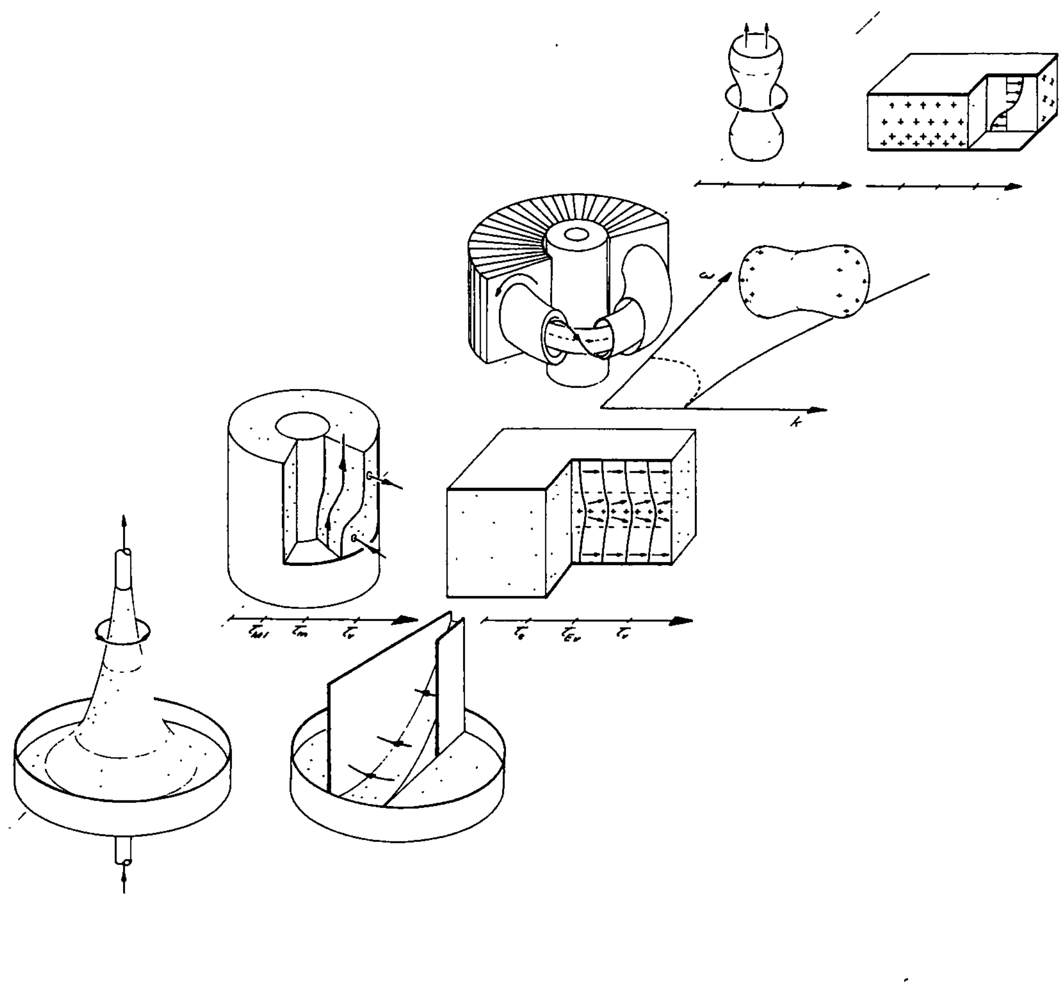
8

Statics and Dynamics of Systems Having a Static Equilibrium



Prob. 8.3.1 In the fringing region near the edges of the electrodes (at a distance large compared to the electrode spacing) the electric field is

$$\bar{E} = -\frac{V_o}{2\pi r}\bar{i}_\theta \tag{1}$$

This field is unaltered if the dielectric assumes a configuration that is essentially independent of θ. In that case, the electric field is everywhere tangential to the interface, continuity of tangential $\bar{E}$ is satisfied and there is no normal $\bar{E}$ (and hence $\bar{D}$) to be concerned with. In the force density and stress-tensor representation of Eq. 3.7.19 (Table 3.10.1) there is no electric force density in the homogeneous bulk of the liquid. Thus, Bernoulli's equation applies without a coupling term. With the height measured from the fluid level outside the field region, points (a) and (b) just above and below the interface at an arbitrary point are rel ted to the pressure at infinity by

$$p_a + \rho_a g \xi = p_\infty \tag{2}$$

$$p_b + \rho_b g \xi = p_\infty \tag{3}$$

The pressure at infinity has been taken as the same in each fluid because there is no surface force density acting in that field-free region. At the interfacial position denoted by (a) and (b), stress equilibrium in the normal direction requires that

$$[\![p]\!]\delta_{nj}n_j = [\![T_{nj}]\!]n_j \tag{4}$$

Thus, if $\rho_a << \rho_b$, it follows from Eqs. 2-4 that

$$\rho_b g \xi = [\![T_{nj}]\!]n_j \quad ; \quad T_{nj} = E_n D_j - \delta_{nj}W' = -\delta_{nj}W' \tag{5}$$

To evaluate the coenergy density, W', use is made of the constitutive law.

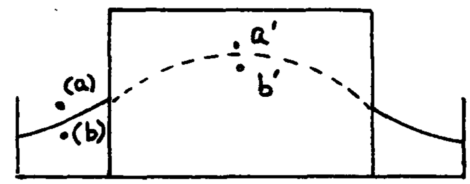
$$W' = \int_0^E D\,\delta E = \int_0^E \left(\epsilon_o E_\theta + \frac{E_\theta}{\alpha_1\sqrt{\alpha_2^2 + E_\theta^2}}\right) dE_\theta = \frac{1}{2}\epsilon_o E_\theta^2 + \frac{2\sqrt{\alpha_2^2 + E_\theta^2}}{2\alpha_1} - \frac{\alpha_2}{\alpha_1} \tag{6}$$

Thus, Eq. 5 can be solved for the interfacial position.

$$\xi = \frac{1}{\rho_b g}[\![-W']\!] = \frac{1}{\rho_b g \alpha_1}\left\{\sqrt{\alpha_2^2 + \left(\frac{V_o}{2\pi r}\right)^2} - \alpha_2\right\} \tag{7}$$

<u>Prob. 8.3.2</u> Because the liquid is homogeneous, the electromechanical coupling is, according to Eq. 3.8.14 of Table 3.10.1, confined to the interface. To evaluate the stress, note that

$$W' = \int_0^E D\,\delta E = \frac{1}{2}\epsilon_0 E^2 + \frac{\alpha_1}{\alpha_2}\ln\cosh \alpha_2 E$$



Hence, with points positioned as shown in the figure, Bernoulli's equation requires that

$$-p_a = -p_{a'} \tag{2}$$

$$p_b + \rho g \xi_0 = p_{b'} + \rho g \xi \tag{3}$$

and stress balance at the two interfacial positions requires that

$$p_a = p_b \tag{4}$$

$$-p_{a'} + p_{b'} = -\frac{\alpha_1}{\alpha_2}\ln\cosh(\alpha_2 E) \tag{5}$$

Addition of these last four expressions eliminates the pressure. Substitution for E with $V_o/s(z)$ then gives the required result

$$\xi - \xi_0 = \frac{\alpha_1}{\rho g \alpha_2}\ln\cosh\left(\frac{\alpha_2 V_o}{s(z)}\right) \tag{6}$$

Note that the simplicity of this result depends on the fact that regardless of the interfacial position, the electric field at any given z is simply the voltage divided by the spacing.

<u>Prob. 8.4.1</u> (a) From Table 2.18.1, the normal flux density at the surface of the magnets is related to A by $B_x = B_o \cos ky = \partial A/\partial y$. There are no magnetic materials below the magnets, so their fields extend to $x \to -\infty$. It follows that the imposed magnetic field has the vector potential (z directed)

$$A = \frac{B_o}{k} \sin ky \; e^{k(x-d)} \tag{1}$$

Given that $\xi = \xi_o$ at y=0 where A=0, Eq. 8.4.18 is adapted to the case at hand.

hand.

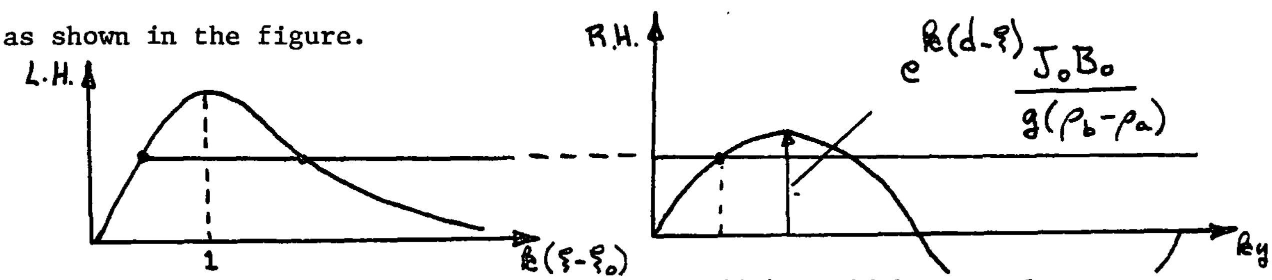
$$\lambda = -\frac{B_o}{k} \sin ky \; e^{k(\xi - d)} \tag{2}$$

and it follows from Eq. 8.4.19 with $\hat{\xi}_\infty \to \hat{\xi}_c$ that

$$\xi = \xi_o + \frac{J_o B_o \sin ky}{kg(\rho_b - \rho_a)} \; e^{k(\xi - d)} \tag{3}$$

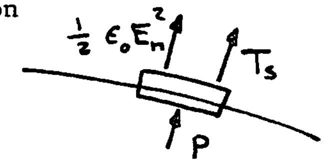
Variables can be regrouped in this expression to obtain the given $\xi(y)$.

(b) Sketches of the respective sides of the implicit expression are as shown in the figure.



The procedure (either graphically or numerically) would be to select a y, evaluate the expression on the right, and then read off the deformation relative to ξ_o from the expression as represented on the left. The peak in the latter curve comes at $k(\xi - \xi_o) = 1$ where its value is 1/e. If the two solutions are interpreted as being stable and unstable to left and right respectively, it follows that if the peak in the curve on the right is just high enough to make these solutions join, there should be an instability. This critical condition follows as

$$J_o B_o / g(\rho_b - \rho_a) = \exp[-k(d - \xi_o) - 1]$$



Prob. 8.4.2 (a) Stress equilibrium in the normal direction at the interface requires that

$$p + \frac{1}{2}\epsilon_o E_n^2 - \gamma\,\nabla\cdot\bar{n} = 0 \qquad (1)$$

The normal vector is related to the interfacial deflection by

$$\bar{n} = \left(\bar{i}_x - \frac{\partial\xi}{\partial y}\bar{i}_y\right)\left[1 + \left(\frac{\partial\xi}{\partial y}\right)^2\right]^{-\frac{1}{2}} \qquad (2)$$

In the long-wave limit, the electric field at the interface is essentially

$$E_n \simeq -\frac{V}{d-\xi} \qquad (3)$$

Finally, Bernoulli's equation evaluated at the interface where the height, is ξ becomes

$$p + \rho g\xi = 0 + \rho g b \Rightarrow p = \rho g(b-\xi) \qquad (4)$$

These last three expressions are substituted into Eq. 1 to give the required relation

$$\gamma\frac{d}{dy}\left\{\left[1+\left(\frac{d\xi}{dy}\right)^2\right]^{-\frac{1}{2}}\frac{d\xi}{dy}\right\} + \frac{1}{2}\epsilon_o\frac{V^2}{(d-\xi)^2} - \rho g(\xi - b) = 0 \qquad (5)$$

(b) For small perturbations of ξ from b, let $\xi = b + \xi'$ where ξ' is "small". Then, the linearized form of Eq. 5 is

$$\gamma\frac{d^2\xi'}{dy^2} + \frac{1}{2}\epsilon_o V^2\left[\frac{1}{(d-b)^2} + \frac{2\xi'}{(d-b)^3}\right] - \rho g\xi' = 0 \qquad (6)$$

With the "drive" put on the right, this expression is

$$\frac{d^2\xi'}{dy^2} - \frac{\xi'}{l_y^2} = \frac{-\epsilon_o V^2}{2(d-b)^2} \qquad (7)$$

where

$$l_y \equiv \left[\frac{\rho g}{\gamma} - \frac{\epsilon_o V^2}{\gamma(d-b)^3}\right]^{-\frac{1}{2}} \qquad (8)$$

is real to insure stability of the interface. To satisfy the asymptotic condition as $y\to\infty$, the increasing exponential must be zero. Thus, the

Prob. 8.4.2 (cont.)

combination of particular and homogeneous solutions that satisfies the boundary condition at y=0 is

$$\xi' = \frac{\epsilon V^2 \ell_y}{2(d-b)^2}\left(1 - e^{-y/\ell_y}\right) \tag{8}$$

(c) The multiplication of Eq. 5 by $u \equiv d\xi/dy$ gives

$$u\frac{d}{dy}\left\{(1+u^2)^{-1/2}u\right\} + \frac{dP}{dy} = 0 \tag{9}$$

where

$$P \equiv \frac{1}{2}\left[\epsilon_o \frac{V^2}{d-\xi} - \rho g(\xi-b)^2\right]$$

To integrate, define

$$v = (1+u^2)^{-1/2}u \tag{10}$$

so that

$$u = (1-v^2)^{-1/2}v \tag{11}$$

Then, Eq. 9 can be written as

$$\frac{v}{\sqrt{1-v^2}}dv + dP = 0 \tag{12}$$

and integration gives

$$-\sqrt{1-v^2} + P = \text{const.} \tag{13}$$

This expression can be written in terms of $d\xi/dy \equiv u$ by using Eq. 10.

$$\frac{-1}{\sqrt{1+\left(\frac{d\xi}{dy}\right)^2}} + P = \text{const.} \tag{14}$$

Because $d\xi/dy \rightarrow 0$ as $\xi \rightarrow \xi_o$, the constant is $P(\xi_o) - 1$ and Eq. 14 becomes

$$\frac{1}{\sqrt{1+\left(\frac{d\xi}{dy}\right)^2}} = -P(\xi_o) + P(\xi) + 1 \tag{15}$$

Solution for $d\xi/dy$ leads to the integral expression

$$\int_b^{\xi} \frac{d\xi}{\sqrt{[1+P(\xi_o)-P(\xi)]^{-2}-1}} = \int_0^y dy \tag{16}$$

Note that the lower limit is set by the boundary condition at y=0.

Prob. 8.6.1 In view of Eq. 31 from problem solution 7.9.2, the requirement that $\hat{v}_r^d = 0$ be zero with $\alpha = R$ but $\beta \to 0$ shows that if $\hat{p}^d$ is to be finite then

$$f_o(0, R, \gamma) = 0 \tag{1}$$

provided that $\omega \neq \pm 2\Omega$. By the definition of this function, given in Table 2.16.2, this is the statement that

$$-j\gamma \frac{J_o'(j\gamma R)}{J_o(j\gamma R)} = 0 = \frac{-j\gamma J_1(j\gamma R)}{J_o(j\gamma R)} \tag{2}$$

So the eigenvalue problem is reduced to finding the roots, X_{ok}, of

$$J_1(j\gamma R) = 0 \tag{3}$$

In view of the definition of γ , the eigenfrequencies are then written in terms of these roots by solving

$$\gamma^2 = -\frac{X_{ok}^2}{R^2} \equiv k^2 \left[1 - \frac{(2\Omega)^2}{\omega^2}\right] \tag{4}$$

for ω .

$$\omega_o = \frac{\pm 2\Omega}{\sqrt{1 + \frac{X_{ok}^2}{(kR)^2}}} \tag{5}$$

(b) According to this dispersion equation, waves having the same frequency have wavenumbers that are negatives. Thus, waves traveling in the $\pm z$ directions can be superimposed to obtain standing pressure waves that vary as $\cos kz$. According to Eq. 14, if p is proportional to $\cos kz$ then $v_z \propto \sin kz$ and the conditions that $v_z(0) = 0$, $v_z(\ell) = 0$ are satisfied if $k = n\pi/\ell, n = 0, 1, 2, \cdots$. For these modes, which satisfy both longitudinal and transverse boundary conditions, the resonance frequencies are therefore

$$\omega_{ok} = \frac{\pm 2\Omega}{\sqrt{1 + \frac{X_{ok}^2 \ell^2}{(n\pi R)^2}}} \tag{6}$$

Problem 8.7.1 The total potential, distinguished from the perturbation potential by a prime, is $\Phi' = -E_o y + \Phi$. Thus,

$$\frac{D\Phi'}{Dt} = \frac{\partial \Phi'}{\partial t} + \bar{v}\cdot\nabla\Phi' = \frac{\partial \Phi}{\partial t} + v_x \frac{\partial \Phi}{\partial x} + v_y\left(-E_o + \frac{\partial \Phi}{\partial y}\right) = 0 \tag{1}$$

to linear terms, this becomes

$$\frac{\partial \Phi}{\partial t} - E_o v_y = 0 \tag{2}$$

which will be recognized as the limit $\sigma \to \infty$ of Eq. 8.7.6 integrated twice on x.

Problem 8.7.2 What is new about these laws is the requirement that the current linked by a surface of fixed identity be conserved. In view of the generalized Leibnitz rule, Eq. 2.6.4 and Stoke's Theorem, Eq. 2.6.3, integral condition (a) requires that

$$\frac{d}{dt}\int_S \bar{J}_f\cdot\bar{n}\,da = \oint_S\left[\frac{\partial \bar{J}_f}{\partial t} + (\nabla\cdot\bar{J}_f)\bar{v}\right]\cdot\bar{n}\,da + \int_S \nabla\times(\bar{J}_f\times\bar{v})\cdot\bar{n}\,da \tag{3}$$

The laws are MQS, so $\bar{J}_f$ is solenoidal and it follows from Eq. 3 that

$$\frac{\partial \bar{J}_f}{\partial t} - \nabla\times(\bar{v}\times\bar{J}_f) = 0 \tag{4}$$

With the understanding that ρ is a constant, and that $\bar{B} = \mu_o\bar{H}$, the remaining laws are standard.

