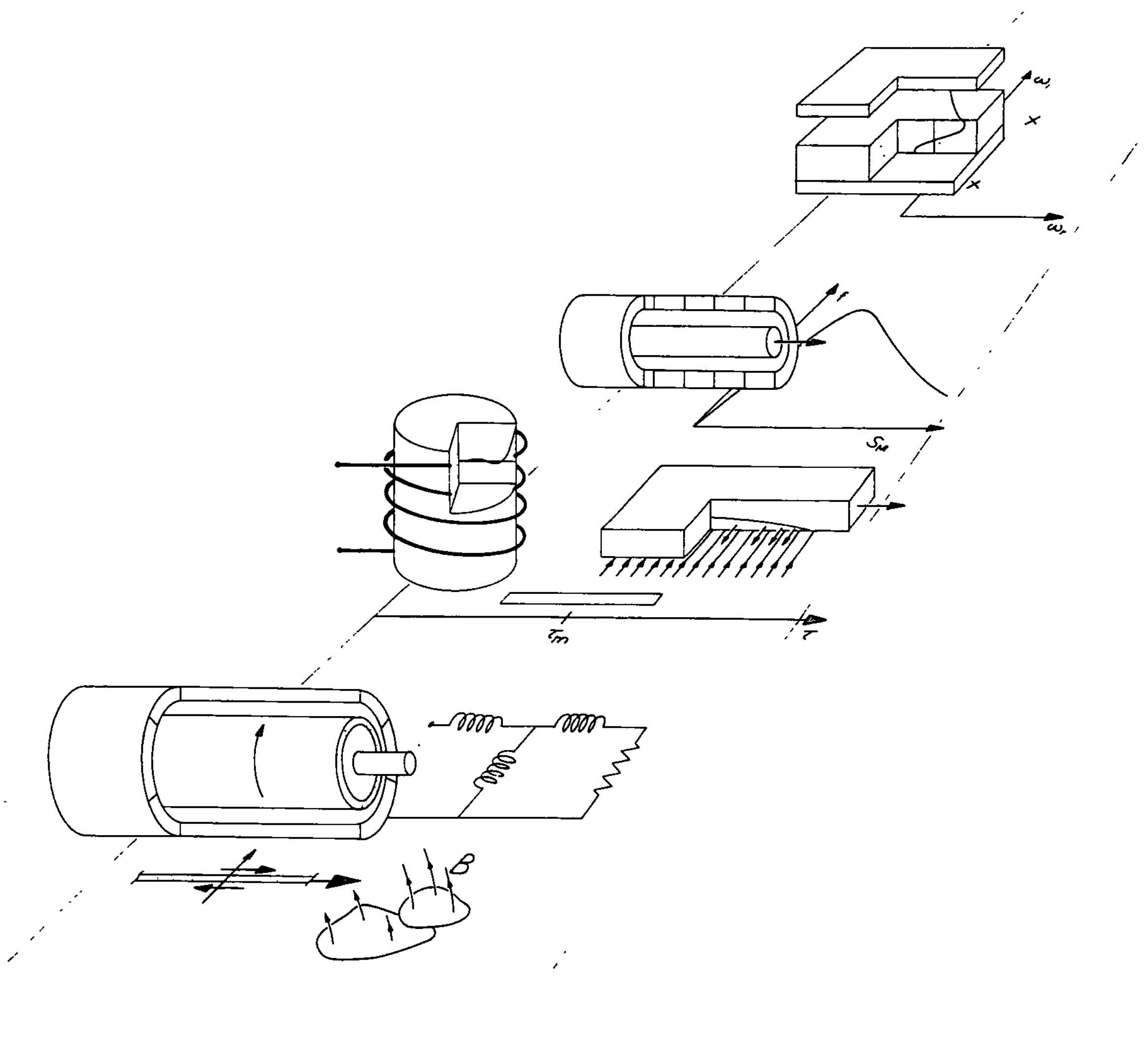
6

Magnetic Diffusion and Induction Interactions



Prob. 6.2.1 (a) The zero order fields follow from current continuity and Ampere's law,

$$\bar{J} = J_o \bar{i}_x \; ; \; \bar{E} = (J_o/\sigma)\, \bar{i}_x \tag{1}$$

$$-\frac{\partial H_y}{\partial z} = J_o \Rightarrow H_y = -\frac{i}{d\ell} z \tag{2}$$

where d is the length in the y direction.

Thus, the magnetic energy storage is

$$\frac{1}{2} L i^2 = \frac{da}{2}\mu \int_{-\ell}^{0} H_y^2 \, dz = \frac{1}{2}\left(\frac{\mu a \ell}{3d}\right) i^2 \tag{3}$$

from which it follows that the inductance is $L = \mu a \ell / 3d$.

With this zero order H_y substituted on the right in Eq. 7, it follows that

$$\frac{\partial^2 H_{y1}}{\partial z^2} = -\frac{\mu\sigma}{d\ell} z \frac{di}{dt} \tag{4}$$

Two integrations bring in two integration functions, the second of which is zero because $H_y=0$ at $z=0$.

$$H_{y1} = \frac{-\mu\sigma z^3}{6 d\ell} \frac{di}{dt} + f(t) z \tag{5}$$

So that the current at $z=-\ell$ on the plate at $x=0$ is $i(t)$, the function $f(t)$ is evaluated by making $H_{y1}=0$ there

$$f = \frac{\mu\sigma\ell}{6d} \frac{di}{dt} \tag{6}$$

Thus, the zero plus first order fields are

$$H_y = -\frac{i}{d\ell} z + \frac{di}{dt} \frac{\mu\sigma\ell}{6d}\left(z - \frac{z^3}{\ell^2}\right) \tag{7}$$

The current density implied by this follows from Ampere's law

$$J_x = -\frac{\partial H_y}{\partial z} = \frac{i}{d\ell} - \frac{\mu\sigma\ell}{6d}\left(1 - \frac{3z^2}{\ell^2}\right)\frac{di}{dt}$$

Finally, the voltage at the terminals is evaluated by recognizing from Ohm's law that $v = aE_z = aJ_x(-\ell)/\sigma$. Thus, Eq. 8 gives

$$v = Ri + L\frac{di}{dt} \tag{9}$$

Prob. 6.2.1 (cont.)

where $L=\mu l a/3d$ and $R=\sigma/\sigma d l$.

Prob. 6.3.1 For the cylindrical rotating shell, Eq. 6.3.2 becomes

$$\frac{1}{r}\left[\frac{\partial K_z}{\partial\theta}-\frac{\partial(K_\theta r)}{\partial z}\right]=-\sigma_s\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\theta}\right)B_r \tag{1}$$

and Eq. 6.3.3 becomes

$$\frac{1}{r}\frac{\partial K_\theta}{\partial\theta}+\frac{\partial K_z}{\partial z}=0 \tag{2}$$

The desired result involves $[\![H_\theta]\!]$, which in view of Ampere's law is K_z. So, between these two equations, K_θ is eliminated by operating on Eq. 1 with $r\partial(\ )/\partial\theta$ and adding to Eq. 2 operated on by $r^2\partial(\ )/\partial z$.

$$\left(r^2\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial\theta^2}\right)K_z=-r\sigma_s\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\theta}\right)B_r \tag{3}$$

Then, because $K_z=[\![H_\theta]\!]$, the desired result, Eq. (b) of Table 6.3.1, is obtained.

Prob. 6.3.2 Equation 6.3.2 becomes

$$(\nabla\times\bar{K}_f)_r=-\sigma_s\frac{\partial B_r}{\partial t}+\sigma_s[\nabla\times(\bar{v}\times\bar{B})]_r \tag{1}$$

or, in cylindrical coordinates

$$\frac{1}{a}\frac{\partial K_z}{\partial\theta}-\frac{\partial K_\theta}{\partial z}=-\sigma_s\left(\frac{\partial B_r}{\partial t}+U\frac{\partial B_r}{\partial z}\right) \tag{2}$$

Equation 6.3.3 is

$$\nabla_\Sigma\cdot\bar{K}_f=\frac{1}{a}\frac{\partial K_\theta}{\partial\theta}+\frac{\partial K_z}{\partial z}=0 \tag{3}$$

while Eq. 6.3.4 requires that

$$[\![H_\theta]\!]=K_z\ ;\ -[\![H_z]\!]=K_\theta \tag{4}$$

The $\partial/\partial z$ of Eq. 2 and $\partial/\partial\theta$ of Eq. 3 then combine (to eliminate $\partial^2K_z/\partial\theta\partial z$) to give

$$-\left(\frac{1}{a^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right)K_\theta=\sigma_s\frac{\partial}{\partial z}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial z}\right)B_r \tag{5}$$

Substitution for K_θ from Eq. 4b then gives Eq. c of Table 6.3.1.

Prob. 6.3.3   Interest is in the radial component of Eq. (2) evaluated at r= d .

$$\frac{1}{d^2\sin\theta}\left[\frac{\partial(K_\phi d\sin\theta)}{\partial\theta}-\frac{\partial(K_\theta d)}{\partial\phi}\right]=-\sigma_s\left(\frac{\partial B_r}{\partial t}+\Omega\frac{d\sin\theta}{d\sin\theta}\frac{\partial B_r}{\partial\phi}\right) \qquad (1)$$

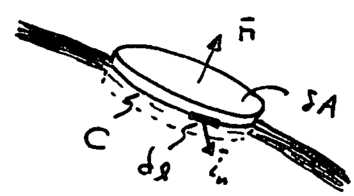
In spherical coordinates, Eq. 3 becomes

$$\frac{1}{d^2\sin\theta}\left[\frac{\partial}{\partial\theta}(K_\theta d\sin\theta)+\frac{\partial}{\partial\phi}(K_\phi d)\right]=0$$

To eliminate K_ϕ , multiply Eq. 2 by $\frac{\partial}{\partial\theta}\sin\theta$ and subtract Eq. 1 operated on by $\partial/\partial\phi$. Because Eq. 4 shows that $[\![H_\phi]\!]=-K_\theta$, Eq. (d) of Table 6.3.1 follows. To obtain Eq. (e) of Table 6.3.1, operate on Eq. (1) with $d\frac{\partial}{\partial\theta}(\sin^2\theta)$, on Eq. (2) with $\frac{\partial}{\partial\phi}(d\sin\theta)$ and add the latter to the former. Then use Eq. (4) to replace K_ϕ with $[\![H_\theta]\!]$.

Prob. 6.3.4 Gauss' law for $\bar{B}$ in integral form is applied to a pill-box enclosing a section of the sheet. The box has the thickness Δ of the sheet and an incremental area δA in the plane of the sheet. With C defined as a contour following the intersection of the sheet and the box, the integral law requires that

$$\Delta\mu\oint_C\bar{H}\cdot\bar{i}_n\,dl+\delta A[\![B_n]\!]=0 \qquad (1)$$



The surface divergence is defined as

$$\nabla_\Sigma\cdot H\equiv\lim_{\delta A\to 0}\frac{1}{\delta A}\oint_C\bar{H}\cdot\bar{i}_n\,dl \qquad (2)$$

Under the assumption that the tangential field intensity is continuous through the sheet, Eq. 1 therefore becomes the required boundary condition.

$$\Delta\mu\,\nabla_\Sigma\cdot\bar{H}+[\![B_n]\!]=0 \qquad (3)$$

In cartesian coordinates and for a planar sheet, $\bar{H}=-\nabla\psi$ and Eq. 3 becomes

$$-\Delta\mu\left[\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right]+[\![B_x]\!]=0 \qquad (4)$$

In terms of complex amplitudes, this is equivalent to

$$\Delta\mu k^2\tilde{\psi}+[\tilde{B}_x^a-\tilde{B}_x^b]=0 \qquad (5)$$

Prob. 6.3.4 (cont.)

From Table 2.16.1, the transfer relations for a layer of arbitrary thickness are

$$\begin{bmatrix} \tilde{B}_x^a \\ \tilde{B}_x^b \end{bmatrix} = \mu k \begin{bmatrix} -\coth k\Delta & \frac{1}{\sinh k\Delta} \\ -\frac{1}{\sinh k\Delta} & \coth k\Delta \end{bmatrix} \begin{bmatrix} \tilde{\psi}^a \\ \tilde{\psi}^b \end{bmatrix} \tag{6}$$

Subtraction of the second expression from the first gives

$$\tilde{B}_x^a - \tilde{B}_x^b = \mu k \left[\frac{1 - \cosh k\Delta}{\sinh k\Delta}\right](\tilde{\psi}^a + \tilde{\psi}^b) \tag{7}$$

In the long-wave limit, cosh $k\Delta \rightarrow 1 + (k\Delta)^2/2$ and sinh $k\Delta \rightarrow k\Delta$ so this expression becomes

$$\tilde{B}_x^a - \tilde{B}_x^b = -\mu \Delta k^2 \frac{(\tilde{\psi}^a + \tilde{\psi}^b)}{2} \tag{8}$$

continuity of tangential $\bar{H}$ requires that $\tilde{\psi}^a \rightarrow \tilde{\psi}^b$, so that this expression agrees with Eq. 5.

