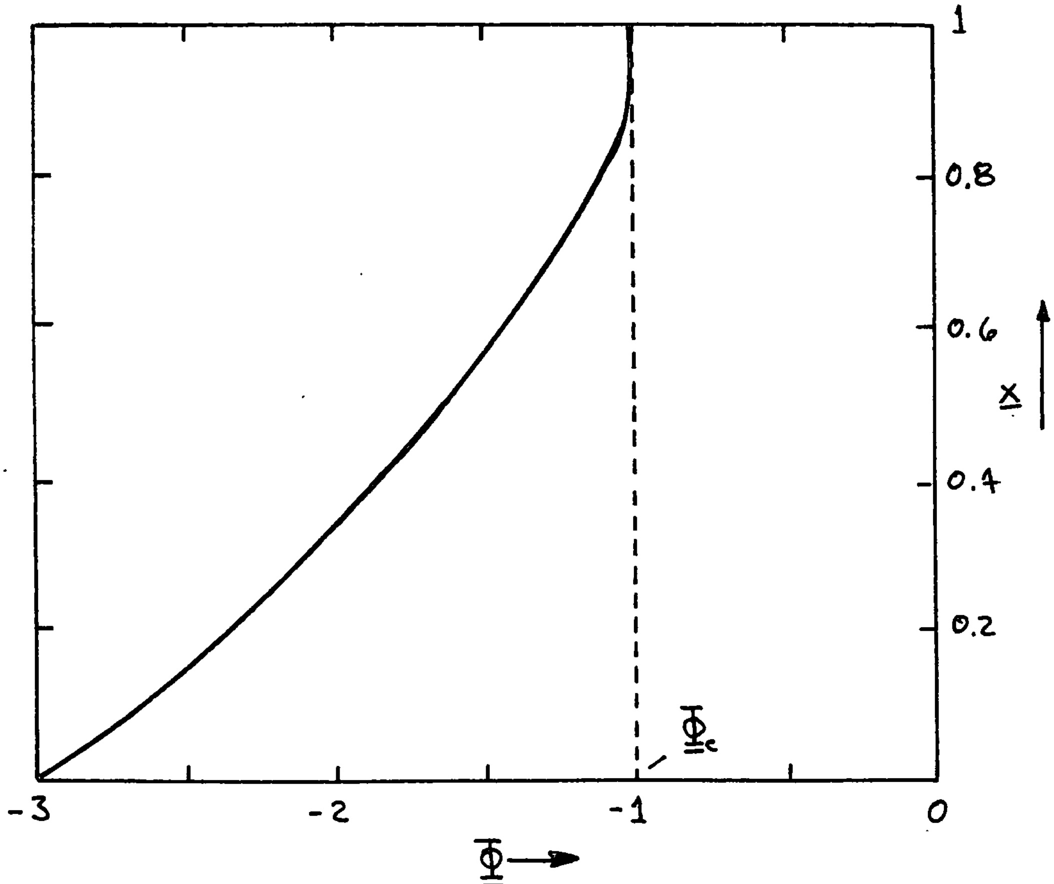


Fig. P10.8.1. Potential distribution over half of distance between parallel boundaries having zeta potentials $\zeta=-3$.

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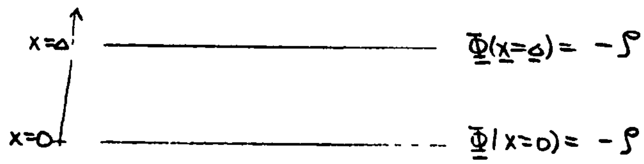
PROBLEM SET 11

3 (10.8.1)



$x = \underline{x}\,\delta_D \qquad \Phi = \underline{\Phi}\,\frac{kT}{q}$

$\Delta = \underline{\Delta}\,\delta_D \qquad \delta_D = \sqrt{\frac{\epsilon kT}{2q^2 n_o}}$

a. IN NORMALIZED TERMS, THE POTENTIAL DISTRIBUTION ACROSS THE ELECTROLYTE IS GIVEN BY

$$\frac{d^2\underline{\Phi}}{d\underline{x}^2} = \sinh(\underline{\Phi})$$

FOR $\underline{\Phi} \ll 1$, $\sinh\underline{\Phi} \simeq \underline{\Phi}$ $\Rightarrow$ $\frac{d^2\underline{\Phi}}{d\underline{x}^2} - \underline{\Phi} = 0$

THIS DIFFERENTIAL EQUATION HAS SOLUTIONS OF THE FORM: $\Phi \sim e^{\pm \underline{x}}$, $\sinh(\underline{x})$, $\cosh(\underline{x})$

IMPOSING THE POTENTIALS AT THE BOUNDARIES GIVES

$$\underline{\underline{\underline{\Phi} = \frac{-\varsigma}{\sinh(\underline{\Delta})}\left[\sinh(\underline{x}) - \sinh(\underline{x}-\underline{\Delta})\right]}} \left(= -\varsigma\,\frac{\cosh\left(\underline{x}-\frac{\underline{\Delta}}{2}\right)}{\cosh\left(\frac{\underline{\Delta}}{2}\right)}\right)$$

