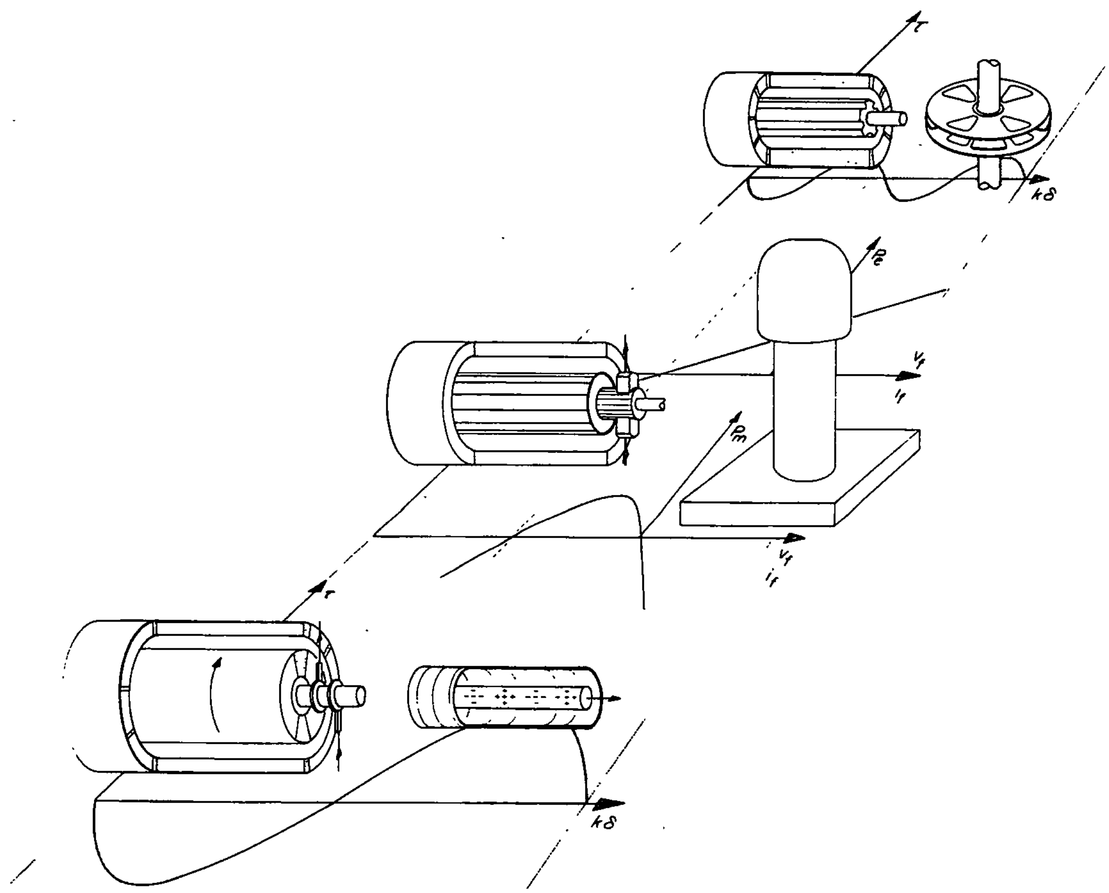
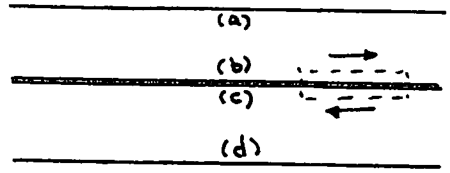
4

Electromechanical Kinematics: Energy-Conversion Models and Processes





Prob. 4.3.1 With the positions as shown in the sketch, the required force is

$$f_z = \frac{A}{2}\,\mathrm{Re}\,\tilde{B}_x^b\left[\tilde{H}_z^b - \tilde{H}_z^c\right]^* \tag{1}$$

With the objective of finding $\tilde{B}_x^b$, first observe that the boundary conditions are.

$$\tilde{H}_z^a = \tilde{K}^s \;;\; -\tilde{H}_z^b + \tilde{H}_z^c = \tilde{K}^r \;;\; \tilde{B}_x^b = \tilde{B}_x^c \;;\; -\tilde{H}_z^d = \tilde{K}^s \tag{2}$$

and the transfer relations of Table 2.16.1 applied to the respective regions require that

$$\begin{bmatrix} \tilde{B}_x^a \\ \tilde{B}_x^b \end{bmatrix} = \mu_0 k \begin{bmatrix} -\coth kd & \dfrac{1}{\sinh kd} \\ \dfrac{-1}{\sinh kd} & \coth kd \end{bmatrix} \begin{bmatrix} \dfrac{\tilde{K}^s}{jk} \\ \dfrac{\tilde{H}_z^b}{jk} \end{bmatrix} ; \quad \begin{bmatrix} \tilde{B}_x^c \\ \tilde{B}_x^d \end{bmatrix} = \mu_0 k \begin{bmatrix} -\coth kd & \dfrac{1}{\sinh kd} \\ \dfrac{-1}{\sinh kd} & \coth kd \end{bmatrix} \begin{bmatrix} \dfrac{\tilde{H}_z^c}{jk} \\ \dfrac{-\tilde{K}^s}{jk} \end{bmatrix} \tag{3}$$

Here, Eqs. 2a and 2d have already been used, as has also the relation $\tilde{H}_z = jk\tilde{\Psi}$

In view of Eq. 2c, Eqs. 3 are used to write

$$\tilde{B}_x^b = \mu_0 k\left[\frac{-\tilde{K}^s}{jk\sinh kd} + \frac{\tilde{H}_z^b \coth kd}{jk}\right] = \tilde{B}_x^c = \mu_0 k\left[-\frac{\tilde{H}_z^c \coth kd}{jk} - \frac{\tilde{K}^s}{jk \sinh kd}\right] \tag{4}$$

and it is concluded that

$$\tilde{H}_z^c = -\tilde{H}_z^b \tag{5}$$

This relation could be argued from the symmetry. In view of Eq. 2b, it follows that

$$\tilde{H}_z^b = -\frac{\tilde{K}^r}{2} \tag{6}$$

so that the required normal flux on the rotor surface follows from Eq. 2b as

$$\tilde{B}_x^b = \mu_0 k\left[\frac{-\tilde{K}^s}{\sinh kd\; jk} - \coth kd\,\frac{\tilde{K}^r}{2jk}\right] \tag{7}$$

Finally, evaluation of Eq. 1 gives

$$f_z = -\frac{A}{2}\,\mathrm{Re}\,\tilde{B}_x^b(\tilde{K}^r)^* = -\frac{\mu_0 A}{2}\,\mathrm{Re}\,\frac{j\tilde{K}^s(\tilde{K}^r)^*}{\sinh kd} \tag{8}$$

This result is identical to Eq. 4.3.4a, so the results for parts (b) and (c) will be the same as Eqs. 4.3.9a.

Prob. 4.3.2 Boundary conditions on the stator and rotor surfaces are

$$\tilde{H}_z^s = \tilde{K}^s \tag{1}$$

$$\tilde{B}_x^r = \tilde{B}^r \tag{2}$$

where

$$\tilde{K}^s = -j K_o^s e^{j\omega t} \tag{3}$$

$$\tilde{B}^r = B_o^r e^{jk(Ut+\delta)} \tag{4}$$

From Eq. (a) of Table 2.16.1, the air gap fields are therefore related by

$$\begin{bmatrix} \tilde{B}_x^s \\ \tilde{B}_x^r \end{bmatrix} = \mu_o k \begin{bmatrix} -\coth kd & \dfrac{1}{\sinh kd} \\ \dfrac{-1}{\sinh kd} & \coth kd \end{bmatrix} \begin{bmatrix} \dfrac{\tilde{K}^s}{jk} \\ \dfrac{\tilde{H}_z^r}{jk} \end{bmatrix} \tag{5}$$

In terms of these complex amplitudes, the required force is

$$f_z = \frac{A}{4} \operatorname{Re} \tilde{B}_x^r \tilde{H}_z^{r*} \tag{6}$$

From Eq. 5b,

$$\tilde{H}_z^r = jk \tanh kd \left(\frac{\tilde{B}_x^r}{\mu_o k} + \frac{\tilde{K}_z^s}{jk \sinh kd} \right) \tag{7}$$

Introduced into Eq. 6, this expression gives

$$f_z = \frac{A}{4} \frac{1}{\cosh kd} \operatorname{Re} \tilde{K}_z^{s*} \tilde{B}_x^r \tag{8}$$

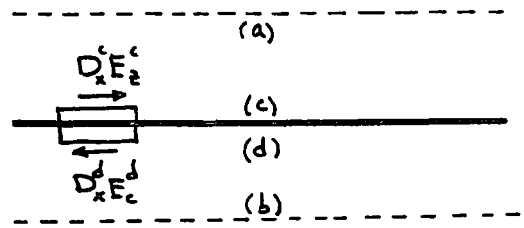
For the particular distributions of Eqs. 3 and 4,

$$\begin{aligned} f_z &= \frac{A}{4} \frac{1}{\cosh kd} \operatorname{Re} \left(j K_o^s e^{-j\omega t} \right) \left(B_o^r e^{jk(Ut+\delta)} \right) \\ &= -\frac{A}{4} \frac{1}{\cosh kd} K_o^s B_o^r \sin[(kU-\omega)t + k\delta] \end{aligned} \tag{9}$$

Under synchronous conditions, this becomes

$$f_z = -\frac{A K_o^s B_o^r}{4 \cosh kd} \sin k\delta$$

<u>Prob. 4.3.3</u> With positions as designated in the sketch, the total force per unit area is



$$\langle f_z \rangle_z = \langle D_x^c E_z^c - D_x^d E_z^d \rangle_z A \quad (1)$$

$$= \frac{1}{2} Re\left(\tilde{D}_x^c \tilde{E}_z^{c*} - \tilde{D}_x^d \tilde{E}_z^{d*} \right) A$$

With the understanding that the surface charge on the sheet is a given quantity, boundary conditions reflecting the continuity of tangential electric field at the three surfaces and that Gauss' law be satisfied through the sheet are

$$\tilde{\Phi}^a \text{ given};\ \tilde{\Phi}^b \text{ given};\ \tilde{\Phi}^c = \tilde{\Phi}^d;\ \tilde{D}_x^c - \tilde{D}_x^d = \tilde{\sigma}_f \text{ given} \quad (2)-(4)$$

Bulk relations are given by Table 2.16.1. In the upper region

$$\begin{bmatrix} \tilde{D}_x^a \\ \tilde{D}_x^c \end{bmatrix} = \epsilon_o k \begin{bmatrix} -\coth kd & \dfrac{1}{\sinh kd} \\ -\dfrac{1}{\sinh kd} & \coth kd \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^a \\ \tilde{\Phi}^c \end{bmatrix} \quad (5)$$

and in the lower

$$\begin{vmatrix} \tilde{D}_x^d \\ \tilde{D}_x^b \end{vmatrix} = \epsilon_o k \begin{bmatrix} -\coth kd & \dfrac{1}{\sinh kd} \\ -\dfrac{1}{\sinh kd} & \coth kd \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^d \\ \tilde{\Phi}^b \end{bmatrix} \quad (6)$$

In view of Eq. 4, Eq. 1 becomes

$$\langle f_z \rangle_z = \frac{1}{2} Re\left[-jk \tilde{\sigma}_f \tilde{\Phi}^{c*} \right] A \quad (7)$$

so what is now required is the amplitude $\tilde{\Phi}^c$. The surface charge, given by Eq. 4, as the difference $\tilde{D}_x^c - \tilde{D}_x^d$, follows in terms of the potentials from taking the difference of Eqs. 5b and 6a. The resulting expression is solved for

$$\tilde{\Phi}^c = \frac{\tilde{\sigma}_f}{2\epsilon k \coth kd} + \frac{\tilde{\Phi}^a + \tilde{\Phi}^b}{2 \cosh kd} \quad (8)$$

Substituted into Eq. 7 (where the self terms in $\tilde{\sigma}_f \tilde{\sigma}_f^*$ are imaginary and can therefore be dropped) the force is expressed in terms of the given excitations.