<u>Prob. 6.3.5</u> The boundary condition reflecting the solenoidal nature of the flux density is determined as in Prob. 6.3.4 except that the integral over the sheet cross-section is not simply a multiplication by the thickness. Thus,

$$\mu \oint_C \left[\int_0^\Delta \bar{H}\cdot\bar{i}_n \, dx\right] d\ell + \delta A [\![B_n]\!] = 0 \tag{1}$$

is evaluated using $\bar{H}_t = \bar{H}_t^b + \frac{x}{\Delta}(\bar{H}_t^a - \bar{H}_t^b)$. To that end, observe that

$$\int_0^\Delta \bar{H}\cdot\bar{i}_n \, dx = \bar{H}_t^b\cdot\bar{i}_n \Delta + \tfrac{1}{2}\Delta(\bar{H}_t^a - \bar{H}_t^b)\cdot\bar{i}_n = \Delta\langle \bar{H}_t\rangle\cdot\bar{i}_n \tag{2}$$

so that Eq. 1 becomes

$$\frac{1}{\delta A}\oint_C \langle H_t\rangle\cdot\bar{i}_n \, d\ell + [\![B_n]\!] = 0 \tag{3}$$

In the limit this becomes the required boundary condition.

$$\mu\Delta \nabla_\Sigma \cdot \langle \bar{H}\rangle + [\![B_n]\!] = 0 \tag{4}$$

With the definition

$$\bar{K}_f \equiv \int_0^\Delta \bar{J}\, dx \tag{5}$$

and the assumption that contributions to the line integration of $\bar{H}$ through the sheet are negligible compared to those tangential, Ampere's law still requires

Prob. 6.3.5(cont.)

that

$$\bar{n} \times [\![\bar{H}]\!] = \bar{K}_f \tag{6}$$

The combination of Faraday's and Ohm's laws, Eq. 6.2.3, is integrated over the sheet cross-section.

$$\int_0^\Delta (\nabla \times \bar{J}_f)_n \, dx = \sigma \int_0^\Delta \left\{ -\frac{\partial B_n}{\partial t} + [\nabla \times (\bar{v} \times \bar{B})]_n \right\} dx \tag{7}$$

This reduces to

$$(\nabla \times \bar{K}_f)_n = -\sigma \Delta \left[\frac{\partial}{\partial t} + U \frac{\partial}{\partial y} \right] \langle B_n \rangle \tag{8}$$

where evaluation using the presumed constant plus linear dependence for B_n shows that

$$\int_0^\Delta B_n \, dx = \Delta \langle B_n \rangle \tag{9}$$

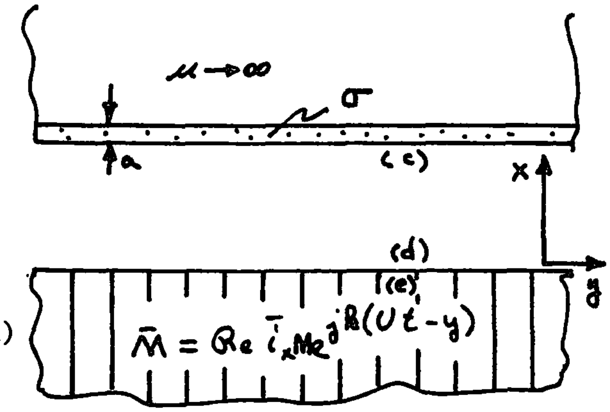
It is still true that

$$\nabla_\Sigma \cdot \bar{K}_f = 0 \tag{10}$$

To eliminate K_y, the y derivative of Eq. 9 is added to the z derivative of Eq. 10 and the z component of Eq. 6 is in turn used to replace K_z. Thus, the second boundary condition becomes

$$\left[\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] [\![H_y]\!] = -\sigma \Delta \frac{\partial}{\partial y} \left[\frac{\partial}{\partial t} + U \frac{\partial}{\partial y} \right] \langle B_x \rangle \tag{11}$$

Note that this is the same as given in Table 6.3.1 provided B_n is taken as the average.



Prob. 6.4.1 For solutions of the form $\exp j(\omega t - ky)$ where $\omega = kU$, let $\bar{H} = -\nabla\psi$. Then, boundary conditions begin with the conducting sheet

$$-\frac{\partial^2 H_y^c}{\partial y^2} = -\sigma_s \frac{\partial}{\partial y}\left(\frac{\partial}{\partial t}\right) B_x^c \qquad (1)$$

or, in terms of complex amplitudes,

$$k^2 \hat{H}_y^c = -\sigma_s \omega k \hat{H}_x^c \Rightarrow \mu_o \hat{H}_x^c = -\frac{jk^2}{\sigma_s \omega} \hat{\psi}^c. \qquad (2)$$

At this same boundary the normal flux density is continuous, but because the region above is infinitely permeable, this condition is implicite to Eq. 1.

At the interface of the moving magnetized member,

$$n \times [\![\hat{H}]\!] = 0 \Rightarrow \hat{\psi}^d = \hat{\psi}^e \qquad (3)$$

and

$$\bar{n}\cdot[\![\mu_o \bar{H}]\!] = -\bar{n}\cdot[\![\mu_o \bar{M}]\!] = Re\, \mu_o M e^{j(\omega t - ky)} \Rightarrow \hat{H}_x^d - \hat{H}_x^e = M \qquad (4)$$

and because the lower region is an infinite half space, $\psi \to 0$ as $x \to -\infty$.

Bulk relations reflecting Laplace's equation in the air-gap are (from Table 2.16.1 with $B_x \to \mu_o H_x$)

$$\begin{bmatrix} \hat{H}_x^c \\ \hat{H}_x^d \end{bmatrix} = k \begin{bmatrix} -\coth kd & \frac{1}{\sinh kd} \\ \frac{-1}{\sinh kd} & \coth kd \end{bmatrix} \begin{bmatrix} \hat{\psi}^c \\ \hat{\psi}^d \end{bmatrix} \qquad (5)$$

In the lower region, $\nabla\cdot\mu_o \bar{M} = 0$, so again $\nabla^2\psi = 0$ and the transfer relation (which represents a solution of $\bar{H} = -\nabla\psi$ where $\nabla^2\psi = 0$ with $\mu \to \mu_o$ and hence $B_x \to \mu_o H_x$. Of course, in the actual problem, $B_x = \mu_o(H_x + M_x)$)is

$$\mu_o \hat{H}_x^e = -\mu_o k \hat{\psi}^e \qquad (6)$$

Looking ahead, what is desired is

$$\langle T_y \rangle_t = -\frac{\mu_o}{2} Re\, \hat{H}_y^c \hat{H}_x^{c*} = -\frac{\mu_o k}{2} Re\, j \hat{\psi}^c \hat{H}_x^{c*} \qquad (7)$$

From Eq. 2 (again with $\hat{H}_y^c = jk\hat{\psi}^c$)

$$\langle T_y \rangle_t = \frac{k}{2} Re\, \hat{\psi}^c \left(\frac{k^2}{\sigma_s \omega}\right) \hat{\psi}^{e*} \qquad (8)$$

Prob. 6.4.1 (cont.)

To solve for $\hat{\psi}^c$, plug Eqs. 2 and 3 into Eq. 5a

$$\begin{bmatrix} \dfrac{jk^2}{\mu_o\sigma_s\omega} - k\coth kd & \dfrac{k}{\sinh kd} \\ \dfrac{-k}{\sinh kd} & k(1+\coth kd) \end{bmatrix} \begin{bmatrix} \hat{\psi}^c \\ \hat{\psi}^e \end{bmatrix} = \begin{bmatrix} 0 \\ M \end{bmatrix} \quad (9)$$

The second of these follows by using Eqs. 3,4 and 6 in Eq. 5b. Thus,

$$\hat{\psi}^c = \frac{Mk}{\sinh kd\left\{\left[\dfrac{k^2}{\sinh^2 kd} - k^2\coth kd(1+\coth kd)\right] + \dfrac{jk^2}{\mu_o\sigma_s\omega}(1+\coth kd)k\right\}} \quad (10)$$

Thus with $\underline{U} \equiv \mu_o\sigma_s U$, Eq. 8 becomes

$$\langle T_y\rangle_t = \frac{\mu_o M^2}{2\sinh^2 kd}\,\frac{\underline{U}}{\sqrt{\underline{U}^2\left[\dfrac{1}{\sinh^2 kd} - \coth kd(1+\coth kd)\right]^2 + (1+\coth kd)^2}} \quad (11)$$

To make $\langle T_y\rangle_t$ proportional to U, design the device to have

$$\underline{U}^2\left[\frac{1}{\sinh^2 kd} - \coth kd(1+\coth kd)\right]^2 \ll (1+\coth kd)^2 \quad (12)$$

In which case

$$\langle T_y\rangle_t = \frac{\mu_o M^2(\mu_o\sigma_s U)}{2\sinh^2 kd(1+\coth kd)} \quad (13)$$

so that the force per unit area is proportional to the velocity of the rotor.

Prob. 6.4.2 For the circuit, loop equations are

$$\begin{bmatrix} j\omega(L_1+M) & -j\omega M \\ -j\omega M & j\omega(L_2+M)+\frac{R}{a_m} \end{bmatrix}\begin{bmatrix} \hat{i}_a \\ \hat{i}_b \end{bmatrix} = \begin{bmatrix} \hat{v}_a \\ 0 \end{bmatrix} \tag{1}$$

Thus,

$$\hat{i}_a = \frac{\hat{v}_a\left[j\omega(L_2+M)+\frac{R}{a_m}\right]}{j\omega(L_1+M)\left[j\omega(L_2+M)+\frac{R}{a_m}\right]+\omega^2 M^2} \tag{2}$$

and written in the form of Eq. 6.4.17, this becomes

$$\hat{v}_a = \left\{ j\omega(L_1+M) - j\omega a_m \frac{\left[j\omega M^2 R + \omega^2 M^2 (L_2+M) a_m\right]}{R^2\left[1+\frac{\omega^2 (L_2+M)^2 a_m^2}{R^2}\right]} \right. \tag{3}$$

where comparison with Eq. 6.4.17 shows that

$$\frac{a_m}{R}\,\omega(L_2+M) = S_m \coth kd \tag{4}$$

$$L_1 + M = \frac{w\ell N_a^2 \mu_o}{2k} \coth kd \tag{5}$$

$$a_m \omega M^2/R = S_m w\ell N_a^2 \mu_o / 2k \sinh^2 kd \tag{6}$$

These three conditions do not uniquely specify the unknowns. But, add to them the condition that $L_1 = L_2$ and it follows from Eq. 6 that

$$\frac{a_m}{R} = \frac{S_m}{\sinh^2 kd}\,\frac{w\ell^2 N_a^2 \mu_o}{4\pi\omega M^2} \tag{7}$$

so that Eq. 4 becomes an expression that can be solved for M

$$M = \frac{w N_a^2 \mu_o \ell^2}{4\pi \sinh kd} \tag{8}$$

and Eq. 5 then gives

$$L_1 = L_2 = \frac{w\ell^2 N_a^2 \mu_o}{4\pi}\left[\coth kd - \frac{1}{\sinh kd}\right] = \frac{w\ell^2 N_a^2 \mu_o}{4\pi} \tanh\left(\frac{kd}{2}\right) \tag{9}$$

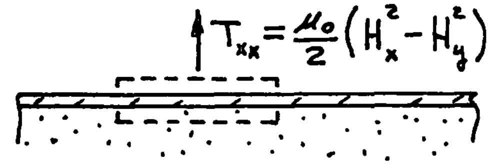
Finally, a return to Eq. 7 gives

$$\frac{a_m}{R} = \frac{S_m}{\omega}\,\frac{4\pi}{w\ell^2 N_a^2 \mu_o} \tag{10}$$

These parameters check with those from the figure.