AT THE MIDPLANE, $\underline{x} = \frac{\underline{\Delta}}{2}$, $\underline{\Phi} = \underline{\Phi}_c$ $\Rightarrow$ $\underline{\Phi}_c = \frac{-\varsigma}{\sinh(\underline{\Delta})}\left[\sinh\left(\frac{\underline{\Delta}}{2}\right) - \sinh\left(\frac{-\underline{\Delta}}{2}\right)\right] = -\varsigma\,\frac{2\sinh\left(\frac{\underline{\Delta}}{2}\right)}{2\sinh\left(\frac{\underline{\Delta}}{2}\right)\cosh\left(\frac{\underline{\Delta}}{2}\right)}$

$$\Rightarrow \underline{\underline{\underline{\Phi}_c = \frac{-\varsigma}{\cosh\left(\frac{\underline{\Delta}}{2}\right)}}}$$

b. IN GENERAL $\frac{d^2\underline{\Phi}}{d\underline{x}^2} = \sinh(\underline{\Phi})$ OR $\frac{d^2\underline{\Phi}}{d\underline{x}^2} - \sinh(\underline{\Phi}) = 0$

MULTIPLICATION BY $\frac{d\underline{\Phi}}{d\underline{x}}$ GIVES $\frac{d^2\underline{\Phi}}{d\underline{x}^2}\frac{d\underline{\Phi}}{d\underline{x}} - \sinh(\underline{\Phi})\frac{d\underline{\Phi}}{d\underline{x}} = 0$

NOW, NOTICE THAT $\frac{d}{d\underline{x}}\left(\frac{1}{2}\left(\frac{d\underline{\Phi}}{d\underline{x}}\right)^2\right) = \frac{d\underline{\Phi}}{d\underline{x}}\frac{d^2\underline{\Phi}}{d\underline{x}^2}$

AND $\frac{d}{d\underline{x}}\left[\cosh(\underline{\Phi})\right] = \sinh(\underline{\Phi})\frac{d\underline{\Phi}}{d\underline{x}}$

$$\Rightarrow \frac{d}{d\underline{x}}\left[\frac{1}{2}\left(\frac{d\underline{\Phi}}{d\underline{x}}\right)^2 - \cosh(\underline{\Phi})\right] = 0$$

OR $$\frac{1}{2}\left(\frac{d\underline{\Phi}}{d\underline{x}}\right)^2 - \cosh(\underline{\Phi}) = C_1$$

DUE TO THE SYMMETRY OF THE PROBLEM, $\frac{d\underline{\Phi}}{d\underline{x}} = 0$ AT THE MIDPLANE, WHERE $\underline{\Phi} = \underline{\Phi}_c$ $\Rightarrow$ $C_1 = -\cosh(\underline{\Phi}_c)$

(OVER)

THIS YIELDS $\frac{1}{2}\left(\frac{d\Phi}{dx}\right)^2 = \cosh(\Phi) - \cosh(\Phi_c)$

OR $\frac{d\Phi}{dx} = \pm\sqrt{2[\cosh(\Phi) - \cosh(\Phi_c)]}$

INTEGRATION GIVES: $\int_0^x dx' = \pm\int_{-\rho}^{\Phi} \frac{d\Phi'}{\sqrt{2[\cosh(\Phi') - \cosh(\Phi_c)]}}$

$$\therefore \quad x = \pm\int_{-\rho}^{\Phi} \frac{d\Phi'}{\sqrt{2[\cosh(\Phi') - \cosh(\Phi_c)]}}$$

WITH THE + SIGN USED FOR $0 \le x < \frac{\Delta}{2}$ AND THE − SIGN USED FOR $\frac{\Delta}{2} < x \le \Delta$. THIS SEPARATION IS NECESSARY TO MAINTAIN THE "SYMMETRY", AND BECAUSE THE FUNCTIONAL TERM IN THE INTEGRAL GOES TO INFINITY AT $\Phi = \Phi_c$ OR $x = \frac{\Delta}{2}$

c. GIVEN $\Delta \equiv \hat{\delta}_D$, IT WOULD SEEM REASONABLE TO USE THE EQUATION IN PART b TO FIND Φ_c, BY FIRST GUESSING Φ_c, THEN NUMERICALLY SOLVING THE INTEGRAL TO $x = \frac{\Delta}{2}$. THE RESULT WOULD THEN BE USED TO MODIFY THE Φ_c TO WITHIN A GIVEN ERROR BY REPEATING THE PROCESS. UNFORTUNATELY, AT $x = \frac{\Delta}{2}$, $\Phi = \Phi_c \Rightarrow$ THE FUNCTION INSIDE THE INTEGRAL BLOWS UP (GOES TO INFINITY), SO A SIMPLE TRAPEZOIDAL INTEGRATION COULD LEAD TO NUMERICAL ERRORS. TO SIDESTEP THIS DIFFICULTY, THE DERIVATIVE OF THE POTENTIAL WILL BE USED IN A FINITE DIFFERENCE TECHNIQUE. WHILE NUMERICAL DIFFERENTIATION IS NOT A RECOMMENDED PROCEDURE IN GENERAL, THE FUNCTIONS ARE SMOOTH ENOUGH IN THIS CASE TO ALLOW THIS SOLUTION.