Prob. 4.3.3(cont.)2

$$\langle f_z \rangle_z = \frac{1}{2} k\, \mathrm{Re}\left[-j\tilde{\sigma}_f \frac{\tilde{\hat{\Phi}}^{a*} + \tilde{\hat{\Phi}}^{b*}}{2\cosh kd} \right] A \tag{9}$$

b) Translation of the given excitations into complex amplitudes gives

$$\begin{aligned} \tilde{\sigma}_f &= -\sigma_o e^{j\omega t} e^{-jk\delta} \\ \tilde{\hat{\Phi}}^a &= V_o e^{j\omega t} \\ \tilde{\hat{\Phi}}^b &= \pm V_o e^{j\omega t} \end{aligned} \tag{10}$$

Thus, with the even excitation, where $\Phi^a = \Phi^b$

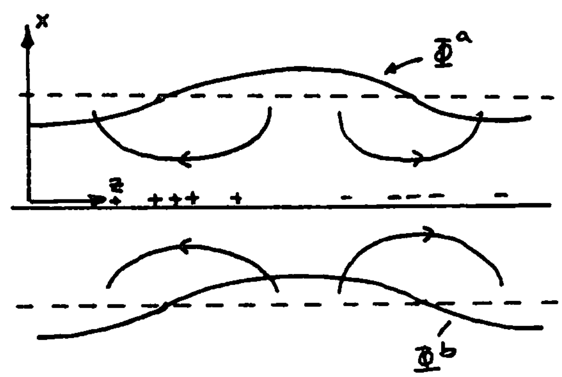
$$\langle f_z \rangle_z = -\frac{k V_o \sigma_o A}{2\cosh kd} \sin k\delta \tag{11}$$

and with the odd excitation, $\langle f_z \rangle_z = 0$.

c) This is a specific case from part (b) with $\omega = 0$ and $\delta = \lambda/4$. Thus,

$$\langle f_z \rangle_z = -\frac{k V_o \sigma_o A}{2\cosh kd} \tag{12}$$

The sign is consistent with the sketch of charge distribution on the sheet and electric field due to the potentials on the walls sketched.



Prob. 4.4.1 a) In the rotor, the magnetization, $\bar{M}$, is specified. Also, it is uniform, and hence has no curl. Thus, within the rotor,

$$\nabla\times\bar{B} \equiv \nabla\times[\mu_0(\bar{H}+\bar{M})] = \nabla\times\mu_0\bar{H} = 0 \tag{1}$$

Also, of course, $\bar{B}$ is solenoidal.

$$\nabla\cdot\bar{B} = 0 \Rightarrow \bar{B} = \nabla\times\bar{A} \tag{2}$$

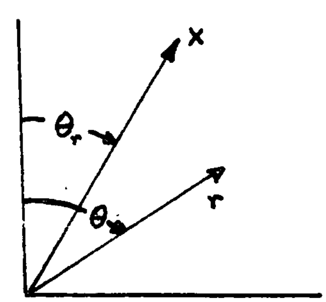
So, the derivation of transfer relations between $\bar{B}$ and $\bar{A}$ is the same as in Sec. 2.19 so long as $\mu_0\bar{H}$ is identified with $\bar{B}$.

b) The condition on the jump in normal flux density is as usual. However, with $\bar{M}$ given, Ampere's law requires that $\bar{n}\times[\![\bar{H}]\!] = \bar{K}_f$ and this can be rewritten using the definition of $\bar{B}$, $\bar{B}=\mu_0(\bar{H}+\bar{M})$. Thus, the boundary condition becomes

$$\bar{n}\times[\![\bar{B}]\!] = \mu_0\bar{K}_f + \mu_0\bar{n}\times[\![\bar{M}]\!] \tag{3}$$

where the jump in tangential $\bar{B}$ is related to the given surface current and given jump in magnetization.

c) With these background statements, the representation of the fields, solution for the torque and determination of the electrical terminal relation follows the usual pattern. First, represent the boundary conditions in terms of the given form of excitation. The magnetization can be written in complex notation, perhaps most efficiently, with the following reasoning. Use x as a cartesian coordinate rotated to the rotor axis angle, as shown in the figure. Then, if the gradient is pictured for the moment in cartesian coordinates, it can be seen that the uniform vector field $M_o\bar{i}_x$ is represented by



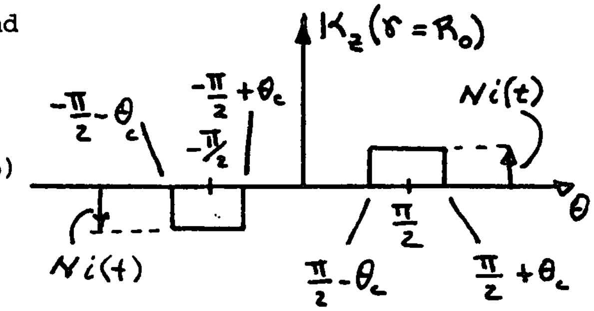
$$\bar{M} = -\nabla\Theta \ ; \ \Theta = -M_o x \tag{4}$$

Prob. 4.4.1(cont.)

Observe that $x = r\cos(\theta - \theta_r)$ and it follows from Eq. 4 that $\bar{M}$ is written in the desired Fourier notation as

$$\bar{M} = \nabla M_o r \cos(\theta-\theta_r) = \nabla\left\{\frac{M_o r}{2}\left[e^{j(\theta-\theta_r)} + e^{-j(\theta-\theta_r)}\right]\right\} \tag{5}$$
$$= \frac{M_o}{2}\left\{\bar{i}_r\left[e^{-j\theta_r}e^{j\theta} + e^{j\theta_r}e^{-j\theta}\right] + \bar{i}_\theta\left[j e^{-j\theta_r}e^{j\theta} - j e^{j\theta_r}e^{-j\theta}\right]\right\}$$

Next, the stator currents are represented in complex notation. The distribution of surface current is as shown in the figure and represented in terms of a Fourier series.



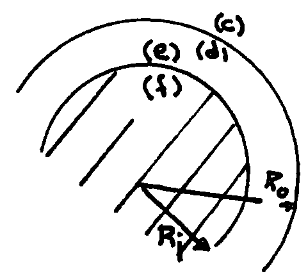
$$\bar{H} = \sum_{m=-\infty}^{+\infty} \tilde{\bar{H}}_m e^{-jm\theta} \tag{6}$$

The coefficients are given by (Eq.2.15.8)

$$\tilde{K}_{zn}(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_z(\theta,t)e^{jn\theta}d\theta = \frac{2Ni}{\pi n} j \sin\left(\frac{n\pi}{2}\right) \sin(n\theta_o) \tag{7}$$

Thus, because superposition can be used throughout, it is possible to determine the fields by considering the boundary conditions as applying to the complex Fourier amplitudes.

Boundary conditions reflecting Eq. 2 at each of the interfaces (designated as shown in the sketch) are,



$$\tilde{A}_n^c = \tilde{A}_n^d \tag{8}$$

$$\tilde{A}_n^e = \tilde{A}_n^f \tag{9}$$

while those representing Eq. 3 at each interface are

$$-\tilde{B}_{\theta n}^d = \mu_o \tilde{K}_{zn} \tag{10}$$

$$\tilde{B}_{\theta n}^e - \tilde{B}_{\theta n}^f = -\mu_o \tilde{M}_{\theta n} = -\left[j e^{-j\theta_r}\delta_{-1n} - j e^{j\theta_r}\delta_{1n}\right]\frac{M_o \mu_o}{2} \tag{11}$$

That $\bar{H}=0$ in the infinitely permeable stator is reflected in Eq. 10. Thust, Eq. 8 is not required to determine the fields in the gap and in the rotor.

Prob. 4.4.1(cont.)

In the gap and within the rotor, the transfer relations (Eqs. (c) of Table 2.19.1) apply

$$\begin{bmatrix} \tilde{B}^d_{\theta m} \\ \tilde{B}^e_{\theta m} \end{bmatrix} = \begin{bmatrix} f_m(R_i,R_o) & g_m(R_o,R_i) \\ g_m(R_i,R_o) & f_m(R_o,R_i) \end{bmatrix} \begin{bmatrix} \tilde{A}^d_m \\ \tilde{A}^e_m \end{bmatrix} \tag{12}$$

$$\tilde{B}^f_{\theta m} = f_m(0,R_i)\tilde{A}^f_m \tag{13}$$

Before solving these relations for the Fourier amplitudes, it is well to look ahead and see just which ones are required. To determine the torque, the rotor can be enclosed by any surface within the air-gap, but the one just inside the stator has the advantage that the tangential field is specified in terms of the driving current, Eq. 10. For that surface (using Eq. 3.9.17 and the orthogonality relation for space averaging the product of Fourier series, Eq. 2.15.17),

$$\begin{aligned} \tau &= R_o(2\pi R_o d)\langle T^d_{\theta r}\rangle_\theta = 2\pi R^2 d\,\langle H^d_r B^d_\theta\rangle_\theta \\ &= 2\pi R_o^2 d\sum_{m=-\infty}^{+\infty} \tilde{H}^d_{rm}\tilde{B}^{e*}_{\theta m} \end{aligned} \tag{14}$$

Because $\tilde{B}^e_{\theta m}$ is known, it is $\tilde{H}^d_{rm}$ that is required where $\tilde{H}^d_{rm} = -jm\tilde{A}^d_m/\mu_o R_o$.

