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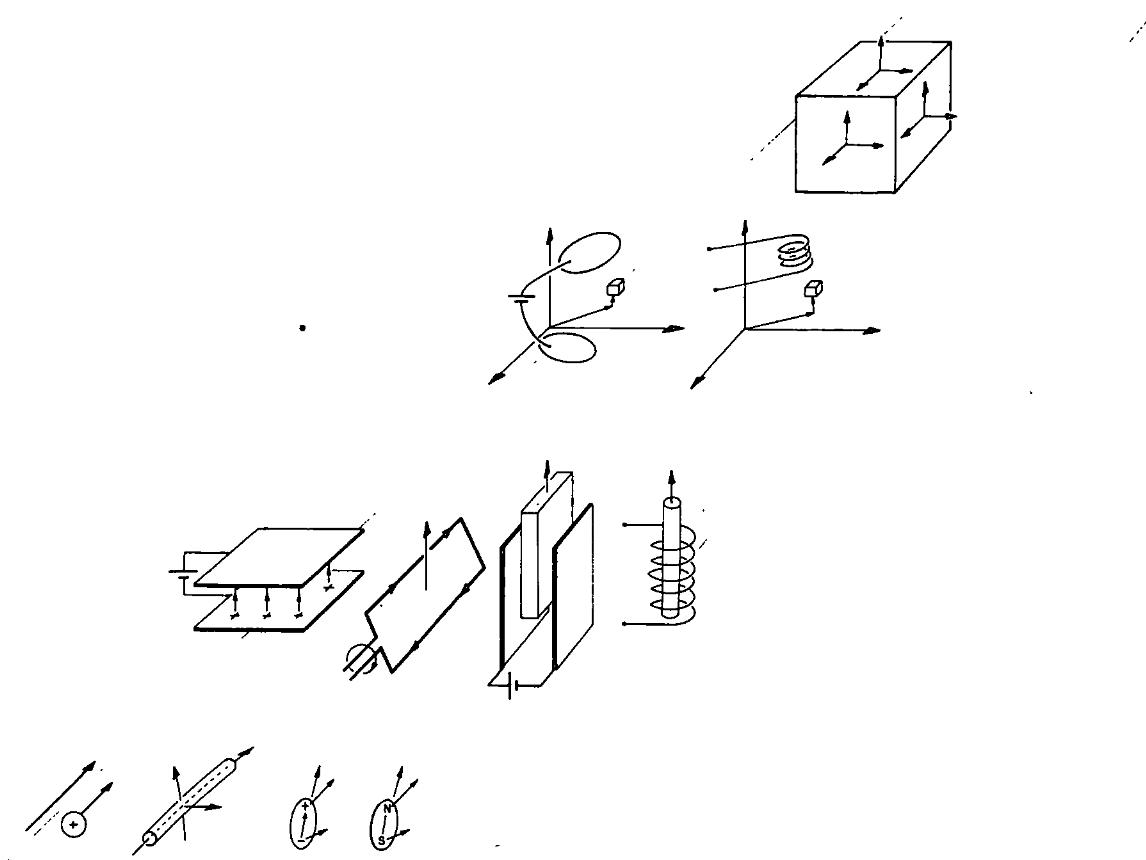
Electromagnetic Forces, Force Densities and Stress Tensors



Prob. 3.3.1 With inertia included but $\bar{H}=0$, Eqs. 3 become

$$m_+ \frac{d\bar{v}_+}{dt} = -m_+ \nu_+ \bar{v}_+ + q_+ \bar{E} \tag{1}$$

$$m_- \frac{d\bar{v}_-}{dt} = -m_- \nu_- \bar{v}_- - q_- \bar{E}$$

With an imposed $\bar{E} = \mathcal{R}e \exp j\omega t$, the response to these linear equations takes the form $\bar{v}_\pm = \mathcal{R}e\, \hat{\bar{v}}_\pm \exp j\omega t$. Substitution into Eqs. 1 gives

$$\hat{\bar{v}}_\pm = \frac{q_\pm \hat{\bar{E}}}{m_\pm (\nu_\pm + j\omega)} \tag{2}$$

Thus, for the effect of inertia to be ignorable

$$\nu_\pm >> \omega \tag{3}$$

In terms of the mobility $b_\pm \equiv q_\pm / m_\pm \nu_\pm$, Eq. 3 requires that

$$q_\pm / b_\pm m_\pm >> \omega = 2\pi f \tag{4}$$

For copper, evaluation gives

$$(1.76 \times 10^{11})/(2\pi)(3\times 10^{-3}) = 9.34 \times 10^{12}\ Hz >> f \tag{5}$$

At this frequency the wavelength of an electromagnetic wave is $\lambda = c/f = 3\times 10^8 / 9.34 \times 10^{12}$, which is approaching the optical range $(32\mu m)$.

