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Solutions Manual for Continuum Electromechanics

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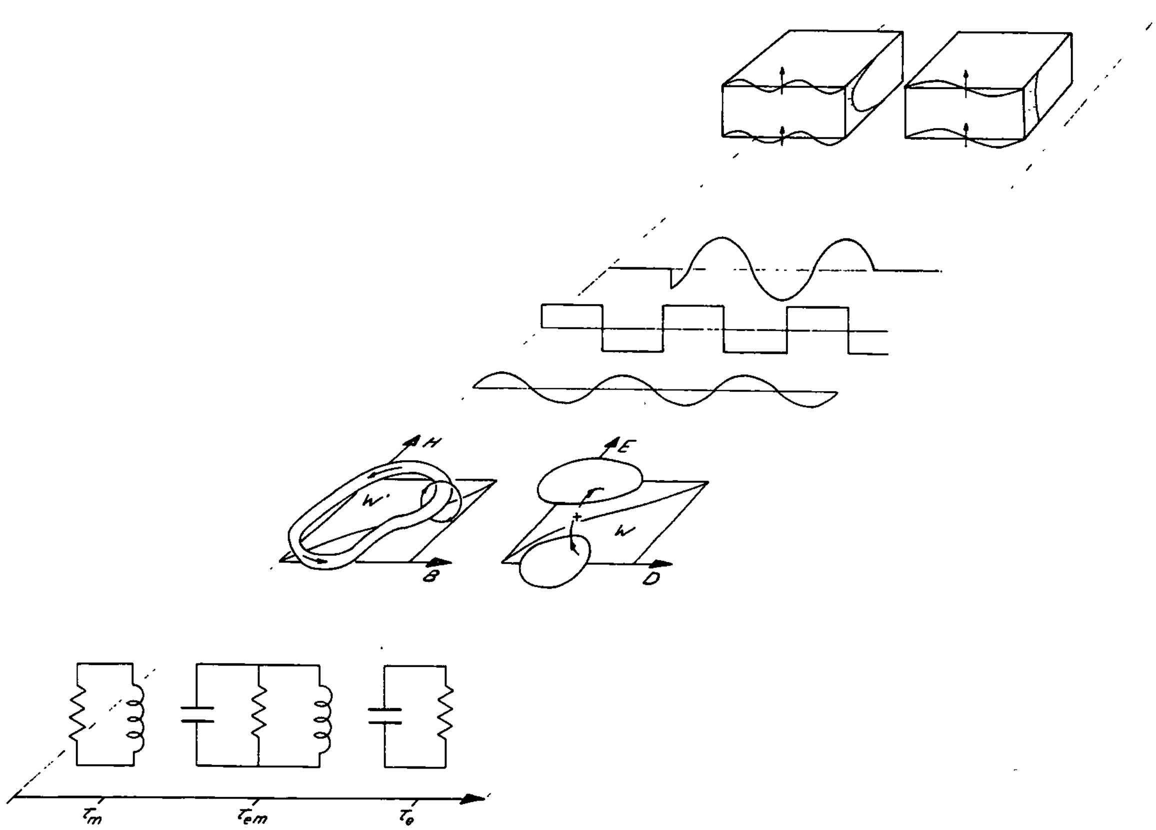
Electrodynamic Laws, Approximations and Relations



Prob. 2.3.1 a) In the free space region between the plates, $\bar{J}_v=\bar{P}=\bar{M}=0$ and Maxwell's equations, normalized in accordance with Eqs. 2.3.4b are

$$\nabla \times \bar{E} = -\frac{\partial \bar{H}}{\partial t} \qquad (1)$$

$$\nabla \times \bar{H} = \beta \frac{\partial \bar{E}}{\partial t} \qquad (2)$$

$$\nabla \cdot \bar{E} = 0 \qquad (3)$$

$$\nabla \cdot \bar{H} = 0 \qquad (4)$$

For fields of the form given, these reduce to just two equations.

$$\frac{\partial E_x}{\partial z} = -\frac{\partial H_y}{\partial t} \qquad (5)$$

$$\frac{\partial H_y}{\partial z} = -\beta \frac{\partial E_x}{\partial t} \qquad (6)$$

Here, the characteristic time is taken as $1/\omega$ so that time dependences $\exp j\omega t$ take the form

$$E_x = \mathrm{Re}\, \hat{E}_x(z)\, e^{jt} \;;\; H_y = \mathrm{Re}\, \hat{H}_y(z)\, e^{jt} \qquad (7)$$

For the time-rate expansion, the dependent variables are expanded in $\beta = \omega^2 \mu\epsilon \ell^2$

$$\hat{E}_x = \sum_{n=0}^{\infty} \hat{E}_{xn} \beta^n \;;\; H_y = \mathrm{Re}\, \hat{H}_{yn} \beta^n \qquad (8)$$

so that Eqs. 5 and 6 become

$$\frac{\partial}{\partial z}\left[\sum_{n=0}^{\infty} \hat{E}_{xn} \beta^n\right] = -j\left[\sum_{n=0}^{\infty} \hat{H}_{yn} \beta^n\right] \qquad (9)$$

$$\frac{\partial}{\partial z}\left[\sum_{n=0}^{\infty} \hat{H}_{yn} \beta^n\right] = -j\beta\left[\sum_{n=0}^{\infty} \hat{E}_{xn} \beta^n\right] \qquad (10)$$