Subtract Eq. 13 from Eq. 12b and use the result to evaluate Eq. 11. Then, in view of Eq. 9 the first of the following two relations follow.

$$\begin{bmatrix} g_m(R_i,R_o) & f_m(R_o,R_i)-f_m(0,R_i) \\ f_m(R_i,R_o) & g_m(R_o,R_i) \end{bmatrix} \begin{bmatrix} \tilde{A}^d_m \\ \tilde{A}^e_m \end{bmatrix} = \begin{bmatrix} -\tilde{M}_{om} \\ \mu_o\tilde{K}_{zm} \end{bmatrix} \tag{15}$$

The second relation comes from Eqs. 12a and 10. From these two equations in two unknowns the required amplitude follows

$$\tilde{A}^d_m = \frac{-\tilde{M}_{om}\,g_m(R_o,R_i) - \mu_o\tilde{K}_{zm}\left[f_m(R_o,R_i) - f_m(0,R_i)\right]}{D_m} \tag{16}$$

where $D_m \equiv g_m(R_i,R_o)g_m(R_o,R_i) - f_m(R_i,R_o)\left[f_m(R_o,R_i) - f_m(0,R_i)\right]$

Prob. 4.4.1(cont.)

Evaluation of the torque, Eq. 14, follows by substitution of $\tilde{H}^d_{rm}$ as determined by Eq. 16 and $\tilde{B}^d_{\theta m}$ as given by Eq. 10.

$$T_z = 2\pi R_o^2 d \sum_{m=-\infty}^{+\infty} \Big\{ -\frac{jm\, g_m(R_o,R_i)}{R_o D_m} \tilde{M}_{\theta m} \tilde{K}^*_{zm} - \frac{jm}{R_o} \mu_o \frac{\tilde{K}_{zm}\tilde{K}^*_{zm}}{D_m} \left[f_m(R_o,R_i) - f_m(0,R_i) \right] \Big\} \tag{17}$$

The second term involves products of the stator excitation amplitudes and it must therefore be expected that this term vanishes. To see that this is so, observe that $\tilde{K}_{zm}\tilde{K}^*_{zm}$ is positive and real and that f_m and g_m are even in m. Because of the m appearing in the series it then follows that the m term cancels with the -m term in the series. The first term is evaluated by using the expressions for $\tilde{M}_{\theta m}$ and $\tilde{K}^*_{zm}$ given by Eqs. 10 and 11. Because there are only two Fourier amplitudes for the magnetization, the torque reduces to simply

$$T_z = -4\mu_o R_o d M_o \sin\theta_o K \sin\theta_r N i(t) \tag{18}$$

where

$$K = g_1(R_o,R_i) \Big/ \left\{ g_1(R_i,R_o) g_1(R_o,R_i) - f_1(R_i,R_o)\left[f_1(R_o,R_i) - f_1(0,R_i) \right] \right\}$$

From the definitions of g_m and f_m, it can be shown that $K=R_i^2/R_o$, so that the final answer is simply

$$T_z = -4\mu_o R_i^2 d M_o \sin\theta_o \sin\theta_r N i(t) \tag{19}$$

Note that this is what is obtained if a dipole moment is defined as the product of the uniform volume magnetization multiplied over the rotor volume and directed at the angle θ_r.

$$|\bar{m}| = \pi R_i^2 d M_o \tag{20}$$

in a uniform magnetic field associated with the m=1 and m=-1 modes,

$$|\bar{H}| = \frac{4N i(t)}{\pi} \sin\theta_o \tag{21}$$

with the torque evaluated as simply $\bar{T} = \mu_o \bar{m} \times \bar{H}$. (Eq. 2, Prob. 3.6.2)

Prob. 4.4.1(cont.)

The flux linked by turns at the position θ having the span $R_o d\theta$ is

$$\Phi_\lambda = [N R_o (d\theta)][A^d(\theta) - A^d(\theta+\pi)]d \tag{22}$$

Thus, the total flux is

$$\lambda = \int_{\frac{\pi}{2}-\theta_o}^{\frac{\pi}{2}+\theta_o} \Phi_\lambda d\theta = \int_{\frac{\pi}{2}-\theta_o}^{\frac{\pi}{2}+\theta_o} d N R_o \sum_{m=-\infty}^{+\infty} \tilde{A}_m^d (1 - e^{-jm\pi}) e^{-jm\theta} d\theta \tag{23}$$

The exponential is integrated to give

$$\lambda = 4 d N R_o \sum_{m=-\infty}^{+\infty} \tilde{A}_m^d \frac{j}{m} e^{-jm\pi} \sin\left(\frac{m\pi}{2}\right) \sin m\theta_o \tag{24}$$

where the required amplitude, $\tilde{A}_m^d$, is given by Eq. 16. Substitution shows that

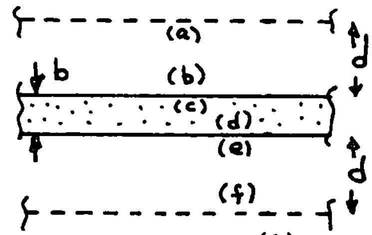
$$\lambda = L i(t) + A_r \mu_o M_o \cos\theta_r \tag{25}$$

where

$$L = \frac{8}{\pi} N^2 \mu_o R_o d \sum_{\substack{m=-\infty \\ (\text{odd})}}^{+\infty} \left(\frac{\sin m\theta_o}{m}\right)^2 \frac{[f_m(0, R_i) - f_m(R_o, R_i)]}{[g_m(R_i, R_o)/R_i]}$$

and

$$A_r = 4 N R_i^2 d \sin\theta_o$$



Prob. 4.6.1 With locations as indicated by the sketch, the boundary conditions are written in terms of complex amplitudes as

$$\tilde{\Phi}^a = \tilde{V}_o;\ \tilde{\Phi}^b = \tilde{\Phi}^c;\ \tilde{D}_x^b = \tilde{D}_x^c;\ \tilde{\Phi}^d = \tilde{\Phi}^e;\ \tilde{D}_x^d = \tilde{D}_x^e;\ \tilde{\Phi}^f = \tilde{V}_o \tag{1}$$

Because of the axial symmetry, the analysis is simplified by recognizing that

$$\tilde{\Phi}^f = \tilde{\Phi}^a\ ;\ \tilde{D}_x^f = -\tilde{D}_x^a \tag{2}$$

This makes it possible to write the required force as

$$f_z = A\left\langle E_z^a D_x^a - E_z^f D_x^f \right\rangle_z = A\,\mathrm{Re}\left(-jk\tilde{\Phi}^{a*}\tilde{D}_x^a\right) = A\,\mathrm{Re}\left(-jk\tilde{V}_o^*\tilde{D}_x^a\right) \tag{3}$$

The transfer relations for the beam are given by Eq. 4.5.18, which becomes

$$\begin{bmatrix} \tilde{\Phi}^c \\ \tilde{\Phi}^d \end{bmatrix} = \frac{1}{\epsilon_o k}\begin{bmatrix} -\coth kb & \dfrac{1}{\sinh kb} \\ \dfrac{-1}{\sinh kb} & \coth kb \end{bmatrix}\begin{bmatrix} \tilde{D}_x^c \\ \tilde{D}_x^d \end{bmatrix} + \sum_{i=0}^{\infty} \frac{\tilde{\rho}_i}{\epsilon_o(\nu_i^2 + k^2)}\begin{bmatrix} (-1)^i \\ 1 \end{bmatrix} \tag{4}$$

These also apply to the air-gap, but instead use the inverse form from Table 2.16.1.

$$\begin{bmatrix} \tilde{D}_x^a \\ \tilde{D}_x^b \end{bmatrix} = \epsilon_o k\begin{bmatrix} -\coth kd & \dfrac{1}{\sinh kd} \\ \dfrac{-1}{\sinh kd} & \coth kd \end{bmatrix}\begin{bmatrix} \tilde{\Phi}^a \\ \tilde{\Phi}^b \end{bmatrix} \tag{5}$$

From the given distribution of ρ it follows that only one Fourier mode is required (because of the boundary conditions chosen for the modes).

$$\Pi_i = \begin{cases} 1 & i = 0 \\ 0 & i \neq 0 \end{cases} \Rightarrow \tilde{\rho}_i = \begin{cases} \tilde{\rho}_o & i = 0 \\ 0 & i \neq 0 \end{cases}\ ;\ \nu_i = 0 \tag{6}$$

With the boundary and symmetry conditions incorporated, Eqs. 4 and 5 become

$$\begin{bmatrix} \tilde{D}_x^a \\ \tilde{D}_x^b \end{bmatrix} = \epsilon_o k\begin{bmatrix} -\coth kd & \dfrac{1}{\sinh kd} \\ \dfrac{-1}{\sinh kd} & \coth kd \end{bmatrix}\begin{bmatrix} \tilde{V}_o \\ \tilde{\Phi}^b \end{bmatrix} \tag{7}$$

Prob. 4.6.1 (cont)

$$\begin{bmatrix}\tilde{\Phi}^b\\ \\ \tilde{\Phi}^b\end{bmatrix} = \frac{1}{\epsilon_0 k}\begin{bmatrix}-\coth kb & \dfrac{1}{\sinh kb}\\ \\ \dfrac{-1}{\sinh kb} & \coth kb\end{bmatrix}\begin{bmatrix}\tilde{D}_x^b\\ \\ \tilde{D}_x^b\end{bmatrix} + \frac{\tilde{\rho}_o}{\epsilon_o k^2} \qquad (8)$$

These represent four equations in the three unknowns $\tilde{D}_x^a, \tilde{D}_x^b, \tilde{\Phi}^b$. They are redundent because of the implied symmetry. The first three equations can be written in the matrix form (9)

$$\begin{bmatrix}-1 & 0 & \dfrac{\epsilon_o k}{\sinh kd}\\ \\ 0 & -1 & \epsilon_o k \coth kd\\ \\ 0 & -\dfrac{1}{\epsilon_o k}\left(\coth kb + \dfrac{1}{\sinh kb}\right) & -1\end{bmatrix}\begin{bmatrix}\tilde{D}_x^a\\ \\ \tilde{D}_x^b\\ \\ \tilde{\Phi}^b\end{bmatrix} = \begin{bmatrix}\epsilon_o k \coth kd\, \tilde{V}_o\\ \\ \dfrac{\epsilon_o k \tilde{V}_o}{\sinh kd}\\ \\ -\dfrac{\tilde{\rho}_o}{\epsilon_o k^2}\end{bmatrix}$$