Prob. 3.5.1 (a) The cross-derivative of Eq. 9 gives the reciprocity condition

$$\frac{\partial q_1}{\partial v_2} = \frac{\partial^2 w'}{\partial v_1 \partial v_2} = \frac{\partial q_2}{\partial v_1} \tag{1}$$

from which it follows that $C_{12} = C_{21}$.

(b) The coenergy found in Prob. 2.13.1 can be used with Eq. 3.5.9 to find the two forces.

Prob. 3.5.1 (cont.)

$$f_1 = \frac{\partial w'}{\partial \xi_1} = \frac{1}{2} v_1^2 \frac{\partial C_{11}}{\partial \xi_1} + v_1 v_2 \frac{\partial C_{21}}{\partial \xi_1} + \frac{1}{2} v_2^2 \frac{\partial C_{22}}{\partial \xi_1} \tag{2}$$

$$f_2 = \frac{\partial w'}{\partial \xi_2} = \frac{1}{2} v_1^2 \frac{\partial C_{11}}{\partial \xi_2} + v_1 v_2 \frac{\partial C_{21}}{\partial \xi_2} + \frac{1}{2} v_2^2 \frac{\partial C_{22}}{\partial \xi_2} \tag{3}$$

The specific dependences of these capacitances on the displacements are determined in Prob. 2.11.1. Thus, Eqs. 2 and 3 become

$$f_1 = d\epsilon_o \left[\frac{1}{2} v_1^2 \left(\frac{1}{b-\xi_2} - \frac{1}{b}\right) + \frac{v_1 v_2}{b} + \frac{1}{2} v_2^2 \left(\frac{1}{\xi_2} - \frac{1}{b}\right)\right] \tag{4}$$

$$f_2 = d\epsilon_o \left[\frac{1}{2} v_1^2 \frac{\xi_1}{(b-\xi_2)^2} - \frac{1}{2} v_2^2 \frac{\xi_1}{\xi_2^2}\right] \tag{5}$$

<u>Prob. 3.5.2</u> (a) The system is electrically linear, so $w' = \frac{1}{2} Cv^2$, where C is the charge per unit voltage on the positive electrode. Note that throughout the region between the electrodes, $E=v/d$. Hence,

$$w' = \frac{1}{2} v^2 \left[\frac{a w \epsilon_o}{d} + \xi \frac{w}{d}(\epsilon - \epsilon_o)\right] \tag{1}$$

(b) The force due to polarization tending to pull the slab into the region between the electrodes is then

$$f = \frac{\partial w'}{\partial \xi} = wd\,(\epsilon - \epsilon_o)\left(\frac{v}{d}\right)^2 \tag{2}$$

The quantity multiplying the cross-sectional area of the slab, wd, can alternatively be thought of as a pressure associated with the Kelvin force density on dipoles induced in the fringing field acting over the cross-section (Sec. 3.6) or as the result of the Korteweg-Helmholtz force density (Sec. 3.7). The latter is confined to a surface force density acting over the cross-section dw, at the dielectric-free space interface. Either viewpoint gives the same net force.

<u>Prob. 3.5.3</u> From Eq. 9 and the coenergy determined in Prob. 2.13.2,

$$f = \frac{\partial w'(v,\xi)}{\partial \xi} = \qquad + \frac{d}{a_1}\left[(a_2^2 b^2 + v^2)^{\frac{1}{2}} - a_2 b\right] \tag{1}$$

Prob. 3.5.4 (a) Using the coenergy function found in Prob. 2.14.1, the radial surface force density follows as

$$T_r = \frac{1}{2\pi\xi d}\frac{\partial w'}{\partial \xi} = \frac{\mu_o i_1 i_2}{d^2} + \frac{\mu_o i_2^2}{2d^2} \tag{1}$$

(b) A similar calculation using the λ's as the independent variables first requires that $w(\lambda_1, \lambda_2, \xi)$ be found, and this requires the inversion of the inductance matrix terminal relations, as illustrated in Prob. 2.14.1. Then, because the ξ dependence of w is more complicated than of w', the resulting expression is more cumbersome to evaluate.

$$T_r = \frac{-1}{2\pi\xi d}\frac{\partial w}{\partial \xi} = \frac{-1}{2\pi^2\mu_o\xi}\left\{\frac{\xi\lambda_1^2}{(a^2-\xi^2)^2} - \frac{2\xi\lambda_1\lambda_2}{(a^2-\xi^2)^2} + \frac{a^2(2\xi^2-a^2)}{(a^2-\xi^2)^2\xi^3}\lambda_2^2\right\} \tag{2}$$

However, if it is one of the λ's that is contrained, this approach is perhaps worthwhile.