Problem 8.7.3 Note that $\bar{v}$ and $\bar{J}_f$ are automatically solenoidal if they take the given form. The x component of Eq. (c) from Prob. 8.7.2 is also an identity while the y and z components are

$$\frac{\partial J_y}{\partial t} - J_o\frac{\partial v_y}{\partial x} = 0 \tag{1}$$

$$\frac{\partial J_z}{\partial t} - J_o\frac{\partial v_z}{\partial x} = 0 \tag{2}$$

Similarly, the x component of Eq. (d) from Prob. 8.7.2 is an identity while the y and z components are

$$\rho\frac{\partial v_y}{\partial t} = B_o J_z + \eta\frac{\partial^2 v_y}{\partial x^2} \tag{3}$$

$$\rho\frac{\partial v_z}{\partial t} = -B_o J_y + \eta\frac{\partial^2 v_z}{\partial x^2} \tag{4}$$

Because $\bar{B}$ is imposed, Ampere's Law is not required unless perturbations in the magnetic field are of interest.

Prob. 8.7.3(cont.)

In terms of complex amplitudes $v_y = \mathrm{Re}\,\hat{v}_y \exp j\omega t$, Eqs. 1 and 2 show that

$$\hat{J}_y = -j\frac{J_o\gamma}{\omega}\hat{v}_y\,;\quad \hat{J}_z = -j\frac{J_o\gamma}{\omega}\hat{v}_z\,;\quad \hat{v}_y = A e^{\gamma x} \tag{5}$$

Substituted into Eqs. 3 and 4, these relations give

$$\begin{bmatrix} (\eta\gamma^2 - j\omega\rho) & -j\dfrac{J_o\gamma B_o}{\omega} \\ j\dfrac{J_o\gamma B_o}{\omega} & (\eta\gamma^2 - j\omega\rho) \end{bmatrix} \begin{bmatrix} \hat{v}_y \\ \hat{v}_z \end{bmatrix} = 0 \tag{6}$$

The dispersion equation follows from setting the determinant of the coefficients equal to zero.

$$(\eta\gamma^2 - j\omega\rho)\frac{\omega}{\gamma} = \pm J_o B_o \tag{7}$$

with the normalization $T_v \equiv \Delta^2\rho/\eta$, $T_{MV} \equiv \eta/J_o B_o \Delta$, $\underline{\gamma} = \gamma\Delta$

it follows that

$$\gamma = \pm\gamma_{\substack{1\\2}}\;;\quad \gamma_{\substack{1\\2}} \equiv \left[\frac{1}{2}\frac{1}{\omega T_{MV}} \pm \sqrt{\left(\frac{1}{2\omega T_{MV}}\right)^2 + j\omega T_v}\,\right] \tag{8}$$

Thus, solutions take the form

$$\hat{v}_z = \hat{A}_1 e^{\gamma_1 x} + \hat{A}_2 e^{-\gamma_1 x} + \hat{A}_3 e^{\gamma_2 x} + \hat{A}_4 e^{-\gamma_2 x} \tag{9}$$

From Eq. 6(a) and the dispersion equation, Eq. 8, it follows from Eq. 9 that

$$\hat{v}_y = j\hat{A}_1 e^{\gamma_1 x} - j\hat{A}_2 e^{-\gamma_1 x} + j\hat{A}_3 e^{\gamma_2 x} - j\hat{A}_4 e^{-\gamma_2 x} \tag{10}$$

The shear stress can be written in terms of these same coefficients using Eq. 9.

$$\hat{S}_{zx} = \eta\left(\gamma_1\hat{A}_1 e^{\gamma_1 x} - \gamma_1\hat{A}_2 e^{-\gamma_1 x} + \gamma_2\hat{A}_3 e^{\gamma_2 x} - \gamma_2\hat{A}_4 e^{-\gamma_2 x}\right) \tag{11}$$

Similarly, from Eq. 10,

$$\hat{S}_{yx} = \eta\left(j\gamma_1\hat{A}_1 e^{\gamma_1 x} + j\gamma_1\hat{A}_2 e^{-\gamma_1 x} + j\gamma_2\hat{A}_3 e^{\gamma_2 x} + j\gamma_2\hat{A}_4 e^{-\gamma_2 x}\right) \tag{12}$$

Evaluated at the respective α and β surfaces, where x = Δ and x=0, Eqs. 9 and 10 show that

Prob. 8.7.3(cont.)

$$\begin{bmatrix} \hat{v}_y^\alpha \\ \hat{v}_y^\beta \\ \hat{v}_z^\alpha \\ \hat{v}_z^\beta \end{bmatrix} = [Q_{ij}] \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \\ \hat{A}_3 \\ \hat{A}_4 \end{bmatrix}; Q_{ij} \equiv \begin{bmatrix} je^{\underline{\gamma}_1} & -je^{-\underline{\gamma}_1} & je^{\underline{\gamma}_2} & -je^{-\underline{\gamma}_2} \\ j & -j & j & -j \\ e^{\underline{\gamma}_1} & e^{-\underline{\gamma}_1} & e^{\underline{\gamma}_2} & e^{-\underline{\gamma}_2} \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{13}$$

Similarly, from Eqs. 11 and 12, evaluation at the surfaces gives

$$\begin{bmatrix} \hat{S}_{yx}^\alpha \\ \hat{S}_{yx}^\beta \\ \hat{S}_{zx}^\alpha \\ \hat{S}_{zx}^\beta \end{bmatrix} = [U_{ij}] \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \\ \hat{A}_3 \\ \hat{A}_4 \end{bmatrix}; U_{ij} \equiv \eta \begin{bmatrix} j\gamma_1 e^{\underline{\gamma}_1} & j\gamma_1 e^{-\underline{\gamma}_1} & j\gamma_2 e^{\underline{\gamma}_2} & j\gamma_2 e^{-\underline{\gamma}_2} \\ j\gamma_1 & j\gamma_1 & j\gamma_2 & j\gamma_2 \\ \gamma_1 e^{\underline{\gamma}_1} & -\gamma_1 e^{-\underline{\gamma}_1} & \gamma_2 e^{\underline{\gamma}_2} & -\gamma_2 e^{-\underline{\gamma}_2} \\ \gamma_1 & -\gamma_1 & \gamma_2 & -\gamma_2 \end{bmatrix} \tag{14}$$

The transfer relations follow from inversion of 13 and multiplication with 14

$$\begin{vmatrix} \hat{S}_{yx}^\alpha \\ \hat{S}_{yx}^\beta \\ \hat{S}_{zx}^\alpha \\ \hat{S}_{zx}^\beta \end{vmatrix} = [W_{ij}] \begin{vmatrix} \hat{v}_y^\alpha \\ \hat{v}_y^\beta \\ \hat{v}_z^\alpha \\ \hat{v}_z^\beta \end{vmatrix}; [W_{ij}] = [Q_{ij}]^{-1}[U_{ij}] \tag{15}$$

All required here are the temporal eigen-frequencies with the velocities constrained to zero at the boundaries. To this end, Eq. 13 is manipulated to take the form (note that $A_1 e^{x} + A_2 e^{-x} \equiv (A_1 + A_2)\cosh x + (A_1 - A_2)\sinh x$)*

$$\begin{bmatrix} \hat{v}_y^\alpha \\ \hat{v}_y^\beta \\ \hat{v}_z^\alpha \\ \hat{v}_z^\beta \end{bmatrix} = \begin{bmatrix} j\cosh\underline{\gamma}_1 & j\sinh\underline{\gamma}_1 & j\cosh\underline{\gamma}_2 & j\sinh\underline{\gamma}_2 \\ j & 0 & j & 0 \\ \sinh\underline{\gamma}_1 & \cosh\underline{\gamma}_1 & \sinh\underline{\gamma}_2 & \cosh\underline{\gamma}_2 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{A}_1 - \hat{A}_2 \\ \hat{A}_1 + \hat{A}_2 \\ \hat{A}_3 - \hat{A}_4 \\ \hat{A}_3 + \hat{A}_4 \end{bmatrix} \tag{16}$$

The condition that the determinant of the coefficients vanish is then

$$\cosh\underline{\gamma}_1 \cosh\underline{\gamma}_2 - \sinh\underline{\gamma}_1 \sinh\underline{\gamma}_2 \equiv \cosh(\underline{\gamma}_1 - \underline{\gamma}_2) = 1 \tag{17}$$

*Transformation suggested by Mr. Rick Ehrlich.

Prob. 8.7.3(cont.)

This expression is identical to $\cos j(\underline{\gamma}_1 - \underline{\gamma}_2) = 1$ and therefore has solutions

$$j(\underline{\gamma}_1 - \underline{\gamma}_2) = 2n\pi\,, \quad n = 0, 1, 2 \cdots \tag{18}$$

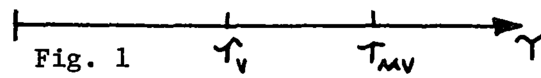
With the use of Eq. 8, an expression for the eigenfrequencies follows

$$2j\left[\left(\frac{1}{2\omega\tau_{MV}}\right)^2 + j\omega\tau_v\right]^{1/2} = 2n\pi \tag{19}$$

Manipulation and substitution $s \equiv j\omega$ shows that this is a cubic in s.

$$s^3\tau_v + (n\pi)^2 s^2 - \frac{1}{4\tau_{MV}^2} = 0 \tag{20}$$

If the viscosity is high enough that inertial effects can be ignored, the ordering of characteristic times is as shown in Fig. 1

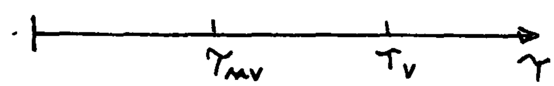


Then, there are two roots to Eq. 20 given by setting $\tau_v = 0$ and solving for s.

$$s = \pm 1/2\tau_{MV} n\pi \tag{21}$$

Thus, there is an instability having a growth rate typified by the magneto-viscous time $2n\pi\tau_{MV}$.

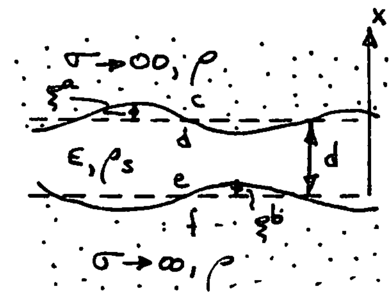
In the opposite extreme, where inertial effects are dominant, the ordering of times is as shown in Fig. 2 and the middle term in Eq. 20 is negligible compared to the other two. In this case,



$$s = 1/(4\tau_{MV}^2\tau_v)^{1/3} = \left(\frac{J_o^2 B_o^2}{4\eta\rho}\right)^{1/3} \tag{22}$$

Note that substitution back into Eq. 20 shows that the approximation is in fact self-consistent. The system is again unstable, this time with a growth rate determined by a time that is between τ_v and τ_{MV}.

<u>Prob. 8.7.4</u> The particle velocity is simply $U = bE = 2a\epsilon E^2/\eta$. Thus, the time required to traverse the distance 2a is $2aU = \eta/\epsilon E^2$.



Prob. 8.10.1 With the designations indicated in the figure, first consider the bulk relations. The perturbation electric field is confined to the insulating layer, where

$$\begin{bmatrix} \hat{e}_x^d \\ \hat{e}_x^e \end{bmatrix} = k \begin{bmatrix} -\coth kd & \frac{1}{\sinh kd} \\ \frac{-1}{\sinh kd} & \coth kd \end{bmatrix} \begin{bmatrix} \hat{\Phi}^d \\ \hat{\Phi}^e \end{bmatrix} \tag{1}$$

The transfer relations for the mechanics are applied three times. Perhaps it is best to first write the second of the following relations, because the transfer relations for the infinite half spaces (with it understood that $k > 0$) follow as limiting cases of the general relations.

$$\hat{p}^c = \frac{j\omega\rho}{k}\hat{v}_x^c = -\frac{\omega^2\rho}{k}\hat{\xi}^a \tag{2}$$

$$\begin{bmatrix} \hat{p}^d \\ \hat{p}^e \end{bmatrix} = \frac{j\omega\rho_s}{k} \begin{bmatrix} -\coth kd & \frac{1}{\sinh kd} \\ \frac{-1}{\sinh kd} & \coth kd \end{bmatrix} \begin{bmatrix} \hat{v}_x^d \\ \hat{v}_x^e \end{bmatrix} = -\frac{\omega^2\rho_s}{k} \begin{bmatrix} -\coth kd & \frac{1}{\sinh kd} \\ \frac{-1}{\sinh kd} & \coth kd \end{bmatrix} \begin{bmatrix} \hat{\xi}^a \\ \hat{\xi}^b \end{bmatrix} \tag{3}$$

$$\hat{p}^f = -\frac{j\omega\rho}{k}\hat{v}^f = \frac{\omega^2\rho}{k}\hat{\xi}^b \tag{4}$$

Now, consider the boundary conditions. The interfaces are perfectly conducting, so

$$\bar{n}\times\bar{E} = 0 \Rightarrow -E_0\frac{\partial\xi}{\partial z} = e_z \tag{5}$$

In terms of the potential, this becomes

$$\hat{\Phi}^a = E_0\hat{\xi}^a \tag{6}$$

Similarly,

$$\hat{\Phi}^b = E_0\hat{\xi}^b \tag{7}$$

Stress equilibrium for the x direction is

$$[\![p]\!]n_x = [\![T_{xj}]\!]n_j - \gamma\nabla\cdot\bar{n}\,n_x \tag{8}$$

In particular,

$$(\Pi_c + p'^c) - (\Pi_d + p'^d) = -\frac{\epsilon}{2}(E_0 + e_x)^2 + \gamma\left(\frac{\partial^2\xi}{\partial y^2} + \frac{\partial^2\xi}{\partial z^2}\right) \tag{9}$$

Hence, in terms of complex amplitudes, stress equilibrium for the upper interface is

Prob. 8.10.1(cont.)

$$-\hat{p}^c + \hat{p}^d - \epsilon E_o \hat{e}_x^d - k^2\gamma\hat{\xi}^a = 0 \tag{10}$$

Similarly, for the lower interface,

$$-\hat{p}^e + \hat{p}^f + \epsilon E_o \hat{e}_x^e - k^2\gamma\hat{\xi}^b = 0 \tag{11}$$

Now, to put these relations together and obtain a dispersion equation, insert Eqs. 5 and 6 into Eq. 1. Then, Eqs. 1-4 can be substituted into Eqs. 9 and 10, which become

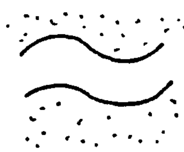
$$\begin{bmatrix} \left[\frac{\omega^2\rho}{k} + \frac{\omega^2\rho_s}{k}\coth kd + \epsilon E_o^2 k\coth kd - k^2\gamma\right] & \left[\frac{-\omega^2\rho_s}{k\sinh kd} - \frac{\epsilon E_o^2 k}{\sinh kd}\right] \\ \left[\frac{-\omega^2\rho_s}{k\sinh kd} - \frac{\epsilon_o E_o^2 k}{\sinh kd}\right] & \left[\frac{\omega^2\rho}{k} + \frac{\omega^2\rho_s}{k}\coth kd + \epsilon_o E_o^2 k\coth kd - k^2\gamma\right] \end{bmatrix}\begin{bmatrix}\hat{\xi}^a \\ \hat{\xi}^b\end{bmatrix} = 0 \tag{12}$$

For the kink mode ($\xi^a = \xi^b$), both of these expressions are satisfied if

$$\frac{\omega^2}{k}\left(\rho + \rho_s\coth kd - \frac{\rho_s}{\sinh kd}\right) + \epsilon E_o^2 k\left[\coth kd - \frac{1}{\sinh kd}\right] - k^2\gamma = 0 \tag{13}$$

With the use of the identity $(\cosh u - 1)/\sinh u = \tanh u/2$, this expression reduces to

$$\frac{\omega^2}{k}\left(\rho + \rho_s\tanh\frac{kd}{2}\right) = \gamma k^2 - \epsilon E_o^2 k\tanh\frac{kd}{2} \tag{14}$$



For the sausage mode ($\xi^a = -\xi^b$), both are satisfied if

$$\frac{\omega^2}{k}\left(\rho + \rho_s\coth kd + \frac{\rho_s}{\sinh kd}\right) + \epsilon E_o^2 k\left[\coth kd + \frac{1}{\sinh kd}\right] - k^2\gamma = 0 \tag{15}$$

and because $(\cosh u + 1)/\sinh u = \coth u/2$

$$\frac{\omega^2}{k}\left(\rho + \rho_s\coth\frac{kd}{2}\right) = \gamma k^2 - \epsilon E_o^2 k\coth\frac{kd}{2} \tag{16}$$

In the limit $kd \ll 1$, Eqs. 14 and 16 become

$$\frac{\omega^2}{k}\left(\rho + \rho_s\frac{kd}{2}\right) = \left(\gamma - \frac{\epsilon E_o^2 d}{2}\right)k^2 \tag{17}$$

$$\frac{\omega^2}{k}\left(\rho + \frac{2\rho_s}{kd}\right) = \gamma k^2 - \frac{2\epsilon E_o^2}{d} \tag{18}$$

Prob. 8.10.1(cont.)

Thus, the effect of the electric field on the kink mode is equivalent to having a field dependent surface tension with $\gamma \to \gamma - \epsilon E_o^2 d/2$

The sausage mode is unstable at $k \to 0$ (infinite wavelength) with $E_o=0$ while the kink mode is unstable at $E_o = \sqrt{2\gamma/\epsilon d}$. If the insulating liquid filled in a hole between regions filled by high conductivity liquid, the hole boundaries would limit the values of possible k's. Then there would be a threshold value of E_o.