Equating like powers of β results in a hierarchy of expressions

$$\frac{\partial \hat{E}_{xn}}{\partial z} = -j\, \hat{H}_{yn} \qquad (11)$$

$$\frac{\partial \hat{H}_{yn}}{\partial z} = -j\, \hat{E}_{x(n-1)} \qquad (12)$$

Boundary conditions on the upper and lower plates are satisfied identically. (No tangential $\bar{E}$ and no normal $\bar{B}$ at the surface of a perfect conductor.) At $z=0$ where there is also a perfectly conducting plate, $E_x=0$. At $z=-\ell$, Ampere's law requires that $i/w=H_y$ (boundary condition, 2.10.21). (Because $w \gg s$, the magnetic field intensity outside the region between the plates is negligible compared to that inside.) With the characteristic magnetic field taken as I_o/w, where $i(t) = \underline{i}(t)\ I_o$, it follows that the normalized boundary conditions are

$$\hat{E}_x(0) = 0 \;;\; \hat{H}_y(-1) = 1 \qquad (13)$$

Prob. 2.3.1 (cont)

The zero order Eq. 12 requires that

$$\frac{\partial \hat{H}_{y0}}{\partial z} = 0 \qquad (13)$$

and reflects the nature of the magnetic field distribution in the static limit $\beta \rightarrow 0$. The boundary condition on H_y, Eq. 13, evaluates the integration constant.

$$\hat{H}_{y0} = 1 \qquad (14)$$

The electric field induced through Faraday's law follows by using this result in the zero order statement of Eq. 11. Because what is on the right is independent of z, it can be integrated to give

$$\hat{E}_{x0} = -jz \qquad (15)$$

Here, the integration constant is zero because of the boundary condition on E_x, Eq. 13. These zero order fields are now used to find the first order fields The n=1 version of Eq. 12 with the right hand side evaluated using Eq. 15 can be integrated. Because the zero order fields already satisfy the boundary conditions, it is clear that all higher order terms must vanish at the appropriate boundary, E_{xn} at z=0 and H_{yn} at z=1. Thus, the integration constant is evaluated and

$$\hat{H}_{y1} = -\frac{1}{2}(z^2-1) \qquad (16)$$

This expression is inserted into Eq. 11 with n=1, integrated and the constant evaluated to give

$$\hat{E}_{x1} = j\frac{1}{2}\left(\frac{1}{3}z^3-z\right) \qquad (17)$$

If the process is repeated, it follows that

$$\hat{H}_{y2} = \frac{1}{4}\left(\frac{1}{6}z^4- \quad z^2+\frac{5}{6}\right) \qquad (18)$$

$$\hat{E}_{x2} = -j\frac{1}{4}\left(\frac{1}{30}z^5-\frac{1}{3}z^3+\frac{5}{6}z\right) \qquad (19)$$

so that, with the coefficients defined by Eqs. 15-19, solutions to order are

$$\hat{E}_x = E_{x0}+E_{x1}\beta + E_{x2}\beta^2;\ \hat{H}_y = \hat{H}_{y0}+\hat{H}_{y1}\beta+\hat{H}_{y2}\beta^2 \qquad (20)$$

Prob. 2.3.1(cont.)

Note that the surface charge on the lower electrode, as well as the surface current density there, are related to the fields between the electrodes by

$$\sigma_f = E_x \quad ; \quad K_z = H_y \tag{21}$$

The respective quantities on the upper electrode are the negatives of these quantities. (Gauss' law and Ampere's law). With Eqs. 7 used to recover the time dependence, what have been found to second order in β are the normalized fields

$$E_x = z\left[1 - \tfrac{1}{2}\left(\tfrac{1}{3}z^2 - 1\right)\beta + \tfrac{1}{4}\left(\tfrac{1}{30}z^4 - \tfrac{1}{3}z^2 + \tfrac{5}{6}\right)\beta^2\right]\sin t = \sigma_f \tag{22}$$

$$H_y = \left[1 - \tfrac{1}{2}\left(z^2 - 1\right)\beta + \tfrac{1}{4}\left(\tfrac{1}{6}z^4 - z^2 + \tfrac{5}{6}\right)\beta^2\right]\cos t = K_z \tag{23}$$

The dimensioned forms follow by identifying

$$E_x = \frac{\mu_o \omega l I_o}{w} \tag{24}$$

e) Now, consider the exact solutions. Eqs. 7 substituted into Eqs. 5 and 6 give

$$\frac{d^2\hat{H}_y}{dz^2} + \beta\hat{H}_y = 0 \tag{25}$$

$$\hat{E}_x = \frac{j}{\beta}\frac{d\hat{H}_y}{dz} \tag{26}$$

Solutions that satisfy these expressions as well as Eqs. 13 are

$$\hat{H}_y = \cos(\sqrt{\beta}\, z)/\cos\sqrt{\beta} \tag{27}$$

$$\hat{E}_x = \frac{-j}{\sqrt{\beta}}\sin(\sqrt{\beta}\, z)/\cos\sqrt{\beta} \tag{28}$$

These can be expanded to second order in β as follows.

$$\hat{H}_y \cong \frac{1 - \frac{1}{2}\beta z^2 + \frac{1}{4!}\beta^2 z^4 + \cdots}{1 - \frac{1}{2}\beta + \frac{1}{4!}\beta^2 + \cdots} \tag{29}$$

$$\cong \left(1 - \tfrac{1}{2}\beta z^2 + \tfrac{1}{4!}\beta^2 z^4\right)\left(1 - \left(-\tfrac{1}{2}\beta + \tfrac{1}{4!}\beta^2\right) + \left(-\tfrac{1}{2}\beta + \tfrac{1}{4!}\beta^2\right)^2\right)$$

$$= \left(1 - \tfrac{1}{2}\beta z^2 + \tfrac{1}{24}\beta^2 z^4\right)\left(1 + \tfrac{1}{2}\beta + \tfrac{5}{24}\beta^2 - \tfrac{1}{24}\beta^3 + \tfrac{1}{576}\beta^4\right)$$