Prob. 6.4.3 The force on the "stator" is the negative of that on the "rotor".

$$\langle f_x\rangle_t = -\frac{p\ell w}{2}\frac{\mu_o}{4}\,Re\Big\{\hat{H}^r_{x+}\hat{H}^{r*}_{x+} - \hat{H}^r_{y+}\hat{H}^{r*}_{y+} + \hat{H}^r_{x-}\hat{H}^{r*}_{x-} - \hat{H}^r_{y-}\hat{H}^{r*}_{y-}\Big\} \qquad (1)$$



In the following, the response is found for the ± waves separately, and then these are combined to evaluate Eq. 1. From Eq. 6.4.9,

$$\hat{H}^r_{x\pm} = \mp j\left[\frac{\hat{K}^A_\pm}{\sinh kd} + \coth kd\,\hat{H}^r_{y\pm}\right] \qquad (2)$$

So that

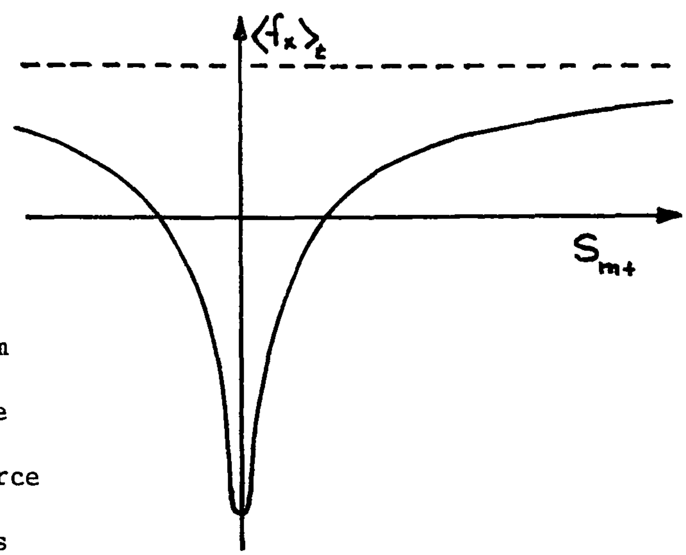
$$|\hat{H}^r_{x\pm}|^2 - |\hat{H}^r_{y\pm}|^2 = \Big\{\frac{|\hat{K}^A_\pm|}{\sinh^2 kd} + \frac{\coth kd}{\sinh kd}\left[\hat{K}^A_\pm\hat{H}^{r*}_{y\pm} + \hat{K}^{A*}_\pm\hat{H}^r_{y\pm}\right] + (\coth^2 kd - 1)|\hat{H}^r_{y\pm}|^2\Big\} \qquad (3)$$

Now, use is made of Eq. 6.4.6 to write Eq. 3 as

$$|\hat{H}^r_{x\pm}|^2 - |\hat{H}^r_{y\pm}|^2 = \frac{|\hat{K}^A_\pm|^2}{\sinh^2 kd}\left\{1 - \frac{S_{m\pm}}{1 + S^2_{m\pm}\coth^2 kd}\right\} \qquad (4)$$

So, in general

$$\langle f_x\rangle_t = -\frac{p\ell w}{8}\frac{\mu_o}{\sinh^2 kd}\left\{\frac{|\hat{K}^A_+|^2(1 - S^2_{m+})}{1 + S^2_{m+}\coth^2 kd} + \frac{|\hat{K}^A_-|^2(1 - S^2_{m-})}{1 + S^2_{m-}\coth^2 kd}\right\} \qquad (5)$$



With two-phase excitation (a pure traveling wave) the second term does not contribute and the dependence of the normal force on S_m is as shown to the right. At low frequency (from the conductor frame of reference) the magnetization force prevails (the force is attractive). For high frequencies

Prob. 6.4.3 (cont.)

($|S_m| > 1$) the force is one of repulsion, as would be expected for a force associated with the induced currents.

With single phase excitation, the currents are as given by Eq. 6.4.18

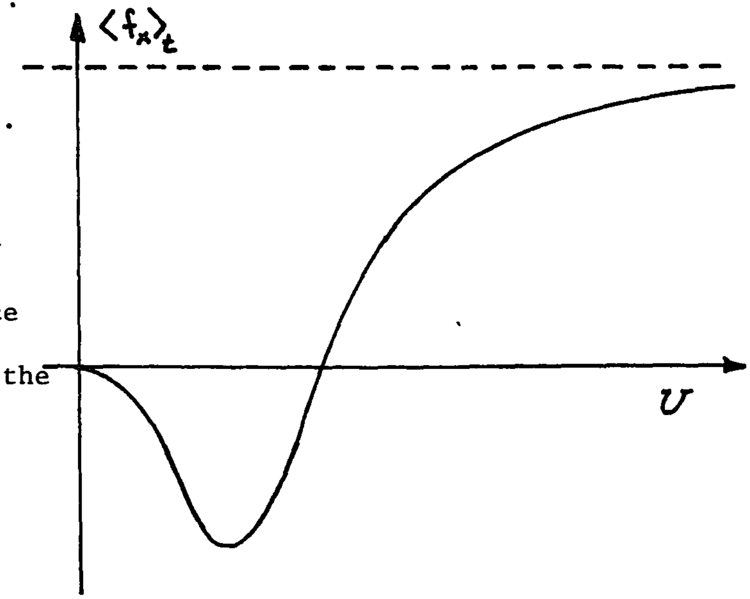
$$\hat{K}_+^a = \hat{K}_-^a = \frac{1}{2} N_a \hat{i}_a \tag{6}$$

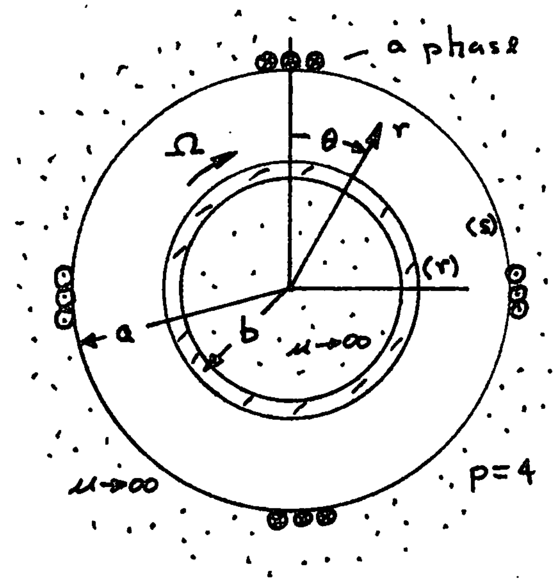
and Eq. 5 becomes

$$\langle f_x \rangle_t = \frac{p l w}{32} \frac{\mu_o N_a^2 |\hat{i}_a|}{\sinh^2 kd} \left\{ \frac{S_{m+}^2 - 1}{1 + S_{m+}^2 \coth^2 kd} + \frac{S_{m-}^2 - 1}{1 + S_{m-}^2 \coth^2 kd} \right\} \tag{7}$$

where $S_{m\pm} = \mu_o \sigma_s (\omega \mp kU)/k$.

The dependence of the force on the speed is illustrated by the figure. Making the velocity large is equivalent to making the frequency high, so at high velocity the force tends to be one of repulsion. In the neighborhood of the synchronous condition there is little induced current and the force is one of attraction.





<u>Prob. 6.4.4</u> Two-phase stator currents are represented by

$$K_z^s = \mathrm{Re}\,\hat{i}_a e^{j\omega t} N_a \cos\left(\frac{\theta p}{2}\right) + \hat{i}_b e^{j\omega t} N_b \cos\left[\left(\frac{p\theta}{2}\right) - \frac{\pi}{2}\right] \tag{1}$$

and this expression can be written in terms of complex amplitudes as

$$K_z^s = \mathrm{Re}\left[\hat{K}_+^s e^{j(\omega t - m\theta)} + \hat{K}_-^s e^{j(\omega t + m\theta)}\right] \tag{2}$$

where

$$\hat{K}_\pm^s = \frac{1}{2}\left(\hat{i}_a N_a + \hat{i}_b N_b e^{\pm j\frac{\pi}{2}}\right)$$

Boundary conditions are written using designations shown in the figure. At the stator surface,

$$\hat{H}_\theta^s = -\hat{K}^s \tag{3}$$

while at the rotor surface (Eq. b, Table 6.3.1)

$$\frac{m^2}{b^2}\hat{H}_\theta^r = \frac{\sigma_s m}{b}(\omega - m\Omega)\hat{B}_r^r \Rightarrow \hat{H}_\theta^r = \sigma_s(\omega - m\Omega)(-j\hat{A}^r) \tag{4}$$

In the gap, the vector potential is used to make calculation of the terminal relations more convenient. Thus, Eq. d of Table 2.19.1 is

$$\begin{bmatrix} \hat{A}^s \\ \hat{A}^r \end{bmatrix} = \mu_o \begin{bmatrix} F_m(b,a) & G_m(a,b) \\ G_m(b,a) & F_m(a,b) \end{bmatrix} \begin{bmatrix} \hat{H}_\theta^s \\ \hat{H}_\theta^r \end{bmatrix} \tag{5}$$

To determine $\hat{H}_\theta^r$, write Eq. 5b using Eq. 3 for $\hat{H}_\theta^s$ and Eq. 4 for $\hat{A}^r$.

$$\frac{\hat{H}_\theta^r}{-j\sigma_s(\omega - m\Omega)} = -\mu_o G_m(b,a)\hat{K}^s + \mu_o F_m(a,b)\hat{H}_\theta^r \tag{6}$$

This expression is solved and rationalized to give

$$\hat{H}_{\theta\pm}^r = \frac{\hat{K}_\pm^s G_m(b,a)\mu_o\sigma_s(\omega \mp m\Omega)\left[j + F_m(a,b)\mu_o\sigma_s(\omega \mp m\Omega)\right]}{1 + F_m^2(a,b)\left[\mu_o\sigma_s(\omega \mp m\Omega)\right]^2} \tag{7}$$

Here, $\hat{H}_{\theta r}$ is written by replacing $m \to -m$ and recognizing that F_m and G_m are even in m.

Prob. 6.4.4 (cont.)

The torque is

$$\langle T\rangle_t = 2\pi b^2 w\,\frac{1}{2}\,\mathrm{Re}\left[\hat{B}^r_{r+}(\hat{H}^r_{\theta+})^* + \hat{B}^r_{r-}(\hat{H}^r_{\theta-})^*\right] \tag{8}$$

which in view of Eq. 5b and $\hat{B}_r = -jm\hat{A}/r$ becomes

$$\langle T\rangle_t = \pi b^2 w\,\mathrm{Re}\left[\frac{-jm}{b}\hat{A}^r_+(\hat{H}^r_{\theta+})^* + \frac{jm}{b}\hat{A}^r_-(\hat{H}^r_{\theta-})^*\right] \tag{9}$$

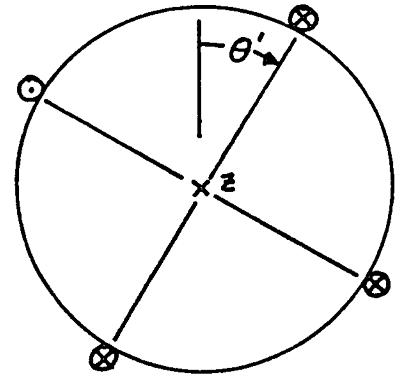
$$= \pi b^2 w\,\mathrm{Re}\left[\frac{jm\mu_o}{b}\hat{K}^A_+ G_m(b,a)\hat{H}^{r*}_{\theta+} - \frac{jm}{b}\mu_o\hat{K}^A_- G_m(b,a)(\hat{H}^r_{\theta-})^*\right]$$

Finally, with the use of Eq. 7,

$$\langle T\rangle_t = \pi b w m\mu_o G_m^2(b,a)\left\{\frac{|\hat{K}^a_+|^2\mu_o\sigma_s(\omega-m\Omega)}{1+F_m^2(a,b)[\mu_o\sigma_s(\omega-m\Omega)]^2} - \frac{|\hat{K}^a_-|^2\mu_o\sigma_s(\omega+m\Omega)}{1+F_m^2(a,b)[\mu_o\sigma_s(\omega+m\Omega)]^2}\right\} \tag{10}$$

where m=p/2. This expression is similar in form to Eq. 6.4.11.

Prob. 6.4.5 From Eq. (f), Table 2.18.1



$$\Phi_\lambda = \frac{pw}{2}\left[A^a(\theta') - A^a(\theta' + \frac{2\pi}{p})\right] \quad (1)$$

Because $A^a(\theta' + \frac{4\pi}{p}) = A^a(\theta')$, the flux linked by the total coil is just $p/2$ times that linked by the turns having the positive current in the z direction at θ' and returned at $\theta' + \pi/p$.