USING FINITE DIFFERENCES: $\frac{d\Phi}{dx} = \pm\sqrt{2[\cosh(\Phi) - \cosh(\Phi_c)]} \equiv \pm f(\Phi, \Phi_c)$

BUT $\frac{d\Phi}{dx} \simeq \frac{\Phi(x+\Delta x) - \Phi(x)}{\Delta x} \Rightarrow \Phi(x+\Delta x) \simeq \Phi(x) \pm f(\Phi(x), \Phi_c)\,\Delta x$ (A

NOW, AN INITIAL $\Phi_c = 0$ (WHICH IS THE MAXIMUM Φ) IS GUESSED, THEN EQ. (A) IS ITERATED UPON (WITH $\Phi(x=0) = -\rho$ AND Δx KNOWN) UNTIL $x = \frac{\Delta}{2}$, SO THAT $\Phi_c' = \Phi(x = \frac{\Delta}{2})$. THIS Φ_c' IS COMPARED TO Φ_c TO SEE IF THE DIFFERENCE IS SMALL. IF IT IS NOT, THEN THE PROCESS CAN BE REPEATED, WITH Φ_c REPLACED BY Φ_c'. ONCE Φ_c IS KNOWN, (A) CAN BE USED TO FIND $\Phi(x)$

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PROBLEM SET 11

3. (10.8.1) CONTINUED.

THIS ALGORITHM IS IMPLEMENTED BY THE PROGRAM LISTED ON THE FOLLOWING PAGES (AND IN PART d.)

AS A CHECK FOR THE PROGRAM'S COMPUTATION OF Φ_c, THE RESULTS OF PART a WERE USED

e.g. $\Phi_c \simeq -\dfrac{\rho}{\cosh(\frac{\Delta}{2})}$ FOR Φ SMALL.

AS A TEST, $\rho = 0.1$ AND $\Delta = 1$.

FROM PART a, $\Phi_c = -0.0887$.

FROM THE PROGRAM:

# OF STEPS	Φ_c	% DIFFERENCE
21	-0.0839	5.4%
63	-0.0859	3.2%
189	-0.0865	2.5%

AS ANOTHER TEST, $\rho = 0.025$ AND $\Delta = 1$

FROM PART a, $\Phi_c = -0.0222$

FROM MY PROGRAM, WITH 101 STEPS, $\Phi_c = -0.0216$

$\Rightarrow$ 2.7% DIFFERENCE.

IN BOTH CASES, THE FRACTIONAL NUMERICAL ERROR IN Φ_c IS 0.001 (0.1%), A SPECIFIED BY MY PROGRAM.

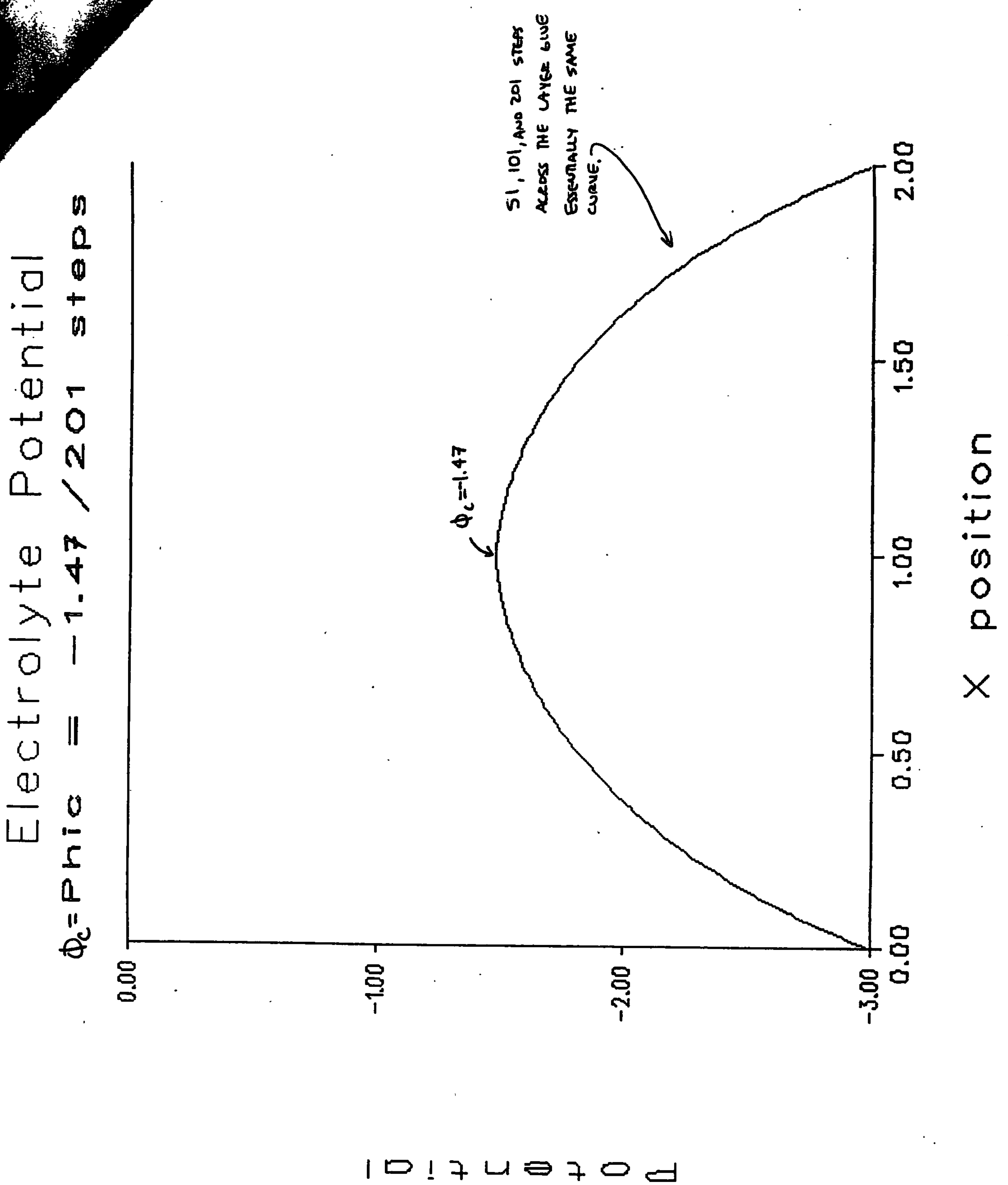
THESE TESTS LEAD ME TO BELIEVE THAT THE ALGORITHM DOES WORK SATISFACTORILY, EVEN WITH A SMALL NUMBER OF POINTS

d. WITH $\rho = 3$ AND $\Delta = 2$, THE PROGRAM WAS RUN AGAIN. IN THIS CASE, THE FOLLOWING VALUES OF Φ_c WERE FOUND:

# OF STEPS	Φ_c
51	-1.40
101	-1.45
201	-1.47

THIS INDICATES A CONVERGENCE OF $\Phi_c \simeq \underline{-1.5}$.

A PLOT OF THE POTENTIAL DISTRIBUTION IS ON THE NEXT PAGE.

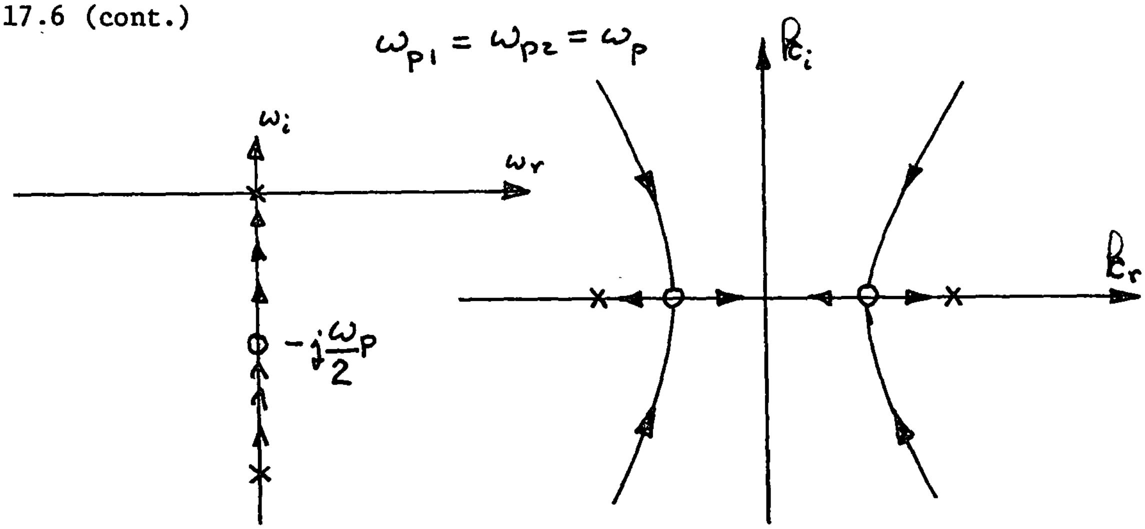


Electrolyte Potential
ϕ_c=Phic = -1.47 / 201 steps
51, 101, AND 201 STEPS ACROSS THE LAYER GIVE ESSENTIALLY THE SAME CURVE.
ϕ_c=-1.47
0.00
-1.00
-2.00
-3.00
0.00
0.50
1.00
1.50
2.00
X position
Potential

```
      program Zeta_Potentials

      integer istep,imid
      real*4 delta,delx,phi(9999),phic,phierr,zeta,perror
      common istep,delta,delx,phi,phic,phierr,zeta

      call input
    2 delx = delta/real(istep-1)
      imid = 1 + istep/2
      phic = 0.0
    3 continue
c
c     CALCULATE THE VALUE OF PHIc

      do 4 i=1,imid-1
          phi(i+1) = phi(i) + delx * sqrt(2*(cosh(phi(i))-cosh(phic)))
    4 continue
c
c     DETERMINE IF THE UNCERTAINTY IN PHIc IS LESS THAN THE ERROR
      perror = (phi(imid)-phic)/(phic + 1.0e-06)
      if(abs(perror).gt.abs(phierr)) then
          phic = phi(imid)
          goto 3
      endif
c
c     PREPARE AND SEND THE DATA TO THE OUTPUT FILE
      do 5 i=1,imid-1
          phi(istep-i+1)=phi(i)
    5 continue
      call output
      STOP 'GOOD BYE'
      END


      SUBROUTINE INPUT
      integer istep
      real*4 delta,delx,phi(9999),phic,phierr,zeta
      common istep,delta,delx,phi,phic,phierr,zeta
c
c     INPUT THE NECESSARY PARAMETERS FOR THE PLOT
    8 write(*,*) 'Enter the zeta potential:'
       read(*,*,err=8) zeta
    9 write(*,*) 'Enter the normalized distance:'
       read(*,*,err=9) delta
   10 write(*,*) 'Enter the (odd) number of steps across the layer:'
       read(*,*,err=10) istep
   11 write(*,*) 'Enter the error fraction for the midplane phi:'
       read(*,*,err=11) phierr
      phi(1) = - zeta
      RETURN
      END
```

```
      SUBROUTINE OUTPUT
      integer istep
      real*4 delta,delx,phi(9999),phic,phierr,zeta,x
      common istep,delta,delx,phi,phic,phierr,zeta

      WRITE THE DESIRED DATA TO AN OUTPUT FILE, READY FOR ENABLE TO PLOT
      open(unit=6,file='e:zeta.out',status='new')
      write(6,*) 'The potential parameters are'
      write(6,9500) istep,delta,phic,zeta,phierr
 9500 format(' Steps=  ', ',',I5,/,' Delta=  ', ',',F10.4,/,
     &       ' Phi_c=  ', ',',F10.4,/,' Zeta=  ', ',',F10.4,/,
     &       ' Error=  ', ',', F10.4)
      write(6,*) ' X position    Phi(x)  '
      do 100 i=1,istep
          x = real(i-1) * delx
          write(6,9510) x,phi(i)
  100 continue
 9510 format(' ',F10.5,',',F10.5)
      close(unit=6)
      RETURN
      END
```

Prob. 11.17.6 (cont.)