Prob. 8.11.1 (a) In static equilibrium, $\bar{H}$ is tangential to the interface and hence not affected by the liquid. Thus, $\bar{H} = \bar{i}_\theta H_o(R/r)$ where $H_o=I/2\pi R$. The surface force densities due to magnetization and surface tension are held in equilibrium by the pressure jump $(\mu_a \equiv \mu_o, \ \mu_b \equiv \mu)$

$$\Pi_a - \Pi_b = -\frac{1}{2}(\mu_a - \mu_b)H_o^2 - \frac{\gamma}{R} \tag{1}$$

(b) Perturbation boundary conditions at the interface are, at $r = R+\xi$

$$\bar{n}\cdot[\![\mu\bar{H}]\!] = \left(\bar{i}_r - \frac{1}{R}\frac{\partial\xi}{\partial\theta}\bar{i}_\theta - \frac{\partial\xi}{\partial z}\bar{i}_z\right)\cdot\left([\![\mu h_r]\!]\bar{i}_r + [\![\mu(H_o\frac{R}{r} + h_\theta)]\!]\bar{i}_\theta + [\![\mu h_z]\!]\bar{i}_z\right) \tag{2}$$

which to linear terms requires

$$[\![\mu\hat{h}_r]\!] = -j[\![\mu]\!]\frac{H_o m}{R}\hat{\xi} \tag{3}$$

and $\bar{n}\times[\![\bar{H}]\!]=0$ which to linear terms requires that $[\![h_\theta]\!]=0$ and $[\![h_z]\!]=0$

These are represented by

$$[\![\hat{\psi}]\!] = 0 \tag{4}$$

With $n_j[\![p]\!] = [\![T_{rj}]\!]n_j + T_s n_r$, stress equilibrium for the interface requires that

$$[\![p]\!] = -\frac{1}{2}[\![\mu(H_o\frac{R}{R+\xi} + h_\theta)^2]\!]_{R+\xi} - \gamma\nabla\cdot\bar{n} \tag{5}$$

To linear terms, this expression becomes Eq. (1) and

$$[\![\hat{p}]\!] = [\![\mu]\!]\frac{H_o^2}{R}\hat{\xi} - [\![\mu]\!]H_o j\frac{m\hat{\psi}^a}{R} + \frac{\gamma}{R^2}[(1-m^2)-(kR)^2]\hat{\xi} \tag{6}$$

where use has been made of $\hat{h}_\theta = jm\hat{\psi}/R$

Perturbation fields are assumed to decay to zero as $r\to\infty$ and to be finite at r = 0. Thus, bulk relations for the magnetic field are (Table 2.16.2)

Prob. 8.11.1 (cont.)

$$\hat{\psi}^a = \hat{h}_r^a / f_m(\infty, R) \tag{7}$$

$$\hat{\psi}^b = \hat{h}_r^b / f_m(0, R) \tag{8}$$

From Eqs. (3) and (4) together with these last two expressions, it follows that

$$\hat{\psi}^a = -jm[\![\mu]\!] H_o \hat{\xi} / R[\mu_a f_m(\infty, R) - \mu_b f_m(0, R)] \tag{9}$$

This expression is substituted into Eq. (6), along with the bulk relation for the perturbation pressure, Eq. (f) of Table 7.9.1, to obtain the desired dispersion equation.

$$-\omega^2 \rho F_m(0,R) = (\mu_b - \mu_a)\frac{H_o^2}{R} + m^2 [\![\mu]\!]^2 H_o^2 \frac{1}{R^2[\mu_a f_m(\infty,R) - \mu_b f_m(0,R)]} \tag{10}$$
$$- \frac{\gamma}{R^2}\left[(1-m^2) - (kR)^2\right]$$

(c) Remember (from Sec. 2.17) that $F_m(0,R)$ and $f_m(0,R)$ are negative while $f_m(\infty,R)$ is positive. For $\mu_b > \mu_a$, the first "imposed field" term on the right stabilizes. The second "self-field" term stabilizes regardless of the permeabilities, but only influences modes with finite m. Thus, sausage modes can "exchange" with no change in the self-fields. Clearly, all modes $m \neq 0$ are stable. To stabilize the m=0 mode,

$$(\mu_b - \mu_a)\frac{H_o^2}{R} > \frac{\gamma}{R^2} \tag{11}$$

(d) In the m=0 mode the mechanical deformations are purely radial. Thus, the rigid boundary introduced by the magnet does not interfere with the motion. Also, the perturbation magnetic field is zero, so there is no difficulty satisfying the field boundary conditions on the magnet surface. (Note that the other modes are altered by the magnet). In the long wave limit, Eq. 2.16.28 gives $F_o(0,R) = f_o^{-1}(0,R) \rightarrow (-k^2 R/2)^{-1}$ and hence, Eq. (10) becomes simply

$$\omega^2 = (\mu - \mu_o)\frac{H_o^2}{\rho} k^2 \tag{12}$$

Thus, waves propagate in the z direction with phase velocity $\sqrt{(\mu - \mu_o) H_o^2/\rho}$

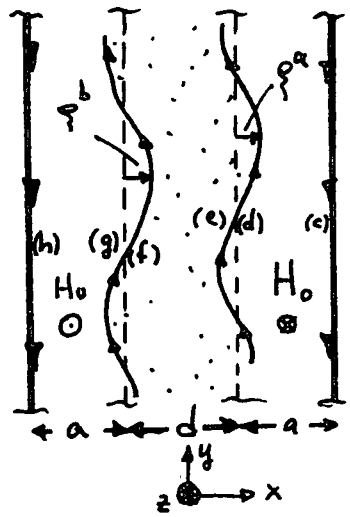
Prob. 8.11.1 (cont.)

Resonances occur when the longitudinal wavenumbers are multiples of n

Thus, the resonance frequencies are

$$f_n = \frac{n H_o}{2\ell}\sqrt{\frac{\mu-\mu_o}{\rho}} \tag{13}$$

<u>Prob. 8.12.1</u> In the vacuum regions to either side of the fluid sheet the magnetic fields take the form



$$\bar H = -H_o\bar i_z + \bar h \tag{1}$$

$$\bar H = H_o\bar i_z + \bar h \tag{2}$$

where $\bar h = -\nabla\psi$.

In the regions to either side, the mass density is negligible, and so the pressure there can be taken as zero. In the fluid, the pressure is therefore

$$p = \frac{1}{2}\mu_o H_o^2 + \mathrm{Re}\,\hat p\, e^{j(\omega t - k_y y - k_z z)} \tag{3}$$

where p is the perturbation associated with departures of the fluid from static equilibrium. Boundary conditions reflect the electromechanical coupling and are consistent with fields governed by Laplace's equation in the vacuum regions and fluid motions governed by Laplace's equation in the layer. That is one boundary condition on the magnetic field at the surfaces bounding the vacuum, and one boundary condition on the fluid mechanics at each of the deformable interfaces. First, because $\bar n\cdot\bar B = 0$ on the perfectly conducting interfaces,

$$\hat h_x^c = 0 \tag{4}$$

$$\left[\bar i_x - \frac{\partial\xi}{\partial y}\bar i_y - \frac{\partial\xi}{\partial z}\bar i_z\right]\cdot\left[-H_o\bar i_z + \bar h\right] = 0 \Rightarrow \hat h_x^d = j k_z \hat\xi^a H_o \tag{5}$$

$$\hat h_x^g = -j k_z \hat\xi^b H_o \tag{6}$$

$$\hat h_x^h = 0 \tag{7}$$

In the absence of surface tension, stress balance requires that

$$[\![p]\!]\, n_x = [\![T_{xj}]\!]\, n_j \Rightarrow \tag{8}$$

In particular, to linear terms at the right interface

$$\hat p^e = -\mu_o H_o \hat h_z^d = -j k_z \mu_o H_o \hat\psi^d \tag{9}$$

Prob. 8.12.1(cont.)

Similarly, at the left interface

$$\hat{p}^f = \mu_o H_o \hat{h}_z^g = jk\mu_o H_o \hat{\psi}^g \tag{10}$$

In evaluating these boundary conditions, the amplitudes are evaluated at the unperturbed position of the interface. Hence, the coupling between interfaces through the bulk regions can be represented by the transfer relations. For the fields, Eqs. (a) of Table 2.16.1 (in the magnetic analogue) give

$$\begin{bmatrix} \hat{\psi}^c \\ \hat{\psi}^d \end{bmatrix} = \frac{1}{k}\begin{bmatrix} -\coth ka & \frac{1}{\sinh ka} \\ \frac{-1}{\sinh ka} & \coth ka \end{bmatrix}\begin{bmatrix} \hat{h}_x^c \\ \hat{h}_x^d \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} \hat{\psi}^g \\ \hat{\psi}^h \end{bmatrix} = \frac{1}{k}\begin{bmatrix} -\coth ka & \frac{1}{\sinh ka} \\ \frac{-1}{\sinh ka} & \coth ka \end{bmatrix}\begin{bmatrix} \hat{h}_x^g \\ \hat{h}_x^h \end{bmatrix} \tag{12}$$

For the fluid layer, Eqs. (c) of Table 7.9.1 become

$$\begin{bmatrix} \hat{p}^e \\ \hat{p}^f \end{bmatrix} = \frac{j\omega\rho}{k}\begin{bmatrix} -\coth kd & \frac{1}{\sinh kd} \\ \frac{-1}{\sinh kd} & \coth kd \end{bmatrix}\begin{bmatrix} \hat{v}_x^e \\ \hat{v}_x^f \end{bmatrix} \tag{13}$$

Because the fluid has a static equilibrium, at the interfaces, $\hat{v}_x^e = j\omega\hat{\xi}^a$, $\hat{v}_x^f = j\omega\hat{\xi}^b$.

It sounds more complicated then it really is to make the following substitutions. First, Eqs. 4-7 are substituted into Eqs. 11 and 12. In turn, Eqs. 11b and 12a are used in Eqs. 9 and 10. Finally these relations are entered into Eqs. 13 which are arranged to give

$$\begin{bmatrix} -\frac{\omega^2\rho}{k}\coth kd + \mu_o \frac{H_o^2 k_z^2}{k}\coth ka & \frac{\omega^2\rho}{k}\frac{1}{\sinh kd} \\ -\frac{\omega^2\rho}{k}\frac{1}{\sinh kd} & \frac{\omega^2\rho}{k}\coth kd - \mu_o\frac{H_o^2 k_z^2}{k}\coth ka \end{bmatrix}\begin{bmatrix} \hat{\xi}^a \\ \hat{\xi}^b \end{bmatrix} = 0 \tag{14}$$

For the kink mode, note that setting $\hat{\xi}^a = \hat{\xi}^b$ insures that both of Eqs. 14 are satisfied if*

$$* \tanh\tfrac{1}{2}u = \frac{\cosh u - 1}{\sinh u} = \frac{\sinh u}{\cosh u + 1}$$

Prob. 8.12.1(cont.)

$$\frac{\omega^2\rho}{k}\tanh\frac{kd}{2} = \frac{\mu_o H_o^2 k_z^2}{k}\coth ka \tag{15}$$

Similarly, if $\hat{\xi}^a = -\hat{\xi}^b$, so that a sausage mode is considered, both equations are satisfied if

$$\frac{\omega^2\rho}{k}\coth\frac{kd}{2} = \frac{\mu_o H_o^2 k_z^2}{k}\coth ka \tag{16}$$

These last two expression comprise the dispersion equations for the respective modes. It is clear that both of the modes are stable. Note however that perturbations propagating in the y direction (k_z=0) are only neutrally stable. This is the "interchange" direction discussed with Fig. 8.12.3. Such perturbations result in no change in the magnetic field between the fluid and the walls and in no change in the surface current. As a result, there is no perturbation magnetic surface force density tending to restore the interface.

Problem 8.12.2

Stress equilibrium at the interface requires that

$$-\Pi - P_d' + P_e' - T_{rr}\Big|_{R+\xi} = 0 \Rightarrow \hat{p}^d = -\mu_o H_o^2 \frac{\hat{\xi}}{R} + \mu_o H_o \hat{h}_\theta^e \; ; \; \Pi = \frac{1}{2}\mu_o H_o^2 \tag{1}$$

Also, at the interface flux is conserved, so

$$\bar{n}\cdot\bar{H}\Big|_{R+\xi} = 0 \Rightarrow \hat{h}_r^e = -j\frac{H_o m}{R}\hat{\xi} \tag{2}$$

While at the inner rod surface

$$\hat{h}_r^f = 0 \tag{3}$$

At the outer wall,

$$\hat{\xi}^c = 0 \tag{4}$$

Bulk transfer relations are

$$\begin{bmatrix} \hat{p}^c \\ \hat{p}^d \end{bmatrix} = -\rho\omega^2 \begin{bmatrix} F_m(R,a) & G_m(a,R) \\ G_m(R,a) & F_m(a,R) \end{bmatrix} \begin{bmatrix} 0 \\ \hat{\xi} \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} \hat{h}_\theta^e \\ \hat{h}_\theta^f \end{bmatrix} = \frac{jm}{R} \begin{bmatrix} F_m(b,R) & G_m(R,b) \\ G_m(b,R) & F_m(R,b) \end{bmatrix} \begin{bmatrix} \hat{h}_r^e \\ 0 \end{bmatrix} \tag{6}$$

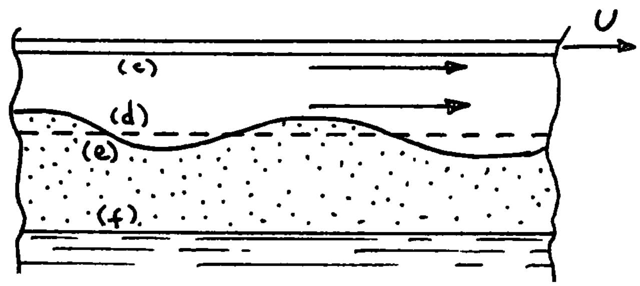
The dispersion equation follows by substituting Eq. (1) for $\hat{p}^d$ in Eq. (5b) with $\hat{h}_\theta^e$ substituted from Eq. (6a). On the right in Eq. (5b), Eq. (2) is substituted. Hence,

$$\frac{-\mu_o H_o^2}{R}\hat{\xi} + \mu_o H_o jm F_m(b,R)\left(\frac{-jH_o m}{R}\right)\hat{\xi} = -\rho\omega^2 F_m(a,R)\hat{\xi} \tag{7}$$

Thus, the dispersion equation is

$$\omega^2 = \frac{\mu_o H_o^2}{\rho R F_m(a,R)}\left[1 - \frac{m^2}{R} F_m(b,R)\right] \tag{8}$$

From the reciprocity energy conditions discussed in Sec. 2.17, $F_m(a,R) > 0$ and $F_m(b,R) < 0$, so Eq. 8 gives real values of ω regardless of k. The system is stable.



Problem 8.12.3 In static equilibrium $\bar{v}=0$,

$$\Pi_a - \Pi_b = -\frac{1}{2}\mu_o H_o^2$$ and

$$p = \Pi_b - \rho g x \tag{1}$$

With positions next to boundaries denoted as shown in the figure, boundary conditions from top to bottom are as follows. For the conducting sheet backed by an infinitely permeable material, Eq. (a) of Table 6.3.1 reduces to

$$k^2 \hat{h}_y^c = -\mu_o \sigma_s k_y (\omega - k_y U)\hat{h}_x^c \tag{2}$$

The condition that the normal magnetic flux vanish at the deformed interface is to linear terms

$$\hat{h}_x^d + j k_y H_o \hat{\xi} = 0 \tag{3}$$

The perturbation part of the stress balance equation for the interface is

$$-\hat{p}^e = -\mu_o H_o \hat{h}_y^d - k^2 \gamma \hat{\xi} - \rho g \hat{\xi} \tag{4}$$

In addition, continuity and the definition of the interface require that $\hat{v}_x = j\omega\hat{\xi}$

Finally, the bottom is rigid, so

$$\hat{v}_x^f = 0 \tag{5}$$

Bulk relations for the perturbations in magnetic field follow from Eqs. (a) of Table 2.16.1

$$\begin{bmatrix} \hat{h}_x^c \\ \hat{h}_x^d \end{bmatrix} = \frac{k}{jk_y}\begin{bmatrix} -\coth ka & \dfrac{1}{\sinh ka} \\ \dfrac{-1}{\sinh ka} & \coth ka \end{bmatrix}\begin{bmatrix} \hat{h}_y^c \\ \hat{h}_y^d \end{bmatrix} \tag{6}$$

where $\hat{h}_y = jk_y\hat{\psi}$ has been used.

Problem 8.12.3 (cont.)