In terms of the complex amplitudes

$$\Phi_\lambda = \frac{pw}{2} Re\left[\hat{A}_+^a e^{j(\omega t - m\theta')} + \hat{A}_-^a e^{j(\omega t + m\theta')} - \hat{A}_+^a e^{j(\omega t - m\theta' - \pi)} - \hat{A}_-^a e^{j(\omega t + m\theta' + \pi)}\right] \quad (2)$$

$$= pw\, Re\left[\hat{A}_+^a e^{-jm\theta'} + \hat{A}_-^a e^{jm\theta'}\right] e^{j\omega t}$$

so

$$\lambda_a = \int_{-\pi/p}^{\pi/p} \Phi_\lambda N_a \cos\left(\frac{\theta' p}{2}\right) a\, d\theta' \quad (3)$$

or

$$\lambda_a = \frac{N_a p w a}{2} Re \int_{-\frac{\pi}{p}}^{\frac{\pi}{p}} \left[\hat{A}_+^a e^{-j\frac{p\theta'}{2}} + \hat{A}_-^a e^{j\frac{p\theta'}{2}}\right]\left[e^{j\frac{p\theta'}{2}} + e^{-j\frac{p\theta'}{2}}\right] d\theta' e^{j\omega t} \quad (4)$$

The only terms contributing are those independent of θ'

$$\lambda_a = \frac{N_a p w a}{2} Re\left[\hat{A}_+^a + \hat{A}_-^a\right] e^{j\omega t} \quad (5)$$

Substitution from Eqs. 5a and 7 from Prob. 6.4.4 then gives

Prob. 6.4.5 (cont.)

$$\lambda_a = \frac{N_a p w a}{2} Re\Big\{ -\mu_o F_m(b,a)\hat{K}_+^a - \mu_o F_m(b,a)\hat{K}_-^a$$

$$+ \frac{\mu_o G_m(a,b)\hat{K}_+^a G_m(b,a)\mu_o\sigma_s(\omega - m\Omega)[j + F_m(a,b)\mu_o\sigma_s(\omega - m\Omega)]}{1 + F_m^2(a,b)[\mu_o\sigma_s(\omega - m\Omega)]^2} \tag{6}$$

$$- \frac{\mu_o G_m(a,b)\hat{K}_-^a G_m(b,a)\mu_o\sigma_s(\omega + m\Omega)[j + F_m(a,b)\mu_o\sigma_s(\omega + m\Omega)]}{1 + F_m^2(a,b)[\mu_o\sigma_s(\omega + m\Omega)]^2}$$

For two phase excitation $\hat{K}_+^a = \frac{1}{2} N_a \hat{i}_a$, $\hat{K}_-^a = 0$

this becomes

$$\lambda_a = Re\,\hat{\lambda}_a e^{j\omega t} \tag{7}$$

where

$$\hat{\lambda}_a = \frac{\mu_o N_a^2 p w a b}{4}\hat{i}_a \Big\{ -\frac{F_m(b,a)}{b} + \frac{G_m(a,b)G_m(b,a)}{b^2}\cdot \left[\frac{S_m\left[j + \frac{F_m(a,b)}{b}S_m\right]}{1 + \frac{F_m^2(a,b)}{b^2}S_m^2}\right]\Big\}$$

$$S_m \equiv \mu_o\sigma_s b(\omega - m\Omega)$$

For the circuit of Fig. 6.4.3,

$$\hat{v}_a = j\omega\hat{\lambda}_a = j\omega\Big\{(L_1 + M) - \frac{[\omega^2 M^2(L_2 + M) + j\omega M^2\frac{R}{r}]}{\omega^2(L_2+M)^2 + (\frac{R}{r})^2}\Big\}$$

$$= j\omega\Big\{(L_1 + M) - \omega M^2\frac{r}{R}\frac{[j + \omega(L_2 + M)\frac{r}{R}]}{[1 + (\frac{r}{R})^2\omega^2(L_2+M)^2]}\Big\} \tag{8}$$

Prob. 6.4.5 (cont.)

compared to Eq. 7 with $\mathcal{A}_m \equiv S_m/\mu_o\sigma_s b\omega = (1-\frac{m\Omega}{\omega})$ this expression gives

$$L_1 + M = \frac{-\mu_o N_a^2 p w b a}{4}\,\frac{F_m(b,a)}{b} \tag{9}$$

$$\omega(L_2+M)\frac{A}{R} = \frac{F_m(a,b)}{b} = \frac{(L_2+M)}{R\mu_o\sigma_s b} \tag{10}$$

$$-\frac{M^2}{R\mu_o\sigma_s b} = \frac{\mu_o N_a^2 p w a b}{4}\,\frac{G_m(a,b)\,G_m(b,a)}{b^2} \tag{11}$$

Assume $L_1 = L_2$ and Eqs. $\underset{9}{2}$ and $\underset{10}{3}$ then give

$$\frac{F_m(a,b)}{b} = -\frac{N_a^2 p w a}{R\sigma_s 4}\,\frac{F_m(b,a)}{b} \tag{12}$$

from which it follows that

$$R = -\frac{N_a^2 p w a}{4\sigma_s}\,\frac{F_m(b,a)}{F_m(a,b)} \tag{13}$$

Note from Eq. (b) of Table 2.16.2 that $F_m(b,a)/F_m(a,b) = -a/b$ so Eq. 6 becomes

$$R = \frac{N_a^2 p w a^2}{4\sigma_s b} \tag{14}$$

From this and Eq. 4 it follows that

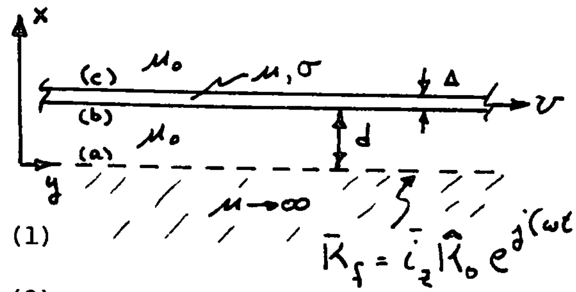
$$M = \frac{\mu_o N_a^2 p w a^2}{4}\sqrt{\frac{-G_m(a,b)\,G_m(b,a)}{ab}} \tag{15}$$

Note that $G_m(a,b) = -G_m(b,a)\,b/a$, so this can also be written as

$$M = \frac{\mu_o N_a^2 p w a b\; G_m(b,a)}{4b} \tag{16}$$

Finally, from Eqs. 2 and 9

$$L_1 = L_2 = \frac{\mu_o N_a^2 p w b a}{4}\left\{-\frac{F_m(b,a)}{b} - \frac{G_m(b,a)}{b}\right\} \tag{17}$$



Prob. 6.4.6 In terms of the cross-section shown, boundary conditions from Prob. 6.3.5 require that

$$-j\frac{k\mu\Delta}{2}(\hat{H}_y^c + \hat{H}_y^b) + (\hat{B}_x^c - \hat{B}_x^b) = 0 \tag{1}$$

$$-k^2(\hat{H}_y^c - \hat{H}_y^b) + \sigma\Delta k(\omega - kU)(\hat{B}_x^a + \hat{B}_x^b) \tag{2}$$

In addition, the fields must vanish as $x \to \infty$ and at the current sheet

$$\hat{H}_y^a = jk\hat{\psi}^a = \hat{K}_o \Rightarrow \hat{\psi}^a = \hat{K}_o/jk \tag{3}$$

Bulk conditions require that

$$\hat{B}_x^c = \mu_o k \hat{\psi}^c \quad (k > 0) \tag{4}$$

$$\begin{bmatrix} \hat{B}_x^b \\ \hat{B}_x^a \end{bmatrix} = \mu_o k \begin{bmatrix} -\coth k\Delta & \dfrac{1}{\sinh k\Delta} \\ \dfrac{-1}{\sinh k\Delta} & \coth k\Delta \end{bmatrix} \begin{bmatrix} \hat{\psi}^b \\ \dfrac{\hat{K}_o}{jk} \end{bmatrix} \tag{5}$$

In terms of the magnetic potential, Eqs. 1 and 2 are

$$\frac{k^2\mu\Delta}{2}(\hat{\psi}^c + \hat{\psi}^b) + (\hat{B}_x^c - \hat{B}_x^b) = 0 \tag{6}$$

$$-jk^2(\hat{\psi}^c - \hat{\psi}^b) + \sigma\Delta(\omega - kU)(\hat{B}_x^c + \hat{B}_x^b) = 0 \tag{7}$$

These two conditions are now written using Eqs. 3 and 5a to eliminate B_x^c and B_x^b.

$$\begin{bmatrix} k\mu_o\left(\dfrac{k\Delta\mu}{2\mu_o} + 1\right) & k\mu_o\left(\dfrac{k\Delta\mu}{2\mu_o} + \coth kd\right) \\ k^2\left[-j + \dfrac{\sigma\Delta(\omega - kU)\mu_o}{k}\right] & k^2\left[j - \dfrac{\sigma\Delta(\omega - kU)\mu_o}{k}\coth kd\right] \end{bmatrix} \begin{bmatrix} \hat{\psi}^c \\ \hat{\psi}^b \end{bmatrix} = \begin{bmatrix} \dfrac{\mu_o\hat{K}_o}{j\sinh kd} \\ \dfrac{j\sigma\Delta(\omega - kU)\mu_o\hat{K}_o}{\sinh kd} \end{bmatrix} \tag{8}$$

From these expressions it follows that

$$\tilde{\psi}^c = \frac{\hat{K}_o}{k\sinh kd} \frac{\left\{1 + j\dfrac{\mu_o\sigma\Delta}{k}(\omega - kU)\left(-\dfrac{\mu\Delta k}{2\mu_o}\right)\right\}}{\left\{j\left[(1 + \coth kd) + \dfrac{k\Delta\mu}{\mu_o}\right] - \dfrac{\sigma\mu_o\Delta(\omega - kU)}{k}\left[2\coth kd + \dfrac{k\Delta\mu}{2\mu_o}(1 + \coth kd)\right]\right\}} \tag{9}$$

Prob. 6.4.6(cont.)

In the limit where $\mu \to \mu_o$, having $\mu_o \sigma \Delta (\omega - kU)/k >> 1$ results in Eq. 9 becoming

$$\hat{\psi}^c \to \frac{-\hat{K}_o}{k \sinh kd} \Big/ \left[\frac{\mu_o \sigma \Delta (\omega - kU)}{k} \, 2 \coth kd\right] \tag{10}$$

Thus, as $\mu_o \sigma \Delta (\omega - kU)/k$ is raised, the field is shielded out of the region above the sheet by the induced currents.

In the limit where $\sigma \to 0$, for $(k\Delta)\mu/\mu_o >> 1$, Eq. 9 becomes

$$\hat{\psi}^c = \frac{\hat{K}_o}{k \sinh kd} \Big/ j\left(\frac{k\Delta\mu}{\mu_o}\right) \tag{11}$$

and again as $k\Delta\mu/\mu_o$ is made large the field is shielded out. (Note that by the requirements of the thin sheet model, $k\Delta << 1$, so μ/μ_o must be very large to obtain this shielding.)

With $k\Delta\mu/\mu_o$ finite, the numerator as well as the denominator of Eq. 9 becomes large as $\mu_o \sigma \Delta (\omega - kU)/k$ is raised. The conduction current shielding tends to be compromised by having a magnetizable sheet. This conflict should be expected, since the conduction current shields by making the normal flux density vanish. By contrast, the magnetizable sheet shields by virtue of tending to make the tangential field intensity zero. The tendency for the magnetization to duct the flux density through the sheet is in conflict with the effect of the induced current, which is to prevent a normal flux density.

Prob. 6.4.7 For the given distribution of surface current, the Fourier transform of the complex amplitude is

$$\hat{K}^s = \hat{K}_o \int_0^\ell e^{j(k-\beta)} dy = \frac{\hat{K}_o [e^{j(k-\beta)\ell} - 1]}{j(k-\beta)} \tag{1}$$

It follows from Eq. 5.16.8 that the desired force is

$$\langle f_y \rangle_t = \frac{w}{2} Re \int_{-\infty}^{+\infty} \hat{B}_x^r (\hat{H}_y^r)^* dz = \frac{w}{4\pi} Re \int_{-\infty}^{+\infty} \hat{B}_x^r (\hat{H}_y^r)^* dk \tag{2}$$

In evaluating the integral on k, observe first that Eq. 6.4.9 can be used to evaluate $\hat{B}_x^r$.

$$\langle f_y \rangle_t = \frac{-w}{4\pi} Re \int_{-\infty}^{+\infty} j\mu_o \left[\frac{\hat{K}^s}{\sinh kd} + \coth kd \, \hat{H}_y^r \right] (\hat{H}_y^r)^* dk \tag{3}$$

Because the integration is over real values of k only, it is clear that the second term of the two in brackets is purely imaginary and hence makes no contribution. With Eq. 6.4.6 used to substitute for $\hat{H}_y^r$, the expression then becomes

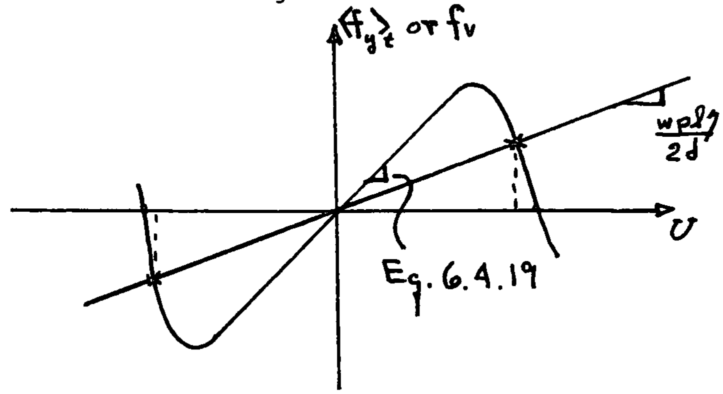
$$\langle f_y \rangle_t = \frac{w}{4\pi} \mu_o \int_{-\infty}^{+\infty} \frac{|\hat{K}^s|^2 S_m \, dk}{\sinh^2 kd \,(1 + S_m^2 \coth^2 kd)} \tag{4}$$

The magnitude $|\hat{K}^s|$ is conveniently found from Eq. 1 by first recognizing that

$$\hat{K}^s = \frac{2j\hat{K}_o}{j(k-\beta)} \left[\frac{e^{j\frac{(k-\beta)\ell}{2}} - e^{-j\frac{(k-\beta)\ell}{2}}}{2j} \right] e^{j\frac{(k-\beta)\ell}{2}} = 2j\hat{K}_o \frac{\sin\left[\frac{(k-\beta)\ell}{2}\right]}{j(k-\beta)} e^{j\frac{(k-\beta)\ell}{2}} \tag{5}$$

Substitution of this expression into Eq. 4 finally results in the integral given in the problem statement.