In using Cramer's rule for finding $\tilde{D}_x^a$ (required to evaluate Eq. 3) note that terms proportional to $\tilde{V}_o$ will make no contribu on when inserted into Eq. 3 (all coefficients in Eq. 9 are real), so there is no need to write these terms out. Thus,

$$\tilde{D}_x^a = [\ ]\tilde{V}_o + \frac{\tilde{\rho}_o G}{k}\ ;\ G = \left[\sinh kd + \cosh kd\left(\coth kb + \frac{1}{\sinh kb}\right)\right]^{-1} \qquad (10)$$

and Eq. 3 becomes

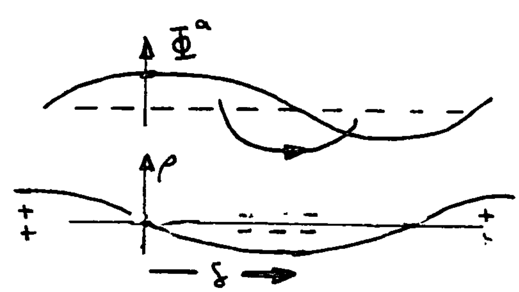
$$f_z = A G\, \mathrm{Re}\left[-j \tilde{V}_o^{*} \tilde{\rho}_o\right] \qquad (11)$$

b) In the particular case where

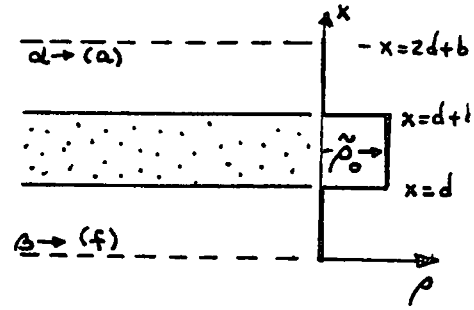
$$\tilde{V}_o = V_o e^{j\omega t}\ ;\ \tilde{\rho}_o = -\rho_o e^{j(\omega t + k\delta)} \qquad (12)$$

the force given by Eq. 11 reduces to

$$f_z = -A G V_o \rho_o \sin k\delta \qquad (13)$$



The sketch of the wall potential and the beam charge when t=0 suggests that indeed the force should be zero if δ and be negative if $0 < k\delta < \pi$



Prob. 4.6.1 (cont.)

c) With the entire region represented by the relations of Eq. 4, the charge distribution to be represented by the modes is that of the sketch. With $\Delta \equiv b + 2d$ and $\Pi_i = \cos\frac{i\pi}{\Delta}x$, Eq. 4.5.17 gives the mode amplitudes.

$$\tilde{\rho}_i = \frac{2}{\Delta}\int_d^{d+b} \tilde{\rho}_o \cos\frac{i\pi}{\Delta}x\,dx = \frac{2\tilde{\rho}_o}{i\pi}\left[\sin\frac{i\pi}{\Delta}(d+b) - \sin\frac{i\pi d}{\Delta}\right]\Bigg|_{i\neq 0};\quad \tilde{\rho}_i = \frac{\tilde{\rho}_o b}{\Delta}\Bigg|_{i=0} \tag{14}$$

So, with the transfer relations of Eq. 4.5.18 applied to the entire region,

$$\begin{bmatrix}\tilde{V}_o\\ \tilde{V}_o\end{bmatrix} = \frac{1}{\epsilon_o k}\begin{bmatrix}-\coth k\Delta & \frac{1}{\sinh k\Delta}\\ -\frac{1}{\sinh k\Delta} & \coth k\Delta\end{bmatrix}\begin{bmatrix}\tilde{D}_x^a\\ \tilde{D}_x^f\end{bmatrix} + \sum_{i=0}^{\infty}\frac{\tilde{\rho}_i}{\epsilon_o\left[\left(\frac{i\pi}{\Delta}\right)^2 + k^2\right]}\begin{bmatrix}(-1)^i\\ 1\end{bmatrix} \tag{15}$$

Symmetry requires that $\tilde{D}_x^a = -\tilde{D}_x^f$, which is consistent with both of Eqs. 15 reducing to the same thing. That is, the modal amplitudes are zero for i odd. From either equation it follows that

$$\tilde{V}_o = \frac{1}{\epsilon_o k}\left[-\coth k\Delta - \frac{1}{\sinh k\Delta}\right]\tilde{D}_x^a + \sum_{i=0}^{\infty}\frac{\tilde{\rho}_i}{\epsilon_o\left[\left(\frac{i\pi}{\Delta}\right)^2 + k^2\right]} \tag{16}$$

The terms multiplying V_o are not written out because they make no contribution to the force.

$$\tilde{D}_x^a = (\quad)\tilde{V}_o + \frac{\epsilon_o k \sinh k\Delta}{(\cosh k\Delta + 1)}\sum_{\substack{i=0\\ (\text{even})}}^{\infty}\frac{\tilde{\rho}_i}{\varepsilon\left[\left(\frac{i\pi}{\Delta}\right)^2 + k^2\right]} \tag{17}$$

Thus, the force is evaluated using as surfaces of integration surfaces at (a) and (f).

$$f_z = \frac{1}{2}A\,\mathrm{Re}\left(-jk\tilde{\Phi}^{a*}\tilde{D}_x^a + jk\tilde{\Phi}^{f*}\tilde{D}_x^f\right) = Ak\,\mathrm{Re}\left(-j\tilde{V}_o^*\tilde{D}_x^a\right) \tag{18}$$

$$= -\frac{Ak^2\epsilon_o \sinh k\Delta}{\cosh k\Delta + 1}\,\mathrm{Re}\sum_{\substack{i=0\\ \text{even}}}^{\infty}\frac{j\tilde{V}_o^*\tilde{\rho}_i}{\varepsilon\left[\left(\frac{i\pi}{\Delta}\right)^2 + k^2\right]}$$

Prob. 4.6.1 (cont)

In terms of the z-t dependence given by Eq. 12, this force is

$$f_z = -\frac{A k^2 \epsilon_o \sinh k\Delta}{\cosh k\Delta + 1}\left\{\frac{b}{\Delta \epsilon k^2} + \sum_{\substack{i=2\\ \text{even}}}^{\infty} \frac{2\left[\sin\frac{i\pi}{\Delta}(d+b) - \sin\frac{i\pi d}{\Delta}\right]}{\epsilon i\pi\left[\left(\frac{i\pi}{\Delta}\right)^2 + k^2\right]}\right\} V_o \rho_o \sin k\delta \tag{19}$$

<u>Prob. 4.8.1</u> a) The relations of Eq. 9 are applicable in the case of the planar layer provided the coefficients F_m and G_m are identified by comparing Eq. 8 to Eq. (b) of Table 2.19.1.

$$F_m(\beta,\alpha) = -F_m(\alpha,\beta) \rightarrow -\frac{\coth k\Delta}{k} \; ; \; G_m(\alpha,\beta) = -G_m(\beta,\alpha) \rightarrow \frac{1}{k \sinh k\Delta} \tag{1}$$

Thus, the transfer relations are as given in the problem.

b) The given forms of A_p and J_z are substituted into Eq. 4.8.3a to show that

$$\frac{d^2 \Pi_i}{dx^2} + \nu_i^2 \Pi_i = 0 \tag{2}$$

where

$$\nu_i^2 = \frac{\mu \tilde{J}_i^2}{\tilde{A}_i} - k^2 \tag{3}$$

Solutions to Eq. that have zero derivatives on the boundaries (and hence make $H_{yp}=0$ on the α and β surfaces) are

$$\Pi_i = \cos \nu_i x \; ; \; \nu_i = \frac{i\pi}{\Delta}, \; i = 0, 1, 2, \ldots \tag{4}$$

From Eq. 3 it then follows that

$$\tilde{A}_i \Pi_i = \frac{\mu \tilde{J}_i \cos \nu_i x}{\left(k^2 + \left(\frac{i\pi}{\Delta}\right)^2\right)} \tag{5}$$

Substitution into the general transfer relation found in part (a) then gives the required transfer relation from part (b).

In view of the Fourier modes selected to represent the x dependence, Eq. 4, the Fourier coefficients are

$$\underset{i \neq 0}{\tilde{J}_i} = \frac{2}{\Delta}\int_0^{\Delta} \tilde{J}_z(x) \cos\left(\frac{\pi i}{\Delta}x\right) dx \; ; \; \tilde{J}_o = \frac{1}{\Delta}\int_0^{\Delta} \tilde{J}_z(x)\, dx \tag{6}$$

Prob. 4.9.1 Because of the step function dependence of the current density on y, it is generally necessary to use a Fourier series representation (rather than complex amplitudes). The positions just below the stator current sheet and just above the infinitely permeable "rotor" material are designated by (a) and (b) respectively. Then, in terms of the Fourier amplitudes, the force per unit y-z area is

$$T_y = \langle H_y^a B_x^a \rangle_y = \sum_{m=-\infty}^{+\infty} \tilde{H}_{ym}^{*} \tilde{B}_{xm} = \sum_{m=-\infty}^{+\infty} j \tilde{K}_m^{s*} k_m \tilde{A}_m^a \tag{1}$$

The stator excitation is represented as a Fourier series by writing it as

$$K_z^s = \frac{\tilde{K}^s}{2} e^{-jk_1 y} + \frac{\tilde{K}^{s*}}{2} e^{jk_1 y} = \sum_{m=-\infty}^{+\infty} \tilde{K}_m^s e^{-jk_m y}; \quad \tilde{K}_m^s = \frac{1}{2}(\tilde{K}^s \delta_{1m} + \tilde{K}^{s*} \delta_{-1m}) \tag{2}$$

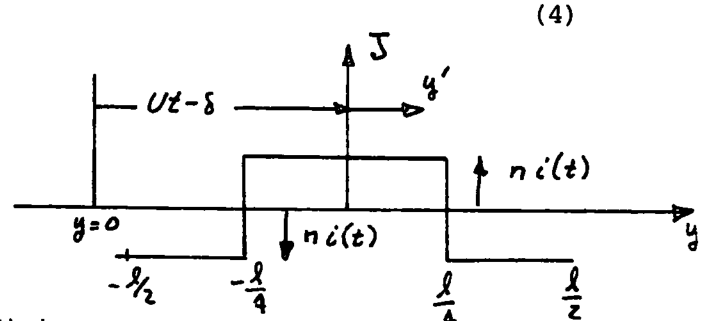
The "rotor" current density is written so as to be consistent with the adaptation of the transfer relations of Prob. 4.8.1 to the Fourier representation.

$$J = \sum_{m=-\infty}^{+\infty} \sum_{p=0}^{\infty} \tilde{J}_{mp}(t) \cos \gamma_p x \, e^{-jk_m y} \tag{3}$$

Here, the expansion on p accounting for the x dependence reduces to just the p=0 term, so Eq. 3 becomes

$$J = \sum_{m=-\infty}^{+\infty} \tilde{J}_{mo}(t) e^{-jk_m y} \tag{4}$$

The coefficients J_{mo} are determined by the y dependence, sketched in the figure. First, expand in terms of the series



$$J = \sum_{m=-\infty}^{+\infty} \tilde{J}'_{mo} e^{-jk_m y'} \tag{5}$$

where $y'=y-(Ut-\delta)$. This gives the coefficients

$$\tilde{J}'_{mo} = \frac{2ni(t)}{\pi m} \sin\left(\frac{m\pi}{2}\right) \tag{6}$$.