See, Briggs, R.J., Electron-Stream Interaction With Plasmas, M.I.T. Press (1964) pp 32-34 and 42-44.

Prob. 10.9.1 (a) In using Eq. (a) of Table 9.3.1, the double layer is assumed to be inside the boundaries. (This is by contrast with the use made of this expression in the text, where the electrokinetics was represented by a slip boundary condition at the walls, and there was no interaction in the bulk of the fluid.) Thus, $v^{\alpha}=0, v^{\beta}=0$ and $T_{yx} = \epsilon E_y d\Phi/dx$. Because the potential has the same value on each of the walls, the last integral is zero.

$$\int_0^{\Delta} T_{yx}\,dx = \int_0^{\Delta} \epsilon E_y \frac{d\Phi}{dx}\,dx = \epsilon E_y[\Phi(\Delta) - \Phi(0)] = 0 \qquad (1)$$

and the next to last integral becomes

$$\int_0^{x} T_{yx}\,dx = \epsilon E_y[\Phi(x) - \Phi(0)] = \epsilon E_y[\Phi(x) + \zeta] \qquad (2)$$

Thus, the velocity profile is a superposition of the parabolic pressure driven flow and the potential distribution biased by the zeta potential so that it makes no contribution at either of the boundaries.

(b) If the Debye length is short compared to the channel width, then $\underline{\Phi}$ =0 over most of the channel. Thus, Eqs. 1 and 2 inserted into Eq. (a) of Table 9.3.1 give the profile, Eq. 10.9.5.

(c) Division of Eq. (a) of Table 9.3.1 evaluated using Eqs. 1 and 2 by $\epsilon E_y kT/q\eta$ gives the desired normalized form. For example, if $\underline{\zeta}$= -3 and $\underline{\Delta}$ =2, the electrokinetic contribution to the velocity profile is as shown in Fig. P10.8.1.

Prob. 10.9.2 (a) To find S_{yx}, note that from Eq. (a) of Table 9.3.1 with the wall velocities taken as $\epsilon \zeta E_y/\eta$

$$v_x = \frac{\epsilon \zeta E_y}{\eta} + \frac{\Delta^2}{2\eta}\frac{\partial p'}{\partial y}\left[\left(\frac{x}{\Delta}\right)^2 - \frac{x}{\Delta}\right] \qquad (1)$$

Thus, the stress is

$$S_{yx} = \eta\frac{\partial v_y}{\partial x} = \frac{\Delta}{2}\frac{\partial p'}{\partial y}\left(\frac{2x}{\Delta} - 1\right) \qquad (2)$$

This expression, evaluated at x=0, combines with Eqs. 10.9.11 and 10.9.12 to give the required expression.

(b) Under open circuit conditions, where the wall currents

Prob. 10.9.2 (cont.)

due to the external stress are returned in the bulk of the fluid and where the generated voltage also gives rise to a negative slip velocity that tends to decrease E_y, the generated potential is gotten by setting i in the given equation equal to zero and solving for E_y and hence v.

$$v = \frac{(\zeta \Delta\epsilon/\eta)\,\Delta p}{\left[\Delta\sigma + \dfrac{2\rho_o \zeta^2 \epsilon \delta_o}{\eta(RT/\zeta)}\right]} \tag{3}$$

Prob. 10.10.1 In Eq. 10.9.12, what is $(\zeta\epsilon\delta_D/2)E_y$ compared to $\delta_D^2 S_{yx}$? To approximate the latter, note that $S_{yx} \sim \eta U/R$, where from Eq. 10.10.10, U is at most $(\epsilon\zeta/\eta)E_o$. Thus, the stress term is of the order of $\delta_D^2 \epsilon\zeta/R$ and this is small compared to the surface current driven by the electric field if $R \gg \delta_D$.

Prob. 10.10.2 With the particle constrained and the fluid motionless at infinity, U=0 in Eq. 10.10.9. Hence, with the use of Eq. 10.10.7, that expression gives the force.

$$f_z = \frac{6\pi R\epsilon\zeta E_o}{1 + \dfrac{\sigma_s}{\sigma R}} \tag{1}$$

The particle is pulled in the same direction as the liquid in the diffuse part of the double layer. For a positive charge, the fluid flows from south to north over the surface of the particle and is returned from north to south at a distance on the order of R from the particle.

Prob. 10.10.3 Conservation of charge now requires that

$$-\sigma\frac{\partial\Phi}{\partial r} + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left[(\sigma_s E_\theta + \beta S_{\theta r})\sin\theta\right] = 0 \tag{1}$$

with K_θ again taking the form of Eq. 10.10.4. Using the stress functions with θ dependence defined in Table 7.20.1, Eq. 1 requires that

$$-\sigma\left(-E_o - \frac{2A}{R^3}\right) + \frac{2}{R}\left[\sigma_s\left(-E_o + \frac{A}{R^3}\right) + \beta\,\tilde{S}^{\beta}_{\theta r}\right] = 0 \tag{2}$$

Prob. 10.10.3 (cont.)