The mechanical perturbation bulk relations follow from Eqs. (c) of Table 7.9.1

$$\begin{bmatrix} \hat{p}^e \\ \\ \hat{p}^f \end{bmatrix} = \frac{j\omega\rho}{k} \begin{bmatrix} -\coth kb & \frac{1}{\sinh kb} \\ \\ \frac{-1}{\sinh kb} & \coth kb \end{bmatrix} \begin{bmatrix} \hat{v}_x^e \\ \\ \hat{v}_x^f \end{bmatrix} \tag{7}$$

where

$$\hat{v}_x^e = j\omega\hat{\xi} \tag{8}$$

Equations 2 and 6a give

$$\hat{H}_y^c = \frac{j\mu_o\sigma_s k(\omega - k_y U)\hat{H}_y^d}{\sinh ka\left[k^2 + j\sigma_s\mu_o k(\omega - k_y U)\coth ka\right]} \tag{9}$$

This expression combines with Eqs. 3 and 6b to show that

$$kH_o\hat{\xi} = \frac{k}{k_y}\left\{\frac{-j\mu_o\sigma_s k(\omega - kU)}{\sinh^2 ka\left[k^2 + j\sigma_s\mu_o k(\omega - k_y U)\coth ka\right]} + \coth ka\right\}\hat{H}_y^d \tag{10}$$

Thus, the stress balance equation, Eq. 4, can be evaluated using $\hat{H}_y^d$ from Eq. 10 along with $\hat{p}^e$ from Eq. 7a, Eq. 5 and Eq. 8. The coefficient of $\hat{\xi}$ is the desired dispersion equation.

$$\begin{aligned} \frac{\omega^2\rho}{k}\coth kb = {}& \rho g + k^2\gamma \\ & + \mu_o H_o^2\frac{k_y^2}{k}\tanh ka\left\{\frac{1 + j\frac{\mu_o\sigma_s}{k}(\omega - k_y U)\coth ka}{1 + j\frac{\mu_o\sigma_s}{k}(\omega - k_y U)\tanh ka}\right\} \end{aligned} \tag{11}$$

Prob. 8.12.4 The development of this section leaves open the configuration beyond the radius r=a. Thus, it can be readily adapted to include the effect of the lossy wall. The thin conducting shell is represented by the boundary condition of Eq. (b) from Table 6.3.1.

$$j\left(\frac{m^2}{a^2}+k^2\right)(\hat{\psi}^e-\hat{\psi}^b)=\sigma_s\mu_o\omega\,\hat{h}_r^b \tag{1}$$

where (e) denotes the position just outside the shell. The region outside the shell is free space and described by the magnetic analogue of Eq. (b) from Table 2.16.2.

$$\hat{\psi}^e=F_m(\infty,a)\hat{h}_r^e=F_m(\infty,a)\hat{h}_r^b \tag{2}$$

Equations 8.12.4a and 8.12.7 combine to represent what is "seen" looking inward from the wall.

$$\hat{\psi}^b=F_m(R,a)\hat{h}_r^b-j\,G_m(a,R)\left(\frac{mH_t}{R}+kH_a\right)\hat{\xi} \tag{3}$$

Thus, substitution of Eqs. 2 and 3 into Eq. 1 gives

$$\hat{h}_r^b=\frac{G_m(a,R)\left(\frac{mH_t}{R}+kH_a\right)\hat{\xi}}{j\left[F_m(\infty,a)-F_m(R,a)\right]-\mu_o\sigma_s\omega)/\left(\frac{m^2}{a^2}+k^2\right)} \tag{4}$$

Finally, this expression is inserted into Eq. 8.12.11 to obtain the desired dispersion equation.

$$\begin{aligned}\omega^2\rho F_m(0,R)=&\frac{\mu_oH_t^2}{R}-\mu_o\left(\frac{m}{R}H_t+kH_a\right)^2F_m(a,R)\\&-j\frac{\mu_o\left(\frac{m}{R}H_t+kH_a\right)^2G_m(R,a)\,G_m(a,R)}{j\left[F_m(\infty,a)-F_m(R,a)\right]-\mu_o\sigma_s\omega)/\left(\frac{m^2}{a^2}+k^2\right)}\end{aligned} \tag{5}$$

The wall can be regarded as perfectly conducting provided that the last term is negligible compared to the one before it. First, the conduction term in the denominator must dominate the energy storage term.

$$\frac{\mu_o\sigma_s|\omega|}{\left(\frac{m^2}{a^2}+k^2\right)}>F_m(\infty,a)-F_m(R,a)>0 \tag{6}$$

Prob. 8.12.4(cont.)

Second, the last term is then negligible if

$$\frac{\mu_o \sigma_s |\omega|}{(\frac{m^2}{a^2}+k^2)} > -G_m(R,a)G_m(a,R)/F_m(a,R) > 0 \qquad (7)$$

In general, the dispersion equation is a cubic in ω and describes the coupling of the magnetic diffusion mode on the wall with the surface Alfvén waves propagating on the perfectly conducting column. However, in the limit where the wall is highly resistive, a simple quadratic expression is obtained for the damping effect of the wall on the surface waves. With the second term in the denominator small compared to the first, $(a+b)^{-1} \simeq a^{-1} - b/a^2$ and

$$-\rho F_m(0,R)(j\omega)^2 + B(j\omega) + K = 0 \qquad (8)$$

where an effective spring constant is

$$K = \frac{-\mu_o H_t^2}{R} + \mu_o\left(\frac{m}{R}H_t + kH_a\right)^2 F_m(a,R) + \frac{\mu_o\left(\frac{m}{R}H_t + kH_a\right)^2 G_m(R,a)G_m(a,R)}{F_m(\infty,a) - F_m(R,a)} \qquad (9)$$

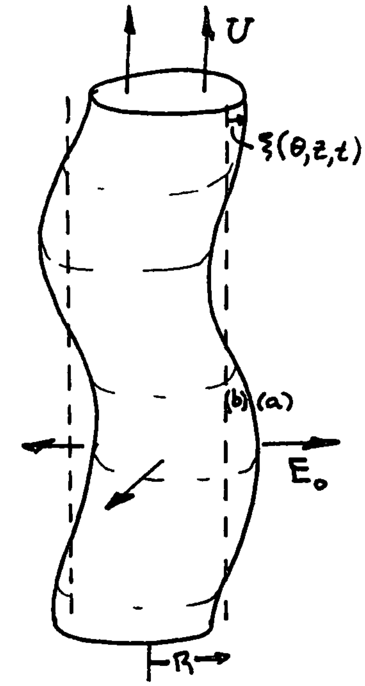
and an effective damping coefficient is

$$B = \frac{-\mu_o\left(\frac{m}{R}H_t + kH_a\right)^2 G_m(R,a)G_m(a,R)}{[F_m(\infty,a) - F_m(R,a)]^2}\,\frac{\mu_o\sigma_s}{(\frac{m^2}{a^2}+k^2)} \qquad (10)$$

Thus, the frequencies (given by Eq. 8) are

$$j\omega = \frac{-B \pm \sqrt{B^2 - (-\rho F_m(0,R))4K}}{2[-\rho F_m(0,R)]} \qquad (11)$$

Note that $F_m(0,R) < 0$, $F_m(a,R) > 0$, $F_m(\infty,a) - F_m(R,a) > 0$ and $G_m(R,a)G_m(a,R) < 0$. Thus, the wall produces damping.



Prob. 8.13.1 In static equilibrium, the radial stress balance becomes

$$[\![p]\!] = [\![T_{rr}]\!] - \gamma\left(\frac{1}{R_1}+\frac{1}{R_2}\right) \quad (1)$$

so that the pressure jump under this condition is

$$[\![\Pi]\!] = \frac{1}{2}\epsilon_o E_o^2 - \frac{\gamma}{R} \quad (2)$$

In the region surrounding the column, the electric field intensity takes the form

$$\bar{E} = E_o \frac{R}{r}\bar{i}_r + \bar{e}\ ;\ \bar{e} = -\nabla\Phi \quad (3)$$

while inside the column the electric field is zero and the pressure is given by

$$p = \Pi_b + p'(r,\theta,z,t) = \Pi_b + \mathrm{Re}\,\hat{p}(r)\,e^{j(\omega t - m\theta - kz)} \quad (4)$$

Electrical boundary conditions require that the perturbation potential vanish as r becomes large and that the tangential electric field vanish on the deformable surface of the column.

$$\bar{n}\times\bar{E}\Big|_{r=R+\xi} = 0 \cong \begin{vmatrix} \bar{i}_r & \bar{i}_\theta & \bar{i}_z \\ 1 & -\frac{1}{R}\frac{\partial\xi}{\partial\theta} & -\frac{\partial\xi}{\partial z} \\ E_o\frac{R}{r}+e_r & e_\theta & e_z \end{vmatrix} \Rightarrow e_z = -E_o\frac{\partial\xi}{\partial z} \quad (5)$$

In terms of complex amplitudes, with $\hat{e}_z = jk\hat{\Phi}$,

$$\hat{\Phi}^a = E_o\hat{\xi} \quad (6)$$

Stress balance in the radial direction at the interface requires that (with some linearization) $(p_a' \approx 0)$

$$\Pi_a - \Pi_b - p_b' = \frac{1}{2}\epsilon_o\left[E_o\frac{R}{(R+\xi)} + e_r\right]^2 + (T_s)_r \quad (7)$$

To linear terms, this becomes (Eqs. (f) and (h), Table 7.6.2 for $\bar{T}_s$)

$$\hat{p}_b = \frac{\epsilon_o E_o^2}{R}\hat{\xi} - \epsilon_o E_o\hat{e}_r^a - \frac{\gamma}{R^2}\left(1 - m^2 - (kR)^2\right)\hat{\xi} \quad (8)$$

Bulk relations representing the fields surrounding the column and the fluid within are Eq. (a) of Table 2.16.2 and (f) of Table 7.9.1

Prob. 8.13.1(cont.)

$$\hat{e}_r^a = f_m(\infty,R)\hat{\Phi}^a \tag{10}$$

$$\hat{p}^b = j(\omega - kU)\rho F_m(0,R)\hat{v}_r \tag{11}$$

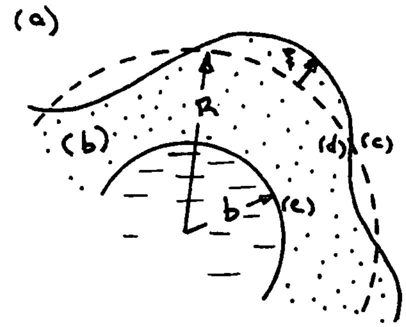
Recall that $\hat{v}_r = j(\omega - kU)\hat{\xi}$, and it follows that Eqs. 9,10 and 6 can be substituted into the stress balance equation to obtain

$$-(\omega - kU)^2\rho F_m(0,R)\hat{\xi} = \frac{\epsilon_o E_o^2}{R}\hat{\xi} - \epsilon_o E_o^2 f_m(\infty,R)\hat{\xi} - \frac{\gamma}{R^2}(1-m^2-k^2R^2)\hat{\xi} \tag{12}$$

If the amplitude is to be finite, the coefficients must equilibrate. The result is the dispersion equation given with the problem.

Problem 8.13.2

The equilibrium is static with the distribution of electric field intensity



$$E_r = \frac{q}{4\pi r^2}\begin{Bmatrix} \frac{1}{\epsilon_0} \\ \frac{1}{\epsilon} \end{Bmatrix}\begin{matrix} ;R<r \\ ;b<r<R \end{matrix} \qquad (1)$$

and difference between equilibrium pressures required to balance the electric surface force density and surface tension

$$\Pi_b - \Pi_a = \frac{2\gamma}{R} - \frac{1}{2}\frac{q^2}{16\pi^2 R^2}\left[\frac{\epsilon - \epsilon_0}{\epsilon\epsilon_0}\right] \qquad (2)$$

With the normal given by Eq. 8.17.18, the perturbation boundary conditions require $\bar n \times [\![\bar E]\!] = 0$ at the interface.

$$\hat\Phi^c - \hat\Phi^d - \hat\xi\frac{q}{4\pi R^2}\left[\frac{\epsilon - \epsilon_0}{\epsilon\epsilon_0}\right] = 0 \qquad (3)$$

that the jump in normal $\bar D$ be zero,

$$\epsilon_0 \hat e_r^c - \epsilon \hat e_r^d = 0 \qquad (4)$$

and that the radial component of the stress equilibrium be satisfied

$$-(\hat p^c - \hat p^d) - \frac{2q^2}{(4\pi)^2 R^5}\frac{(\epsilon - \epsilon_0)}{\epsilon\epsilon_0}\hat\xi + \frac{q}{4\pi R^2}(\hat e_r^c - \hat e_r^d) - \frac{\gamma}{R^2}(n-1)(n+2)\hat\xi = 0 \qquad (5)$$

In this last expression, it is assumed that Eq. (2) holds for the equilibrium stress. On the surface of the solid perfectly conducting core,

$$\hat\xi^e = 0 \quad ; \quad \hat\Phi^e = 0 \qquad (6)$$

Mechanical bulk conditions require (from Eq. 8.12.25) (F(b,R) < 0 for $R > b$

$$\hat p^c = 0 \quad ; \quad \hat p^d = -\omega^2\rho F(b,R)\hat\xi \qquad (7)$$

while electrical conditions in the respective regions require (Eq. 4.8.16)*

$$\varepsilon_0 \hat e_r^c = \frac{\varepsilon_0 (n+1)}{R}\hat\Phi^c \quad ; \quad \varepsilon \hat e_r^d = \epsilon f(b,R)\hat\Phi^d \qquad (1*0$$

Now, Eqs. (7) and (8) are respectively used to substitute for $\hat p^c, \hat p^d, \hat e_r^c$ & $\hat e_r^d$ in Eqs. (5) and (4) to make Eqs. (3)-(5) become the three expressions

* $\lim_{b\to 0} f(b,R) = -n/R$; $f(b,R) < 0$ for $R > b$

Problem 8.13.2 (cont.)

$$\begin{bmatrix} 1 & -1 & \dfrac{-q(\epsilon-\epsilon_o)}{4\pi R^2 \epsilon\epsilon_o} \\ \epsilon_o\dfrac{(n+1)}{R} & -\epsilon f(b,R) & 0 \\ \dfrac{q(n+1)}{4\pi R^3} & -\dfrac{q f(b,R)}{4\pi R^2} & -\omega^2\rho F(b,R)-\dfrac{2q^2(\epsilon-\epsilon_o)}{(4\pi)^2 R^5\epsilon\epsilon_o}-\dfrac{\gamma(n-1)(n+2)}{R^2} \end{bmatrix}\begin{bmatrix}\hat{\Phi}^c \\ \hat{\Phi}^d \\ \hat{\xi}\end{bmatrix}=0 \quad (9)$$

The determinant of the coefficients gives the required dispersion equation which can be solved for the inertial term to obtain

$$-\omega^2\rho F_n(b,R) = \frac{2q^2(\epsilon-\epsilon_o)}{(4\pi)^2 R^5\epsilon\epsilon_o} + \frac{\gamma}{R^2}(n-1)(n+2) + \frac{q^2(\epsilon-\epsilon_o)^2(n+1)f(b,R)}{(4\pi)^2 R^4\epsilon\epsilon_o\left[-\epsilon f(b,R)R+\epsilon_o(n+1)\right]} \quad (10)$$

The system will be stable if the quantity on the right is positive. In the limit $b \ll R$, this comes down to the requirement that for instability

$$\Gamma\frac{\epsilon_o}{\epsilon}\left\{2\frac{(\epsilon-\epsilon_o)}{\epsilon_o} - \left(\frac{\epsilon}{\epsilon_o}-1\right)^2\frac{(n+1)n}{\frac{\epsilon}{\epsilon_o}n+(n+1)}\right\} + (n-1)(n+2) < 0 \quad (11)$$

or

$$\Gamma > \frac{(n-1)(n+2)}{\left[\left(\frac{\epsilon}{\epsilon_o}-1\right)^2\frac{(n+1)n}{\frac{\epsilon}{\epsilon_o}n+n+1} - 2\left(\frac{\epsilon}{\epsilon_o}-1\right)\right]\frac{\epsilon_o}{\epsilon}} \quad (12)$$

where

$$\Gamma \equiv \frac{q^2}{\gamma(4\pi)^2 R^3\epsilon_o}$$

and it is clear from Eq. (11) that for cases of interest, the denominator of Eq. (12) is positive.

Problem 8.13.2 (cont.)

The figure shows how the conditions for incipient instability can be calculated given ϵ/ϵ_o. What is plotted is the right hand side of Eq. (2). In the range where this function is positive, it has an asymptote which can be found by setting the denominator of Eq. (12) to zero

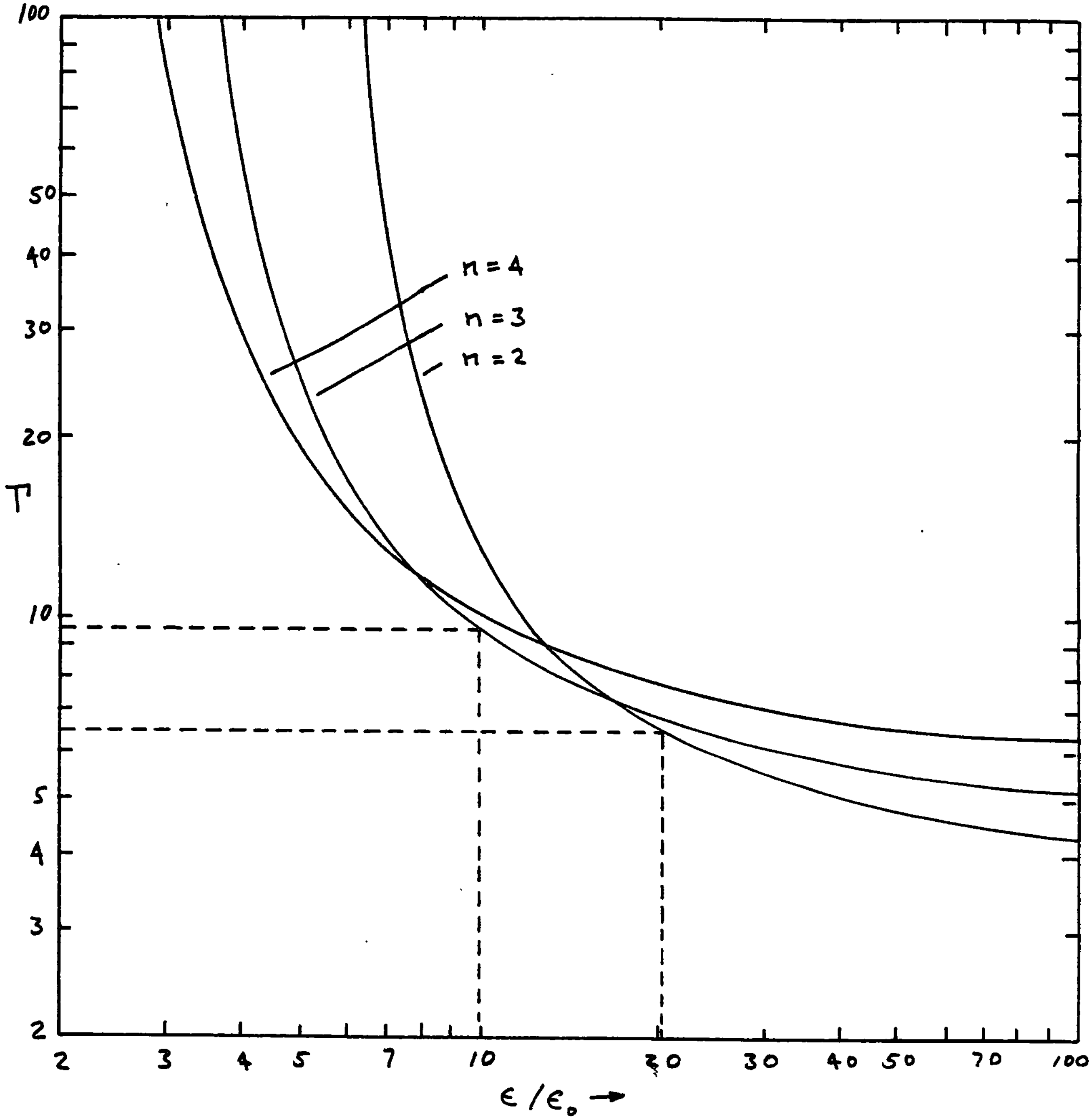
$$\left(\frac{\epsilon}{\epsilon_o}\right)_n = \frac{n^2+3n+2}{n(n-1)} \tag{13}$$

The asymptote in the horizontal direction is the limit of Eq. (12) as $\epsilon/\epsilon_o \to \infty$

$$\Gamma_n = n+2 \tag{14}$$

The curves are for the lowest mode numbers n = 2,3,4 and give an idea of how higher modes would come into play. To use the curves, take ϵ/ϵ_o =20 as an example. Then, it is clear that the first mode to become unstable is n=2 and that instability will occur as the charge is made to exceed about a value such that Γ = 6.5. Similarly, for ϵ/ϵ_o=10, the first mode to become unstable is n=3, and to make this happen, the value of Γ must be Γ=9.6. The higher order modes should be drawn in to make the story complete, but it appears that as ϵ/ϵ_o is reduced, the most critical mode number is increased, as is also the value of Γ required to obtain the instability.