Thus, the coefficients in the y dependent Fourier series, Eq.4, become

$$\tilde{J}_{mo} = \frac{2ni(t)}{\pi m} \sin \frac{m\pi}{2} e^{jk_m(Ut-\delta)} \tag{7}$$

Boundary conditions at the (a) and (b) surfaces require that $\tilde{H}_{ym}^b = 0$ and $\tilde{H}_{ym}^a = -\tilde{K}_m^s$. Thus, the first equation in the transfer relation found in Problem 4.8.1 becomes

Prob. 4.9.1(cont.)

$$\tilde{A}_m^a = \frac{\mu_o}{k_m}\coth k_m d\, \tilde{K}_m^s + \frac{\mu_o \tilde{J}_{mo}}{k_m^2} \tag{8}$$

Thus, Eq. 1 can be evaluated. Note that the "self" terms drop out because the coefficient of $\tilde{K}_m^s \tilde{K}_m^{s*}$ is odd in m (the m'th term is cancelled by the -m'th term)

$$T_y = \sum_{m=-\infty}^{+\infty} \frac{j\mu_o}{2\pi m k_m}\left[\tilde{K}^{s*}\delta_{1m} + \tilde{K}^s \delta_{-1m}\right]\sin\left(\frac{m\pi}{2}\right) e^{jk_m(Ut-\delta)} \tag{9}$$

This expression reduces to

$$T_y = \frac{2\mu_o n\, i(t)}{\pi k_1}\left[\frac{\tilde{K}^s e^{-jk_1(Ut-\delta)} - \tilde{K}^{s*} e^{jk_1(Ut-\delta)}}{2j}\right] \tag{10}$$

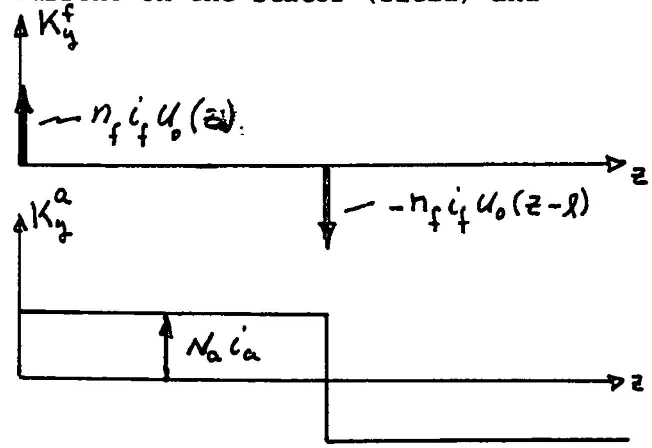
If the stator current is the pure traveling wave

$$K^s = K_o^s \cos(\omega t - k_1 y) \Rightarrow \tilde{K}^s = K_o^s e^{j\omega t} \tag{11}$$

and Eq. (10) reduces to

$$T_y = n\, i(t)\, \frac{\mu_o \ell}{\pi^2} K_o^s \sin\left(\frac{2\pi\delta}{\ell}\right) \tag{12}$$

<u>Prob. 4.10.1</u> The distributions of surface current on the stator (field) and rotor (armature) are shown in the sketches. These are represented as Fourier series having the standard form



$$K_y^f = \sum_{m=-\infty}^{+\infty} \tilde{K}_m^f e^{-jk_m z} \tag{1}$$

with coefficients given by

$$\tilde{K}_m^f = \frac{1}{2l}\int_0^{2l} K_y^f e^{jk_m z} dz \tag{2}$$

It follows that the Fourier amplitudes are

$$\tilde{K}_m^f = \frac{n_f i_f}{2l}\left(1 - e^{-jm\pi}\right) \tag{3}$$

and

$$\tilde{K}_m^a = j\frac{N_a i_a}{m\pi}\left(1 - e^{jm\pi}\right) \tag{4}$$

Boundary conditions at the stator (f) and rotor (a) surfaces are $(\bar{H} = -\nabla\Psi)$

$$H_z^f = K_y^f \Rightarrow \tilde{\Psi}_m^f = \tilde{K}_m^f / jk_m \tag{5}$$

$$H_z^a = -K_y^a \Rightarrow \tilde{\Psi}_m^a = -\tilde{K}_m^a / jk_m \tag{6}$$

Fields in the air-gap are represented by the flux-potential transfer relations (Table 2.16.1)

$$\begin{bmatrix} \tilde{B}_{xm}^f \\ \tilde{B}_{xm}^a \end{bmatrix} = \mu_o \frac{m\pi}{l} \begin{bmatrix} -\coth\left(\frac{m\pi b}{l}\right) & \frac{1}{\sinh\left(\frac{m\pi b}{l}\right)} \\ \frac{-1}{\sinh\left(\frac{m\pi b}{l}\right)} & \coth\left(\frac{m\pi b}{l}\right) \end{bmatrix} \begin{bmatrix} \frac{\tilde{K}_m^f}{jk_m} \\ -\frac{\tilde{K}_m^a}{jk_m} \end{bmatrix} \tag{7}$$

The force is found by evaluating the Maxwell stress over a surface that encloses the rotor with the air-gap part of the surface adjacent to the rotor (where fields are denoted by (a)).

$$f_z = 2ld\langle B_x^a H_z^a\rangle_z = -2ld\langle B_x^a K_y^a\rangle_z = -2ld\sum_{m=-\infty}^{+\infty} \tilde{B}_{xm}^a (\tilde{K}_m^a)^* \tag{8}$$

Prob. 4.10.1(cont.)

In view of the transfer relations, Eqs. 7, this expression becomes

$$f_z = -j2\ell d\mu_o \sum_{m=-\infty}^{+\infty} \frac{\left(j\tilde{K}_m^a\right)^* \tilde{K}_m^f}{\sinh\left(\frac{m\pi b}{\lambda}\right)} \tag{9}$$

In turn, the surface currents are given in terms of the terminal currents by Eqs. 3 and 4. Note that the self-field term makes no contribution because the sum is over terms that are odd in m. That is, for the self-field contribution, the m'th term in the series is cancelled by the -m'th term.

$$f_z = \mu_o d N_a i_a n_f i_f \sum_{m=-\infty}^{+\infty} -\frac{\left(1-e^{-jm\pi}\right)\left(1-e^{-jm\pi}\right)}{m\pi \sinh\left(\frac{m\pi b}{\ell}\right)} \tag{10}$$

This expression reduces to the standard form

$$f_z = -G_m i_a i_f \tag{11}$$

where

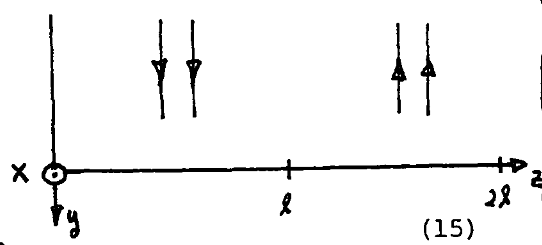
$$G_m = \mu_o d N_a n_f \sum_{m\ \text{odd}}^{\infty} \frac{8}{\pi} \frac{1}{m \sinh\left(\frac{m\pi b}{\ell}\right)} \tag{12}$$

To find the armature terminal relation, Faraday's integral law is written for a contour that is fixed in space and passes through the brushes and instantaneously contiguous conductors.

$$\oint_C (\bar{E} - \bar{v}\times\mu_o\bar{M})\cdot d\bar{\ell} = -\int_S \frac{\partial \bar{B}}{\partial t}\cdot\bar{n}\,da \tag{13}$$

In the conductors, $\bar{M}=0$ and Ohm's law requires that

$$\bar{E} = \frac{\bar{J}}{\sigma} - \bar{v}\times\mu_o\bar{H} \tag{14}$$



The armature winding is wound as in Fig. 4.10.3a with the axes and position of the origin as sketched to the right. Thus, Eq. 13 becomes

$$-v_a + \int_{wire} \frac{\bar{J}}{\sigma}\cdot d\bar{\ell} - \int_{wire} U B_x \bar{i}_y \cdot d\bar{\ell} = -\frac{d}{dt}\int_S B_x\,da \tag{15}$$

Each of the solid conductors in Fig. 4.10.3 carries half of the current. Thus, the second term in Eq. 15 becomes

$$\int_{wire} \frac{\bar{J}}{\sigma}\cdot d\bar{\ell} = \frac{A_a J_y}{A_a \sigma}\ell_a = \frac{i_a}{2}\frac{\ell_a}{\sigma A_a} = R_a i_a\ ;\ R_a \equiv \frac{\ell_a}{2\sigma A_a} \tag{16}$$

Prob. 4.10.1(cont.)

The third "speed-voltage" term in Eq. 15 becomes

$$\int_{wire} U B_x \bar{i}_y \cdot d\bar{l} = d \int_0^{l} U B_x N_a d - d \int_{l}^{2l} U B_x N_a dz \tag{17}$$

and this becomes

$$\int_{wire} U B_x \bar{i}_y \cdot d\bar{l} = d U N_a \left\{ \int_0^{l} \sum_{-\infty}^{+\infty} \tilde{B}^a_{xm} e^{-jk_m z} - \int_{l}^{2l} \sum_{-\infty}^{+\infty} \tilde{B}^a_{xm} e^{-jk_m z} \right\}$$

$$= -4jdUN_a \sum_{\substack{m=-\infty \\ (odd)}}^{+\infty} \frac{\tilde{B}^a_{xm}}{k_m} \tag{18}$$

From the bulk transfer relations, Eq. 7b, this becomes

$$\int_{wire} U B_x \bar{i}_y \cdot d\bar{l} = -4jdUN_a \sum_{\substack{m=-\infty \\ (odd)}}^{+\infty} \frac{\mu_o m\pi}{l k_m} \left\{ \frac{-\frac{n_f i_f}{2l}(1 - e^{-jm\pi})}{\sinh\left(\frac{m\pi b}{l}\right) j k_m} - \frac{\coth\left(\frac{m\pi b}{l}\right)}{k_m} \frac{N_a i_a}{m\pi} (1 - e^{jm\pi}) \right\} \tag{19}$$

The second term makes no contribution because it is odd in m. Thus, the speed-voltage term reduces to

$$\int_{wire} U B_x \bar{i}_y \cdot d\bar{l} = G_m U i_f \tag{20}$$

where G_m is the same as defined by Eq. 12.