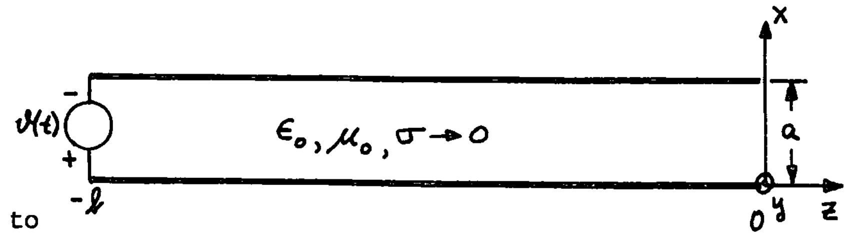
$$\cong 1 - \tfrac{1}{2}\left(z^2 - 1\right)\beta + \tfrac{1}{4}\left(\tfrac{1}{6}z^4 - z^2 + \tfrac{5}{6}\right)\beta^2$$

Prob. 2.3.1(cont.)

$$\hat{E}_x \cong \frac{-j}{\sqrt{\beta}} \frac{(\sqrt{\beta}\,z) - \frac{1}{3!}(\sqrt{\beta}\,z)^3 + \frac{1}{5!}(\sqrt{\beta}\,z)^5 - \cdots}{1 - \frac{1}{2}\beta + \frac{1}{4!}\beta^2 - \cdots} \tag{30}$$

$$\cong -jz\left[1 - \frac{1}{2}\left(\frac{1}{3}z^2 - 1\right)\beta + \frac{1}{4}\left(\frac{1}{30}z^4 - \frac{1}{3}z^2 + \frac{5}{6}\right)\beta^2\right]$$

These expressions thus prove to be the same expansions as found from the time-rate expansion.



Prob. 2.3.2 Assume

$$\bar{E} = \bar{i}_x E_x(z,t)$$
$$\bar{H} = \bar{i}_y H_y(z,t)$$

and Maxwell's equations reduce to

$$\frac{\partial E_x}{\partial z} = -\frac{\partial \mu_o H_y}{\partial t} \quad ; \quad -\frac{\partial H_y}{\partial z} = \frac{\partial \epsilon_o E_x}{\partial t} \tag{1}$$

In normalized form (Eqs. 2.3.5a-2.3.10a) these are

$$\frac{\partial E_x}{\partial z} = -\beta \frac{\partial H_y}{\partial t} \quad ; \quad -\frac{\partial H_y}{\partial z} = \frac{\partial E_x}{\partial t} \tag{2}$$

Let

$$E_x = E_{xo} + \beta E_{x1} + \beta^2 E_{x2} + \cdots$$
$$H_y = E_{yo} + \beta H_{y1} + \beta^2 H_{y2} + \cdots \tag{3}$$

Then, Eqs. 2 become

$$\frac{\partial E_{xo}}{\partial z} + \beta\left[\frac{\partial E_{x1}}{\partial z} + \frac{\partial H_{yo}}{\partial t}\right] + \beta^2\left[\frac{\partial E_{x2}}{\partial z} + \frac{\partial H_{y1}}{\partial t}\right] + \cdots = 0$$
$$\left[\frac{\partial H_{yo}}{\partial z} + \frac{\partial E_{xo}}{\partial t}\right] + \beta\left[\frac{\partial H_{y1}}{\partial z} + \frac{\partial E_{x1}}{\partial t}\right] + \beta^2\left[\frac{\partial H_{y2}}{\partial z} + \frac{\partial E_{x2}}{\partial t}\right] + \cdots = 0 \tag{4}$$

Zero order terms in β require

$$\frac{\partial E_{xo}}{\partial z} = 0 \Rightarrow E_{xo} = E_{xo}(t) = \frac{v(t)}{a}\frac{1}{\mathcal{E}} \tag{5}$$

$$\frac{\partial H_{yo}}{\partial z} = -\frac{\partial E_{xo}}{\partial t} = -\frac{1}{a\mathcal{E}}\frac{dv}{dt} \Rightarrow H_{yo} = -\frac{1}{a\mathcal{E}}\frac{dv}{dt} z \tag{6}$$

Boundary conditions have been introduced to insure $E_x(-\ell,t) = v/a$ and, because $K_z(0,t) = 0$, $H_y(0,t) = 0$

Now consider first order terms.

$$\frac{\partial E_{x1}}{\partial z} = -\frac{\partial H_{yo}}{\partial t} = \frac{1}{a\mathcal{E}}\frac{d^2 v}{dt^2} z \Rightarrow E_{x1} = \frac{1}{a\mathcal{E}}\frac{d^2 v}{dt^2}\frac{1}{2}(z^2 - \ell^2)$$
$$\frac{\partial H_{y1}}{\partial z} = -\frac{\partial E_{x1}}{\partial t} = -\frac{1}{a\mathcal{E}}\frac{d^3 v}{dt^3}\frac{1}{2}(z^2 - \ell^2) \Rightarrow H_{y1} = \frac{-1}{a\mathcal{E}}\frac{d^3 v}{dt^3}\frac{1}{2}\left(\frac{z^3}{3} - z\right) \tag{7}$$

Prob. 2.3.2 (cont.)

The integration functions in these last two functions are determined by the boundary conditions which, because the first terms satisfy the boundary conditions, must satisfy homogeneous boundary conditions; $\underline{E}_{x1}(\underline{z}=-1)=0, \underline{H}_{y1}(0)=0$.
In normalized form, we have

$$E_x = \frac{v(t)}{a\mathcal{E}} + \beta\frac{1}{a\mathcal{E}}\frac{d^2v}{dt^2}\frac{1}{2}(z^2-1)+\cdots$$

$$H_y = -\frac{1}{a\mathcal{E}}\frac{dv}{dt}z - \frac{\beta}{a\mathcal{E}}\frac{d^3v}{dt^3}\frac{1}{2}\left(\frac{z^3}{3}-z\right)+\cdots \tag{8}$$

In unnormalized form

$$E_x = \frac{v(t)}{a} + \frac{\mu_o\epsilon_o}{a}\frac{d^2v}{dt^2}\frac{1}{2}(z^2-\ell^2)+\cdots$$

$$H_y = -\frac{\epsilon_o}{a}\frac{dv}{dt}z - \frac{\mu_o\epsilon_o^2}{a}\frac{d^3v}{dt^3}\frac{1}{2}\left(\frac{z^3}{3}-z\ell^2\right)+\cdots \tag{9}$$