Problem 8.13.2 (cont.)



Prob. 8.14.1 As in Sec. 8.14, the bulk coupling can be absorbed in the pressure. This is because in the bulk the only external force is

$$\bar{F} = -\nabla \mathcal{E} \quad ; \quad \mathcal{E} \equiv q \Phi \tag{1}$$

where $q = Q / \frac{4}{3}\pi R^3$ is uniform throughout the bulk of the drop. Thus, the bulk force equation is the same as for no bulk coupling if $p \rightarrow \Pi \equiv p + q\Phi$. In terms of equilibrium and perturbation quantities,

$$\pi = p_o(r) + q\Phi_o(r) + \mathrm{Re}\,\hat{\pi}(r) P_n^m(\cos\theta) e^{j(\omega t - m\phi)} \tag{2}$$

where $\Pi = p_o(r) + q\Phi_o(r)$ and $\hat{\pi}$ plays the role $\hat{p}$ in the mechanical transfer relations. Note that from Gauss' Law, $\Phi_o = qr^2/6\epsilon_o$, and that because the drop is in static equilibrium, $d\Pi/dr = 0$ and Π is independent of r. Thus, for a solid sphere of liquid, Eq. (i) of Table 7.9.1 becomes

$$\hat{\pi}^b = j\omega \rho_b F_n(0, R) \hat{v}_r^b \tag{3}$$

In the outside fluid, there is no charge density and this same transfer relation becomes

$$\hat{p}^a = j\omega \rho_a F_n(\infty, R) \hat{v}_r^a \tag{4}$$

At each point in the bulk, where deformations leave the charge distribution uniform, the perturbation electric field is governed by Laplace's equation. Thus, Eq. (a) of Table 2.16.3 becomes

$$\hat{e}_r^a = f_n(\infty, R) \hat{\Phi}^a \tag{4}$$

$$\hat{e}_r^b = f_n(0, R) \hat{\Phi}^b \tag{5}$$

Boundary conditions are written in terms of the surface displacement

$$\hat{v}_x^a = \hat{v}_x^b = j\omega \hat{\xi} \tag{6}$$

Prob. 8.14.1 (cont.)

Because there is no surface force density (The permittivity is ϵ_o in each region and there is no free surface charge density.)

$$[\![p]\!]\Big|_{r=R+\xi} = T_s \tag{7}$$

This requires that

$$\Pi_a + R_e \hat{p}^a P_n^m e^{j(\omega t - m\phi)} - \left\{ \Pi_b - q\Phi_o\Big|_{R+\xi} + R_e(\hat{\pi}^b - q\hat{\Phi}^b) P_n^m e^{j(\omega t - m\phi)} \right\} = T_s \tag{8}$$

Continuation of the linearization gives

$$\Pi_a - \Pi_b + q\Phi_o(R) = -\frac{2\gamma}{R} \tag{9}$$

for the static equilibrium and

$$\hat{p}^a - \hat{\pi}^b - \frac{q^2 R}{3\epsilon_o}\hat{\xi} + q\hat{\Phi}^b = -\frac{\gamma}{R^2}(n-1)(n+2)\hat{\xi} \tag{10}$$

for the perturbation. In this last expression. Eq. (1) of Table 7.6.2 has been used to express the surface tension force density on the right.

That the potential is continuous at r=R is equivalent to the condition that $\bar{n} \times [\![\bar{E}]\!] = 0$ there. This requires that

$$\begin{bmatrix} \bar{i}_r & \bar{i}_\theta & \bar{i}_\phi \\ 1 & -\frac{1}{r}\frac{\partial \xi}{\partial \theta} & \frac{-1}{r\sin\theta}\frac{\partial \xi}{\partial \phi} \\ [\![E_o(r) + e_r]\!] & [\![e_\theta]\!] & [\![e_\phi]\!] \end{bmatrix} = 0 \Rightarrow [\![e_\theta]\!] + \frac{1}{R}\frac{\partial \xi}{\partial \theta}[\![E_o]\!] = 0 \tag{11}$$

where the second expression is the ϕ component of the first. It follows from Eq. (11) that

$$-[\![\hat{\Phi}]\!] + \hat{\xi}[\![E_o]\!] = 0 \tag{12}$$

and finally, because $[\![E_o]\!] = 0$

$$\hat{\Phi}^a - \hat{\Phi}^b = 0 \tag{13}$$

The second electrical condition requires that $\bar{n} \cdot [\![\bar{D}]\!] = 0$, which becomes

$$[\![\epsilon_o E_o]\!]\Big|_{R+\xi} + [\![\epsilon_o e_r]\!] = 0 \tag{14}$$

Prob. 8.14.1 (cont.)

Linearization of the equilibrium term gives

$$[\![\frac{dE_o}{dr}]\!] \xi + [\![e_r]\!] = 0 \tag{15}$$

Note that outside, $E_o = R^3 q/3\epsilon_o r^2$ while inside, $E_o = q r/3\epsilon_o$. Thus, Eq. 15 becomes

$$-\frac{q}{\epsilon_o}\hat{\xi} + \hat{e}_r^a - \hat{e}_r^b = 0 \tag{16}$$

Equations 4 and 5, with Eq. 13, enter into Eq. 16 to give

$$-\frac{q}{\epsilon_o}\hat{\xi} + [f_n(\infty,R) - f_n(0,R)]\hat{\Phi}^b = 0 \tag{17}$$

which is solved for $\hat{\Phi}^b$. This can then be inserted into Eq. 10, along with $\hat{p}^a$ and $\hat{\pi}^b$ given by Eqs. 2 and 3 and Eq. 6 to obtain the desired dispersion equation

$$\omega^2[\rho_a F_n(\infty,R) - \rho_b F_n(0,R)] = \frac{\gamma}{R^2}(n-1)(n+2) - \frac{q^2 R}{3\epsilon_o} + \frac{q^2}{\epsilon_o[f_n(\infty,R) - f_n(0,R)]} \tag{18}$$

The functions $F_n(\infty,R) > 0, F_n(0,R) < 0$ and $f_n(\infty,R) - f_n(0,R) = (2n+1)/R$ so it follows that the imposed field (second term on the right) is destabilizing, and that the self-field (the third term on the right) is stabilizing. In spherical geometry, the surface tension term is stabilizing for all modes of interest, $n > 1$.

All modes first become unstable (as Q is raised) as the term on the right in Eq. 18 passes through zero. With $q \equiv Q/\frac{4}{3}\pi R^3$, this condition is therefore $(n \neq 1)$

$$Q^2 = \frac{8}{3}\pi^2 \epsilon_o \gamma R^3 (n+2)(2n+1) \tag{19}$$

The n=0 mode is not allowed because of mass conservation. The n=1 mode, which represents lateral translation, is marginally stable, in that it gives

Prob. 8.14.1 (cont.)

$\omega = 0$ in Eq. 18. The n=1 mode has been excluded from Eq. 19. For $n>0$, Q^2 is a monotonically increasing function of n in Eq. 19, so the first unstable mode is n=2. Thus, the most critical displacement of the interfaces have the three relative surface displacements shown in Table 2.16.3 for P_2^m. The critical charge is

$$Q_c = \sqrt{\frac{160}{3}\pi^2 R^3 \gamma \epsilon_o} = 7.3\pi\sqrt{\epsilon_o \gamma R^3}$$

Note that this charge is slightly lower than the critical charge on a perfectly conducting sphere drop (Rayleigh's limit, Eq. 8.13.11).

Prob. 8.14.2 The configuration is as shown in Fig. 8.14.2 of the text, except that each region has its own uniform permittivity. This complication evidences itself in the linearization of the boundary conditions, which is somewhat more complicated because of the existence of a surface force density due to the polarization.

The x-component of the condition of stress equilibrium for the interface is in general

$$-[\![p]\!]n_x + [\![T_{xj}]\!]n_j + T_s = 0 \tag{1}$$

This expression becomes

$$-[\![-q\Phi_o + \Pi - \rho g x]\!]_{x=\xi} - [\![p']\!]_{x=0} + [\![\frac{\epsilon}{2}(E_o + e_x)^2]\!]_{x=\xi} + \gamma\left(\frac{\partial^2\xi}{\partial y^2} + \frac{\partial^2\xi}{\partial z^2}\right) = 0 \tag{2}$$

Note that $E_o = E_o(x)$, so that there is a perturbation part of E_o^2 evaluated at the interface, namely $\xi E_o\, dE_o/dx$. Thus, with the equilibrium part of Eq. 2 cancelled out, the remaining part is

$$[\![q\frac{d\Phi_o}{dx}]\!]\hat{\xi} + g\hat{\xi}(\rho_a - \rho_b) - (\hat{p}^d - \hat{p}^e) + [\![\epsilon E_o \hat{e}_x]\!] + [\![\epsilon E_o \frac{dE_o}{dx}]\!]\hat{\xi} - \gamma k^2\hat{\xi} = 0 \tag{3}$$

It is the bulk relations written in terms of $\hat{\pi}$ that are available, so this expression is now written using the definition $\hat{p} = \hat{\pi} - q\hat{\Phi}$. Also, $d\Phi_o/dx = -E_o$ and $\epsilon\, dE_o/dx = q$, so Eq. 3 becomes

$$g(\rho_a - \rho_b)\hat{\xi} - [\![\hat{\pi}]\!] + [\![q\hat{\Phi}]\!] + [\![\epsilon E_o \hat{e}_x]\!] - \gamma k^2\hat{\xi} = 0 \tag{4}$$

Prob. 8.14.2 (cont.)

The first of the two electrical boundary conditions is

$$\bar{n} \times [\![\bar{E}]\!] \Big|_{x=\xi} = 0 \Rightarrow [\![e_y]\!] + [\![E_o]\!] \frac{\partial \xi}{\partial y} = 0 \tag{5}$$

and to linear terms this is

$$-[\![\hat{\Phi}]\!] + [\![E_o]\!] \hat{\xi} = 0 \tag{6}$$

The second condition is

$$\bar{n} \cdot [\![\epsilon \bar{E}]\!] \Big|_{x=\xi} = 0 \Rightarrow [\![\epsilon \hat{e}_x]\!] + [\![\epsilon \frac{dE_o}{dx}]\!] \hat{\xi} = 0 \tag{7}$$

By Gauss Law, $\epsilon dE_o/dx = q$ and so this expression becomes

$$[\![\epsilon \hat{e}_x]\!] + [\![q]\!] \hat{\xi} = 0 \tag{8}$$

These three boundary conditions, Eqs. 4, 6 and 8, are three equations in the unknowns $\hat{\xi}, \hat{\pi}^d, \hat{\pi}^e, \hat{\Phi}^d, \hat{\Phi}^e, \hat{e}_x^d, \hat{e}_x^e$. Four more relations are provided by the electrical and mechanical bulk relations, Eqs. 12b, 13a, 14b and 15a, which are substituted into these boundary conditions to give

$$\begin{bmatrix} g(\rho_a - \rho_b) - \gamma k^2 + \frac{\omega^2}{k}(\rho_a \coth ka + \rho_b \coth kb) & q_a + \epsilon_a E_o k \coth ka & -q_b + \epsilon_b E_o k \coth kb \\ [\![E_o]\!] & -1 & 1 \\ [\![q]\!] & \epsilon_a k \coth ka & \epsilon_b k \coth kb \end{bmatrix} \begin{bmatrix} \hat{\xi} \\ \hat{\Phi}^d \\ \hat{\Phi}^e \end{bmatrix} = 0 \tag{9}$$

This determinant reduces to the desired dispersion equation.

Prob. 8.14.2 (cont.)

$$\frac{\omega^2}{k}(\rho_a \coth ka + \rho_b \coth kb) = g(\rho_b - \rho_a) + \gamma k^2 + E_a q_a - E_b q_b$$
$$+ \frac{(q_a - q_b)^2}{\epsilon_a \coth ka + \epsilon_b \coth kb} - \frac{2(\epsilon_b - \epsilon_a)(q_a E_a \coth kb + q_b E_b \coth ka)}{\epsilon_a \coth ka + \epsilon_b \coth kb} \qquad (10)$$
$$- \frac{k(\epsilon_a - \epsilon_b)^2 E_a E_b}{\epsilon_a \tanh kb + \epsilon_b \tanh ka}$$

In the absence of convection, the first and second terms on the right represent the respective effects of gravity and capillarity. The third term on the right is an imposed field effect of the space charge, due to the interaction of the space charge with fields that could largely be imposed by the electrodes. By contrast, the fourth term, which is also due to the space-charge interaction, is proportional to the square of the space-charge discontinuity at the interface, and can, therefore, be interpreted as a self-field term, where the interaction is between the space charge and the field produced by the space charge. This term is present, even if the electric field intensity at the interface were to vanish. The fifth and sixth terms are clearly due to polarization, since they would not be present if the permittivities were equal. In the absence of any space-charge densities, only the sixth term would remain, which always tends to destabilize the interface. However, by contrast with the example of Sec. 8.10, the fifth term is one due to both the polarizability and the space charge. That is, E_a and E_b include effects of the space-charge. (See "Space-Charge Dynamics of Liquids", Phys. Fluids, 15 (1972), p. 1197.)

Problem 8.15.1

Because the force density is a pure gradient, Equation 7.8.11 is applicable. With $B_o = \mu_o I/2\pi r = -\frac{\partial A}{\partial r}$, it follows that $A = -(\mu_o I/2\pi r)\ln(r/R)$ so that $\mathcal{E} = -J_o A$ and Equation 7.8.11 becomes

$$p = \Pi - \frac{J_o \mu_o I}{2\pi}\ln\left(\frac{r}{R}\right) + \rho\frac{\partial\Theta}{\partial t} \tag{1}$$

Note that there are no self-fields giving rise to a perturbation field, as in Section 8.14. There are also no surface currents, so the pressure jump at the interface is equilibrated by the surface tension surface force density.

$$\Pi_a - \Pi_b = -\frac{\gamma}{R} \tag{2}$$

while the perturbation requires that

$$\frac{J_o \mu_o I}{2\pi}\ln\left(\frac{R+\xi}{R}\right) - p'^b = \gamma\left[\frac{\xi}{R^2} + \frac{1}{R^2}\frac{\partial^2\xi}{\partial\theta^2}\right] \tag{3}$$

Linearization of the first term on the left $(\ln(1+x) \sim x)$, substitution to obtain complex amplitudes and use of the pressure-velocity relation for a column of fluid from Table 7.9.1 then gives an expression that is homogeneous $\hat{\xi}D(\omega,m) = 0$. Thus the dispersion equation, $D(\omega,m) = 0$, is

$$-\omega^2\rho F_m(0,R) = -\frac{\gamma}{R^2}(1-m^2) + \frac{\mu_o J_o I}{2\pi R} \tag{4}$$

(c) Recall from Section 2.17 that $F_m(0,R) < 0$ and that the m = 0 mode is excluded because there is no z dependence. Surface tension therefore only tends to stabilize. However, in the m = 1 mode (which is a pure translation of the column) it has no effect and stability is determined by the electromechanical term. It follows that the m = 1 mode is unstable if $J_o I < 0$. Higher order modes become unstable for $-J_o I = (m^2-1)2\pi\gamma/\mu_o R$. Conversely,

Problem 8.15.1 (cont.)

all modes are stable if $J_o I > 0$. With J_o and I of the same sign, the $\bar{J} \times \mu_o \bar{H}$ force density is radially inward. The uniform current density fills regions of fluid extending outward providing an incremental increase in the pressure (say at r = R) of the fluid at any fixed location. The magnetic field is equivalent in its effect to a radially directed gravity that is inward if $J_o I > 0$.