To evaluate the right hand side of Eq. 15, observe that the flux linked by turns in the range z'+dz' to z' is

$$\left(d \int_{z'}^{z'+l} B^a_x \, dz \right) N_a dz' \tag{21}$$

so that altogether the flux linked is

$$\int_S B_x da = \int_0^{l} \left[d \int_{z'}^{z'+l} B^a_x dz \right] N_a dz' \tag{22}$$

Expressed in terms of the Fourier series, this becomes

$$\int_S B_x da = -\frac{4 N_a d l^2}{\pi^2} \sum_{\substack{-\infty \\ (odd)}}^{+\infty} \frac{\tilde{B}^a_{xm}}{m^2} \tag{23}$$

The normal flux at the armature is expressed in terms of the terminal currents by using Eqs. 15b and 3 and 4.

$$\int_S B_x da = -\frac{4 N_a d l \mu_o}{\pi} \sum_{\substack{-\infty \\ (odd)}}^{+\infty} \frac{1}{m} \left\{ \frac{-\left(\frac{n_f i_f}{2l}\right)(1 - e^{-jm\pi})}{\sinh\left(\frac{m\pi b}{l}\right) j k_m} - \frac{\coth\left(\frac{m\pi b}{l}\right) N_a i_a}{m\pi \, k_m} (1 - e^{jm\pi}) \right\} \tag{24}$$

Prob. 4.10.1(cont.)

The first term in this expression is odd in m and makes no contribution. Thus, it reduces to simply

$$\int_S B_x\, da = L_a i_a \tag{25}$$

where

$$L_a \equiv \frac{16 N_a^2 d \ell^2 \mu_o}{\pi^3} \sum_{\substack{1\\(\text{odd})}}^{\infty} \frac{\coth\left(\frac{m\pi b}{\ell}\right)}{m^3} \tag{26}$$

So, the armature terminal relation is in the classic form

$$v_a = R_a i_a + L_a \frac{di_a}{dt} - G_m U i_f \tag{27}$$

where R_a, L_a and G_m are defined by Eqs. 16, 26 and 12.

The use of Faraday's law for the field winding is similar but easier because it is not in motion. Equation 13 written for a path through the field winding becomes

$$-v_f + R_f i_f = -\frac{d}{dt}\int B_x^f\, da \tag{28}$$

The term on the right is written in terms of the Fourier series and the integral carried out to obtain

$$\int B_x^f\, da = dn_f \int_0^\ell B_x^f\, dz = dn_f \int_0^\ell \sum_{-\infty}^{+\infty} \tilde{B}_{xm}^f e^{-jk_m z}\, dz \tag{29}$$

Substitution of Eqs. 3 and 4 gives

$$\int B_x^f da = dn_f \mu_o \sum_{-\infty}^{+\infty} (e^{-jm\pi} - 1) \left\{ \frac{-n_f i_f (1 - e^{-jm\pi})}{jk_m 2\ell} \coth\left(\frac{m\pi b}{\ell}\right) - \frac{N_a i_a (1 - e^{jm\pi})}{(jk_m)^2 \sinh\left(\frac{m\pi b}{\ell}\right)} \right\} \tag{30}$$

The last term vanishes because it is odd in m. Thus,

$$\int B_x^f da = L_f i_f \; ; \; L_f \equiv \frac{4\mu_o d n_f^2}{\pi} \sum_{\substack{1\\(\text{odd})}}^{\infty} \frac{\coth m\pi b/\ell}{m} \tag{31}$$

and the field terminal relation, Eq. 28, becomes

$$v_f = R_f i_f + L_f \frac{di_f}{dt} \tag{32}$$

Prob. 4.12.1 The divergence and curl relations for $\bar{E}$ require that

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{\partial E_z}{\partial z} = 0 \qquad (1)$$

$$\frac{\partial E_r}{\partial z} + \frac{\partial E_z}{\partial r} = 0 \qquad (2)$$

Because $E_r=0$ on the z axis, the first term in Eq. 2, the condition that the curl be zero, is small in the neighborhood of the z axis. Thus,

$$\frac{\partial E_z}{\partial r} \approx 0 \Rightarrow E_z \simeq E_z(z) \qquad (3)$$

and Eq. 1 requires that

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_r) = -\frac{dE_z}{dz} \qquad (4)$$

Integration of this expression on r can be carried out because the right-hand side is only a function of z. Because $E_r=0$ at r=0, it follows that

$$E_r = -\frac{1}{2}r\, dE_z/dz \qquad (5)$$

Now, if it is recognized that $E_z = -d\Phi/dz$ without approximation, it follows that Eq. 5 is the required expression for E_r.

Prob. 4.13.1 Using the same definitions of surface variables and potential difference as used in the text, $(\hat{\xi}^s = \hat{\xi}_o, \hat{\xi}^r \equiv \hat{\xi}_o e^{jk\delta})$

$$V = Re\,\hat{V}_o e^{j\omega t}\,;\ \xi_s = Re\,\hat{\xi}^s e^{-jkz}\,;\ \xi_r = Re\,\hat{\xi}^r e^{j(2\omega t - kz)} \tag{1}$$

At each of the electrode surfaces, the constant potential boundary condition requires that

$$n \times \bar{E} = 0 \Rightarrow E_z = -E_x \frac{\partial \xi}{\partial z} \tag{2}$$

For example, at the rotor surface,

$$E_z(x=0) + \frac{\partial E_z}{\partial x}\Big|_{x=0} \xi_r = -E_x \frac{\partial \xi_r}{\partial z} \Rightarrow E_z^r = -\frac{\partial}{\partial z}(E_x^r \xi) \tag{3}$$

where the irrotational nature of $\bar{E}$ is exploited to write the second equation. Thus, the conditions at the perturbed electrode surfaces are related to those in fictitious planes x=0 and x=d for the rotor and stator respectively as

$$E_z^r = -\frac{\partial}{\partial z}(E_x^r \xi^r) \Rightarrow \Phi^r = -E_x^r \xi^r \tag{4}$$

$$E_z^s = -\frac{\partial}{\partial z}(E_x^s \xi^s) \Rightarrow \Phi^s = -E_x^s \xi^s + Re\,\hat{V}_o e^{j\omega t} \tag{5}$$

First, find the net force on a section of the rotor having length ℓ in the y direction and $2\pi/k$ in the z direction at some arbitrary instant in time.

$$f_z = \epsilon_o \ell \frac{2\pi}{k} \langle E_x E_z \rangle_z \tag{6}$$

The periodicity condition, together with the fact that there is no material in the air-gap, and hence no force density there, require that Eq. 6 can be integrated in any x plane and the same answer will be obtained. Although not physically meaningful, the integration is mathematically correct if carried out in the plane x=0 (the rotor plane). For convenience, that is what will be done here.

By way of finding the quantities required to evaluate Eqs. 4 and 5, it follows from Eqs. 1 that

$$E_x^s \xi^s = \frac{1}{4d}\left(\hat{V}_o e^{j\omega t} + \hat{V}_o^* e^{-j\omega t}\right)\left(\hat{\xi}^s e^{-jkz} + \hat{\xi}^{s*} e^{jkz}\right) \tag{7}$$

$$= \frac{1}{4d}\left\{\left[\hat{V}_o \hat{\xi}^s e^{j(\omega t - kz)} + (\hat{V}_o \hat{\xi}^s)^* e^{-j(\omega t - kz)}\right] + \left[\hat{V}_o \hat{\xi}^{s*} e^{j(\omega t + kz)} + (\hat{V}_o \hat{\xi}^{s*})^* e^{-j(\omega t + kz)}\right]\right\}$$

Prob. 4.13.1 (cont.)

and that

$$E_x^r \xi^r = \frac{1}{4d}\Big\{\Big[\hat{V}_o \hat{\xi}^r e^{j(3\omega t - kz)} + (\hat{V}_o \hat{\xi}^r)^* e^{-j(3\omega t - kz)}\Big] + \Big[(\hat{V}_o^* \hat{\xi}^r) e^{j(\omega t - kz)} + (\hat{V}_o^* \hat{\xi}^r)^* e^{-j(\omega t - kz)}\Big]\Big\} \tag{8}$$

Thus, these last two equations can be written in the complex amplitude form

$$E_x^s \xi^s = \frac{1}{2d} \mathrm{Re}\Big[(\hat{V}_o \hat{\xi}^s e^{j\omega t}) e^{-jkz} + (\hat{V}_o^* \hat{\xi}^s e^{-j\omega t}) e^{-jkz}\Big] \tag{9}$$

$$E_x^r \xi^r = \frac{1}{2d} \mathrm{Re}\Big[(\hat{V}_o \hat{\xi}^r e^{3j\omega t}) e^{-jkz} + (\hat{V}_o^* \hat{\xi}^r e^{j\omega t}) e^{-jkz}\Big] \tag{10}$$

The transfer relations, Eqs. a of Table 2.16.1, relate variables in this form evaluated in the fictitious stator and rotor planes.

$$\begin{bmatrix} \tilde{E}_x^s \\ \tilde{E}_x^r \end{bmatrix} = k \begin{bmatrix} -\coth kd & \dfrac{1}{\sinh kd} \\ \dfrac{-1}{\sinh kd} & \coth kd \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^s \\ \tilde{\Phi}^r \end{bmatrix} \tag{11}$$

It follows that

$$E_x^r = \mathrm{Re}\Big\{\frac{\hat{V}_o}{d} e^{j\omega t} - \frac{k}{2d \sinh kd}\Big[\hat{V}_o \hat{\xi}^s e^{j\omega t} + \hat{V}_o^* \hat{\xi}^s e^{-j\omega t}\Big] e^{-jkz} + \frac{k \coth kd}{2d}\Big[\hat{V}_o \hat{\xi}^r e^{3j\omega t} + \hat{V}_o^* \hat{\xi}^r e^{j\omega t}\Big] e^{-jkz} \tag{12}$$

Also, from Eq. 4,

$$E_z^r = \mathrm{Re} - \frac{jk}{2d}\Big[\hat{V}_o \hat{\xi}^r e^{3j\omega t} + \hat{V}_o^* \hat{\xi}^r e^{j\omega t}\Big] e^{-jkz} \tag{13}$$