(c) Evaluation of Eq. 2 with $\lambda_2 = 0$ gives the surface force density if the inner ring completely excludes the flux.

$$T_r = \frac{-\lambda_1^2}{2\pi^2\mu_o(a^2-\xi^2)^2} \tag{3}$$

Note that according to either Eq. 1 or 3, the inner coil is compressed, as would be expected by simply evaluating $\bar{J}_f \times \mu_o \bar{H}$. To see this from Eq. 1, note that if $\lambda_2=0$, then $i_1=-i_2$.

Prob. 3.6.1 Force equilibrium for each element of the static fluid is

$$\nabla p = \bar{F} = \nabla\left[\frac{1}{2}(\epsilon-\epsilon_o)E^2\right] \tag{1}$$

where the force density due to gravity could be included, but would not contribute to the discussion. Integration of Eq. (1) from the outside interface (a) to the lower edge of the slab (b)(which is presumed well within the electrodes) can be carried out without regard for the details

Prob. 3.6.1 (cont.)

of the field by using Eq. 2.6.1.

$$\int_a^b \nabla p \cdot d\bar{l} = \int_a^b \nabla\left[\tfrac{1}{2}(\epsilon-\epsilon_o)E^2\right]\cdot d\bar{l} \Rightarrow P_b - P_a = \tfrac{1}{2}(\epsilon-\epsilon_o)\left[E_a^2 - E_b^2\right] \quad (2)$$

Thus, the pressure acting upward on the lower extremity of the slab is

$$P_b = \tfrac{1}{2}(\epsilon-\epsilon_o)E^2 \quad (3)$$

which gives a force in agreement with the result of Prob. 3.5.2, found using the lumped parameter energy method.

$$f = w\,d\,P_b = w\,d\,\tfrac{1}{2}(\epsilon-\epsilon_o)E^2 \quad (4)$$

Prob. 3.6.2 With the charges comprising the dipole respectively at $\bar{r}_+$ and $\bar{r}_-$, the torque is

$$\bar{\tau} = \bar{r}_+ \times q\bar{E}(\bar{r}_+) - \bar{r}_- \times q\bar{E}(\bar{r}_-) \quad (1)$$

Expanding about the position of the negative charge, $\bar{r}_-$,

$$\bar{\tau} \cong (\bar{r}_- + \bar{d}) \times \left[q\bar{E}(\bar{r}_-) + q\bar{d}\cdot\nabla\bar{E}\right] - \bar{r}_- \times q\bar{E}(\bar{r}_-) \quad (2)$$

To first order in $\bar{d}$ this becomes the desired expression.

The torque on a magnetic dipole could be found by using an energy argument for a discrete system, as in Sec. 3.5. Forces and displacements would be replaced by torques and angles. However, because of the complete analogy summarized by Eqs. 8-10, $\bar{H} \leftrightarrow \bar{E}$ and $\bar{P} \leftrightarrow \mu_o\bar{M}$ This means that $\bar{p} \leftrightarrow \mu_o\bar{m}$ and so the desired expression follows directly from Eq. 2.

Prob. 3.7.1 Demonstrate that for a constitutive law implying no interaction the Korteweg-Helmholtz force density

$$\bar{F} = \rho_f\bar{E} + \bar{P}\cdot\nabla\bar{E} + \nabla\left(\tfrac{1}{2}\epsilon_o\bar{E}\cdot\bar{E} + W - \bar{E}\cdot\bar{D} - \sum_{i=1}^{m}\frac{\partial W}{\partial \alpha_i}\alpha_i\right) \quad (1)$$

becomes the Kelvin force density. That is, $(\quad) = 0$. Let $\chi_e = c\rho$, $\alpha_1 = \rho$ and evaluate $(\quad)$

$$W = \int \bar{E}\cdot\delta\bar{D} = \frac{D^2}{2\epsilon_o(1+\chi_e)} = \frac{\bar{E}\cdot\bar{D}}{2} \quad (2)$$