Prob. 6.4.8 From Eq. 7.13.1, the viscous force retarding the motion of the rotor is



$$f_v = \frac{w p \ell}{2}\left(\frac{\eta U}{d}\right) \tag{1}$$

Thus, the balance of viscous and magnetic forces is represented graphically as shown in the sketch. The slope of the magnetic force curve near the origin is given by Eq. 6.4.19. As the magnetic field is raised, the static equilibrium at the origin becomes one with U either positive or negative as the slopes of the respective curves are equal at the origin. Thus, instability is incipient as

$$\frac{kd}{\sinh^2 kd} R_M \frac{[R_M^2 \coth^2 kd - 1]}{[R_M^2 \coth^2 kd + 1]^2} > \omega \tau_{MV} \tag{2}$$

where $R_M = \omega \tau_m$, $\tau_m \equiv \mu_o \sigma_s / k$, $\tau_{MV} = \eta / \mu_o H_o^2$.

Prob. 6.5.1 The z component of Eq. 6.5.3 is written with $\bar{v} = \Omega r \bar{i}_\theta$ and $\bar{A} = A(r, \theta, t)\bar{i}_z$ by recognizing that

$$\nabla \times \bar{A} = \frac{1}{r}\frac{\partial A}{\partial \theta}\bar{i}_r - \frac{\partial A}{\partial r}\bar{i}_\theta \tag{1}$$

so that

$$\bar{v} \times \nabla \times \bar{A} = \begin{bmatrix} \bar{i}_r & \bar{i}_\theta & \bar{i}_z \\ 0 & \Omega r & 0 \\ \frac{1}{r}\frac{\partial A}{\partial \theta} & -\frac{\partial A}{\partial r} & 0 \end{bmatrix} = -\bar{i}_z \Omega \frac{\partial A}{\partial \theta} \tag{2}$$

Thus, because the z component of the vector Laplacian in polar coordinates is the same as the scalar Laplacian, Eq. 6.5.8 is obtained from Eq. 6.5.3

$$\frac{1}{\mu\sigma}\nabla^2 A = \frac{\partial A}{\partial t} + \Omega \frac{\partial A}{\partial \theta} \tag{3}$$

Solutions $A = \mathcal{R}e\, \hat{A}(r) \exp j(\omega t - m\theta)$ are introduced into this expression to obtain

$$\frac{1}{\mu\sigma}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\hat{A}}{dr}\right) - \frac{m^2}{r^2}\hat{A}\right] = j(\omega - m\Omega)\hat{A} \tag{4}$$

which becomes Eq. 6.5.9

$$\frac{d^2\hat{A}}{dr^2} + \frac{1}{r}\frac{d\hat{A}}{dr} - \left(\gamma^2 + \frac{m^2}{r^2}\right)\hat{A} = 0 \tag{5}$$

where

$$\gamma^2 \equiv j\mu\sigma(\omega - m\Omega)$$

Compare this to Eq. 2.16.19 and it is clear that the solution is the linear combination of $H_m(j\gamma r)$ and $J_m(j\gamma r)$ that make

Prob. 6.5.1 (cont.)

$$\hat{A}(\alpha) = \hat{A}^{\alpha} \qquad \hat{A}(\beta) = \hat{A}^{\beta} \quad .$$

This can be accomplished by writing two equations in the two uknown coefficients of H_m and J_m or by inspection as follows. The "answer" will look like

$$\hat{A}(r) = \hat{A}^{\alpha}\left[\frac{(\qquad)}{(\qquad)} H_m(j\gamma r) + \frac{(\qquad)}{(\qquad)} J_m(j\gamma r)\right] + \hat{A}^{\beta}\left[\frac{(\qquad)}{(\qquad)} H_m(j\gamma r) + \frac{(\qquad)}{(\qquad)} J_m(j\gamma r)\right] \tag{6}$$

The coefficients of the first term must be such that the combination multiplying $\hat{A}^{\alpha}$ vanishes where $r=\beta$ (because there, the answer cannot depend on $\hat{A}^{\alpha}$). To this end, make them $J_m(j\gamma\beta)$ and $H_m(j\gamma\beta)$ respectively. The denominator is then set to make the coefficient of $\hat{A}^{\alpha}$ unity where $r = \alpha$. Similar reasoning sets the coefficient of $\hat{A}^{\beta}$. The result is

$$\hat{A}(r) = \hat{A}^{\alpha}\frac{[H_m(j\gamma r)J_m(j\gamma\beta) - J_m(j\gamma r)H_m(j\gamma\beta)]}{[H_m(j\gamma\alpha)J_m(j\gamma\beta) - J_m(j\gamma\alpha)H_m(j\gamma\beta)]} + \hat{A}^{\beta}\frac{[H_m(j\gamma r)J_m(j\gamma\alpha) - J_m(j\gamma r)H_m(j\gamma\alpha)]}{[H_m(j\gamma\beta)J_m(j\gamma\alpha) - J_m(j\gamma\beta)H_m(j\gamma\alpha)]} \tag{7}$$

The tangential $\bar{H}$, $H_\theta = -(\partial A/\partial r)/\mu$ so it follows from Eq. 7 that

$$\hat{H}_\theta = -\frac{j\gamma}{\mu}\left\{\hat{A}^{\alpha}\frac{[H_m'(j\gamma r)J_m(j\gamma\beta) - J_m'(j\gamma r)H_m(j\gamma\beta)]}{[H_m(j\gamma\alpha)J_m(j\gamma\beta) - J_m(j\gamma\alpha)H_m(j\gamma\beta)]} + \hat{A}^{\beta}\frac{[H_m'(j\gamma r)J_m(j\gamma\alpha) - J_m'(j\gamma r)H_m(j\gamma\alpha)]}{[H_m(j\gamma\beta)J_m(j\gamma\alpha) - J_m(j\gamma\beta)H_m(j\gamma\alpha)]}\right\} \tag{8}$$

Prob. 6.5.1 (cont.)

Evaluation of this expression at $r = \alpha$ gives $\hat{H}_\theta^\alpha$

$$\hat{H}_\theta^\alpha = \frac{1}{\mu}\left\{ f_m(\beta,\alpha,\gamma)\hat{A}^\alpha + g_m(\alpha,\beta,\gamma)\hat{A}^\beta \right\} \tag{9}$$

where

$$f_m(\beta,\alpha,\gamma) = j\gamma \frac{[J_m'(j\gamma\alpha)H_m(j\gamma\beta) - H_m'(j\gamma\alpha)J_m(j\gamma\beta)]}{[H_m(j\gamma\alpha)J_m(j\gamma\beta) - J_m(j\gamma\alpha)H_m(j\gamma\beta)]}$$

and

$$g_m(\alpha,\beta,\gamma) = \frac{j}{\pi\alpha} \frac{[J_m'(j\gamma\alpha)H_m(j\gamma\alpha) - H_m'(j\gamma\alpha)J_m(j\gamma\alpha)]}{[H_m(j\gamma\beta)J_m(j\gamma\alpha) - J_m(j\gamma\beta)H_m(j\gamma\alpha)]}$$

Of course, Eq. 9 is the first of the desired transfer relations, the first of Eqs. (c) of Table 6.5.1. The second follows by evaluating Eq. 9 at $r = \beta$. Note that these definitions are consistent with those given in Table 2.16.2 with $k \rightarrow \gamma$. Because γ generally differs according to the region being described, it is included in the argument of the function.

To determine Eq. (d) of Table 6.5.1, these relations are inverted. For example, by Kramer's rule

$$F_m(\beta,\alpha,\gamma) = \frac{1}{\mu} \frac{f_m(\alpha,\beta,\gamma)}{\frac{1}{\mu^2}\left[f_m(\beta,\alpha,\gamma)f_m(\alpha,\beta,\gamma) - g_m(\beta,\alpha,\gamma)g_m(\alpha,\beta,\gamma)\right]} \tag{10}$$

Prob. 6.5.2 By way of establishing the representation, Eqs. g and h of Table 2.18.1 define the scalar component of the vector potential.

$$\bar{B} = -\frac{1}{r}\frac{\partial \Lambda}{\partial z}\bar{i}_r + \frac{1}{r}\frac{\partial \Lambda}{\partial r}\bar{i}_z \quad ; \quad \Lambda \equiv Ar \tag{1}$$

$$\bar{A} = \bar{i}_\theta \, A(r, z, t) \tag{2}$$

Thus, the θ component of Eq. 6.5.3 requires that (Appendix A)

$$\frac{1}{\mu\sigma}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rA)\right) + \frac{\partial^2 A}{\partial z^2}\right] = \frac{\partial A}{\partial t} + U\frac{\partial A}{\partial z} \tag{3}$$

In terms of the complex amplitude, this requires that

$$\frac{d^2\hat{A}}{dr^2} + \frac{1}{r}\frac{d\hat{A}}{dr} - \left(\gamma^2 + \frac{1}{r^2}\right)\hat{A} = 0 \tag{4}$$

where $\gamma^2 = k^2 + j(\omega - kU)\mu\sigma$. The solution to this expression satisfying the appropriate boundary conditions is Eq. 156.14.15. In view of Eq.1 ,

$$H_z = \frac{B_z}{\mu} = \frac{1}{\mu r}\frac{\partial \Lambda}{\partial r} \tag{5}$$

Observe from Eq. 2.16.26d (evaluated using m=0) that $uR_1' + R_1 = (uR_1)' = uR_0$ where R_m can be either J_m or H_m and the prime indicates a derivative with respect to the argument. Thus, with Eq. 6.5.15 used to evaluate Eq. 5, it follows that

$$H_z = \frac{j\gamma}{\mu}\left\{\hat{A}^\alpha \frac{[H_1(j\gamma\beta)J_0(j\gamma r) - J_1(j\gamma\beta)H_0(j\gamma r)]}{[H_1(j\gamma\beta)J_1(j\gamma\alpha) - J_1(j\gamma\beta)H_1(j\gamma\alpha)]} + \hat{A}^\beta \frac{[J_1(j\gamma\alpha)H_0(j\gamma r) - H_1(j\gamma\alpha)J_0(j\gamma r)]}{[J_1(j\gamma\alpha)H_1(j\gamma\beta) - H_1(j\gamma\alpha)J_1(j\gamma\beta)]}\right\} \tag{6}$$

Further, observe that(Eq. 2.16.26c) $J_1(j\gamma x) = -J_0'(j\gamma x)$ so, Eq. 6 becomes

$$H_z = -\frac{\gamma^2}{\mu}\left\{\frac{\hat{A}^\alpha}{j\gamma}\frac{[J_0'(j\gamma\beta)H_0(j\gamma r) - H_0'(j\gamma\beta)J_0(j\gamma r)]}{[H_0'(j\gamma\beta)J_0'(j\gamma\alpha) - J_0'(j\gamma\beta)H_0'(j\gamma\alpha)]} \right.$$
$$\left. \frac{\hat{A}^\beta}{j\gamma}\frac{[H_0'(j\gamma\alpha)J_0(j\gamma r) - J_0'(j\gamma\alpha)H_0(j\gamma r)]}{[J_0'(j\gamma\alpha)H_0'(j\gamma\beta) - H_0'(j\gamma\alpha)J_0'(j\gamma\beta)]}\right\} \tag{7}$$

This expression is evaluated at $r=\alpha$ and $r=\beta$ respectively to obtain the equations e of Table 6.5.1. Because Eqs. e and f take the same form as Eqs. b and a respectively of Table 2.16.2, the inversion to obtain Eqs. f has already been shown.