Thus, the space average called for with Eq. 6 becomes

$$f_z = \frac{\epsilon_o \ell 2\pi}{k} \frac{1}{2} \mathrm{Re}\Big[\tilde{E}_x^r (\tilde{E}_z^r)^*\Big] \tag{14}$$

which, with the use of Eqs. 12 and 13, is

$$f_z = \frac{\epsilon_o \ell \pi}{4k} \mathrm{Re}\Big\{\frac{-jk^2}{d^2 \sinh kd}\Big[\hat{V}_o \hat{V}_o^* \hat{\xi}^s \hat{\xi}^{r*} e^{-2j\omega t} + \hat{V}_o \hat{V}_o^* \hat{\xi}^s \hat{\xi}^{r*} e^{-4j\omega t} + \hat{V}_o \hat{V}_o^* \hat{\xi}^s \hat{\xi}^{r*} + \hat{V}_o^* \hat{V}_o \hat{\xi}^s \hat{\xi}^{r*} e^{-2j\omega t}\Big] - \frac{jk^2}{d^2} \coth kd \Big[2\hat{V}_o \hat{\xi}^r \hat{V}_o^* \hat{\xi}^{r*} + (\hat{\xi}^r)^2 (\hat{V}_o^2 e^{2j\omega t} + \hat{V}_o^{*2} e^{-2j\omega t})\Big] \tag{15}$$

The self terms (in $\hat{\xi}^r \cdot \hat{\xi}^r$) either are imaginary or have no time average. The terms in $\hat{\xi}^r \cdot \hat{\xi}^s$ also time-average to zero, except for the term that is

Prob. 4.13.1 (cont.)

independent of time. That term makes the only contribution to the time-average expression $(\hat{\xi}^s = \hat{\xi}_o\, , \hat{\xi}^r = \hat{\xi}_o e^{jk\delta})$

$$\langle f_z \rangle_t = \frac{\epsilon_o \ell \pi k}{4 d^2 \sinh kd} |\hat{V}_o|^2 \, Re \, j \hat{\xi}_o \hat{\xi}_o^* e^{-jk\delta} \tag{16}$$

In the long-wave limit $kd \ll 1$, this result becomes

$$\langle f_z \rangle_t = \frac{\epsilon_o \ell \pi}{4 d^3} |\hat{V}_o|^2 |\xi_o|^2 \sin k\delta \tag{17}$$

which is in agreement with Eq. 4.13.12.

Prob. 4.13.2 For purposes of making a formal quasi-one-dimensional expansion, field variables are normalized such that

$$H_x = H_o \underline{H}_x \quad , x = d\,\underline{x} \quad , \xi = d\,\underline{\xi} \quad , f_z = \mu_o\left(\frac{d}{\lambda}\right) H_o^2 \underline{f}_z \qquad (1)$$
$$H_z = H_o\left(\frac{d}{\lambda}\right) \underline{H}_z \quad , z = \lambda\,\underline{z} \quad , \psi = H_o d\,\underline{\psi} \quad ,$$

The MQS conditions that the field intensity be irrotational and solenoidal in the air gap then require that

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = 0 \qquad (2)$$

$$\frac{\partial H_x}{\partial x} + \left(\frac{d}{\lambda}\right)^2 \frac{\partial H_z}{\partial z} = 0 \qquad (3)$$

If all field quantities are expanded as series in $\gamma \equiv (d/\lambda)^2$,

$$H_x = \sum_{i=0}^{\infty} H_{xi}\gamma^i \quad , H_z = \sum_{i=0}^{\infty} H_{zi}\gamma^i \qquad (4)$$

then, the equations become

$$\frac{\partial H_{xi}}{\partial z} = \frac{\partial H_{zi}}{\partial x} \quad , \frac{\partial H_{xi}}{\partial x} = -\frac{\partial H_{z(i-1)}}{\partial z} \qquad (5)$$

The lowest order field follows from the first two equations

$$\frac{\partial H_{xo}}{\partial x} = 0 \qquad (6)$$

$$\frac{\partial H_{zo}}{\partial x} = \frac{\partial H_{xo}}{\partial z} \qquad (7)$$

It follows that

$$H_x \cong H_{xo} = f(z,t) \qquad (8)$$

$$H_z \cong H_{zo} = x\frac{\partial f}{\partial z} + g(z,t) \qquad (9)$$

Boundary conditions at the stator and rotor surfaces respectively are

$$H_z^a = K_y(z,t) = K_o \sin(\omega t - kz) \qquad (10)$$

$$\bar{n} \times \bar{H}(x=\xi) = 0 \qquad (11)$$

In terms of the magnetic potential, these conditions are

$$\psi^a = -\left(\frac{\lambda}{d}\right)\frac{1}{2\pi} \cos\left[2\pi(t - z)\right] \qquad (12)$$

$$\psi(x=\xi) = 0 \qquad (13)$$

where variables are normalized such that $H_o = K_o$, $t = \tau\,\underline{t}$ $(\tau \equiv 2\pi/\omega)$.

Prob. 4.13.2(cont.)

Integration of $\bar{H}=-\nabla\Psi$ between the rotor and stator surfaces shows that

$$-H_o d\,\underline{\Psi}^a = \int_{-1+\underline{\xi}}^{\mathcal{D}} H_o\, \underline{H}_x\, d\, d\, \underline{x} \tag{14}$$

In view of Eq. 8,

$$\Psi^a = -\int_{-1+\xi}^{\mathcal{D}} H_x dx = -(1-\xi) f \tag{15}$$

and so the integration function f(z,t) is determined.

$$f(z,t) = \frac{-\Psi^a}{1-\xi} = \frac{\lambda}{d}\frac{1}{2\pi}\frac{\cos[2\pi(t-z)]}{1-\xi} \tag{16}$$

From Eqs. 8 and 9 it follows that

$$H_z^a \cong H_{zo}^a = \left[x\frac{\partial f}{\partial z} + g\right]_{x=0} = K_y = \sin[2\pi(t-z)] \tag{17}$$

so that

$$g = K_y \tag{18}$$

Actually, this result is not required to find the force, but it does complete the job of finding the zero order fields as given by Eqs. 8 and 9.

To find the force at any instant, it is necessary to carry out an integration of the magnetic shear stress over the lower surface of the stator.

$$\langle f_z \rangle_z = \int_0^1 H_x^a H_z^a dz \tag{19}$$

Evaluation gives

$$\langle f_z \rangle_z = \int_0^1 f(z,t)\left(\frac{\lambda}{d}\right) K_y\, dz \tag{20}$$

$$= \int_0^1 \left\{\frac{\lambda}{d}\frac{1}{2\pi}\frac{\cos[2\pi(t-z)]}{1-\xi_o\cos[4\pi(Ut-(z-\delta)]}\right\}\left\{\frac{\lambda}{d}\sin[2\pi(t-z)]\right\}dz$$

$$= \left(\frac{\lambda}{d}\right)^2\frac{1}{4\pi}\int_0^1 \frac{\sin[4\pi(t-z)]\,dz}{1-\xi_o\cos[4\pi(Ut-(z-\delta))]} \equiv F(t,\delta)$$

The time average force (per unit area then follows as

$$\langle\langle f_z\rangle_z\rangle_t = \int_0^1 F(t,\delta)\,dt \tag{21}$$

In the small amplitude limit, this integration reduces to $(\xi \ll 1)$

Prob. 4.13.2(cont.)

$$\langle f_z \rangle_z = \frac{1}{4\pi}\left(\frac{\lambda}{d}\right)^2 \int_0^1 \sin[4\pi(t-z)][1+\xi_o \cos[4\pi(Ut-(z-\delta))]]dz \qquad \underline{\underline{22}}$$

$$= \frac{1}{4\pi}\left(\frac{\lambda}{d}\right)^2 \int_0^1 \xi_o \sin[4\pi(t-z)]\cos[4\pi(Ut-(z-\delta))]dz$$

$$= -\frac{1}{4\pi}\left(\frac{\lambda}{d}\right)^2 \xi_o \int_0^1 \sin^2[4\pi(t-z)]\sin\{4\pi[(U-1)t+\delta]\}dz$$

$$= -\frac{1}{8\pi}\left(\frac{\lambda}{d}\right)^2 \xi_o \sin[(U-1)t+\delta]$$

Thus, the time average force is in general zero. However, for the synchronous condition, where $\underline{U} \equiv [U/\lambda][2\pi/\omega] = 1,$

it follows that the time average force per unit area is

$$\langle\langle f_z \rangle_z\rangle_t = -\frac{1}{8\pi}\left(\frac{\lambda}{d}\right)^2 \xi_o \sin\delta \qquad \underline{\underline{23}}$$

In dimensional form, this expression is

$$\langle f_z \rangle_z = \frac{-\mu_o K_o^2 k \xi_o}{4(kd)^2}\sin(2k\delta) \qquad (24)$$

and the same as the long wave limit of Eq. 4.3.27, which as kd $\rightarrow$ 0, becomes

$$\langle f_z \rangle_z = \frac{-\mu_o K_o^2 k \xi_o}{4\sinh^2 kd}\sin 2k\delta \rightarrow -\frac{\mu_o K_o^2 k\xi_o}{4(kd)^2}\sin 2k\delta \qquad (25)$$

In fact it is possible to carry out the integration called for with Eq. 20 provided interest is in the synchronous condition. In that case and Eq. 20 reduces to $\left(G \equiv (d/\lambda)^2 4\pi F\right)$

$$G = \frac{1}{4\pi}\int_a^{a+4\pi} \frac{\sin\varsigma\cos(4\pi\delta) - \cos\varsigma\sin 4\pi\delta}{1-\xi_o\cos\varsigma}d\varsigma \qquad \underline{\underline{26}}$$

where

$$\varsigma \equiv 4\pi(t-z) + 4\pi\delta, \quad a \equiv 4\pi(t+\delta) - 4\pi$$

In turn, this expression becomes

$$G = \frac{\cos 4\pi\delta}{4\pi}\int_a^{a+4\pi}\frac{\sin\varsigma}{1-\xi_o\cos\varsigma}d\varsigma - \frac{\sin 4\pi\delta}{4\pi}\int_a^{a+4\pi}\frac{\cos\varsigma\, d\varsigma}{1-\xi_o\cos\varsigma} \qquad \underline{\underline{27}}$$

The first integral vanishes, as can be seen from

$$\int_a^{a+4\pi}\frac{\sin\varsigma}{1-\xi_o\cos\varsigma}d\varsigma = -\int_a^{a+4\pi}\frac{d(\cos\varsigma)}{1-\xi_o\cos\varsigma} = \frac{1}{\xi_o}\ln\left[1-\xi_o\cos\varsigma\right]_a^{a+4\pi} = \frac{1}{\xi_o}\ln[1] = 0 \qquad (28)$$

Prob. 4.13.2(cont.)