Thus,

$$\frac{\partial W}{\partial \rho} = \frac{\partial W}{\partial \chi_e}\frac{\partial \chi_e}{\partial \rho} = c\left[\frac{-D^2}{2\epsilon_o(1+\chi_e)^2}\right] = -\frac{c\,\epsilon_o E^2}{2} \quad (3)$$

Prob. 3.7.1 (cont.)

so that

$$-\frac{\partial W}{\partial \rho}\rho = \chi_e \frac{\epsilon_o}{2} E^2 \tag{4}$$

and

$$(\ \ ) = \left(\frac{\bar{E}\cdot\bar{D}}{2} + \frac{\epsilon_o}{2}E^2 - \bar{E}\cdot\bar{D} + \frac{\chi_e\epsilon_o}{2}E^2\right)$$

$$= -\frac{\epsilon_o E^2}{2}(1+\chi_e) + \frac{\epsilon_o}{2}E^2 + \chi_e\frac{\epsilon_o}{2}E^2 \tag{5}$$

<u>Prob. 3.9.1</u> In the expression for the torque, Eq. 3.9.16,

$$\bar{r} = x\bar{i}_x + y\bar{i}_y + z\bar{i}_z \tag{1}$$

so that it becomes

$$\bar{T} = \int_V [\bar{i}_x(yF_3 - zF_2) + \bar{i}_y(zF_1 - xF_3) + \bar{i}_3(xF_2 - yF_1)]dV \tag{2}$$

Because $F_i = \partial T_{ij}/\partial x_j$

$$\bar{T} = \int_V \left[\bar{i}_x\left(y\frac{\partial T_{3j}}{\partial x_j} - z\frac{\partial T_{2j}}{\partial x_j}\right) + \bar{i}_y\left(z\frac{\partial T_{1j}}{\partial x_j} - x\frac{\partial T_{3j}}{\partial x_j}\right) + \bar{i}_z\left(x\frac{\partial T_{2j}}{\partial x_j} - y\frac{\partial T_{1j}}{\partial x_j}\right)\right]dV$$

$$= \int_V \Big[\bar{i}_x\Big(\frac{\partial}{\partial x_j} yT_{3j} - \underbrace{T_{3j}\frac{\partial y}{\partial x_j}}_{T_{32}} - \frac{\partial}{\partial x_j} zT_{2j} + \underbrace{T_{2j}\frac{\partial z}{\partial x_j}}_{T_{23}}\Big) \tag{3}$$

$$+ \bar{i}_y\Big(\frac{\partial}{\partial x_j} zT_{1j} - \underbrace{T_{1j}\frac{\partial z}{\partial x_j}}_{T_{13}} - \frac{\partial}{\partial x_j} xT_{3j} + \underbrace{T_{3j}\frac{\partial x}{\partial x_j}}_{T_{31}}\Big)$$

$$+ \bar{i}_z\Big(\frac{\partial}{\partial x_j} xT_{2j} - \underbrace{T_{2j}\frac{\partial x}{\partial x_j}}_{T_{21}} - \frac{\partial}{\partial x_j} yT_{1j} + \underbrace{T_{1j}\frac{\partial y}{\partial x_j}}_{T_{12}}\Big)\Big]dV$$

Prob. 3.9.1 (cont.)

Because $T_{ij} = T_{ji}$ (symmetry)

$$\bar{\Upsilon} = \int_V \frac{\partial}{\partial x_j}\left[\bar{i}_x\left(y T_{3j} - z T_{2j}\right) + \bar{i}_y\left(z T_{1j} - x T_{3j}\right) + \bar{i}_z\left(x T_{2j} - y T_{1j}\right)\right] dV \tag{4}$$

From the tensor form of Gauss' theorem, Eq. 3.8.4, this volume integral becomes the surface integral

$$\bar{\Upsilon} = \oint_S \left[\bar{i}_x\left(y T_{3j} - z T_{2j}\right) + \bar{i}_y\left(z T_{1j} - x T_{3j}\right) + \bar{i}_z\left(x T_{2j} - y T_{1j}\right)\right] n_j \, da \tag{5}$$

$$= \oint_S \bar{r} \times \left(\bar{\bar{T}} \cdot \bar{n}\right) da$$

Prob. 3.10.1 Using the product rule,

$$\bar{F} = \frac{1}{2}\epsilon \nabla(\bar{E}\cdot\bar{E}) = \nabla\left(\frac{1}{2}\epsilon \bar{E}\cdot\bar{E}\right) - \frac{1}{2}\bar{E}\cdot\bar{E}\nabla\epsilon \tag{1}$$

The first term takes the form $\nabla\pi$ while the second agrees with Eq. 3.7.22 if $\rho_f = 0$.