Prob. 6.6.1 For the pure traveling wave, Eq. 6.7.7 reduces to

$$\langle S_d \rangle_{yt} = -\tfrac{1}{2}(\omega - kU)\,\mathrm{Re}\, j\left[\hat{A}^b(\hat{H}_y^b)^* - \hat{A}^c(\hat{H}_y^c)^*\right] \tag{1}$$

The boundary condition represented by Eq. 6.6.3 makes the second term zero while Eq. 6.6.5b shows that the remaining expression can also be written as

$$\langle S_d \rangle_{yt} = -\tfrac{1}{2}(\omega - kU)\,\mathrm{Re}\left\{\frac{j\mu_o}{k}\left[\frac{\hat{K}_+^a}{\sinh kd} + \coth kd\,\hat{H}_y^b\right]\right\}(\hat{H}_y^b)^* \tag{2}$$

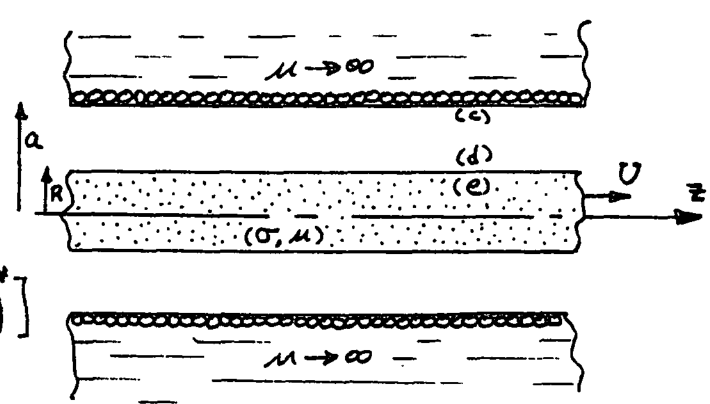
The "self" term therefore makes no contribution. The remaining term is evaluated by using Eq. 6.6.9.

$$\langle S_d \rangle_{yt} = \tfrac{1}{2}(\omega - kU)\frac{\mu_o}{k}\frac{|\hat{K}_+^a|^2}{\sinh^2 kd}\,\mathrm{Re}\frac{j}{\left[\frac{k}{\gamma^*}\frac{\mu}{\mu_o}\coth\gamma^* a + \coth kd\right]} \tag{3}$$

Prob. 6.6.2 (a) To obtain the drive in terms of complex amplitudes, write the cosines in complex form and group terms as forward and backward traveling waves. It follows that

$$\hat{K}_\pm^a = \hat{i}_a\frac{N_a}{2} + \hat{i}_b\frac{N_b}{2}e^{j\frac{2\pi}{3}} + \hat{i}_c\frac{N_c}{2}e^{j\frac{4\pi}{3}} \tag{1}$$

To determine the time average force, the rod is enclosed by a circular cylindrical surface having radius R and axial length ℓ . Boundary locations are as indicated in the diagram. Using the theorem of Eq. 5.16.4, it follows that



$$\langle f_z \rangle_t = 2\pi R\ell\langle B_r^d H_z^d\rangle_{zt} \tag{2}$$

$$= \pi R\ell\,\mathrm{Re}\left[\hat{B}_{r+}^d(\hat{H}_{z+}^d)^* - \hat{B}_{r-}^d(\hat{H}_{z-}^d)^*\right]$$

With the use of Eqs. (e) from Table 2.19.1 to represent the air-gap fields

Prob. 6.6.2 (cont.)

the "self" terms are dropped and Eq. (2) becomes

$$\langle f_z \rangle_t = \frac{\mu_0 \pi R l}{k} Re\left[j g_0(R,a,k)(\hat{H}^d_{z+})^* - j g_0(R,a,-k)(\hat{H}^d_{z-})^* \right] \tag{3}$$

So, $\hat{H}^d_{z\pm}$ is desired. To this end observe that boundary and jump conditions are

$$\hat{H}^c_z = \hat{K}^a \tag{4}$$

$$\hat{H}^d_z = \hat{H}^e_z \tag{5}$$

$$\hat{A}^d = \hat{A}^e \Rightarrow \hat{\Lambda}^d = \hat{\Lambda}^e \tag{6}$$

It follows from Eqs. (f) of Table 6.5.1 applied to the air-gap and to the rod that

$$\frac{\hat{\Lambda}^e}{R} = -\frac{\mu}{\gamma^2} f_0(0,R,\gamma)\hat{H}^e_z = -\frac{\mu_0}{k^2} g_0(R,a,k)\hat{K}^a - \frac{\mu_0}{k^2} f_0(a,R,k)\hat{H}^d_z \tag{7}$$

Hence,

$$\hat{H}^d_{z\pm} = \hat{H}^e_{z\pm} = \frac{-g_0(R,a,\pm k)\hat{K}^a}{f_0(a,R,\pm k) - \frac{k^2}{\gamma_\pm^2}\frac{\mu}{\mu_0} f_0(0,R,\pm\gamma)} \; ; \gamma \equiv \sqrt{k^2 + j\mu\sigma(\omega \mp kU)} \tag{8}$$

<u>Prob. 6.6.3</u> The Fourier transform of the excitation surface current is

$$\hat{K}^a = \hat{K}_0 \frac{e^{j(k-\beta)l} - 1}{j(k-\beta)} = \frac{2\hat{K}_0 e^{j\frac{(k-\beta)l}{2}}}{k-\beta} \sin\left[\frac{(k-\beta)l}{2}\right] \tag{1}$$

In terms of the Fourier transforms, Eq. 5.16.8 shows that the total force is

$$\langle f_y \rangle_t = \frac{w}{4\pi} Re \int_{-\infty}^{+\infty} (\hat{B}^b_x)^* \hat{H}^b_y \, dk \tag{2}$$

In view of Eq. 6.6.5b, this expression becomes

$$\langle f_y \rangle_t = \frac{w}{4\pi} Re \int_{-\infty}^{+\infty} \mu_0 j \frac{(\hat{K}^a)^*}{\sinh kd} \hat{H}^b_y \, dk \tag{3}$$

where the term in $\hat{H}^b_y(\hat{H}^b_y)^*$ has been eliminated by taking the real part. With the use of Eq. 6.6.9, this expression becomes

$$\langle f_y \rangle_t = \frac{-w\mu_0}{4\pi} Re \int_{-\infty}^{+\infty} \frac{j\,|\hat{K}^a|^2 dk}{\sinh^2 kd\left[\frac{k}{\gamma}\frac{\mu}{\mu_0}\coth\gamma a + \coth kd\right]} \tag{4}$$

With the further substitution of Eq. 1, the expression stated with the problem is found.

Prob. 6.7.1 It follows from Eq. 6.7.7 that the power dissipation (per unit y-z area) is

$$P_d \equiv \langle S_d \rangle_{yt} = -\frac{1}{2}(\omega - kU)\,Re\,j\left[\hat{A}^\alpha (\hat{H}_y^\alpha)^* - \hat{A}^\beta (\hat{H}_y^\beta)^*\right] \tag{1}$$

The time average mechanical power output (again per unit y-z area) is the product of the velocity U and the difference in magnetic shear stress acting on the respective surfaces

$$P_m = \frac{1}{2}\,Re\left[\hat{B}_x^\alpha (\hat{H}_y^\alpha)^* - \hat{B}_x^\beta (\hat{H}_y^\beta)^*\right] U \tag{2}$$

Because $\hat{B}_x = -jk\hat{A}$, this expression can be written in terms of the same combination of amplitudes as appears in Eq. 1

$$P_m = -\frac{k}{2}\,Re\,j\left[\hat{A}^\alpha (\hat{H}_y^\alpha)^* - \hat{A}^\beta (\hat{H}_y^\beta)^*\right] \tag{3}$$

Thus, it follows from Eqs. 1 and 3 that

$$E_{ff} \equiv \frac{P_m}{P_m + P_d} = \frac{U}{(\omega/k)} \tag{4}$$

From the definition of s,

$$\frac{U}{(\omega/k)} = 1 - s \tag{5}$$

so that

$$E_{ff} = 1 - s \tag{6}$$

Prob. 6.7.2

The time average and space average power dissipation per unit y-z area is given by Eq. 6.7.7. For this example n=1 and

$$\langle S_d \rangle_{yt} = -\mathrm{Re}\, j\frac{(\omega - kU)}{2} \hat{A}^b (\hat{H}_y^b)^* \tag{1}$$

$$= \mathrm{Re}\, j\frac{(\omega - kU)}{2} (\hat{A}^b)^* \hat{H}_y^b$$

because $\hat{H}_y^\beta = \hat{H}_y^d = 0.$

From Eq. 6.5.5b

$$\langle S_d \rangle_{yt} = \mathrm{Re}\, j\frac{(\omega - kU)}{2}\frac{\mu_o}{k}\left[\frac{\hat{K}^{a*}}{\sinh kd}\hat{H}_y^b\right] \tag{2}$$

where, in expressing $\hat{A}^b$, the term in $\hat{H}_y^b$ has been dropped because the real part is taken.

In view of Eq. 6.6.9, this expression becomes

$$\langle S_d \rangle_{yt} = -\mathrm{Re}\, j\frac{(\omega - kU)}{2}\frac{\mu_o}{k}\frac{|\hat{K}_+^a|^2}{\sinh^2 kd\left[\frac{k}{\gamma}\frac{\mu}{\mu_o}\coth\gamma a + \coth kd\right]} \tag{3}$$

Note that it is only because $\gamma \equiv \sqrt{(ka)^2 + jS_m}/a$ is complex that this function has a non-zero value.

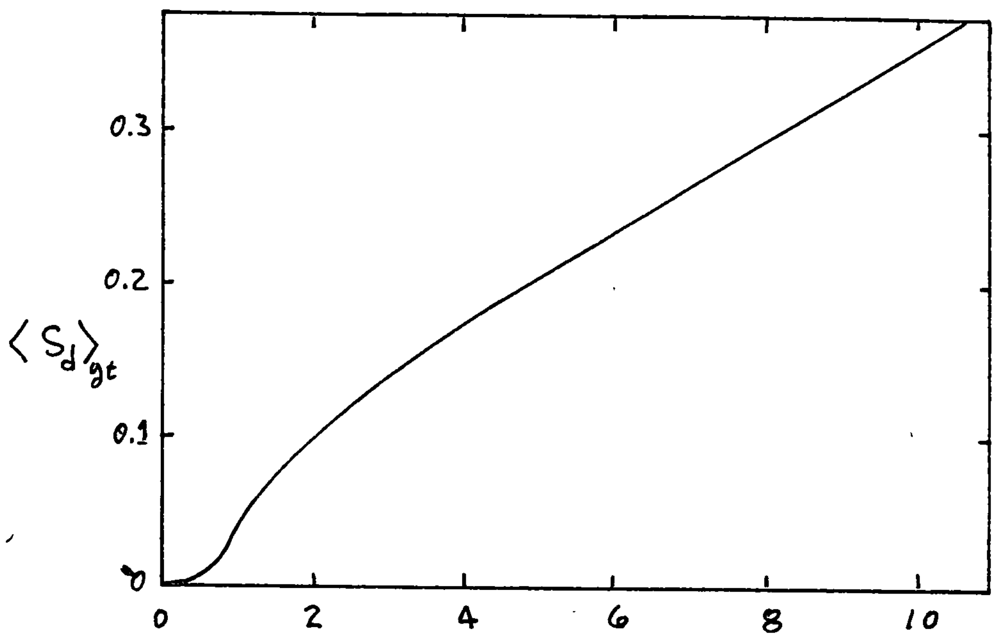
In terms of $S_m \equiv \mu\sigma a^2(\omega - kU)$

$$\langle S_d \rangle_{yt} = -\mathrm{Re}\frac{S_m \mu_o}{2\mu\sigma a^2}\frac{|\hat{K}_+^a|^2}{k\sinh^2 kd}\left\{\frac{j}{\left[\frac{k}{\gamma}\frac{\mu}{\mu_o}\coth\gamma a + \coth kd\right]}\right\} \tag{4}$$

Note that the term in $\{\ \}$ is the same function as represents the S_m dependence of the time average force/unit area, Fig. 6.6.2. Thus, the dependence

Prob. 6.7.2 (cont.)

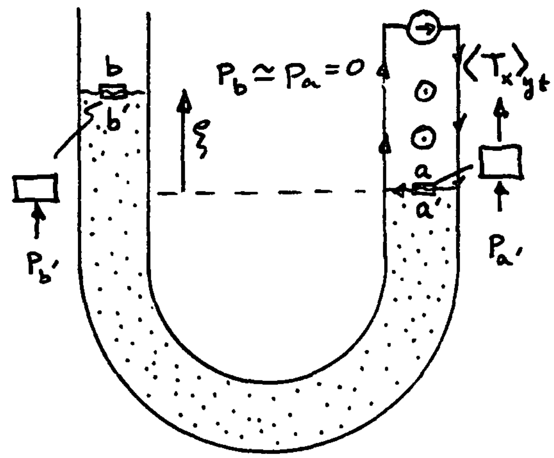
of $\langle S_d \rangle_{yt}$ on S_m is the function shown in that figure multiplied by S_m .