Compare these series to the exact solutions, which by inspection are

$$E_x = \frac{v_o}{a}\frac{\cos\frac{\omega}{c}z}{\cos\frac{\omega}{c}\ell}\cos\omega t \cong \frac{v_o}{a}\cos\omega t\left[1-\frac{1}{2}\frac{\omega^2}{c^2}(z^2-\ell^2)+\cdots\right]$$

$$H_y = \frac{v_o}{a\mu_o c}\frac{\sin\left(\frac{\omega}{c}z\right)}{\cos\left(\frac{\omega}{c}\ell\right)}\sin\omega t \simeq \frac{v_o}{a}\sqrt{\frac{\epsilon_o}{\mu_o}}\left[\frac{\omega z}{c}+\frac{1}{2}\left(\frac{\omega}{c}\right)^3\left(\ell^2 z-\frac{1}{3}z^3\right)+\cdots\right]$$

Thus, the formal expansion gives the same result as a series expansion of the exact solution. Note that what is being expanded is

$$\left(\frac{\omega}{c}z\right)^2 = \left[\frac{\sqrt{\mu_o\epsilon_o}\,\ell}{1/\omega}\left(\frac{z}{\ell}\right)\right]^2 \equiv \beta\left(\frac{z}{\ell}\right)^2$$

The quasi-static equations are Eqs. 5 and 6 in unnormalized form, which respectively represent the one-dimensional forms of $\nabla\times\bar{E}=0$ and conservation

Prob. 2.3.2 (cont.)

of charge ($H_y \leftrightarrow K_z$ in lower electrode), give the zero order solutions. Conservation of charge on electrode gives linearly increasing K_z which is the same as H_y.

Prob. 2.3.3 In the volume of the Ohmic conductor, Eqs. 2.2.1-2.2.5, with $\bar{P}=\bar{M}=\bar{v}=0$) become

$$\nabla\times\bar{E} = -\mu_o\frac{\partial\bar{H}}{\partial t} \tag{1}$$

$$\nabla\times\bar{H} = \epsilon_o\frac{\partial\bar{E}}{\partial t} + \sigma\bar{E} \tag{2}$$

$$\nabla\cdot\epsilon_o\bar{E} = \rho_f \tag{3}$$

$$\nabla\cdot\mu_o\bar{H} = 0 \tag{4}$$

Fields are now assumed that are transverse to their spatial dependence, z, that satisfy the boundary conditions on the electrodes at x=0 and x=a (no tangential $\bar{E}$ or normal $\bar{H}$) and that have the same temporal dependence as the excitation.

$$\bar{E} = \bar{E}(z,t) = \bar{i}_x\, Re[\hat{E}_x(z)\exp j\omega t] \tag{5}$$

$$\bar{H} = \bar{H}(z,t) = \bar{i}_y\, Re[\hat{H}_y(z)\exp j\omega t] \tag{6}$$

It follows that $\rho_f=0$ and that all components of Eqs. 1 and 2 are identically satisfied except the y component of Eq. 1 and the x component of Eq. 2, which require that

$$\frac{d\hat{E}_x}{dz} = -j\omega\mu_o\hat{H}_y \tag{7}$$

$$-\frac{d\hat{H}_y}{dz} = (\sigma+j\omega\epsilon_o)\hat{E}_x \tag{8}$$

Transverse fields are solenoidal, so Eqs. 3 and 4 are identically satisfied with $\rho_f=0$. (See Sec. 5.10 for a discussion of why $\rho_f=0$ in the volume of a uniform conductor. Note that the arguments given there can be applied to a conductor at rest without requiring that the system be EQS.)

Elimination of E_x between Eqs. 8 and 7 shows that

$$\frac{d^2\hat{H}_y}{dz^2} + k^2\hat{H}_y = 0\ ;\quad k^2 \equiv \omega^2\mu_o\epsilon_o - j\omega\mu_o\sigma \tag{9}$$

and in terms of $\hat{H}_y$, $\hat{E}_x$ follows from Eq. 8.

$$\hat{E}_x = \frac{-1}{\sigma+j\omega\epsilon_o}\frac{d\hat{H}_y}{dz} \tag{10}$$

b) Solutions to Eq. 9 take the form

$$\hat{H}_y = H_+e^{-jkz} + H_-e^{jkz} \tag{11}$$

Prob. 2.3.3(cont.)

In terms of these same coefficients, H_+ and H_-, it follows from Eq. 10 that

$$\hat{E}_x = \frac{-jk}{\sigma + j\omega\epsilon_o}\left[H_+ e^{-jkz} - H_- e^{jkz}\right] \tag{12}$$

Because the electrodes are very long in the y direction compared to the spacing a, and because fringing fields are ignored at z=0, the magnetic field outside the region between the perfectly conducting electrodes is essentially zero. It follows from the boundary condition required by Ampere's law at the respective ends (Eq. 21 of Table 2.10.1) that

$$H_y(0,t) = 0 \Rightarrow \hat{H}_y(0) = 0 \tag{13}$$

$$H_y(-\ell,t) = Re\, \hat{K} \exp j\omega t \Rightarrow \hat{H}_y(-\ell) = \hat{K} \tag{14}$$

Thus, the two coefficients in Eq. 11 are evaluated and the expressions of Eqs. 11 and 12 become those given in the problem statement.