In index notation,

$$F_i = \frac{1}{2}\epsilon \frac{\partial}{\partial x_i}\left(E_k E_k\right) \tag{2}$$

where ε is a spatially varying function.

$$F_i = \epsilon E_k \frac{\partial E_k}{\partial x_i} \tag{3}$$

Because $\nabla \times \bar{E} = 0$,

$$F_i = \epsilon E_k \frac{\partial E_i}{\partial x_k} = \frac{\partial}{\partial x_k}\left(\epsilon E_k E_k\right) - E_i \frac{\partial \epsilon E_k}{\partial x_k} \tag{4}$$

Because $\rho_f = \nabla\cdot\epsilon\bar{E} = 0$, the last term is absent. The first term takes the required form $\partial T_{ik}/\partial x_k$.

Prob. 3.10.2 From Eqs. 2.13.11 and 3.7.19,

$$W' = \int \bar{D}\cdot\delta\bar{E} = \int (d_1 E + d_2 E^2)\delta E = \frac{1}{2}d_1 E^2 + \frac{d_2}{4}E^4; \; T_{ij} = E_i D_j - \delta_{ij} W' \tag{1}$$

Thus, the force density is $(\partial E_i/\partial x_j = \partial E_j/\partial x_i , \partial D_j/\partial x_j = 0)$

$$F_i = \frac{\partial T_{ij}}{\partial x_j} = \frac{\partial E_j}{\partial x_i} D_j - \frac{\partial W'}{\partial x_i} = -\frac{1}{2}\bar{E}\cdot\bar{E}\frac{\partial d_1}{\partial x_i} - \frac{1}{4}(\bar{E}\cdot\bar{E})^2 \frac{\partial d_2}{\partial x_i} \tag{2}$$

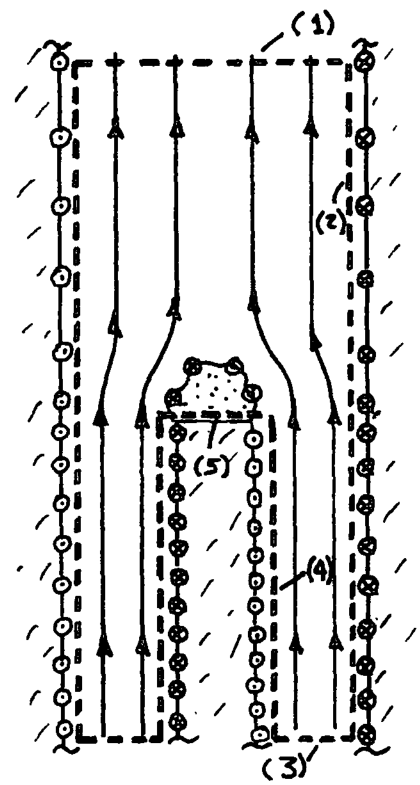
The Kelvin stress tensor, Eq. 3.6.5, differs from Eq. 1b only by the term in δ_{ij}, so the force densities can only differ by the gradient of a pressure.

Prob. 3.10.3

(a) The magnetic field is "trapped" in the region between tubes. For an infinitely long pair of coaxial conductors, the field in the annulus is uniform. Hence, because the total flux $\pi a^2 B_o$ must be constant over the length of the system, in the lower region

$$B_z = \frac{a^2 B_o}{a^2 - b^2} \tag{1}$$

(b) The distribution of surface current is as sketched below. It is determined by the condition that the magnetic flux at the extremities be as found in (a) and by the condition that the normal flux density on any of the perfectly conducting surfaces vanish.



(c) Using the surface force density $\bar{K} \times \langle\bar{B}\rangle$, it is reasonable to expect the net magnetic force in the z direction to be downward.