Prob. 6.8.1 Equations 6.8.10 and 6.8.11 are directly applicable. The skin depth is short, so $\hat{H}_y^\beta$ is negligible. Elimination of $\hat{H}_y^\alpha$ between the two expressions gives

$$\langle T_x \rangle_{yt} = -\frac{\mu_o}{4}(2\sigma\delta)\langle S_d \rangle_{yt} = -\sqrt{\frac{\mu_o\sigma}{2\omega}}\,\langle S_d \rangle_{yt} \tag{1}$$

where $\langle S_d \rangle_{yt}$ is the time average power dissipated per unit area of the interface. Force equilibrium at the interfaces can be pictured from the control volumes shown.



$$P_{b'} = 0 \tag{2}$$

$$\langle T_x \rangle_{yt} + P_{a'} = 0 \tag{3}$$

Bernoulli's equation relates the pressures at the interfaces inside the liquid.

$$P_{a'} = P_{b'} + \rho g \xi \tag{4}$$

Elimination of the p's between these last three expressions then gives

$$\langle T_x \rangle_{yt} = -\rho g \xi \tag{5}$$

So, in terms of the power dissipation as given by Eq. 1, the "head" is

$$\xi = \frac{1}{\rho g}\sqrt{\frac{\mu_o\sigma}{2\omega}}\,\langle S_d \rangle_{yt} \tag{6}$$

Prob. 6.9.1

With

$$\xi = \frac{x}{2}\sqrt{\frac{\mu\sigma}{t'}} \tag{1}$$

$$\frac{\partial}{\partial t'} f(\xi) = \frac{df}{d\xi}\frac{\partial \xi}{\partial t'} = -\frac{x}{4}\sqrt{\mu\sigma}\, t^{-\frac{3}{2}} \frac{df}{d\xi} \tag{2}$$

and

$$\frac{\partial f}{\partial x} = \frac{df}{d\xi}\frac{\partial \xi}{\partial x} = \frac{1}{2}\sqrt{\mu\sigma}\, t'^{-\frac{1}{2}} \frac{df}{d\xi} \tag{3}$$

Taking this latter derivative again gives

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{2}\sqrt{\mu\sigma}\, t'^{-\frac{1}{2}} \frac{d^2 f}{d\xi^2}\frac{\partial \xi}{\partial x} = \frac{1}{4}\mu\sigma t^{-1} \frac{d^2 f}{d\xi^2} \tag{4}$$

Thus, Eq. 6.9.3 becomes

$$\frac{1}{\mu\sigma} t'^{-1} \frac{d^2 H_y}{d\xi^2} = -x\sqrt{\mu\sigma}\, t'^{-\frac{3}{2}} \frac{dH_y}{d\xi} \tag{5}$$

or,

$$\frac{d^2 H_y}{d\xi^2} + 2\frac{x}{2}\sqrt{\frac{\mu\sigma}{t'}} \frac{dH_y}{d\xi} = 0 \tag{6}$$

In view of the definition of ξ, Eq. 1, this expression is the same as Eq. 6.9.7.

Prob. 6.9.2 (a) The field in the liquid metal is approximated by Eq. 6.9.1 with U=0. Thus, the field is computed as though it had no y dependence and is simply

$$H_y = \mathcal{R}e\, \hat{H}_y\, e^{\frac{x}{\delta}} e^{j(\omega t + \frac{x}{\delta})} \tag{1}$$

The amplitude of this field is a slowly varying function of y, however, given by the fact that the flux is essentially trapped in the air-gap. Thus, $\hat{H}_y = a\hat{H}_o/h$ and Eq. (1) becomes

$$H_y = \mathcal{R}e\, \frac{a\hat{H}_o}{h}\, e^{\frac{x}{\delta}} e^{j(\omega t + \frac{x}{\delta})} \tag{2}$$

(b) Gauss' Law can now be used to find H_x. First, observe from Eq. (2) that

$$\frac{\partial H_x}{\partial x} = -\frac{\partial H_y}{\partial y} = \mathcal{R}e\, \frac{a\hat{H}_o}{h^2}\frac{dh}{dy}\, e^{\frac{x}{\delta}} e^{j(\omega t + \frac{x}{\delta})} \tag{3}$$

Then, integration gives H_x

$$H_x = \mathcal{R}e\, \frac{a\hat{H}_o\delta}{1+j}\frac{1}{h^2}\frac{dh}{dy}\, e^{\frac{x}{\delta}} e^{j(\omega t - \frac{x}{\delta})} \tag{4}$$

The integration constant is zero because the field must vanish as $x \to -\infty$.

(c) The time-average shearing surface force density is found by integrating the Maxwell stress tensor over a pill box enclosing the complete skin region.

$$\langle T_y \rangle_t = \frac{1}{2}\mathcal{R}e\,\mu_o \hat{H}_x \hat{H}_y^*\Big|_{x=0} = \frac{\mu_o}{4} a^2 |\hat{H}_o|^2 \delta \frac{dh}{dy} \tag{5}$$

As would be expected, this surface force density goes to zero as either the skin depth or the slope of the electrode vanish.

(d) If Eq. 5 is to be independent of y,

$$\frac{1}{h^3}\frac{dh}{dy} = \text{constant} = \frac{S}{a^3} \tag{6}$$

Integration follows by multiplying by dy

$$\int_a^h \frac{dh}{h^3} = \int_0^y \frac{S}{a^3}\, dy$$

and the given distribution h(y) follows.

Prob. 6.9.2(cont.)

(e) Evaluated using h(y), Eq. 6 becomes

$$\langle T_y \rangle_t = \frac{\mu_o}{4} |\hat{H}_o|^2 \frac{\delta}{a} S \tag{8}$$

Prob. 6.9.3 From Eq. 6.8.11, the power dissipated per unit area is (there is no β surface)

$$\langle S_d \rangle_{yt} = \frac{1}{2\sigma\delta'} |\hat{H}_y^d|^2 \tag{1}$$

where

$$\delta' \rightarrow \sqrt{\frac{2}{|\omega|\mu\sigma}}$$

Thus, Eq. 2 of Prob. 6.9.2 can be exploited to write $H_y(x=0)$ in Eq. 1 as

$$\langle S_d \rangle_{yt} = \frac{1}{2\sigma\delta} |\hat{H}_o|^2 [1 + 2\, S(\tfrac{y}{a})] \tag{2}$$

The total power dissipation per unit depth in the z direction is

$$\int_0^\ell \langle S_d \rangle_{yt} dy = \frac{|\hat{H}_o|^2}{2\sigma\delta} \int_0^\ell [1 - 2\, S(\tfrac{y}{a})] dy = \frac{|\hat{H}_o|^2 \ell}{2\sigma\delta}(1 - \frac{S\ell}{a}) \tag{3}$$

Prob. 6.9.4 Because $\bar{J}_f' = \bar{J}_f$ and $\bar{J}_f' = \sigma \bar{E}'$, the power dissipation per unit y-z area is

$$S_d = \int_{-\infty}^{0} \bar{E}' \cdot \bar{J}_f' \, dx = \int_{-\infty}^{0} \frac{\bar{J}_f \cdot \bar{J}_f}{\sigma} \, dx \tag{1}$$

In the "boundary-layer" approximation, the z component of Ampere's law becomes

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \simeq \frac{\partial H_y}{\partial x} = J_z \tag{2}$$

So that the dissipation density is

$$\frac{J_z^2}{\sigma} \simeq \frac{1}{\sigma} \left(\frac{\partial H_y}{\partial x} \right)^2 \tag{3}$$

In view of Eq. 6.9.8,

$$\begin{aligned} \frac{J_z^2}{\sigma} &= \frac{H_o^2}{\sigma} \left[\frac{\partial}{\partial x} \operatorname{erf}(\xi) \right]^2 = \frac{H_o^2}{\sigma} \left(\frac{2 e^{-\xi^2}}{\sqrt{\pi}} \right)^2 \left(\frac{\partial \xi}{\partial x} \right)^2 \\ &= H_o^2 \frac{\mu}{t' \pi} e^{-2\xi^2} \end{aligned} \tag{4}$$

Note that the only x dependence is now through ξ. Thus,

$$S_d = \frac{\mu H_o^2}{\pi t'} \int_{-\infty}^{0} e^{-2\xi^2} dx = \frac{2\sqrt{2}}{\pi} \frac{\mu H_o^2}{t'} \sqrt{\frac{t'}{\mu\sigma}} \int_{-\infty}^{0} e^{-2\xi^2} d(\xi\sqrt{2}) \tag{5}$$

$$= \sqrt{\frac{2}{\pi}} \frac{\mu H_o^2}{\sqrt{t'\mu\sigma}} \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-(\sqrt{2}\xi)^2} d(\sqrt{2}\xi) = \frac{\mu H_o^2 \sqrt{2}}{\sqrt{\pi \mu \sigma t'}}$$

Prob. 6.9.4 (cont.)

So, for $y > Ut$ where $t' = y/U$

$$S_d = \begin{cases} \dfrac{\mu H_o^2 \sqrt{2}}{\sqrt{\pi \mu \sigma t}} & ; \; y > Ut \\[2ex] \dfrac{\mu H_o^2 \sqrt{2}}{\sqrt{\dfrac{\pi \mu \sigma y}{U}}} & ; \; 0 < y < Ut \end{cases} \tag{6}$$

For $Ut < L$ the total power per unit length in the z direction is

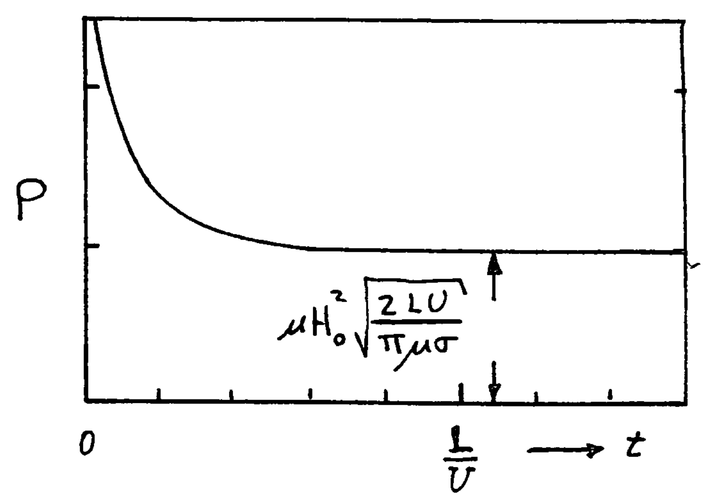
$$P = \int_0^{Ut} \frac{\sqrt{2}\, \mu H_o^2}{\sqrt{\dfrac{\pi \mu \sigma y}{U}}} + \int_{Ut}^{L} \frac{\sqrt{2}\, \mu H_o^2}{\sqrt{\pi \mu \sigma t}}\, dy \tag{7}$$

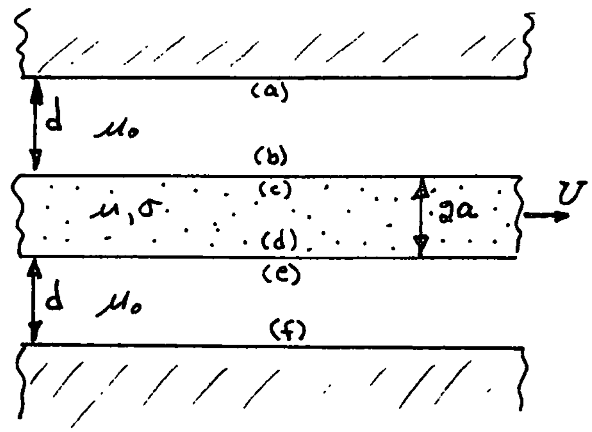
and this becomes

$$P = \frac{\sqrt{2}\, \mu H_o^2}{\sqrt{\pi \mu \sigma}} \left[2\sqrt{U}\sqrt{Ut} + \frac{1}{\sqrt{t}}(L - Ut) \right] \tag{8}$$

$$= \frac{\sqrt{2}\, \mu H_o^2}{\sqrt{\pi \mu \sigma}} \left[U\sqrt{t} + L/\sqrt{t} \right]$$

The time dependence of the total force is therefore as shown in the sketch.





Prob. 6.10.1 Boundary conditions for the eigenmodes are homogeneous. In terms of the designations shown in the sketch,

$$\hat{H}_y^a = 0 \tag{1}$$

$$\hat{H}_y^b = \hat{H}_y^c \tag{2}$$

$$\hat{A}^b = \hat{A}^c \tag{3}$$

$$\hat{H}_y^d = \hat{H}_y^e \tag{4}$$

$$\hat{A}^d = \hat{A}^e$$

(5)

(6)

The bulk conditions are conveniently written with these conditions incorporated from the outset. In all three regions they are as given by Eq. (b) of Table 6.5.1 with suitable identification of properties and dimensions. In the upper air gap, it is the second equation that is required.

$$\hat{A}^b = \frac{\mu_o}{k} \coth kd \; \hat{H}_y^b \tag{7}$$

For the slab

$$\begin{bmatrix} \hat{A}^b \\ \hat{A}^e \end{bmatrix} = \frac{\mu}{\gamma} \begin{bmatrix} -\coth 2\gamma a & \dfrac{1}{\sinh 2\gamma a} \\ \dfrac{-1}{\sinh 2\gamma a} & \coth 2\gamma a \end{bmatrix} \begin{bmatrix} \hat{H}_y^b \\ \hat{H}_y^e \end{bmatrix} \tag{8}$$

while for the lower gap it is the first equation that applies

$$\hat{A}^e = -\frac{\mu_o}{k} \coth kd \; \hat{H}_y^e \tag{9}$$

Now, with Eqs. 7 and 9 used to evaluate Eq. 8, it follows that

$$\begin{bmatrix} -\dfrac{\mu_o}{k} \coth kd - \dfrac{\mu}{\gamma} \coth 2\gamma a & \dfrac{\mu}{\gamma} \dfrac{1}{\sinh 2\gamma a} \\ -\dfrac{\mu}{\gamma} \dfrac{1}{\sinh 2\gamma a} & \dfrac{\mu_o}{\gamma} \coth kd + \dfrac{\mu}{\gamma} \coth 2\gamma a \end{bmatrix} \begin{bmatrix} \hat{H}_y^b \\ \hat{H}_y^e \end{bmatrix} = 0 \tag{10}$$

Note that both of these equations are satisfied if $H_y^b = H_y^e$ so that

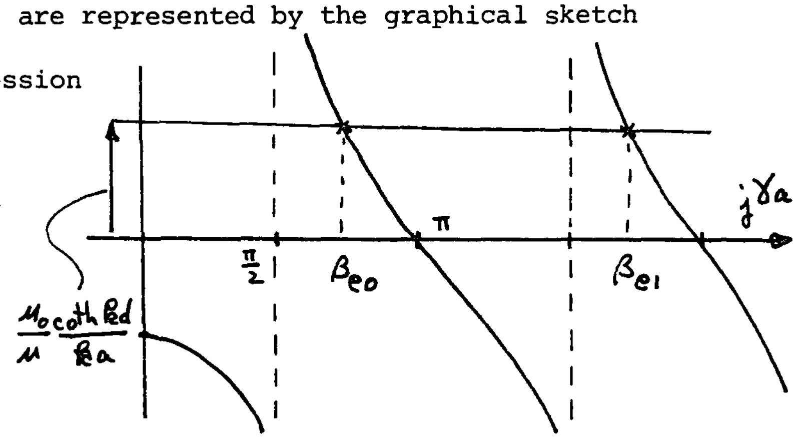
Prob. 6.10.1(cont.)