Problem 8.16.1 In static equilibrium

$$S_{xx} = -p = \begin{cases} -\Pi_o & ; x > 0 \\ -\Pi_o + \rho g x + \frac{1}{2}\epsilon_o E_o^2 ; & x < 0 \end{cases} \tag{1}$$

In the bulk regions, where there is no electromechanical coupling, the stress-velocity relations of Eq. 7.19.19 apply

$$\begin{bmatrix} \hat{S}_{xx}^e \\ \hat{S}_{yx}^e \end{bmatrix} = \eta \begin{bmatrix} \frac{\gamma}{k}(\gamma + k) & -j(\gamma - k) \\ j(\gamma - k) & \gamma + k \end{bmatrix} \begin{bmatrix} \hat{v}_x^e \\ \hat{v}_y^e \end{bmatrix} \tag{2}$$

and the flux-potential relations, Eq. (a) of Table 2.16.1, show that

$$\hat{E}_x^d = k \hat{\Phi}^d \tag{3}$$

The crux of the interaction is represented by the perturbation boundary conditions. Stress equilibrium in the x direction requires that

$$[\![S_{xj}]\!] n_j + [\![T_{xj}]\!] n_j - \gamma_\ell \nabla \cdot \bar{n}\, n_x = 0 \tag{4}$$

With the use of Eq. (d) of Table 7.6.2 and $\hat{\xi} = \hat{v}_x^b / j\omega$, the linearized version of this condition is

$$\frac{j\rho g}{k} \frac{\hat{v}_x^e}{(\omega/k)} + \epsilon_o E_o \hat{e}_x^d + j\gamma_\ell k \frac{\hat{v}_x^e}{(\omega/k)} - \hat{S}_{xx}^e = 0 \tag{5}$$

The stress equilibrium in the y direction requires that

$$[\![S_{yj}]\!] n_j + [\![T_{yj}]\!] n_j - \gamma_\ell (\nabla \cdot \bar{n}) n_y = 0 \tag{6}$$

and the linearized form of this condition is

$$\epsilon_o E_o \hat{e}_y^d - \frac{\epsilon_o E_o^2 k}{\omega} \hat{v}_x^e - \hat{S}_{yx}^e = 0 \tag{7}$$

Prob. 8.16.1 (cont.)

The tangential electric field must vanish on the interface, so

$$\hat{e}_y^d = \frac{E_o k}{\omega}\hat{v}_x^e \tag{8}$$

and from this expression and Eq. 7, it follows that the latter condition can be replaced with

$$\hat{S}_{yx}^e = 0 \tag{9}$$

Equations 2 and 3 combine with Eqs. 5 and 9 to give the homogeneous equations

$$\begin{bmatrix} \frac{j\rho g}{\omega} - j\frac{\epsilon_o E_o^2 k}{\omega} + j\frac{\gamma_a k^2}{\omega} - \eta\frac{\gamma}{k}(\gamma+k) & j\eta(\gamma-k) \\ j(\gamma-k) & (\gamma+k) \end{bmatrix}\begin{bmatrix}\hat{v}_x^e \\ \hat{v}_y^e\end{bmatrix} = 0 \tag{10}$$

Multiplied out, the determinant becomes the desired dispersion equation.

$$j\omega\eta\frac{[k(\gamma-k)^2 - \gamma(\gamma+k)^2]}{k(\gamma+k)} = -(\epsilon_o k E_o^2 - \gamma_a k^2 - \rho g) \tag{11}$$

With the use of the definition $\gamma^2 \equiv k^2 + j\omega\rho/\eta$, this expression becomes

$$-\frac{j\omega\eta}{k}\left(\frac{4k^2\gamma}{\gamma+k} + \frac{j\omega\rho}{\eta}\right) = \rho g + k^2\gamma_a - \epsilon_o k E_o^2 \tag{12}$$

Now, in the limit of low viscosity, $k/\gamma \to 0$ and Eq. 12 become

$$\omega^2\frac{\rho}{k} - j4k\eta\omega - (\rho g + k^2\gamma_a - \epsilon_o k E_o^2) = 0 \tag{13}$$

which can be solved for ω.

$$\omega = j\frac{2k^2\eta}{\rho} \pm\sqrt{-\left(\frac{2k^2\eta}{\rho}\right)^2 + \frac{k}{\rho}\left(\rho g + k^2\gamma_a - \epsilon_o k E_o^2\right)} \tag{14}$$

Note that in this limit, the rate of growth depends on viscosity, but the field for incipience of instability does not.

In the high viscosity limit, $\gamma \approx k + j\omega\rho/2\eta k$ and Eq. 12 become

$$\frac{-j\omega\rho}{k}\left[\frac{4k^2\left(k + \frac{j\omega\rho}{2\eta k}\right)}{2k + \frac{j\omega\rho}{2\eta k}} + \frac{j\omega\rho}{\eta}\right] = \rho g + k^2\gamma_a - \epsilon_o k E_o^2 \tag{15}$$

Prob. 8.16.1 (cont.)

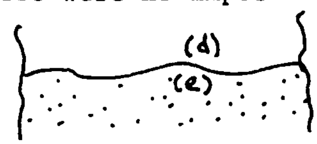
Further expansion of the denominator reduces this expression to

$$\frac{3}{2}\frac{\omega^2\rho}{k} = 2j\omega\eta k + \rho g + k^2\gamma_a - \epsilon_o k E_o^2 \tag{16}$$

Again, viscosity effects the rate of growth, but not the conditions for incipience of instability.

Problem 8.16.2 In static equilibrium, there is no surface current, and so the distribution of pressure is the same as if there were no imposed $\bar{H}$.

$$S_{xx} = -p = \begin{cases} -\Pi_o & ; x>0 \\ -\Pi_o + \rho g x & ; x<0 \end{cases} \tag{1}$$



The perfectly conducting interface is to be modeled by its boundary conditions. The magnetic flux density normal to the interface is taken as continuous.

$$\bar{n}\cdot[\![\bar{B}]\!] = 0 \tag{2}$$

With this understood, consider the consequences of flux conservation for a surface of fixed identity in the interface (Eqs. 2.6.4 and 6.2.4).

$$\frac{d}{dt}\int_S \bar{B}\cdot\bar{n}\,da = \int_S \left[\frac{\partial \bar{B}}{\partial t} + \nabla\times(\bar{B}\times\bar{v})\right]\cdot\bar{n}\,da = 0 \tag{3}$$

Linearized, and in view of Eq. 2, this condition becomes

$$\frac{\partial H_x}{\partial t} = -H_o\frac{\partial v_y}{\partial y} \Rightarrow \hat{H}_x^d = \hat{H}_x^e = H_o k\,\hat{v}_y^e/\omega \tag{4}$$

Bulk conditions in the regions to either side of the interface represent the fluid and fields without a coupling. The stress-velocity conditions for the lower half-space are Eqs. 2.19.19.

$$\begin{bmatrix} \hat{S}_{xx}^e \\ \hat{S}_{yx}^e \end{bmatrix} = \eta \begin{bmatrix} \frac{\gamma}{k}(\gamma+k) & -j(\gamma-k) \\ j(\gamma-k) & \gamma+k \end{bmatrix} \begin{bmatrix} \hat{v}_x^e \\ \hat{v}_y^e \end{bmatrix}$$

While the flux-potential relations for the magnetic fields, Eqs. (a) of Table 2.16.1, reduce to

$$\hat{B}_x^d = \mu_o k\,\hat{\psi}^d = -j\mu_o \hat{H}_y^d \;;\; \hat{B}_x^e = -\mu_o k\,\hat{\psi}^e = j\mu_o \hat{H}_y^e \tag{6}$$

Prob. 8.16.2 (cont.)

Boundary conditions at the interface for the fields are the linearized versions of Eqs. 2 and 4. For the fluid, stress balance in the x direction requires

$$\frac{j\rho g}{\omega}\hat{v}_x^e + j\frac{\gamma_a k^2}{\omega}\hat{v}_x^e - \hat{S}_{xx}^e = 0 \tag{7}$$

where $\hat{v}_x^e = j\omega\hat{\xi}$. Stress balance in the y direction requires

$$-\hat{S}_{yx}^e + 2\mu_o H_o \hat{H}_y^d = 0 \tag{8}$$

$$\begin{bmatrix} j\frac{\rho g}{\omega} + j\frac{\gamma_a k^2}{\omega} - \eta\left(\frac{\gamma}{k}\right)(\gamma+k) & j\eta(\gamma-k) \\ j\eta(\gamma-k) & \eta(\gamma+k) - j2\mu_o H_o^2\frac{k}{\omega} \end{bmatrix}\begin{bmatrix}\hat{v}_x^e \\ \hat{v}_y^e\end{bmatrix} = 0 \tag{9}$$

It follows that the required dispersion equation is

$$\left[\rho g + \gamma_a k^2 + j\eta\left(\frac{\gamma}{k}\right)(\gamma+k)\omega\right]\left[\eta(\gamma+k) - j2\mu_o H_o^2\frac{k}{\omega}\right] - j\eta^2(\gamma-k)^2\omega = 0 \tag{10}$$

In the low viscosity limit, $\gamma \sim \sqrt{j\omega\rho/\eta} + \frac{1}{2}\sqrt{\eta/j\omega\rho}\,k^2$ and therefore the last term goes to zero as $\eta \longrightarrow 0$ so that the equation factors into the dispersion equations for two modes. The first, the transverse mode, is represented by the first term in brackets in Eq. 10, which can be solved to give the dispersion equation for a gravity-capillary mode with no coupling to the magnetic field.

$$\omega^2 = gk + \frac{\gamma_a k^2}{\rho} \tag{11}$$

The second term in brackets becomes the dispersion equation for the mode involving dilatations of the interface.

$$\omega = \omega_c\left[\frac{\sqrt{3}}{2} + \frac{j}{2}\right]; \quad \omega_c \equiv \left[\frac{2\mu_o H_o^2 k}{\sqrt{\eta\rho}}\right]^{2/3} \tag{12}$$

If $\omega > \omega_c$, then in the second term in brackets of Eq. 10, $\eta(\gamma+k) > 2\mu_o H_o^2 k/\omega$ and the dispersion equation is as though there were no electromechanical coupling. Thus, for $\omega >> \omega_c$ the damping effect of viscosity is much as in Problem 8.16.1. In the opposite extreme, if $\omega << \omega_c$, then the second term

Prob. 8.16.2 (cont.)

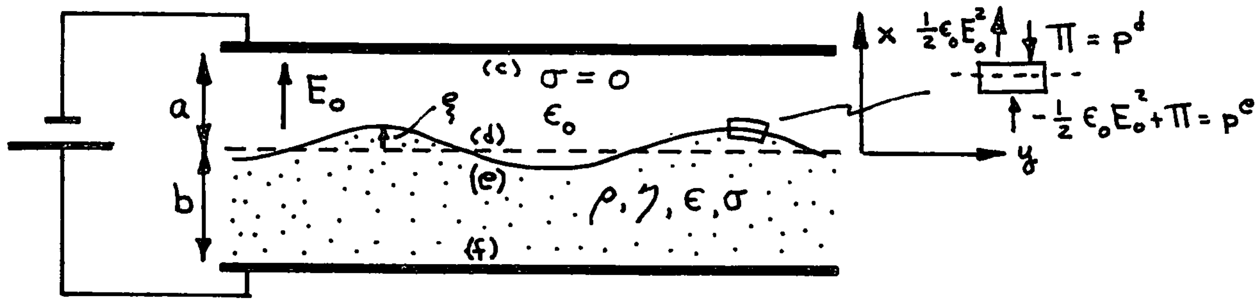
has $\eta(\gamma+k)<<2\mu_o H_o^2 k/\omega$ and is approximated by the magnetic field term. In this case, Eq. 10 is approximated by

$$\rho g + \gamma_a k^2 - \frac{\omega^2 \rho}{\eta} + j\eta\omega + j\frac{\omega^3 \rho \eta}{2\mu_0 H_o^2 k} = 0 \tag{13}$$

In the limit of very high H_o, the last term is negligible and the remainder of the equation can be used to approximate the damping effect of viscosity.

Certainly the model is not meaningful unless the magnetic diffusion time based on the sheet thickness and the wavelength is small compared to times of interest. Suggested by Eq. 6.10.2 in the limit $d \to \infty$ is a typical magnetic diffusion time $\mu\sigma a/k$, where a is the thickness of the "perfectly" conducting layer.

Prob. 8.16.3 A cross-section of the configuration is shown in the figure.



In static equilibrium, the electric field intensity is

$$\bar{E} = \begin{cases} E_o \bar{i}_x & x > 0 \\ 0 & x < 0 \end{cases} \tag{1}$$

and in accordance with the stress balance shown in the figure, the mechanical stress, S_{xx}, reduces to simply the negative of the hydrodynamic pressure.

$$S_{xx} = -p = \begin{cases} -\Pi \\ \rho g x + \frac{1}{2}\epsilon_o E_o^2 - \Pi \end{cases} \tag{2}$$

Electrical bulk conditions reflecting the fact that $\bar{E} = -\nabla\Phi$ where Φ satisfies Laplace's equation both in the air-gap and in the liquid layer are Eqs. (b) from Table 2.16.1. Incorporated at the outset are the boundary conditions $\hat{\Phi}^c = 0$ and $\hat{\Phi}^f = 0$, reflecting the fact that the upper and lower electrodes are highly conducting.

$$\hat{e}_x^d = k \coth ka \, \hat{\Phi}^d \tag{3}$$

$$\hat{e}_x^e = -k \coth kb \, \hat{\Phi}^e \tag{4}$$

The mechanical bulk conditions, reflecting mass conservation and force equilibrium for the liquid, which has uniform mass density and viscosity, are Eqs. 7.20.6. At the outset, the boundary conditions at the lower electrode requiring that both the tangential and normal liquid velocities be zero are incorporated in writing these expressions $(\hat{v}_x^f = 0,\ \hat{v}_y^f = 0)$.

$$\hat{S}_{xx}^e = \eta P_{11} \hat{v}_x^e + \eta P_{13} \hat{v}_y^e \tag{5}$$

$$\hat{S}_{yx}^e = \eta P_{31} \hat{v}_x^e + \eta P_{33} \hat{v}_y^e \tag{6}$$

Prob. 8.16.3 (cont.)

Boundary conditions at the upper and lower electrodes have already been included in writing the bulk relations. The conditions at the interface remain to be written, and of course represent the electromechanical coupling. Charge conservation for the interface, Eq. 23 of Table 2.10.1 and Gauss'law, require that

$$\frac{\partial \sigma_f}{\partial t} = -\nabla_\Sigma \cdot (\sigma_f \bar{v}) - \bar{n} \cdot [\![\sigma \bar{E}]\!] \tag{7}$$

where by Gauss'law $\sigma_f = \bar{n} \cdot [\![\epsilon \bar{E}]\!]$.

Linearized and written in terms of the complex amplitudes, this requires that

$$j\omega(\epsilon_o \hat{e}_x^d - \epsilon \hat{e}_x^e) = jk\epsilon_o E_o \hat{v}_y^e + \sigma \hat{e}_x^e \tag{8}$$

The tangential electric field at the interface must be continuous. In linearized form this requires that

$$[\![e_y]\!] + \frac{\partial \xi}{\partial y} E_o = 0 \tag{9}$$

Because $\hat{\xi} = \hat{v}_x / j\omega$ and $\hat{e}_y = jk\hat{\Phi}$, this condition becomes

$$\hat{\Phi}^d - \hat{\Phi}^e - \frac{\hat{v}_x^e}{j\omega} E_o = 0 \tag{10}$$

In general, the balance of pressure and viscous stresses (represented by S_{ij}), of the Maxwell stress and of the surface tension surface force density, require that

$$[\![S_{ij}]\!] n_j + [\![T_{ij}]\!] n_j + n_i \gamma \frac{\partial^2 \xi}{\partial y^2} = 0 \tag{11}$$

With i=x (the x component of the stress balance) this expression requires that to linear terms

$$[\![S_{xx}]\!] + [\![S_{xy}]\!]\left(-\frac{\partial \xi}{\partial y}\right) + [\![T_{xx}]\!] + [\![T_{xy}]\!]\left(-\frac{\partial \xi}{\partial y}\right) + \gamma \frac{\partial^2 \xi}{\partial y^2} = 0 \tag{12}$$

By virtue of the forsight in writing the equilibrium pressure, Eq. 2, the equilibrium parts of Eq. 12 balance out. The perturbation part requires that

$$-\frac{\rho g}{j\omega}\hat{v}_x^e - \hat{S}_{xx}^e + \epsilon_o E_o \hat{e}_x^d - \frac{\gamma k^2}{j\omega}\hat{v}_x^e = 0 \tag{13}$$

Prob. 8.16.3 (cont.)

With i=y, (the shear component of the stress balance) Eq. 11 requires that

$$[\![S_{yx}]\!] + [\![S_{yy}]\!]_o\left(-\frac{\partial\xi}{\partial y}\right) + [\![T_{yx}]\!] + [\![T_{yy}]\!]_o\left(-\frac{\partial\xi}{\partial y}\right) = 0 \tag{14}$$

Observe that the equilibrium quantities $[\![S_{yy}]\!]_o = -\frac{1}{2}\epsilon_o E_o^2$ and $[\![T_{yy}]\!] = -\frac{1}{2}\epsilon_o E_o^2$ so that this expression reduces to

$$-\hat{S}^e_{yx} - \frac{\epsilon_o E_o^2 k}{\omega}\hat{v}^e_x + jk\epsilon_o E_o \hat{\Phi}^d = 0 \tag{15}$$

The combination of the bulk and boundary conditions, Eqs. 3-6,8,10,13 and 15, comprise eight equations in the unknowns $(\hat{e}^d_x, \hat{e}^e_x, \hat{\Phi}^d, \hat{\Phi}^e, \hat{S}^e_{xx}, \hat{S}^e_{yx}, \hat{v}^e_x, \hat{v}^e_y)$. The dispersion equation will now be determined in two steps. First, consider the "electrical" relations. With the use of Eqs. 3 and 4, Eqs. 8 and 10 become

$$\begin{bmatrix} j\omega k\epsilon_o \coth ka & j\omega\epsilon k\coth kb + \sigma k\coth kb \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \hat{\Phi}^d \\ \hat{\Phi}^e \end{bmatrix} = \begin{bmatrix} jk\epsilon_o E_o \hat{v}^e_y \\ \frac{E_o}{j\omega}\hat{v}^e_x \end{bmatrix} \tag{16}$$

From these two expressions, it follows that

$$\hat{\Phi}^d = \frac{jk\epsilon_o E_o \hat{v}^e_y + \frac{E_o}{j\omega}\hat{v}^e_x(j\omega\epsilon\coth kb + \sigma\coth kb)}{j\omega(\epsilon_o\coth ka + \epsilon\coth kb) + \sigma\coth kb} \tag{17}$$

In terms of $\hat{\Phi}^d$, $\hat{e}^d_x$ is easily written using Eq. 3.