By use of integral tables, the remaining integral can be carried out.

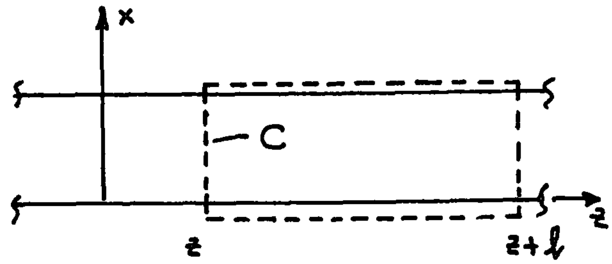
$$G = -\frac{\sin 4\pi\delta}{\xi_o\sqrt{1-\xi_o^2}}\left(1-\sqrt{1-\xi_o^2}\right) \tag{29}$$

In dimensional form, the force per unit area therefore becomes

$$\langle f_z\rangle_t = -\frac{\mu_o K_o^2 \sin 2k\delta}{2\xi_o k\sqrt{1-\left(\frac{\xi_o}{d}\right)^2}}\left[1-\sqrt{1-\left(\frac{\xi_o}{d}\right)^2}\right] \tag{30}$$

Note that under synchronous conditions, the instantaneous force is independent of time, so no time-average is required. Also, in the limit $\xi_o/d \ll 1$, this expression reduces to Eq. 25.

Prob. 4.14.1 Ampere's law and the condition that $\bar{H}$ is solenoidal take the quasi-one-dimensional forms



$$\frac{\partial H_x}{\partial x} = 0 \tag{1}$$

$$\frac{\partial H_z}{\partial x} = \frac{\partial H_x}{\partial z} \tag{2}$$

and it follows that

$$H_x = H_x(z) \tag{3}$$

$$H_z = x\frac{\partial H_x}{\partial z} + f(z,t) \tag{4}$$

The integral form of Ampere's law becomes

$$\oint_C \bar{H}\cdot d\ell = [H_x(z+\ell) - H_x(z)]b = \int_S \bar{J}\cdot\bar{n}\,da = \begin{cases} -n_f i_f - 2N_a(z-\ell/2) \,;\; 0<z<\ell \\ n_f i_f + 2N_a(z-3\ell/2) \,;\; \ell<z<2\ell \end{cases} \tag{5}$$

Because the model represents one closed on itself, $\bar{H}(z+\ell) = -\bar{H}(z)$ and it follows that Eqs. 5 become

$$H_x(z) = \begin{cases} \frac{n_f i_f}{2b} + \frac{N_a i_a}{b}\left(z-\frac{\ell}{2}\right) \,;\; 0<z<\ell \\ -\frac{n_f i_f}{2b} - \frac{N_a i_a}{b}\left(z-\frac{3\ell}{2}\right) \,;\; \ell<z<2\ell \end{cases} \tag{6}$$

and it follows that

$$\frac{\partial H_x}{\partial z} = \begin{cases} \pm N_a i_a/b \,;\; {}^{0}_{\ell}<z<{}^{\ell}_{2\ell} \\ \pm\frac{n_f i_f}{b}u_o\left({}^{z}_{z}-\ell\right) \end{cases} \tag{7}$$

At the rotor surface, where x=0,

$$H_z = \pm N_a i_a \,;\; {}^{0}_{\ell}<z<{}^{\ell}_{2\ell} \tag{8}$$

and so Eq. 7 can be used to deduce that

$$H_z = \begin{cases} \pm N_a i_a\left(\frac{x}{b}-1\right) \,;\; {}^{0}_{\ell}<z<{}^{\ell}_{2\ell} \\ \pm n_f i_f \frac{x}{b}u_o\left({}^{z}_{z}-\ell\right) \mp N_a i_a\left(\frac{x}{b}-1\right) \,;\; z = {}^{0}_{\ell} \end{cases} \tag{9}$$

The force follows from an integration of the stress tensor over the surface of a volume enclosing the rotor with depth d in the y direction and one periodicity length, 2ℓ in the z direction.

$$f = d\int_0^{2\ell} \mu_o H_x H_z\,dz \tag{10}$$

Prob. 4.14.1 (cont.)

This expression is evaluated.

$$f=\mu_o d\left\{\int_0^{\ell}\left[\frac{n_f i_f}{2b}+\frac{N_a i_a}{b}\left(z-\frac{\ell}{2}\right)\right]N_a i_a\left(\frac{x}{b}-1\right)dz+\int_{0^-}^{0^+}\left(-n_f i_f\frac{x}{b}u_o(z)\right)\frac{\ell}{2}\frac{N_a i_a}{b}dz\right. \tag{11}$$

$$\left.+\int_{\ell}^{2\ell}-\left[\frac{n_f i_f}{2b}+\frac{N_a i_a}{b}\left(z-\frac{3\ell}{2}\right)\right]\left[-N_a i_a\left(\frac{x}{b}-1\right)\right]dz+\int_{\ell^-}^{\ell^+}\left[n_f i_f\frac{x}{b}u_o(x-\ell)\right]\left[-\frac{\ell}{2}\frac{N_a i_a}{b}\right]dz\right\}$$

$$=\mu_o d\left[N_a i_a n_f i_f\frac{\ell}{b}\left(\frac{x}{b}-1\right)-N_a i_a n_f i_f\frac{\ell}{b}\frac{x}{b}\right]$$

$$=-N_a n_f i_a i_f \mu_o d\frac{\ell}{b}$$

This detailed calculation is simplified if the surface of integration is pushed to x=0, where the impulses do not contribute and the result is the same as given by Eq. 11.

$$f=-G_m i_f i_a \;;\; G_m\equiv\mu_o\frac{d\ell}{b}N_a n_f \tag{12}$$

Note that this agrees with the result from Prob. 4.10.1, where in the long-wave limit ($b/\ell \ll 1$)

$$G_m\rightarrow\frac{\mu_o d\ell N_a n_f}{b}\sum_{n=1}^{\infty}\frac{8}{\pi^2}\frac{1}{m^2} \tag{13}$$

because

$$\sum_{n=1}^{\infty}\frac{1}{m^2}\rightarrow\frac{\pi^2}{8} \tag{14}$$

To determine the field terminal relation, use Faraday's integral law

$$-v_f+\int_{\text{wire}}\bar{E}\cdot d\ell=-\frac{d}{dt}\lambda_f \;;\; \lambda_f\equiv\int_S\bar{B}\cdot\bar{n}\,da \tag{15}$$

Using the given fields, this expression becomes

$$\lambda_f=n_f\Phi \tag{16}$$

$$\Phi=d\int_0^{\ell}\mu_o H_x\,dz=d\int_0^{\ell}\left[\frac{n_f i_f}{2b}+\frac{N_a i_a}{b}\left(z-\frac{\ell}{2}\right)\right]\mu_o dz=L_f i_f \;;\; L_f=\mu_o d n_f^2\ell/2b$$

This results compares to Eq. 31 of Prob. 4.10.1 where in this limit

$$L_f\rightarrow\frac{4\mu_o \ell d n_f^2}{8b}\sum_{\substack{n=1\\(\text{odd})}}^{\infty}\frac{8}{\pi^2 m^2} \tag{17}$$

The field winding is fixed, so Ohm's law is simply $\bar{J}=\sigma\bar{E}$ and therefore Eq. 15 becomes

$$-v_f+\int_{\text{wire}}\frac{\bar{J}}{\sigma}\cdot d\bar{\ell}=-L_f\frac{di_f}{dt} \tag{18}$$

Because

Prob. 4.14.1 (cont.)

$$R_f \equiv \frac{1}{\sigma}\frac{2n_f d}{A_{wire}} \tag{19}$$

the field equation is

$$v_f = i_f R_f + L_f \frac{di_f}{dt} \tag{20}$$

For the armature the integration is again in the laboratory frame of reference.

The flux linked is

$$\lambda_a = \int_0^\ell \phi(z) N_a \, dz \tag{21}$$

where

$$\phi = \int_z^{z+\ell} \mu_o H_x \, dz' = \int_z^\ell \mu_0\left[\frac{n_f i_f}{2b} + \frac{N_a i_a}{b}\left(z' - \frac{\ell}{2}\right)\right]dz' + \int_\ell^{z+\ell} \mu_0\left[-\frac{n_f i_f}{2b} - \frac{N_a i_a}{b}\left(z' - \frac{3\ell}{2}\right)\right]dz' \tag{22}$$

$$= \frac{\mu_o d}{b}\left[\frac{1}{2} n_f i_f (\ell - 2z) - N_a i_a z(z-\ell)\right]$$

Thus,

$$\lambda_a = L_a i_a; \; L_a \equiv \frac{1}{6}\frac{\mu_o d \ell^3}{b} N_a^2 \tag{23}$$

This compares to the result from Prob. 4.10.1

$$L_a = \frac{N_a^2 d \ell^3 \mu_o}{6b} \sum_{-\infty\,(odd)}^{+\infty} \frac{6 \cdot 16}{\pi^4 m^4} \; ; \; \sum_{\substack{-\infty \\ odd}}^{+\infty} \frac{1}{m^4} = \frac{\pi^4}{6 \cdot 16} \tag{24}$$

For the moving conductors, Ohm's law requires that

$$E_y = \frac{i_a}{A_{wire}\sigma} - v_z \mu_o H_x \tag{25}$$

and so Faraday's law becomes

$$-v_a + d\int_0^\ell N_a E_y \, dz - d\int_\ell^{2\ell} N_a E_y \, dz = -\frac{d}{dt} L_a i_a \tag{26}$$

or

$$-v_a + dN_a\left\{\frac{2\ell i_a}{A_{wire}\sigma} + \int_0^\ell -v_z \mu_o\left[\frac{n_f i_f}{2b} + \frac{N_a i_a}{b}\left(z - \frac{\ell}{2}\right)\right]dz\right. \tag{27}$$

Thus

$$\left. - \int_\ell^{2\ell} -v_z \mu_o\left[-\frac{n_f i_f}{2b} - \frac{N_a i_a}{b}\left(z - \frac{3\ell}{2}\right)\right]dz\right\} = -L_a \frac{di_a}{dt} \tag{28}$$

and finally

$$v_a = i_a R_a - G_m v_z i_f + L_a \frac{di_a}{dt} \tag{29}$$

where

$$R_a = \frac{2\ell d N_a}{A_{wire}\sigma}$$