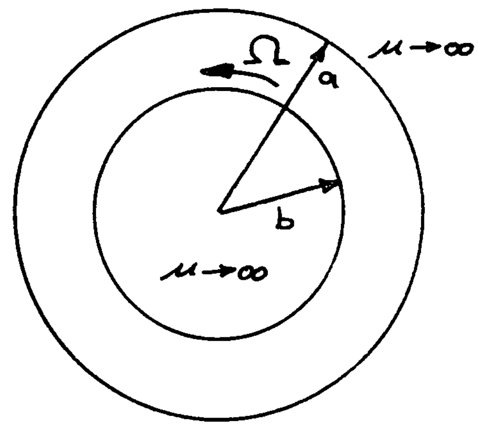
$$-\frac{\mu_o}{k}\coth kd - \frac{\mu}{\gamma}\left(\coth 2\gamma a \mp \frac{1}{\sinh 2\gamma a}\right) = 0 \tag{11}$$

with the upper sign applying. Similarly, if $H_y^b = -H_y^e$, both expressions are satisfied and Eq. 11 is found with the lower sign applying. In this way, it has been shown that the eigenvalue equation that would be obtained by setting the determinant of the coefficients in Eq. 10 equal to zero can be factored into expressions that are given by Eq. 11. Further, it is seen that the roots given by these factors can respectively be identified with the even and odd modes. By using the identity $(\cosh x - 1)/\sinh x = \tanh(x/2)$ and $(\cosh x + 1)/\sinh x = \coth(x/2)$ it follows that the eigenvalue equations can be written as

$$\frac{\mu_o}{\mu}\frac{\coth kd}{ka} = \begin{cases} -\dfrac{\tan j\gamma a}{j\gamma a}; & \text{even} \\ \dfrac{\cot j\gamma a}{j\gamma a}; & \text{odd} \end{cases} \tag{12}$$

so that the expression for the odd solutions is the same as Eq. 6.10.1 with roots given by the graphical solution of Fig. 6.10.2 and eigenfrequencies given by Eq. 6.10.7. The even solutions are represented by the graphical sketch shown. The roots of this expression can be used in Eq. 6.10.7 to obtain the eigenfrequencies for these modes. Note that the dominant mode is odd, as would be expected for the tangential magnetic field associated with a current tending to be uniform over the sheet cross-section.





Prob. 6.10.2 (a) In Eq. (d) of Table 6.5.1, $\hat{H}_\theta^\alpha$ and $\hat{H}_\theta^\beta$ are zero so the determinant of the coefficients is zero. But, the resulting expression can be written out and then factored using the identity footnote to Table 2.16.2. This is the common denominator of the coefficients in the inverse matrix, Eq. (c) of that table. Thus, the required equation is (see Table 2.16.2 for denominators of f_m and g_m to which the determinant is proportional).

$$J_m(j\gamma a)H_m(j\gamma b) - J_m(j\gamma b)H_m(j\gamma a) = 0 \tag{1}$$

This can be written, using the recommended dimensionless parameters, and the definition of H_m in terms of N_m (Eq. 2.16.29) as

$$J_m[j(\gamma a)]\,N_m[j(\gamma a)\lambda] - J_m[j(\gamma a)\lambda]\,N_m[j(\gamma a)] = 0 \tag{2}$$

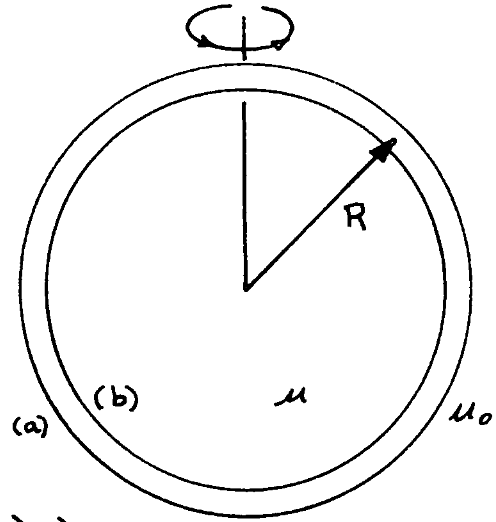
where $\lambda \equiv b/a$ ranges from 0 to 1 and $\gamma a \equiv \sqrt{j\mu\sigma a^2(\omega - m\Omega)}$.

(b) Given $\lambda \equiv b/a$ and the azimuthal wavenumber, m, Eq. 2 is a transcendental equation for the eigenvalues $\gamma a \equiv (\gamma a)_{mn}$ (which turn out to be real). The eigenfrequencies then follow. an

$$\omega_{mn} = m\Omega - j\frac{(\gamma a)_{mn}^2}{\mu\sigma a^2} \tag{3}$$

For example, for m=0 and 1, the roots to Eq. 2 are tabulated (Abramowitz, M. and Stegun, I.A., Handbook of Mathematical Functions, (National Bureau of Standards Applied Math Series, 1964) p. 415.) However, to make use of their tabulation, the eigenvalue should be made γb and the expression written as

$$J_m(j\gamma b)\,N_m\left[(j\gamma b)\tfrac{a}{b}\right] - J_m\left[(j\gamma b)\tfrac{a}{b}\right]N_m(j\gamma b) = 0 \tag{4}$$



Prob. 6.10.3 Solutions are of form

$$\psi = \mathrm{Re}\,\hat{\psi}(r)\,P_n^m \exp j(\omega t - m\phi)$$

(a) The first boundary condition is Eq. d, Table 6.3.1

$$\left(\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\sin\theta + \frac{\partial^2}{\partial\phi^2}\right)[\![H_\phi]\!]$$
$$= -\sigma_s R\sin\theta\frac{\partial}{\partial\phi}\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\phi}\right)B_r^a \qquad (2)$$

With the substitution of the assumed form and $\hat{H}_\phi = jm\hat{\psi}/r\sin\theta$

$$\frac{jm}{R}(\hat{\psi}^a - \hat{\psi}^b)\left[\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}\sin\theta - m^2\right]\frac{P_n^m(\cos\theta)}{\sin\theta} \qquad (3)$$
$$= -\sigma_s R\sin\theta\, m(\omega - m\Omega)\hat{B}_r^a P_n^m(\cos\theta)$$

In view of Eq. 2.16.31a, this becomes

$$-\frac{jm}{R}(\hat{\psi}^a - \hat{\psi}^b)n(n+1) = -\sigma_s R\hat{B}_r^a \qquad (4)$$

The second boundary condition is

$$\hat{B}_r^a = \hat{B}_r^b \qquad (5)$$

Prob. 6.10.3 (cont.)

Bulk relations are (Eq. (d) of Table 2.16.3)

$$\hat{B}_r^a = \frac{\mu_o(n+1)}{R}\hat{\psi}^b \tag{6}$$

for the exterior region and (Eq. (c) of Table 2.16.3)

$$\hat{B}_r^b = -\frac{\mu n}{R}\hat{\psi}^b \tag{7}$$

for the interior region.

These last three expressions, substituted into Eq. 4, then give

$$-\frac{jm}{R}n(n+1)\left[\frac{R}{\mu_o(n+1)} + \frac{R}{\mu n}\right]\hat{B}_r^a = -\sigma_s R m(\omega - m\Omega)\hat{B}_r^a \tag{8}$$

Thus, the desired eigenfrequency expression requires that the coefficients of $\hat{B}_r^a$ be zero. Solved for ω, this gives.

$$\omega = m\Omega + \frac{j}{\sigma_s R\mu_o}\left[n + \frac{(n+1)}{\mu/\mu_o}\right] \tag{9}$$

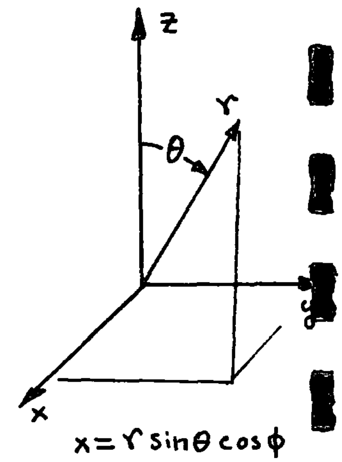
(b) A uniform field in the z direction superimposes on the homogeneous solution a field $\psi = -H_o z = -H_o r\cos\theta$. This has the same θ dependence as the mode m=0, n=1. Thus the mode necessary to satisfy the initial condition is (m,n) = (0,1) (Table 2.16.2) and the eigenfrequency is

$$\omega_{01} = \frac{j}{\sigma_s R\mu_o}\left(1 + \frac{2\mu_o}{\mu}\right) \tag{10}$$

Prob. 6.10.3 (cont.)

The response is a pure decay because there is no dependence of the excitation on the direction of rotation.

(c) With the initial field uniform perpendicular to the z axis there is a ϕ dependence.



$$\psi = -H_o x = -H_o r \sin\theta \cos\phi$$

This is the θ-ϕ dependence of the n=1, m=1 mode (Table 2.16.2).

So

$$\omega_{11} = \Omega + \frac{j}{\sigma_s R \mu_o}\left(1 + \frac{2\mu_o}{\mu}\right) \qquad (11)$$

The decay rate is the same as before, but because the dipole field is now rotating, there is a real part.

Prob. 6.10.4 (a) The temporal modes exist even if the excitation is turned off. Hence, the denominator of Eq. 8 from Prob. 6.6.2 must vanish.

$$\frac{\mu_o}{\mu}\frac{f_o(a,R,k)}{k^2} = \frac{f_o(0,R,\gamma)}{\gamma^2} \qquad (1)$$

(b) It is convenient to group

$$j\mu\sigma(\omega - kU) = S_n \qquad (2)$$

Finding the roots S_n to Eq. 1 is tantamount to finding the desired eigen-frequencies because it then follows from Eq. 2 that

$$\omega_n = \frac{S_n}{j\mu\sigma} + kU \qquad (3)$$

Note that for S_n real both sides of Eq. 1 are real. Thus, a graphical procedure can be used to find these roots.

Prob. 6.10.5 Even with nonuniform conductivity and velocity, Eq. 6.5.3 describes the vector potential. For the z component it follows that

$$\frac{1}{\mu\sigma}\nabla^2 A = \frac{\partial A}{\partial t} + U\frac{\partial A}{\partial y} \tag{1}$$

Thus, the complex amplitude satisfies the equation

$$\frac{d^2 A}{dx^2} - \gamma^2 A = 0\ ;\ \gamma^2(x) \equiv k^2 + j\mu\sigma(x)\left[\omega - kU(x)\right] \tag{2}$$

On the infinitely permeable walls, $H_y=0$ and so

$$\frac{dA}{dx}(\ell) = 0\ ;\ \frac{dA}{dx}(0) = 0 \tag{3}$$

Because Eq. 1 applies over the entire interval $0 < x < a+d \equiv \ell$, there is no need to use a piece-wise continuous representation. Multiply Eq. 2 by another eigenmode, $\hat{A}_m$, and integrate by parts to obtain

$$\hat{A}_m \frac{d\hat{A}_n}{dx}\Big|_0^\ell - \int_0^\ell \left(\frac{d\hat{A}_m}{dx}\frac{d\hat{A}_n}{dx} + \gamma_n^2 \hat{A}_m \hat{A}_n\right) dx = 0 \tag{4}$$

With the roles of m and n reversed, these same steps are carried out and the result subtracted from Eq. 4.

$$\left[\hat{A}_m \frac{d\hat{A}_n}{dx} - \hat{A}_n \frac{d\hat{A}_m}{dx}\right]_0^\ell - \int_0^\ell (\gamma_n^2 - \gamma_m^2)\hat{A}_m \hat{A}_n\, dx = 0 \tag{5}$$

Note that by definition, $\gamma_n^2 - \gamma_m^2 = j\mu\sigma(\omega_n - \omega_m)$

In view of the boundary conditions applying at x=0 and x= ℓ , Eq.   , the required orthogonality condition follows.

$$(\omega_n - \omega_m)\int_0^\ell \sigma(x)\hat{A}_m \hat{A}_n\, dx \tag{6}$$