The remaining two boundary conditions, the stress balance conditions of Eqs. 13 and 15 can now be written in terms of $(\hat{v}^e_y, \hat{v}^e_x)$ alone.

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\begin{bmatrix} \frac{\hat{v}^e_x}{j\omega} \\ \hat{v}^e_y \end{bmatrix} = 0 \tag{18}$$

where

$$M_{11} = -j\omega\eta P_{11} - \rho g - \gamma k^2 + \frac{\epsilon_o E_o^2 k \coth ka \coth kb\,(j\omega\epsilon + \sigma)}{j\omega(\epsilon_o \coth ka + \epsilon \coth kb) + \sigma \coth kb}$$

$$M_{12} = -\eta P_{13} + \frac{j\,\epsilon_o^2 E_o^2 k \coth ka}{j\omega(\epsilon_o \coth ka + \epsilon \coth kb) + \sigma \coth kb}$$

$$M_{21} = -j\omega\eta P_{31} - j\epsilon_o E_o^2 k + \frac{j\,k\epsilon_o E_o^2 (j\omega\epsilon + \sigma)\coth kb}{j\omega(\epsilon_o \coth ka + \epsilon \coth kb) + \sigma \coth kb}$$

$$M_{22} = -\eta P_{33} - \frac{k\,\epsilon_o^2 E_o^2}{j\omega(\epsilon_o \coth ka + \epsilon \coth kb) + \sigma \coth kb}$$

The dispersion equation follows from Eq. 18 as

$$\underline{M}_{11}\underline{M}_{22} - \underline{M}_{12}\underline{M}_{21} = 0 \tag{19}$$

Here, it is convenient to normalize variables such that

$$\begin{aligned} &\underline{\omega} = \frac{\omega b \eta}{\gamma} \;; \quad \underline{a} = \frac{a}{b} \;; \quad \underline{P}_{ij} = b P_{ij} \\ &\underline{\rho} = \frac{\rho g b^2}{\gamma} \;; \quad \underline{U} = \frac{b \epsilon_o E_o^2}{\gamma} \;; \quad \frac{\omega\epsilon}{\sigma} = \underline{\omega}\tau \\ &\underline{k} = kb \;; \quad \tau = \frac{\gamma}{b\eta}\frac{\epsilon}{\sigma} \end{aligned} \tag{20}$$

and to define

$$C = \frac{\epsilon_o}{\epsilon}\coth \underline{k}\underline{a} + \coth \underline{k} \;; \quad R = \coth \underline{k} \;; \quad S = \coth \underline{k}\underline{a} \tag{21}$$

so that in Eq. 19,

$$\begin{aligned} \underline{M}_{11} &= \frac{b^2}{\gamma} M_{11} = -\underline{P}_{11}\, j\underline{\omega} - \underline{\rho} - \underline{k}^2 + \frac{\underline{k}\underline{U} R S (j\underline{\omega}\tau + 1)}{j\underline{\omega}\tau C + R} \\ \underline{M}_{12} &= \frac{b}{\eta} M_{12} = -\underline{P}_{13} + j\frac{\frac{\epsilon_o}{\epsilon}\tau \underline{U}\,\underline{k}\, S}{j\underline{\omega}\tau C + R} \\ \underline{M}_{21} &= \frac{b^2}{\gamma} M_{21} = -\underline{P}_{31}\, j\underline{\omega} - j\underline{U}\underline{k} + j\frac{\underline{k}\underline{U}(j\underline{\omega}\tau + 1) R}{j\underline{\omega}\tau C + R} \\ \underline{M}_{22} &= \frac{b}{\eta} M_{22} = -\underline{P}_{33} - \frac{\underline{k}\underline{U}\frac{\epsilon_o}{\epsilon}\tau}{j\underline{\omega}\tau C + R} \end{aligned} \tag{22}$$

Prob. 8.16.3 (cont.)

If viscous stresses dominate those due to inertia, the P_{ij} in these expressions are independent of frequency. In the following, this approximation of low-Reynolds number flow is understood. (Note that the dispersion equation can be used if inertial effects are included simply by using Eq. 7.19.13 to define the P_{ij}. However, there is then a complex dependence of these terms on the frequency, reflecting the fact that viscous diffusion occurs on time scales of interest.)

With the use of Eqs. 22, Eq. 19 becomes

$$\left\{(j\underline{\omega}rC+R)(-j\underline{\omega}\underline{P}_{11}-\rho-\underline{k}^2)+\underline{k}\underline{U}RS(j\underline{\omega}r+1)\right\}\left\{-\underline{P}_{33}(j\underline{\omega}rC+R)-\underline{k}\underline{U}\frac{\epsilon_o}{\epsilon}r\right\}$$
$$+\left\{\underline{P}_{13}(j\underline{\omega}rC+R)-j\frac{\epsilon_o}{\epsilon}r\underline{U}\underline{k}S\right\}\left\{-(\underline{P}_{31}j\underline{\omega}+j\underline{U}\underline{k})(j\underline{\omega}rC+R)+j\underline{k}\underline{U}(j\underline{\omega}r+1)R\right\}=0 \qquad (23)$$

That this dispersion equation is in general cubic in $j\omega$ reflects the coupling it represents of the gravity-capillary-electrostatic waves, shear waves and the charge relaxation phenomena (the third root).

Consider the limit where charge relaxation is complete on time scales of interest. Then the interface behaves as an equipotential, $r \rightarrow 0$, and Eq. 23 reduces to

$$j\underline{\omega}=\frac{(\underline{k}\underline{U}S-\rho-\underline{k}^2)\underline{P}_{33}}{(\underline{P}_{11}\underline{P}_{33}-\underline{P}_{13}\underline{P}_{31})} \qquad (24)$$

That there is only one mode is to be expected. Charge relaxation has been eliminated (is instantaneous) and because there is no tangential electric field on the interface, the shear mode has as well. Because damping dominates inertia, the gravity-capillary-electrostatic wave is over damped, or grows as a pure exponential. The factor

$$\frac{\underline{P}_{33}}{\underline{P}_{11}\underline{P}_{33}-\underline{P}_{13}\underline{P}_{31}}\equiv f(\underline{k})=\frac{\left(\frac{1}{4}\sinh 2\underline{k}-\frac{\underline{k}}{2}\right)\left(\sinh^2\underline{k}-\underline{k}^2\right)}{\underline{k}\left(\frac{1}{4}\sinh^2 2\underline{k}-\underline{k}^2-\underline{k}^4\right)} \qquad (25)$$

is positive, so the interface is unstable if

$$\underline{U}>(\rho+\underline{k}^2)/S\underline{k} \qquad (26)$$

Prob. 8.16.3 (cont.)

In the opposite extreme, where the liquid is sufficiently insulating that charge relaxation is negligible so that $r \gg 1$, Eq. 23 reduces to a quadratic expression ($P_{13} = -P_{31}$).

$$a(j\omega)^2 + b(j\omega) + c = 0$$

$$a \equiv \underline{P}_{11}\underline{P}_{33} + \underline{P}_{13}^2;\ b = \left[(\underline{\rho} + \underline{k}^2)P_{33} + \underline{U}\underline{k}\left(\frac{\underline{P}_{11}}{C}\frac{\epsilon_o}{\epsilon} - \frac{RS}{C}P_{33}\right) - 2j\frac{\underline{k}\underline{U}\underline{P}_{13}S}{C}\frac{\epsilon_o}{\epsilon}\right];\ c = \frac{\underline{k}\underline{U}}{C}\frac{\epsilon_o}{\epsilon}\left[\underline{\rho} + \underline{k}^2 - \underline{U}\underline{k}S\right] \tag{27}$$

The roots of this expression represent the gravity-capillary-electrostatic and shear modes. In this limit of a relatively insulating layer, there are electrical shear stresses on the interface. In fact these dominate in the transport of the surface charge.

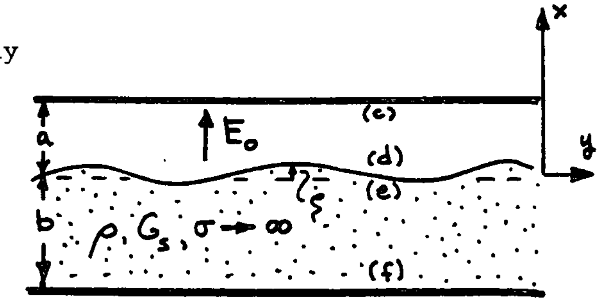
To find the general solution of Eq. 23, it is necessary to write it as a cubic in $j\omega$.

$$(j\omega)^3 + P'(j\omega)^2 + Q'(j\omega) + R' = 0 \tag{28}$$

$$P' = \Big\{2\underline{P}_{11}\underline{P}_{33}rCR + \underline{P}_{33}r^2C\left[C(\underline{\rho} + \underline{k}^2) - \underline{k}\underline{U}RS\right] + r^2C\underline{P}_{11}\underline{k}\underline{U}\frac{\epsilon_o}{\epsilon}$$
$$+ \underline{P}_{13}rC\left[2\underline{P}_{13}R - j\underline{k}\underline{U}r\frac{2\epsilon_o}{\epsilon}S\right]\Big\}\Big/ r^2C^2(\underline{P}_{11}\underline{P}_{33} + \underline{P}_{13}^2)$$

$$Q' = \Big\{\underline{P}_{33}rC\left[(\underline{\rho} + \underline{k}^2)R - \underline{k}URS\right] + \left[R\underline{P}_{11} + rC(\underline{\rho} + \underline{k}^2) - r\underline{k}\underline{U}RS\right]\left[\underline{k}\underline{U}\frac{\epsilon_o}{\epsilon}r + \underline{P}_{33}R\right]$$
$$+ \left[\underline{P}_{13}R - j\frac{\epsilon_o}{\epsilon}r\underline{U}\underline{k}S\right]\left[\underline{P}_{13}R - j\underline{k}r\underline{U}\frac{\epsilon_o}{\epsilon}S\right]\Big\}\Big/ r^2C^2(\underline{P}_{11}\underline{P}_{33} + \underline{P}_{13}^2)$$

$$R' = \Big\{\left[\underline{k}\underline{U}\frac{\epsilon_o}{\epsilon}r + \underline{P}_{33}R\right]\left[(\underline{\rho} + \underline{k}^2)R - \underline{k}\underline{U}RS\right]\Big\}\Big/ r^2C^2(\underline{P}_{11}\underline{P}_{33} + \underline{P}_{13}^2)$$

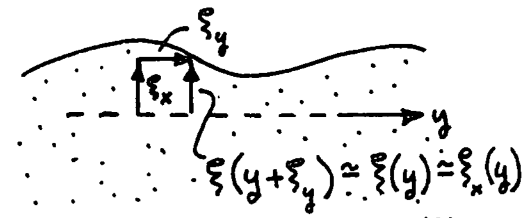


Prob 8.16.4 Because the solid is relatively conducting compared to the gas above, the equilibrium electric field is simply

$$\bar{E} = \begin{cases} E_o \bar{i}_x & x > 0 \\ 0 & x < 0 \end{cases} \tag{1}$$

In the solid, the equations of motion are

$$\rho \frac{\partial^2 \bar{\xi}}{\partial t^2} = -\nabla p + G_s \nabla^2 \bar{\xi} - \rho g \bar{i}_x ; \quad \nabla \cdot \bar{\xi} = 0 \tag{2}$$



where

$$S_{ij} = -p + G_s \left(\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i} \right) \tag{3}$$

It follows from Eq. 2b that

$$\frac{\partial \xi_x}{\partial x} = 0 \Rightarrow \xi_x = \text{const} = 0 \tag{4}$$

so that the static x component of the force equation reduces to

$$\frac{\partial p}{\partial x} = G_s \frac{\partial^2 \xi_x}{\partial x^2} - \rho g \Rightarrow \quad p = \begin{cases} \Pi_a & ; x > 0 \\ \Pi_b - \rho g x & ; x < 0 \end{cases} \tag{5}$$

This expression, together with the condition that the interface be in stress equilibrium, determines the equilibrium stress distribution

$$S_{xx} = -p = \begin{cases} -\Pi_a & ; x > 0 \\ \rho g x - \Pi_a + \frac{1}{2}\epsilon_o E_o^2 & ; x < 0 \end{cases} \tag{6}$$

In the gas above, the perturbation fields are represented by Laplace's equation, and hence the transfer relations (a) of Table 2.16.1

$$\begin{bmatrix} \hat{e}_x^c \\ \hat{e}_x^d \end{bmatrix} = k \begin{bmatrix} -\coth ka & \frac{1}{\sinh ka} \\ -\frac{1}{\sinh ka} & \coth ka \end{bmatrix} \begin{bmatrix} \hat{\Phi}^c \\ \hat{\Phi}^d \end{bmatrix} \tag{7}$$

Perturbation deformations in the solid are described by the analogue transfer relations

$$\begin{bmatrix} \hat{S}_{xx}^e \\ \hat{S}_{xx}^f \\ \hat{S}_{yx}^e \\ \hat{S}_{yx}^f \end{bmatrix} = G_s [P_{ij}] \begin{bmatrix} \hat{\xi}_x^e \\ \hat{\xi}_x^f \\ \hat{\xi}_y^e \\ \hat{\xi}_y^f \end{bmatrix} \qquad \text{where } \gamma \equiv \sqrt{k^2 - \frac{\omega^2 \rho}{G_s}} \tag{8}$$

The interface is described in Eulerian coordinates by $\xi(y,t)$ with this variable related to the deformation of the interface as suggested by the figure.

Prob. 8.16.4 (cont.) Boundary conditions on the fields in the gas recognize that the electrode and the interface are each equipotentials.

$$\hat{\Phi}^c = 0 \tag{9}$$

$$[\bar{n}\times\bar{E}]_{x=\xi} = 0 \Rightarrow \hat{\Phi}^d = E_o\hat{\xi}_x^e \tag{10}$$

Stress equilibrium for the interface is in general represented by

$$[\![S_{ij}]\!] n_j + [\![T_{ij}]\!] n_j = 0 \tag{11}$$

where i is either x or y. To linear terms, the x component requires that

$$-\hat{S}_{xx}^e + \epsilon_o E_o \hat{e}_x^d - \rho g \hat{\xi}_x^e = 0 \tag{12}$$

where the equilibrium part balances out by virtue of the static equilibrium, Eq. 5. The shear component of Eq. 11, i=y, becomes

$$(S_{yx}^d - S_{yx}^e) + (S_{yy}^d - S_{yy}^e)\left(-\frac{\partial\xi}{\partial y}\right) + (T_{yx}^d - T_{yx}^e) + (T_{yy}^d - T_{yy}^e)\left(\frac{-\partial\xi}{\partial y}\right) = 0 \tag{13}$$

Because there is no electrical shear stress on the interface, a fact represented by Eq. 10, this expression reduces to

$$\hat{S}_{yx}^e = 0 \tag{14}$$

In addition, the rigid bottom requires that

$$\hat{\xi}_x^f = 0 \; ; \; \hat{\xi}_y^f = 0 \tag{15}$$

The dispersion equation is now found by writing Eqs. 12 and 14 in terms of $(\hat{\xi}_x^e, \hat{\xi}_y^e)$. To this end, Eq. 8a is substituted for $\hat{S}_{xx}^e$ using Eqs. 15 and $\hat{e}_x^d$ is substituted using Eq. 7b evaluated using Eqs. 9 and 10. This is the first of the two expressions

$$\begin{bmatrix} -G_s P_{11} + \epsilon k \coth ka\, E_o^2 - \rho g & -G_s P_{13} \\ -G_s P_{31} & -G_s P_{33} \end{bmatrix} \begin{bmatrix} \hat{\xi}_x^e \\ \hat{\xi}_y^e \end{bmatrix} = 0 \tag{16}$$

Prob. 8.16.4 (cont.)

The second expression is Eq. 14 evaluated using Eq. 8c for $\hat{S}^e_{yx}$ with Eqs. 15.

It follows from Eq. 16 that the desired dispersion equation is

$$P_{11}P_{33} - P_{33}\frac{\epsilon_o E_o^2 k \coth ka - \rho g}{G_s} - P_{13}P_{31} = 0 \tag{17}$$

where in general, P_{ij} are defined with Eq. 7.19.13 (γ defined with Eq. 8). In the limit where $k^2 >> \omega^2\rho/G_s$, the P_{ij} become those defined with Eq. 7.20.6.