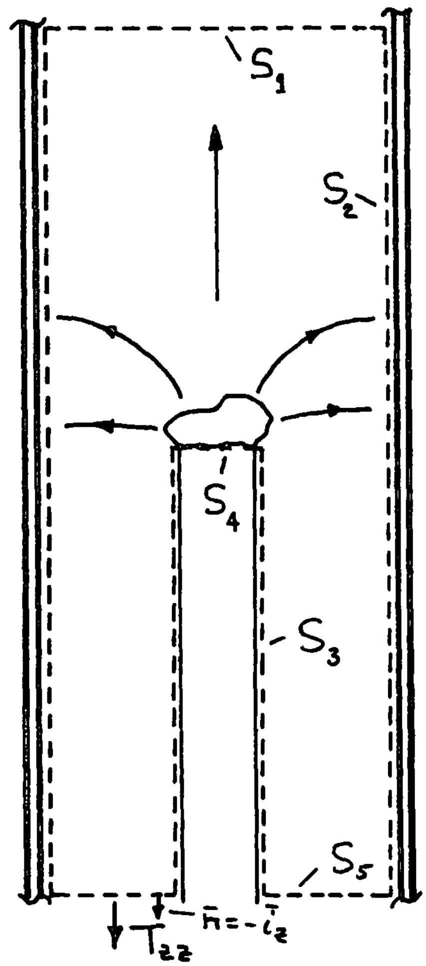
(d) One way to find the net force is to enclose the "blob" by the control volume shown in the figure and integrate the stress tensor over the enclosing surface.

$$f_z = \oint_S T_{zj} n_j da$$

Contributions to this integration over surfaces (4) and (2) (the walls of the inner and outer tubes which are perfectly conducting) vanish because there is no shear stress on a perfectly conducting surface. Surface (5) cuts under the "blob" and hence sustains no magnetic stress. Hence, only surfaces (1) and (3) make contributions, and on them the magnetic flux density is given and uniform. Hence, the net force is

$$f_z = \pi a^2 \left(\frac{B_o^2}{2\mu_o}\right) - \pi(a^2 - b^2)\frac{B_o^2 a^4}{2\mu_o (a^2 - b^2)^2} = -\frac{\pi a^2 B_o^2}{2\mu_o} \frac{b^2}{(a^2 - b^2)} \tag{2}$$

Note that, as expected, this force is negative.



Prob. 3.10.4 The electric field is sketched in the figure. The force on the cap should be upward. To find this force use the surface S shown to enclose the cap. On S_1 the field is zero. On S_2 and S_3 the electric shear stress is zero because it is an equipotential and hence can support no tangential $\bar{E}$. On S_4 the field is zero. Finally, on S_5 the field is that of infinite coaxial conductors.

$$\bar{E} = \bar{i}_r \frac{V_o}{\ln\left(\frac{a}{b}\right)} \frac{1}{r} \tag{1}$$

Thus, the normal electric stress is

$$T_{zz} = -\frac{\epsilon_o}{2} E_r^2 = -\frac{1}{2} \frac{\epsilon_o V_o^2}{\ln^2\left(\frac{a}{b}\right)} \frac{1}{r^2} \tag{2}$$

and the integral for the total force reduces to

$$f_z = \oint_S T_{zj} n_j \, da = -\int_b^a T_{zz} 2\pi r \, dr = \frac{V_o^2 \epsilon_o 2\pi}{2 \ln^2 \frac{a}{b}} \ln \frac{a}{b} = \frac{\pi V_o^2 \epsilon_o}{\ln\left(\frac{a}{b}\right)} \tag{3}$$

Prob. 3.10.5

$$F_i = (\rho_p + \rho_f) E_i = \frac{\partial \epsilon_o E_j}{\partial x_j} E_i = \frac{\partial}{\partial x_j}(\epsilon_o E_i E_j) - \epsilon_o E_j \frac{\partial E_i}{\partial x_j} \tag{1}$$

Because $\frac{\partial E_i}{\partial x_j} = \frac{\partial E_j}{\partial x_i}$, the last term becomes

$$-\epsilon_o E_j \frac{\partial E_i}{\partial x_j} = -\epsilon_o E_j \frac{\partial E_j}{\partial x_i} = -\frac{\partial}{\partial x_i}\left(\frac{1}{2} \epsilon_o E_k E_k\right) \tag{2}$$

Thus

$$F_i = \frac{\partial}{\partial x_j}\left(\epsilon_o E_i E_j - \frac{1}{2} \delta_{ij} \epsilon_o E_k E_k\right) \tag{3}$$

where the quantity in brackets is T_{ij}. Because T_{ij} is the same as any of the T_{ij}'s in Table 3.10.1 when evaluated in free space, use of a surface S surrounding the object to evaluate Eq. 3.9.4 will give a total force in agreement with that predicted by the correct force densities.