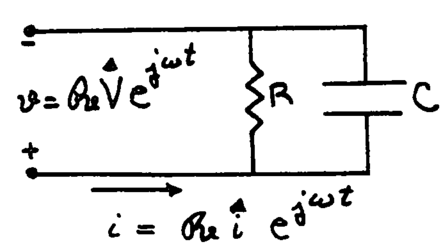
c) Note that

$$k\ell = \sqrt{\omega^2\mu_o\epsilon_o\ell^2 - j\omega\mu_o\sigma\ell^2} = \sqrt{(\omega\tau_{em})^2 - j(\omega\tau_m)^2} \tag{15}$$

so, $|k\ell| \ll 1$ provided that $\omega\tau_{em} \ll 1$ and $\omega\tau_m \ll 1$. To obtain the limiting form of E_x, the exponentials are expanded to first order in $k\ell$. In itself, the approximation does not imply an ordering of the characteristic times. However, if the frequency dependence of E_x expressed by the limiting form is to have any significance, then it is clear that the ordering must be $\tau_m < \tau_{em} < \tau_e$ as illustrated by Fig. 2.3.1 for the EQS approximation.

With the voltage and current defined as $v = E_x(-\ell,t)a$, $i = Kd$, it follows from the limiting form of E_x that

$$\hat{V} = \frac{\hat{I}}{\frac{\sigma\ell d}{a} + j\omega\left(\frac{\epsilon_o\ell d}{a}\right)} \tag{16}$$



This is of the same form as the relation

$$\hat{V} = \frac{\hat{I}}{\frac{1}{R} + j\omega C} \tag{17}$$

found for the circuit shown. Thus, as expected, $C = \epsilon_o\ell d/a$ and $R = a/\sigma\ell d$.

In the MQS approximation, where $\omega\tau_m$ is arbitrary, it is helpful to write Eq. 15 in the form

$$k\ell = \sqrt{-j\omega\tau_m(1+j\omega\tau_e)} \tag{18}$$

The second terms is negligible (the displacement current is small compared to the conduction current) if $\omega\tau_e \ll 1$, in which case

$$k \approx (-1+j)/\delta_m \; ; \; \delta_m \equiv \sqrt{\frac{2}{\omega\mu_o\sigma}} \tag{19}$$

Then, the magnetic field distribution assumes the limiting form

$$H_y = Re\left\{\frac{\hat{K}}{e^{-j\frac{\ell}{\delta_m}}e^{-\frac{\ell}{\delta_m}} - e^{j\frac{\ell}{\delta_m}}e^{\frac{\ell}{\delta_m}}}\left(e^{\frac{z}{\delta_m}}e^{j(\omega t+\frac{z}{\delta_m})} - e^{-\frac{z}{\delta_m}}e^{j(\omega t-\frac{z}{\delta_m})}\right)\right\} \tag{20}$$

That is, Eddy currents induced in the conductor tend to shield out the magnetic field, which tends to be confined to the neighborhood of the current source. The skin depth, δ_m, serves notice that the phenomena accounting for the superimposed decaying waves represented by Eq. 20 is magnetic diffusion. With the exclusion of the displacement current, the dynamics no longer have the attributes of an electromagnetic wave.

It is easy to see that this MQS approximation is valid only if $\omega\tau_e \ll 1$, but how does this imply that $\omega\tau_{em} \ll 1$? Here, the implicite relation between τ_m, τ_e and τ_{em} comes into play. What is considered negligible in Eq. 18 by making $\omega\tau_e \ll 1$ is neglected in the same expression written in terms of τ_{em} and τ_m as Eq. 15 by making $\tau_{em} \ll \tau_m$. Thus, the ordering of characteristic times is $\tau_e < \tau_{em} < \tau_m$, as summarized by the MQS sketch of Fig. 2.3.1.

d) The electroquasistatic equations, Eqs. 2.3.23a-2.3.25a, require that

$$\frac{\partial E_x}{\partial z} = 0 \tag{21}$$

so that E_x is independent of z (uniform) and

$$\frac{d\hat{H}_y}{dz} = -(\sigma + j\omega\epsilon_o)\hat{E}_x \tag{22}$$

It follows that this last expression can be integrated on z with the constant of integration taken as zero because of boundary condition, Eq. 13. That H_y also satisfy Eq. 14 then results in

Prob. 2.3.3(cont.)

$$\hat{E}_x = (\hat{K}/\sigma \ell)/(1 + j\omega\tau_e) \tag{23}$$

which is the same as the EQS limit of the exact solution, Eq. 16.

e) In the MQS limit, where Eqs. 2.3.23a-2.3.25a apply, equations combine to show that H_y satisfies the diffusion equation.

$$\frac{1}{\mu_o\sigma}\frac{\partial^2 H_y}{\partial z^2} = \frac{\partial H_y}{\partial t} \Rightarrow \frac{d^2 \hat{H}_y}{dz^2} = -j\omega\mu_o\sigma \hat{H}_y \tag{24}$$

Formal solution of this expression is the same as carried out in general, and results in Eq. 20.

Why is it that in the EQS limit the electric field is uniform, but that in the MQS limit the magnetic field is not? In the EQS limit, the fundamental field source is ρ_f while for the magnetic field it is $\bar{J}_f$. For this particular problem, where the volume is filled by a uniformly conducting material, there is no accumulation of free charge density, and hence no shielding of $\bar{E}$ from the volume. By contrast, the volume currents can shield the magnetic field from the volume by "skin effect"....the result of having a continuum of inductances and resistances. To have a case study exemplifying how the accumulation of ρ_f (at an interface) can shield out an electric field, consider this same configuration but with the region $0 < x < a$ half filled with conductor ($0 < x < b$) and half free space ($b < x < a$).

Prob. 2.3.4 The conduction constitutive law can be used to eliminate $\bar{E}$ in the law of induction. Then, Eqs. 23b-26b determine $\bar{H}$, $\bar{M}$ and hence $\bar{J}_f$. That the curl of $\bar{E}$ is then specified is clear from the law of induction, Eq. 25b, because all quantities on the right are known from the MQS solution. The divergence of $\bar{E}$ follows by solving the constitutive law for $\bar{E}$ and taking its divergence.

$$\nabla\cdot\bar{E} = \nabla\cdot\left(\frac{\bar{J}_f}{\sigma}\right) - \nabla\times(\bar{v}\times\mu_o\bar{H}) \tag{1}$$

All quantities on the right in this expression have also been found by solving the MQS equations. Thus, both the curl and divergence of $\bar{E}$ are known and $\bar{E}$ is uniquely specified. Given a constitutive law for $\bar{P}$, Gauss Law, Eq. 27b, can be used to evaluate ρ_f.