With the assumption that perturbations having a given wavenumber, k, become unstable by passing into the right half $j\omega$ plane through the origin, it is possible to interprete the roots of Eq. 17 in the limit $\omega \to 0$ as giving the value of $\epsilon_o E_o^2/G_s$ required for instability.

$$\frac{P_{11}P_{33} - P_{13}P_{31}}{P_{33}} = \frac{\epsilon_o E_o^2 k}{G_s}\coth ka - \frac{\rho g}{G_s} \tag{18}$$

In particular, this expression becomes

$$\frac{\epsilon_o E_o^2 k \coth ka}{G_s} - \frac{\rho g}{G_s} = \frac{\left\{\left[\frac{1}{4}\sinh(2kb) + \frac{kb}{2}\right]\left[\frac{1}{4}\sinh 2kb - \frac{kb}{2}\right] - \frac{1}{4}(kb)^4\right\}}{\left[\frac{1}{4}\sinh(2kb) - \frac{kb}{2}\right]\left[\sinh^2 kb - (kb)^2\right]} \tag{19}$$

so that the function on the right depends on kb and a/b. In general, a graphical solution would give the most critical value of kb. Here, the short-wave limit of Eq. 19 is taken, where it becomes

$$\epsilon_o E_o^2 = G_s/4 \tag{20}$$

Problem 8.18.1 For the linear distribution of charge density, the equation is $q = q_e + Dq_e x$. Thus, the upper uniform charge density must have value of $(3d/4)q$ while the lower one must have magnitude of $(d/4)q$. Evaluation gives

$$q_a = q_e + \frac{3}{4} D q_e d \quad ; \quad q_b = q_e + \frac{1}{4} D q_e d \tag{1}$$

The associated equilibrium electric field follows from Gauss' Law and the condition that the potential at x=0 is V_o.

$$E_x = \begin{cases} E_o + \frac{q_a}{\epsilon_o}\left(x - \frac{d}{2}\right) ; & x > \frac{d}{2} \\ E_o + \frac{q_b}{\epsilon_o}\left(x - \frac{d}{2}\right) ; & x < \frac{d}{2} \end{cases} \tag{2}$$

and the condition that the potential be V_o at x=0 and be 0 at x=d.

$$V_o = \int_0^d E_x \, dx = E_o d + (q_a - q_b)\frac{d^2}{8\epsilon_o} \tag{3}$$

With the use of Eqs. 1, this expression becomes

$$E_o = \frac{V_o}{d} - \frac{d^2}{16\epsilon_o} D q_e \tag{4}$$

Similar to Eqs. 1 are those for the mass densities in the layer model.

$$\rho_a = \rho_m + \frac{3}{4} D\rho_m d \quad ; \quad \rho_b = \rho_m + \frac{1}{4} D\rho_m d \tag{5}$$

For the two layer model, the dispersion equation is Eq. 8.14.25, which evaluated using Eqs. 1, 4 and 5, becomes

$$\frac{\omega^2}{k}\rho_m\left(2 + \frac{D\rho_m}{\rho_m} d\right)\coth\left(\frac{kd}{2}\right) = \frac{1}{2}\left[\frac{V_o}{d} D q_e - g D\rho_m\right] d + \frac{(Dq_e)^2 d^3}{\epsilon_o}\left[\frac{1}{8kd \coth\left(\frac{kd}{2}\right)} - \frac{1}{32}\right] \tag{6}$$

In terms of the normalization given with Eq. 8.18.2, this expression becomes

$$\frac{\omega^2}{k}\left(2 + \frac{D\rho_m}{\rho_m}\right)\coth\frac{k}{2} = \frac{1}{2}\left[\frac{\frac{V_o}{d} Dq_e - g D\rho_m}{\frac{|V_o|}{d}|Dq_e|}\right] + S\left[\frac{1}{8k \coth\left(\frac{k}{2}\right)} - \frac{1}{32}\right]\frac{Dq_e}{|Dq_e|} \tag{7}$$

With the numbers $Dq_e/|Dq_e| = 1$, $V_o/|V_o| = 1$, $k = 1$, $D\rho_m = 0$ and $S = 1$, Eq. 7 gives ω =0.349. The weak gradient approximation represented by Eq.

Prob. 8.18.1(cont.)

8.18.10 gives for comparison $\omega = 0.303$ while the numerical result representing the "exact" model, Fig. 8.18.2, gives a frequency that is somewhat higher than the weak gradient result but still lower than the layer model result, about 0.31. The layer model is clearly useful for estimating the frequency or growth rate of the dominant mode.

In the long-wave limit, $k \ll 1$, the weak-gradient imposed field result, Eq. 8.18.10, becomes

$$\omega^2 \rightarrow \frac{k^2 \mathcal{N}}{\pi^2} \tag{8}$$

In the same approximation it is appropriate to set S=0 in Eq. 7, which becomes

$$\omega^2 \rightarrow \frac{k^2 \mathcal{N}}{8} \tag{9}$$

where $D\rho_m \rightarrow 0$. Thus the layer model gives a frequency that is $\pi/\sqrt{8} = 1.11$ times that of the imposed-field weak gradient model.

In the short-wave limit, $k \gg 1$, the layer model predicts that the frequency increases with $\sqrt{k}$. This is in contrast to the dependence typified by Fig. 8.18.4 at short wavelengths with a smoothly inhomogeneous layer. This inadequacy of the layer model is to be expected, because it presumes that the structure of the discontinuity between layers is always sharp no matter how fine the scale of the surface perturbation. In fact, at short enough wavelengths, systems of miscible fluids will have an interface that is smoothly inhomogeneous because of molecular diffusion.

To describe higher order modes in the smoothly inhomogeneous system for wavenumbers that are not extremely short, more layers should be used. Presumably, for each interface, there is an additional pair of modes introduced. Of course, the modes are not identified with a single interface but rather involve the self-consistent deformation of all interfaces. The situation is formally similar to that introduced in Sec. 5.15.

Problem 8.18.2 The basic equations for the magnetizable but insulating inhomogeneous fluid are

$$\rho\left(\frac{\partial\bar{v}}{\partial t}+\bar{v}\cdot\nabla\bar{v}\right)=-\nabla p-\rho g\,\bar{i}_x-\frac{1}{2}H^2\nabla\mu \qquad (1)$$

$$\nabla\cdot\bar{v}=0 \qquad (2)$$

$$\nabla\cdot\mu\bar{h}=0 \qquad (3)$$

$$\nabla\times\bar{h}=0 \qquad (4)$$

$$\frac{D\mu}{Dt}=0 \qquad (5)$$

$$\frac{D\rho}{Dt}=0 \qquad (6)$$

where $\bar{H}=H_s(x)\bar{i}_z+\bar{h}$.

In view of Eq. 4, $\bar{h}=-\nabla\psi$. This means that $\hat{h}_z=-jk_z\hat{\psi}$ and for the present purposes it is more convenient to use $\hat{h}_z$ as a scalar "potential"

$$\hat{h}_x=-\frac{1}{jk_z}D\hat{h}_z\ ;\ \hat{h}_y=\frac{k_y}{k_z}\hat{h}_z \qquad (7)$$

With the definitions $\mu=\mu_s(x)+\mu'$ and $\rho=\rho_s(x)+\rho'$, Eqs. 5 and 6 link the perturbations in properties to the fluid displacement

$$\hat{\mu}=-\frac{\hat{v}_x D\mu_s}{j\omega}\ ;\ \hat{\rho}=-\frac{\hat{v}_x D\rho_s}{j\omega} \qquad (8)$$

Thus, with the use of Eq. 8a and Eqs. 7, the linearized version of Eq. 3 is

$$D(\mu_s D\hat{h}_z)=k^2\mu_s\hat{h}_z+j\frac{k_z^2}{\omega}H_s(D\mu_s)\hat{v}_x\ ;\ k^2\equiv k_y^2+k_z^2 \qquad (9)$$

and this represents the magnetic field, given the mechanical deformation.

To represent the mechanics, Eq. 2 is written in terms of complex amplitudes.

$$D\hat{v}_x=jk_y\hat{v}_y+jk_z\hat{v}_z \qquad (1)$$

and, with the use of Eq. 8b, the x component of Eq. 1 is written in the linearized form

$$[\omega^2\rho_s+gD\rho_s+\tfrac{1}{2}H_s^2D^2\mu_s]\hat{v}_x+\tfrac{1}{2}H_s^2(D\mu_s)D\hat{v}_x-j\omega H_s(D\mu_s)\hat{h}_z=j\omega D\hat{p} \qquad (11)$$

Prob. 8.18.2(cont.)

Similarly, the y and z components of Eq. 1 become

$$j\omega\rho_s \hat{v}_y = j k_y \hat{p} - \frac{1}{2}\frac{k_y}{\omega} H_s^2 (D\mu_s)\hat{v}_x \tag{12}$$

$$j\omega\rho_s \hat{v}_z = j k_z \hat{p} - \frac{1}{2}\frac{k_z}{\omega} H_s^2 (D\mu_s)\hat{v}_x \tag{13}$$

With the objective of making $\hat{v}_x$ a scalar function representing the mechanics, these last two expressions are solved for $\hat{v}_y$ anf $\hat{v}_z$ and substituted into Eq. 10.

$$\omega\rho_s D\hat{v}_x = j k^2 \hat{p} - \frac{1}{2}\frac{k^2}{\omega} H_s^2 (D\mu_s)\hat{v}_x \tag{14}$$

This expression is then solved for $\hat{p}$, and the derivative taken with respect to x. This derivative can then be used to eliminate the pressure from Eq. 11.

$$D[\rho_s(D\hat{v}_x)] - k^2\left[\rho_s - \frac{\mathcal{N}}{\omega^2}\right]\hat{v}_x + j\frac{k^2 H_s (D\mu_s)}{\omega}\hat{h}_z = 0 \tag{15}$$

$$\mathcal{N} \equiv -g D\rho_s + \frac{1}{2}(D\mu_s)D(H_s^2) \cong -g D\rho_s$$

Equations 9 and 15 comprise the desired relations.

In an imposed field approximation where $H_s = H_o$ = constant and the properties have the profiles $\rho_s = \rho_m \exp \beta x$ and $\mu_s = \mu_o \exp \beta x$, Eqs. 9 and 15 become

$$\left[L + \frac{k^2\mathcal{N}}{\rho_s\omega^2}\right]\hat{v}_x + \left[\frac{j k^2 H_o \beta \mu_m}{\rho_m \omega}\right]\hat{h}_z = 0 \tag{16}$$

$$[L]\hat{h}_z + \left[\frac{j k_z^2 H_o \beta}{\omega}\right]\hat{v}_x = 0 \tag{17}$$

where $L \equiv D^2 + \beta D - k^2$

For these constant coefficient equations, solutions take the form $\exp \gamma x$ and $L \rightarrow \gamma^2 + \beta\gamma - k^2$. From Eqs. 16 and 17 it follows that

$$L^2 + \frac{k^2\mathcal{N}}{\rho_s\omega^2}L + \frac{k^2 k_z^2}{\omega^2}\frac{H_o^2\beta^2\mu_m}{\rho_m} = 0 \tag{18}$$

Prob. 8.18.2(cont.)

Solution for L results in

$$L = a \pm b\ ;\ a \equiv \frac{g\beta k^2}{2\omega^2}\ ;\ b \equiv \left[\left(\frac{g\beta k^2}{2\omega^2}\right)^2 - \left(\frac{k k_z}{\omega}\frac{H_o\beta}{\sqrt{\rho_m/\mu_m}}\right)^2\right]^{1/2} \tag{19}$$

From the definition of L, the γ's representing the x dependence follow as

$$\gamma = -\frac{\beta}{2} \pm c_\pm\ ;\ c_\pm \equiv \left[\left(\frac{\beta}{2}\right)^2 + k^2 + a \pm b\right]^{\frac{1}{2}} \tag{20}$$

In terms of these γ's,

$$\hat{v}_x = e^{-\frac{\beta}{2}x}\left[\hat{A}_1 e^{c_+ x} + \hat{A}_2 e^{-c_+ x} + \hat{A}_3 e^{c_- x} + \hat{A}_4 e^{-c_- x}\right] \tag{21}$$

The corresponding $\hat{h}_z$ is written in terms of these same coefficients with the help of Eq. 17

$$\hat{h}_z = -j\frac{k_z^2 H_o\beta}{\omega}\left[\frac{\hat{A}_1 e^{c_+ x}}{a+b} + \frac{\hat{A}_2 e^{-c_+ x}}{a+b} + \frac{\hat{A}_3 e^{c_- x}}{a-b} + \frac{\hat{A}_4 e^{-c_- x}}{a-b}\right]e^{-\frac{\beta}{2}x} \tag{22}$$

Thus, the four boundary conditions require that

$$\begin{bmatrix} e^{c_+ \ell} & e^{-c_+ \ell} & e^{c_- \ell} & e^{-c_- \ell} \\ 1 & 1 & 1 & 1 \\ \frac{e^{c_+ \ell}}{a+b} & \frac{e^{-c_+ \ell}}{a+b} & \frac{e^{c_- \ell}}{a-b} & \frac{e^{-c_- \ell}}{a-b} \\ \frac{1}{a+b} & \frac{1}{a+b} & \frac{1}{a-b} & \frac{1}{a-b} \end{bmatrix}\begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \\ \hat{A}_3 \\ \hat{A}_4 \end{bmatrix} = 0 \tag{23}$$

This determinant is easily reduced by first subtracting the second and fourth columns from the first and third respectively and then expanding by minors.

$$\sinh(c_+\ell)\sinh(c_-\ell)\frac{2b}{a^2-b^2} = 0 \tag{24}$$

Thus, eigenmodes are $c_+\ell = jn\pi$ and $c_-\ell = jn\pi$. The eigenfrequencies follow from Eqs. 19 and 20.

$$\omega_n^2 = \frac{k^2 k_z^2 H_o^2 \beta^2 \mu_m}{K_n^4 \rho_m} - \frac{g\beta k^2}{K_n^2}\ ;\ K_n \equiv \left(\frac{n\pi}{\ell}\right)^2 + \left(\frac{\beta}{2}\right)^2 + k^2 \tag{25}$$

For perturbations with peaks and valleys running perpendicular to the imposed fields, the magnetic field stiffens the fluid. Internal electromechanical waves

Prob. 8.18.2(cont.)

propagate along the lines of magnetic field intensity. If the fluid were confined between parallel plates in the x-z planes, so that the fluid were indeed forced to undergo only two dimensional motions, the field could be used to balance a heavy fluid on top of a light one.... to prevent the gravitational form of Rayleigh-Taylor instability. However, for perturbations with hills and valleys running parallel to the imposed field, the magnetic field remains undisturbed, and there is no magnetic restoring force to prevent the instability. The role of the magnetic field, here in the context of an internal coupling, is similar to that for the hydromagnetic system described in Sec. 8.12 where interchange modes of instability for a surface coupled system were found.

The electric polarization analogue to this configuration might be as shown in Fig. 8.11.1, but with a smooth distribution of ϵ and ρ in the x direction.

Problem 8.18.3 Starting with Eqs. 9 and 15 from Prob. 8.18.2, multiply the first by $\hat{h}_z^*$ and integrate from 0 to ℓ.

$$\int_0^\ell \hat{h}_z^* D(\mu_A D\hat{h}_z)dx - \int_0^\ell k^2\mu_A \hat{h}_z\hat{h}_z^* dx - j\int_0^\ell \frac{k_z^2}{\omega}(D\mu_s)H_A\hat{v}_x\hat{h}_z^* dx = 0 \tag{1}$$

Integration of the first term by parts and use of the boundary conditions on $\hat{h}_z$ gives integrals on the left that are positive definite.

$$-\int_0^\ell \mu_A(D\hat{h}_z)(D\hat{h}_z)^* dx - k^2\int_0^\ell \mu_A\hat{h}_z\hat{h}_z^* dx - j\frac{k_z^2}{\omega}\int_0^\ell (D\mu_A)H_A\hat{v}_x\hat{h}_z^* dx = 0 \tag{2}$$

In summary

$$I_1 = -j\frac{k_z^2}{\omega}\hat{I}_4^* ; I_1 \equiv \int_0^\ell [\mu_A|D\hat{h}_z|^2 + k^2\mu_A|\hat{h}_z|^2]dx, I_4 \equiv \int_0^\ell H_A(D\mu_A)\hat{v}_x^*\hat{h}_z dx \tag{3}$$

Now, multiply Eq. 15 from Prob. 8.18.2 by $\hat{v}_x^*$ and integrate.

$$\int_0^\ell \hat{v}_x^* D\rho_s(D\hat{v}_x)dx - k^2\int_0^\ell \rho_A\hat{v}_x\hat{v}_x^* dx + \frac{k^2}{\omega^2}\int_0^\ell \mathcal{N}\hat{v}_x\hat{v}_x^* dx + j\frac{k^2}{\omega}\int_0^\ell H_A D\mu_A\hat{v}_x^*\hat{h}_z dx = 0 \tag{4}$$

Prob. 8.18.3(cont.)

Integration of the first term by parts and the boundary conditions on $\hat{v}_x$ gives

$$-\int_0^{\ell} \rho_a D\hat{v}_x D\hat{v}_x^{*} dx - k^2 \int_0^{\ell} \rho_a \hat{v}_x \hat{v}_x^{*} dx + \frac{k^2}{\omega^2} \int_0^{\ell} \mathcal{N} \hat{v}_x \hat{v}_x^{*} dx + j\frac{k^2}{\omega} \int_0^{\ell} H_a(D\mu_a)\hat{v}_x^{*} \hat{h}_z dx = 0 \quad (5)$$

and this expression takes the form

$$I_2 - \frac{I_3}{\omega^2} = j\frac{k^2}{\omega}\hat{I}_4 \,;\; I_2 \equiv \int_0^{\ell} (\rho_a |D\hat{v}_x|^2 + k^2 \rho_a |\hat{v}_x|^2) dx \,;\; I_3 = \int_0^{\ell} k^2 \mathcal{N} |\hat{v}_x|^2 dx \quad (6)$$

Multiplication of Eq. 3 by Eq. 6 results in yet another positive definite quantity

$$I_1 I_2 - \frac{I_1 I_3}{\omega^2} = \frac{k^2 k_z^2}{\omega^2} |I_4|^2 \quad (7)$$

and this expression can be solved for the frequency

$$\omega^2 = \frac{k^2 k_z^2 |\hat{I}_4|^2 + I_1 I_3}{I_1 I_2} \quad (8)$$

Because the terms on the right are real, it follows that either the eigenfrequencies are real or they represent modes that grow and decay without oscillation. Thus, the search for eigenfrequencies in the general case can be restricted to the real and imaginary axes of the s plane.

Note that a sufficient condition for stability is $\mathcal{N} > 0$, because that insures that I_3 is positive definite.