Prob. 3.10.6

Showing that the identity holds is a matter of simply writing out the components in cartesian coordinates. The i'th component of the force density is then written using the identity to write $\bar{J}\times\bar{B}$ where $\bar{J} = \nabla\times\bar{H}$.

$$F_i = \frac{\partial H_i}{\partial x_j}B_j - \frac{\partial H_j}{\partial x_i}B_j + \sum_{k=1}^{m}\frac{\partial W}{\partial \alpha_k}\frac{\partial \alpha_k}{\partial x_i} - \frac{\partial}{\partial x_i}\left(\sum_{k=1}^{m}\alpha_k\frac{\partial W}{\partial \alpha_k}\right) \tag{1}$$

In the first term, B_j is moved inside the derivative and the condition $\partial B_j/\partial x_j = \nabla\cdot\bar{B} = 0$ exploited. The third term is replaced by the magnetic analogue of Eq. 3.7.26.

$$F_i = \frac{\partial}{\partial x_j}(H_iB_j) - \frac{\partial H_j}{\partial x_i}B_j + B_j\frac{\partial H_j}{\partial x_i} - \frac{\partial}{\partial x_i}(B_jH_j) + \frac{\partial W}{\partial x_i} - \frac{\partial}{\partial x_i}\sum_{k=1}^{m}\alpha_k\frac{\partial W}{\partial \alpha_k} \tag{2}$$

The second and third terms cancel, so that this expression can be rewritten

$$F_i = \frac{\partial}{\partial x_j}\left[H_iB_j - \delta_{ij}\left(W' + \sum_{k=1}^{m}\alpha_k\frac{\partial W}{\partial \alpha_k}\right)\right];\ W' \equiv \bar{B}\cdot\bar{H} - W \tag{3}$$

and the stress tensor identified as the quantity in brackets.

Problem 3.10.7 The i'th component of the force density is written using the identity of Prob. 2.10.5 to express $\bar{J}_f \times \mu_o\bar{H} = (\nabla\times\bar{H}) \times \mu_o\bar{H}$

$$F_i = \mu_o\left(\frac{\partial H_i}{\partial x_j}H_j\right) - \mu_o\frac{\partial H_j}{\partial x_i}H_j + \mu_o M_j\frac{\partial H_i}{\partial x_j} \tag{1}$$

This expression becomes

$$F_i = \frac{\partial}{\partial x_j}(\mu_o H_iH_j) - H_i\frac{\partial}{\partial x_j}(\mu_o H_j) - \frac{\partial}{\partial x_i}\left(\tfrac{1}{2}\mu_o H_jH_j\right) + \frac{\partial}{\partial x_j}(\mu_o M_jH_i) - H_i\frac{\partial}{\partial x_j}(\mu_o M_j) \tag{2}$$

where the first two terms result from the first term in F_i, the third term results from taking the H_j inside the derivative and the last two terms are an expansion of the last term in F_i. The second and last term combine to give $\nabla\cdot\mu_o(\bar{H}+\bar{M}) \equiv \nabla\cdot\bar{B} = 0$. Thus, with $\bar{B} = \mu_o(\bar{H}+\bar{M})$, the expression takes the proper form for identifying the stress tensor.

$$F_i = \frac{\partial}{\partial x_j}\left[\mu_o(M_j + H_j)H_i - \delta_{ij}\tfrac{1}{2}\mu_o H_kH_k\right] \tag{3}$$

<u>Prob. 3.10.8</u> The integration of the force density over the volume of the dielectric is broken into two parts, one over the part that is well between the plates and therefore subject to a uniform field v/b, and the other enclosing what remains to the left. Observe that throughout this latter volume, the force density acting in the ξ direction is zero. That is, the force density is confined to the interfaces, where it is singular and constitutes a surface force density acting normal to the interfaces. The only region where the force density acts in the ξ direction is on the interface at the right. This is covered by the first integral, and the volume integration can be replaced by an integration of the stress over the enclosing surface. Thus,

$$f = ad\left[-\frac{1}{2}\epsilon_o\left(\frac{v}{b}\right)^2 + \frac{1}{2}\epsilon\left(\frac{v}{b}\right)^2\right] \tag{1}$$

in agreement with the result of Prob. 2.13.2 found using the energy method.