Prob. 2.4.1 For the given displacement vector in Lagrangian coordinates, the velocity follows from Eq. 2.6.1 as

$$\bar{v} = \frac{\partial \bar{\xi}}{\partial t} = -r_o \Omega \sin(\Omega t + \theta_o)\bar{i}_x + \Omega r_o \cos(\Omega t + \theta_o)\bar{i}_y \tag{1}$$

In turn, the acceleration follows from Eq. 2.6.2.

$$\bar{a} = \frac{\partial \bar{v}}{\partial t} = -r_o \Omega^2 \left[\cos(\Omega t + \theta_o)\bar{i}_x + \sin(\Omega t + \theta_o)\bar{i}_y\right] \tag{2}$$

But, in view of Eq. 1, this can also be written in the more familiar form

$$\bar{a} = -\Omega^2 \bar{\xi} \tag{3}$$

Prob. 2.4.2 From Eq. 2.4.4, it follows that in Eulerian coordinates the acceleration is

$$\bar{a} = \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}\right)\bar{i}_x + \left(v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y}\right)\bar{i}_y = -\Omega^2 x \bar{i}_x - \Omega^2 y \bar{i}_y \tag{4}$$

Using coordinates defined in the problem, this is converted to cylindrical form.

$$\bar{a} = -\Omega^2 r\left[\cos\theta(\cos\theta\, \bar{i}_r - \sin\theta\, \bar{i}_\theta) + \sin\theta(\sin\theta\, \bar{i}_r + \cos\theta\, \bar{i}_\theta)\right] \tag{5}$$

Because $\cos^2\theta + \sin^2\theta = 1$, it follows that

$$\bar{a} = -\Omega^2 r\, \bar{i}_r \tag{6}$$

which is equivalent to Eq. 3 of Prob. 2.4.1.

Prob. 2.5.1 By definition, the convective derivative is the time rate of change for an observer moving with the velocity $\bar{v}$, which in this case is $U\bar{i}_x$. Hence,

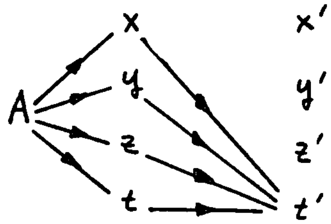
$$\frac{D\Phi}{Dt} = \frac{\partial \Phi'}{\partial t'}$$

and evaluation gives

$$j(\omega - kU)\hat{\Phi} = j\omega'\hat{\Phi}'$$

Because the amplitudes are known to be equal at the same position and time it follows that $\omega - kU = \omega'$. Here, ω is the doppler shifted frequency. The special case where the frequency in the moving frame is zero makes evident why the shift in frequency. In that case $\omega' = 0$ and the moving observer sees a static distribution of Φ that varies sinusoidally with position. The fixed observer sees this distribution moving by with the velocity $U = \omega/k$ and hence observes the frequency kU.

Prob. 2.5.2 To take the derivative with respect to primed variables, say t'; observe in $\bar{A}(x,y,z,t)$, that each variable can in general depend on that variable (say t').



(1)

Thus

$$\frac{\partial A_i}{\partial t'} = \frac{\partial A_i}{\partial t}\frac{\partial t}{\partial t'} + \frac{\partial A_i}{\partial x}\frac{\partial x}{\partial t'} + \frac{\partial A_i}{\partial y}\frac{\partial y}{\partial t'} + \frac{\partial A_i}{\partial z}\frac{\partial z}{\partial t'} \tag{2}$$

From Eq. 1,

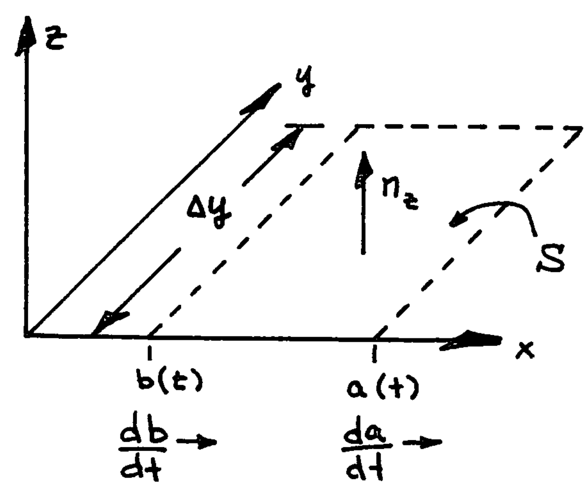
$$\begin{aligned} x &= x' + U_x t' \\ y &= y' + U_y t' \\ z &= z' + U_z t' \\ t &= t' \end{aligned} \quad \Rightarrow \quad \begin{aligned} \frac{\partial t}{\partial t'} &= 1 \\ \frac{\partial x}{\partial t'} &= U_x \\ \frac{\partial y}{\partial t'} &= U_y \\ \frac{\partial z}{\partial z'} &= U_z \end{aligned} \tag{3}$$

so

$$\frac{\partial A_i}{\partial t'} = \frac{\partial A_i}{\partial t}(1) + \frac{\partial A_i}{\partial x}U_x + \frac{\partial A_i}{\partial y}U_y + \frac{\partial A_i}{\partial z}U_z = \frac{\partial A_i}{\partial t} + \bar{U}\cdot\nabla A_i \tag{4}$$

Here, if $\bar{A}$ is a vector then A_i is one of its cartesian components. If $A_i \to \psi$, the scalar form is obtained.