The viscous shear stress can be substituted into this expression using Eq. 10.10.8b with $\tilde{v}_\theta$ given by Eq. 7 and E_θ in turn written using Eq. 10.10.4. Hence,

$$\sigma\left(E_o + \frac{2A}{R^3}\right) + \frac{2}{R}\sigma_s\left(-E_o + \frac{A}{R^3}\right) - \frac{2\beta\eta}{R^2}\left[\frac{3}{2}U + \frac{3\epsilon\varsigma}{\eta}\left(-E_o + \frac{A}{R^3}\right)\right] = 0 \quad (3)$$

This expression can be solved for A/R^3

$$\frac{A}{R^3} = \frac{E_o\left(-\sigma + \frac{2\sigma_s}{R} - \frac{6\beta\epsilon\varsigma}{R^2}\right) + \frac{3\beta\eta}{R^2}U}{2\sigma + \frac{2\sigma_s}{R} - \frac{6\beta\epsilon\varsigma}{R^2}} \quad (4)$$

Substituted into Eq. 10.10.4, this expression determines the potential distribution. With no flow at infinity, the field consists of the uniform imposed field plus a dipole field with moment proportional to A. Note that the terms in β resulting from the shear stress contributions are negligible compared to those in σ_s, provided that $\delta_D \ll R$. With no applied field, the shear stress creates a streaming current around the particle that influences the surrounding potential much as if there were a dipole current source at the origin. The force can be evaluated using Eq. 10.10.9.

$$f_z = -\pi R\eta\left\{\frac{U\left(12\sigma + \frac{12\sigma_s}{R} - \frac{24\beta\epsilon\varsigma}{R^2}\right) - \frac{12\epsilon\varsigma\sigma}{\eta}E_o}{2\sigma + \frac{2\sigma_s}{R} - \frac{6\beta\epsilon\varsigma}{R^2}}\right\} \quad (5)$$

Again, note that, because $\delta_D \ll R$, all terms involving β are negligible. Thus, Eq. 5 reduces to

$$f_z = -6\pi\eta RU + \frac{6\epsilon\varsigma\sigma E_o}{\eta\left(\sigma + \frac{\sigma_s}{R}\right)} \quad (6)$$

which makes it clear that Stoke's drag prevails in the absence of an applied electric field.

Prob. 10.11.1 From Eq. 10.11.6, the total charge of a clean surface is

$$q_d = A\sigma_d \qquad (1)$$

For the Helmholtz layer,

$$\sigma_d = \frac{\epsilon v_d}{\Delta} \qquad (2)$$

Thus, Eq. 10.11.9 gives the coenergy function as

$$W_s' = -\int_{A_o}^{A} \gamma_o \delta A + \epsilon A \int_{\Phi_d}^{v_d} \frac{v_d}{\Delta} \delta v_d = -\gamma_o(A - A_o) + \frac{\epsilon A}{2\Delta}(v_d^2 - \Phi_d^2) \qquad (3)$$

In turn, Eq. 10.11.10 gives the surface tension function as

$$\gamma_e = \gamma_o - \int_{\Phi_d}^{v_d} \frac{\epsilon v_d}{\Delta} \delta v_d = \gamma_o - \frac{\epsilon}{2\Delta}(v_d^2 - \Phi_d^2) \qquad (4)$$

and Eq. 10.11.11 provides the incremental capacitance.

$$C_d = \frac{\partial \sigma_d}{\partial v_d} = \frac{\epsilon}{\Delta} \qquad (5)$$

The curve shown in Fig. 10.11.2b is essentially of the form of Eq. 4. The surface charge density shows some departure from being the predicted linear function of v_d, while the incremental capacitance is quite different from the constant predicted by the Helmholtz model.

Prob. 10.11.2 (a) From the diagram, vertical force equilibrium for the control volume requires that

$$\pi R^2(p^\alpha - p^\beta) + 2\pi R(\gamma_o - \tfrac{1}{2}\epsilon E_\nu^2 \Delta) = 0 \qquad (1)$$

so that

$$p^\alpha - p^\beta = \frac{-2}{R}(\gamma_o - \tfrac{1}{2}\epsilon E_\nu^2 \Delta) \qquad (2)$$

and because $E_\nu = v_d/\Delta$,

$$p^\alpha - p^\beta = -\frac{2}{R}\left(\gamma_o - \frac{1}{2}\epsilon \frac{v_d^2}{\Delta}\right) \qquad (3)$$

Compare this to the prediction from Eq. 10.11.1 (with a clean interface so that $\nabla_\Sigma \to 0$ and with $R_1 = R_2 = R$)

$$p^\alpha - p^\beta = T_r = -\frac{2\gamma_e}{R} \qquad (4)$$

With the use of Eq. 4 from Prob. 10.11.1 with $\Phi_d = 0$, this becomes

Prob. 10.11.2 (cont.)

$$p^{\alpha} - p^{\beta} = -\frac{2}{R^2}\left(\gamma_o - \frac{\epsilon}{2\Delta}v_d\right) \tag{5}$$

in agreement with Eq. 3. Note that the shift from the origin in the potential for maximum γ_e is not represented by the simple picture of the double layer as a capacitor.