<u>Prob. 3.11.1</u> With the substitution $\bar{V} = -\gamma\bar{n}$ (suppress the subscript E), Eq. 1 becomes

$$-\oint_C \gamma\bar{n}\times d\bar{l} = \int_S\left[-\bar{n}\gamma\nabla\cdot\bar{n} - \bar{n}(\bar{n}\cdot\nabla\gamma) + \bar{n}\cdot(\gamma\bar{n}\nabla)\right]da \tag{1}$$

where the first two terms on the right come from expanding $\nabla\cdot\psi\bar{A} = \psi\nabla\cdot\bar{A} + \bar{A}\cdot\nabla\psi$. Thus, the first two terms in the integrand of Eq. 4 are accounted for. To see that the last term in the integrand on the right in Eq. 1 accounts for remaining term in Eq. (4) of the problem, this term is written out in Cartesian coordinates.

$$\begin{aligned}\bar{n}\cdot(\gamma\bar{n}\nabla) = {} & \bar{i}_x\left[n_x\frac{\partial\gamma n_x}{\partial x} + n_y\frac{\partial\gamma n_y}{\partial x} + n_z\frac{\partial\gamma n_z}{\partial y}\right] \\ & + \bar{i}_y\left[n_x\frac{\partial\gamma n_x}{\partial y} + n_y\frac{\partial\gamma n_y}{\partial y} + n_z\frac{\partial\gamma n_z}{\partial y}\right] \\ & + \bar{i}_z\left[n_x\frac{\partial\gamma n_x}{\partial z} + n_y\frac{\partial\gamma n_y}{\partial z} + n_z\frac{\partial\gamma n_z}{\partial z}\right]\end{aligned} \tag{2}$$

Prob. 3.11.1 (cont.)

Further expansion gives

$$\bar{n}\cdot(\gamma\,\bar{n}\nabla) =$$

$$\bar{i}_x\left[n_x^2\frac{\partial\gamma}{\partial x}+n_y^2\frac{\partial\gamma}{\partial x}+n_z^2\frac{\partial\gamma}{\partial x}\right]+\bar{i}_x\left[n_x\gamma\frac{\partial n_x}{\partial x}+n_y\gamma\frac{\partial n_y}{\partial x}+n_z\gamma\frac{\partial n_z}{\partial x}\right] \tag{3}$$

$$+\bar{i}_y\left[n_x^2\frac{\partial\gamma}{\partial y}+n_y^2\frac{\partial\gamma}{\partial y}+n_z^2\frac{\partial\gamma}{\partial y}\right]+\bar{i}_y\left[n_x\gamma\frac{\partial n_x}{\partial y}+n_y\gamma\frac{\partial n_y}{\partial y}+n_z\gamma\frac{\partial n_z}{\partial y}\right]$$

$$+\bar{i}_z\left[n_x^2\frac{\partial\gamma}{\partial z}+n_y^2\frac{\partial\gamma}{\partial z}+n_z^2\frac{\partial\gamma}{\partial z}\right]+\bar{i}_z\left[n_x\gamma\frac{\partial n_x}{\partial z}+n_y\gamma\frac{\partial n_y}{\partial z}+n_z\gamma\frac{\partial n_z}{\partial z}\right]$$

Note that $n_x^2 + n_y^2 + n_z^2 = 1$. Thus, the first third and fifth terms become $\nabla\gamma$.

The second term can be written as

$$\frac{\gamma}{2}\frac{\partial}{\partial x}\left(n_x^2+n_y^2+n_z^2\right)=\frac{\gamma}{2}\frac{\partial}{\partial x}(1)=0 \tag{4}$$

The fourth and sixth terms are similarly zero. Thus, these three terms vanish and Eq. 3 is simply $\nabla\gamma$. Thus, Eq. 1 becomes

$$-\oint_C \gamma\bar{n}\times d\bar{l} = \int_S\left\{-\bar{n}\gamma\nabla\cdot\bar{n}+\left[\nabla\gamma-\bar{n}(\bar{n}\cdot\nabla\gamma)\right]\right\}da \tag{5}$$

With the given alternative ways to write these terms, it follows that Eq. 5 is consistent with the last two terms of Eq. 3.11.8.

Prob. 3.11.2 Use can be made of Eq. 4 from Prob. 3.11.1 to convert the integral over the surface to one over a contour C enclosing the surface.

$$\bar{f} = -\int_C \gamma_E\,\bar{n}\times d\bar{l} \tag{1}$$

If the surface, S, is closed, then the contour, C, must vanish and it is clear that the net contribution of the integration is zero. The double-layer can not produce a net force on a closed surface.