Prob. 2.6.1 For use in Eq. 2.6.4, take as A the given one dimensional function with the surface of integration that shown in the figure. The edges at x=a and x=b have the velocities in the x direction indicated. Thus, Eq. 2.6.4 becomes



$$\Delta y \frac{d}{dt}\int f(x,t)\,dx =$$

$$\Delta y\left[\int \frac{\partial f}{\partial t}dx + \int \frac{\partial f}{\partial z}v_z\,dx\right] + \Delta y\left[f(a)\frac{da}{dt} - f(b)\frac{db}{dt}\right] \tag{1}$$

The second term on the right is zero because A has no divergence. Thus, Δy can be divided out to obtain the given one-dimensional form of Leibnitz' rule.

Prob. 2.6.2 a) By Gauss' theorem,

$$\int_V \nabla\cdot\bar{A}\,dV = \oint_S \bar{A}\cdot\bar{i}_n\,da \tag{1}$$

where on S_1, $\bar{i}_n=\bar{n}$, on S_2, $\bar{i}_n=-\bar{n}$ and on the sides $\bar{i}_n$ has the direction of $-\bar{v}\times d\bar{l}$. Also, $\bar{i}_n da$ integrated between S_1 and S_2 is approximated by $-\bar{v}\Delta t\times d\bar{l}$. Thus, it follows that if all integrals are taken at the same instant in time,

$$\int_V \nabla\cdot\bar{A}\,dV = \int_{S_2}\bar{A}(t)\cdot\bar{n}\,da - \int_{S_1}\bar{A}(t)\cdot\bar{n}\,da - \oint_{C_1}\bar{A}\cdot\bar{v}\Delta t\times d\bar{l} \tag{2}$$

b) At any location,

$$\bar{A}(t+\Delta t) = \bar{A}(t) + \frac{\partial\bar{A}}{\partial t}\Delta t + \cdots \tag{3}$$

Thus, the integral over S_2 when it actually has that location gives

$$\int_{S_2}\bar{A}(t+\Delta t)\cdot\bar{n}\,da = \int_{S_2}\bar{A}(t)\cdot\bar{n}\,da + \int_{S_2}\frac{\partial\bar{A}}{\partial t}\Delta t\cdot\bar{n}\,da + \cdots \tag{4}$$

Because S_2 differs from S_1 by terms of higher order than Δt , the second integral can be evaluated to first order in Δt on S_1.

$$\int_{S_2}\bar{A}(t+\Delta t)\cdot\bar{n}\,da = \int_{S_2}\bar{A}(t)\cdot\bar{n}\,da + \int_{S_1}\frac{\partial\bar{A}}{\partial t}\Delta t\cdot\bar{n}\,da \tag{5}$$

c) For the elemental volume pictured, the height is $\Delta t\,\bar{v}\cdot\bar{n}$, while the area of the base is da , so to first order in Δt , the volume integral reduces to

$$\int_V \nabla\cdot\bar{A}\,dV \cong \int_{S_1}\nabla\cdot\bar{A}\,\Delta t\,\bar{v}\cdot\bar{n}\,da \tag{6}$$

d) What is desired is

$$\frac{d}{dt}\int_S \bar{A}\cdot\bar{n}\,da = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{S_2}\bar{A}(t+\Delta t)\cdot\bar{n}\,da - \int_{S_1}\bar{A}(t)\cdot\bar{n}\,da\right] \tag{7}$$

Substitution from Eq. 5 into this expression gives

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{S_2}\bar{A}(t)\cdot\bar{n}\,da + \int_{S_1}\frac{\partial\bar{A}}{\partial t}(t)\,\Delta t\cdot\bar{n}\,da - \int_{S_1}\bar{A}(t)\cdot\bar{n}\,da\right] \tag{8}$$

The first and last terms on the right can be replaced using Eq. 2

Prob. 2.6.2(cont.)

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_V \nabla\cdot\bar{A}\,dV + \oint_C \bar{A}\cdot\bar{v}\,\Delta t \times d\bar{l} + \int_{S_1} \frac{\partial \bar{A}}{\partial t}\Delta t\cdot\bar{n}\,da\right] \tag{9}$$

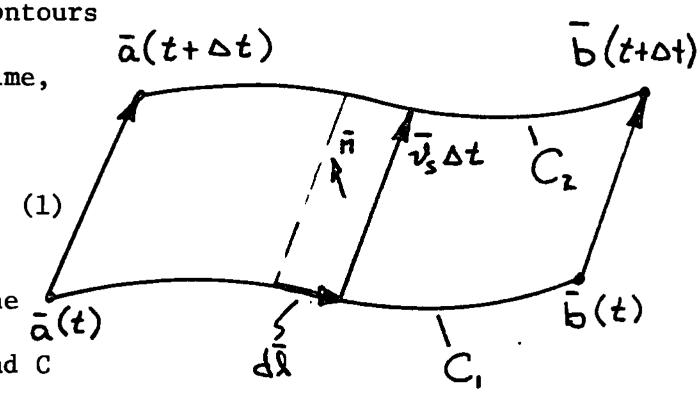
Finally, given that $\bar{A}\cdot\bar{v}\Delta t\times d\bar{l} = \bar{A}\times\bar{v}\Delta t\cdot d\bar{l}$, Eq. 6 is substituted into this expression to obtain

$$\frac{d}{dt}\int_S \bar{A}\cdot\bar{n}\,da = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{S_1}\nabla\cdot\bar{A}\,\Delta t\,\bar{v}\cdot\bar{n}\,da + \int_{S_1}\frac{\partial\bar{A}}{\partial t}\Delta t\cdot\bar{n}\,da + \Delta t\oint_C \bar{A}\times\bar{v}\cdot d\bar{l}\right] \tag{10}$$

With Δt divided out, this is the desired Leibnitz rule generalized to three dimensions.