(b) From Eq. 5 with $R \rightarrow R + \delta\xi$

$$P_o^{\alpha} - P_o^{\beta} + \delta p = -\frac{2}{R+\delta\xi}\left(\gamma_o - \frac{\epsilon}{2\Delta}v_d^2\right) \simeq -\frac{2}{R}\left(\gamma_o - \frac{\epsilon}{2\Delta}v_d^2\right) + \frac{2}{R^2}\left(\gamma_o - \frac{\epsilon v_d^2}{2\Delta}\right)\delta\xi \tag{6}$$

and it follows from the perturbation part of this expression that

$$\delta p = \frac{2}{R^2}\left(\gamma_o - \frac{\epsilon}{2\Delta}v_d^2\right)\delta\xi \tag{7}$$

If the volume "within" the double-layer is preserved, then the thickness of the layer must vary as the radius of the interface is changed in accordance with

$$(\Delta + \delta\Delta)4\pi(R+\delta\xi)^2 = \Delta 4\pi R^2 \Rightarrow \delta\Delta = -\frac{2\Delta\delta\xi}{R} \tag{8}$$

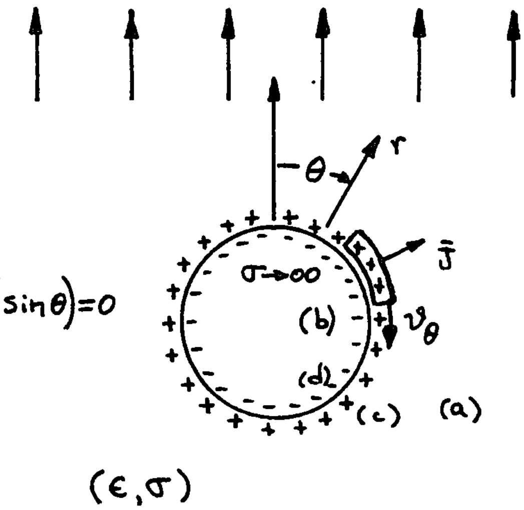
It follows from the evaluation of Eq. 3 with the voltage across the layer held fixed, that

$$\begin{aligned} p^{\alpha} - p^{\beta} + \delta p &= -\frac{2}{R+\delta\xi}\left(\gamma_o - \frac{1}{2}\frac{\epsilon v_d^2}{\Delta+\delta\Delta}\right) \\ &\simeq -\frac{2}{R}\left(\gamma_o - \frac{1}{2}\frac{\epsilon v_d^2}{\Delta}\right) + 2\left(\gamma_o - \frac{1}{2}\frac{\epsilon v_d^2}{\Delta}\right)\frac{\delta\xi}{R^2} - \frac{2}{R}\frac{1}{2}\frac{\epsilon v_d^2}{\Delta^2}\delta\Delta \end{aligned} \tag{9}$$

In view of Eq. 8,

$$\delta p = \frac{2}{R^2}\left(\gamma_o - \frac{1}{2}\frac{\epsilon v_d^2}{\Delta}\right) + \frac{2}{R^2}\frac{\epsilon v_d^2}{\Delta}\delta\xi = \frac{2}{R^2}\left(\gamma_o + \frac{1}{2}\frac{\epsilon v_d^2}{\Delta}\right)\delta\xi \tag{10}$$

What has been shown is that if the volume were actually preserved, then the effect of the potential would be just the opposite of that portrayed by Eq. 7. Thus, Eq. 10 does not represent the observed electrocapillary effect. By contrast with the "volume-conserving" interface, a "clean" interface is one made by simply exposing to each other the materials on each side of the interface.



Prob. 10.12.1 Conservation of charge for the double layer is represented using the volume element shown in the figure.

$$\sigma E_r^c + \nabla_\Sigma \cdot \sigma_d \bar{v} = 0 \Rightarrow -\sigma\left(\frac{\partial \Phi}{\partial r}\right)^c + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(\sigma_d v_\theta^c \sin\theta\right) = 0$$

It is assumed that the drop remains spherical and is biased away from the maximum in the electrocapillary curve at $\sigma_d = \sigma_o$. Thus, with the electric potential around the drop represented by

$$\Phi = -E_o r\cos\theta + \frac{A}{r^2}\cos\theta \tag{2}$$

Eq. 1 becomes

$$-\sigma\left(-E_o\cos\theta - \frac{2A}{R^3}\cos\theta\right) + \sigma_o\frac{2\sin\theta\cos\theta}{R\sin\theta}\tilde{v}_\theta^c = 0$$

and it follows that the θ dependence cancels out so that

$$\frac{2\sigma_o}{R}\tilde{v}_\theta^c + \frac{2\sigma}{R^3}A = -\sigma E_o \tag{3}$$

Normal stress equilibrium requires that

$$S_{rr}^a - S_{rr}^b - \frac{2\gamma_e}{R} = 0 \tag{4}$$

With the equilibrium part of this expression subtracted out, it follows that

$$\tilde{S}_{rr}^a 2\cos\theta - \tilde{S}_{rr}^b 2\cos\theta + \frac{2\sigma_o}{R}\Phi^c = 0 \tag{5}$$

In view of the stress-velocity relations for creep flow, Eqs. 7.21.23 and 7.21.24, this boundary condition becomes

$$-\frac{(6\eta_b + 3\eta_a)}{R}\tilde{v}_\theta^c + \frac{2\sigma_o}{R^3}A + \frac{3}{2R}\eta_a U = 2\sigma_o E_o \tag{6}$$

where additional boundary conditions that have been used are $v_\theta^d = v_\theta^c$ and $v_r^d = v_r^c = 0$. The shear stress balance requires that

$$\tilde{S}_{\theta r}^c \sin\theta - \tilde{S}_{\theta r}^d \sin\theta + \sigma_o E_\theta^c = 0 \tag{7}$$

In view of Eq. 2 and these same stress-velocity relations, it follows that

$$\frac{3}{R}(\eta_a + \eta_b)\tilde{v}_\theta^c - \frac{\sigma_o}{R^3}A + \frac{3\eta_a}{2R}U = -\sigma_o E_o \tag{8}$$

Prob. 10.12.1 (cont.)

Simultaneous solution of Eqs. 3, 6 and 8 for U gives the required relationship between the velocity at infinity, U, and the applied electric field, E_o.

$$U = \frac{\sigma_o R E_o}{2\eta_a + 3\eta_b + \frac{\sigma_o^2}{\sigma}} \tag{9}$$

To make the velocity at infinity equal to zero, the drop must move in the z-direction with this velocity. Thus, the drop moves in a direction that would be consistent with thinking of the drop as having a net charge having the same sign as the charge on the "drop-side" of the double layer.

Prob. 10.12.2 In the sections that have both walls solid, Eq. (a) of Table 9.3.1 applies with $v^\alpha=0$ and $v^\beta=0$.

$$v(x) = \frac{a^2}{2\eta_a}\left(\frac{\partial p}{\partial y}\right)_a^{I}\left[\left(\frac{x}{a}\right)^2 - \frac{x}{a}\right] \tag{1}$$

Integration relates the pressure gradient in the electrolyte (region a) and in this mercury free section (region I) to the volume rate of flow.

$$Q_a^I = w\int_0^a v\,dx = -\frac{a^3 w}{12\eta_a}\left(\frac{\partial p}{\partial y}\right)_a^I \tag{2}$$

Similarly, in the upper and lower sections where there is mercury and electrolyte, these same relations apply with the understanding that for the upper region, x=0 is the mercury interface, while for the mercury, x=b is the interface.

$$v_a(x) = U\left(1-\frac{x}{a}\right) + \frac{a^2}{2\eta_a}\left(\frac{\partial p}{\partial y}\right)_a^{II}\left[\left(\frac{x}{a}\right)^2 - \frac{x}{a}\right] \tag{3}$$

$$v_b(x) = U\frac{x}{b} + \frac{b^2}{2\eta_b}\left(\frac{\partial p}{\partial y}\right)_b^{II}\left[\left(\frac{x}{b}\right)^2 - \frac{x}{b}\right] \tag{4}$$

The volume rates of flow in the upper and lower parts of Section II are then

$$Q_a^{II} = \frac{Uaw}{2} - \frac{a^3 w}{12\eta_a}\left(\frac{\partial p}{\partial y}\right)_a^{II} \tag{5}$$

$$Q_b^{II} = \frac{Ubw}{2} - \frac{b^3 w}{12\eta_b}\left(\frac{\partial p}{\partial y}\right)_b^{II} \tag{6}$$

Because gravity tends to hold the interface level, these pressure gradients

Prob. 10.12.2 (cont.)

need not match. However, the volume rate of flow in the mercury must be zero.

$$Q_b^{II} = 0 \Rightarrow \left(\frac{\partial p}{\partial y}\right)_b^{II} = \frac{6\eta_b U}{b^2} \tag{7}$$

and the volume rates of flow in the electrolyte must be the same

$$Q_a^{I} = Q_a^{II} \Rightarrow \left(\frac{\partial p}{\partial y}\right)_a^{II} = \frac{3\eta_a U}{a^2} \tag{8}$$

Hence, it has been determined that given the interfacial velocity U, the velocity profile in Section II is

$$v_a(x) = U\left\{\left(1 - \frac{x}{a}\right) + \frac{3}{2}\left[\left(\frac{x}{a}\right)^2 - \frac{x}{a}\right]\right\} \tag{9}$$

$$v_b(x) = U\left\{\frac{x}{b} + 3\left[\left(\frac{x}{b}\right)^2 - \frac{x}{b}\right]\right\} \tag{10}$$

Stress equilibrium at the mercury-electrolyte interface determines U. First, observe that the tangential electric field at this interface is approximately

$$E_y = \frac{I}{2\sigma a w} \tag{11}$$

Thus, stress equilibrium requires that

$$\frac{\sigma_o I}{2\sigma a w} + \eta_a \frac{\partial v_a}{\partial x}\bigg|_{x=0} - \eta_b \frac{\partial v_b}{\partial x}\bigg|_{x=b} = 0 \tag{12}$$

where the first term is the double layer surface force density acting in shear on the flat interface. Evaluated using Eqs. 9 and 10, Eq. 12 shows that the interfacial velocity is

$$U = \frac{\sigma_o I}{2\sigma w\left(\frac{5}{2}\eta_a + 4\frac{a}{b}\eta_b\right)} \tag{13}$$

Finally, the volume rate of flow follows from Eqs. 5 and 8 as

$$Q_a = \frac{U a w}{4} \tag{14}$$

Thus, Eqs. 13 and 14 combine to give the required dependence of the electrolyte volume rate of flow as a function of the driving current I.

$$Q_a = \frac{a\left(\frac{\sigma_o}{\sigma}\right) I}{4\left(5\eta_a + 8\eta_b \frac{a}{b}\right)} \tag{15}$$