<u>Prob. 2.6.3</u> Given the geometry of contours C_1 and C_2, if $\bar{A}$ is evaluated at one time, t, Stoke's theorem applies

$$\int_S \nabla\times\bar{A}\cdot\bar{n}\,da = \oint_C \bar{A}\cdot d\bar{l} \tag{1}$$



Here, S is the surface swept out by the open contour during the interval Δt and C is composed of C_1, C_2 and the side segments represented to first order in Δt by $\bar{v}_s(\bar{b}(t), t)\Delta t$ and $\bar{v}_s(\bar{a}(t),t)\Delta t$. Note that for Δt small, $\bar{n} = d\bar{l}\times\bar{v}_s\Delta t$ with $\bar{v}_s$ evaluated at time t. Thus, to linear terms in Δt , Eq. 1 becomes

$$\int_{\bar{a}(t)}^{\bar{b}(t)} \nabla\times\bar{A}\Big|_{C_1,t}\cdot d\bar{l}\times\bar{v}_s\Big|_{C_1,t}\Delta t = \int_{\bar{a}}^{\bar{b}}\bar{A}\Big|_t\cdot d\bar{l} + \bar{A}\cdot\bar{v}_s\Big|_{\bar{b}(t),t}\Delta t - \int_{\bar{a}(t+\Delta t)}^{\bar{b}(t+\Delta t)}\bar{A}\Big|_t\cdot d\bar{l} - \bar{A}\cdot\bar{v}_s\Big|_{\bar{a}(t),t}\Delta t \tag{2}$$

Note that, again to linear terms in Δt,

$$\int_{\bar{a}(t+\Delta t)}^{\bar{b}(t+\Delta t)} \bar{A}(t+\Delta t)\cdot d\bar{l} \cong \int_{\bar{a}(t+\Delta t)}^{\bar{b}(t+\Delta t)} \bar{A}\Big|_t\cdot d\bar{l} + \int_{\bar{a}(t+\Delta t)}^{\bar{b}(t+\Delta t)} \frac{\partial\bar{A}}{\partial t}\Big|_t \Delta t\cdot d\bar{l} \tag{3}$$

Prob. 2.6.3 (cont.)

The first term on the right in this expression is substituted for the third one on the right in Eq. 2, which then becomes

$$\int_{\bar{a}(t)}^{\bar{b}(t)} \nabla\times\bar{A}\Big|_t \cdot d\bar{\ell}\times\bar{v}_s\Big|_{C_1,t}\Delta t = \int_{\bar{a}(t)}^{\bar{b}(t)} \bar{A}\Big|_{C_1,t}\cdot d\bar{\ell} + \bar{A}\cdot\bar{v}_s\Big|_{b(t),t}\Delta t \tag{4}$$

$$- \int_{\bar{a}(t+\Delta t)}^{\bar{b}(t+\Delta t)} \bar{A}(t+\Delta t)\cdot d\bar{\ell} + \int_{\bar{a}(t+\Delta t)}^{\bar{b}(t+\Delta t)} \frac{\partial\bar{A}}{\partial t}\Big|_{C_1,t}\Delta t\cdot d\bar{\ell} - \bar{A}\cdot\bar{v}_s\Big|_{a(t),t}\Delta t$$

The first and third terms on the right comprise what is required to evaluate the derivative. Note that because the integrand of the fourth term is already first order in Δt , the end points can be evaluated when t=t.

$$\frac{d}{dt}\int_{\bar{a}(t)}^{\bar{b}(t)} \bar{A}\cdot d\bar{\ell} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\int_{\bar{a}(t)}^{\bar{b}(t)}\frac{\partial\bar{A}}{\partial t}\Delta t\cdot d\bar{\ell}\right.$$

$$\left. + \bar{A}\cdot\bar{v}_s\Big|_{\bar{b}(t),t}\Delta t - \bar{A}\cdot\bar{v}_s\Big|_{\bar{a}(t),t}\Delta t + \int_{\bar{a}(t)}^{\bar{b}(t)} \nabla\times\bar{A}\cdot\bar{v}_s\times d\bar{\ell}\,\Delta t\right\} \tag{5}$$

The sign of the last term has been reversed because the order of the cross product is reversed. The Δt cancels out on the right-hand side and the expression is the desired generalized Leibnitz rule for a time-varying contour integration.

Prob. 2.8.1 a) In the steady state and in the absence of a conduction current, $\bar{J}_f$, Ampere's law requires that

$$\nabla\times\bar{H} = \nabla\times(\bar{P}\times\bar{v}) \tag{1}$$

so one solution follows by setting the arguments equal.

$$\bar{H} = \bar{P}\times\bar{v} = -\left(\frac{P_o a}{\pi}\right)U\sin\left(\frac{\pi x}{a}\right)\bar{i}_z \tag{2}$$

Because the boundary conditions, H_z(x=±a)=0 are also satisfied, this is the required solution. For different boundary conditions, a "homogeneous" solution would have to be added.

Prob. 2.8.2 (cont.)

b) The polarization current density follows by direct evaluation.

$$\bar{J}_p = \nabla\times(\bar{P}\times\bar{v}) = \rho_o U \cos(\pi x/a)\bar{i}_y \tag{3}$$

Thus, Ampere's law reads

$$\nabla\times\bar{H} = -\frac{\partial H_z}{\partial x}\bar{i}_y = \rho_o U \cos(\pi x/a)\bar{i}_y \tag{4}$$

where it has been assumed that $\partial(\ )/\partial y$ and $\partial(\ )/\partial z = 0$. Integration then gives the same result as in Eq. 2.

c) The polarization charge is

$$\rho_p = -\nabla\cdot\bar{P} = -\frac{\partial P_x}{\partial x} = \rho_o \cos(\pi x/a) \tag{5}$$

and it can be seen that in this case, $\bar{J}_p = U\rho_p\bar{i}_y$. This is a special case because in general the polarization current is

$$\nabla\times(\bar{P}\times\bar{v}) \equiv \bar{P}\nabla\cdot\bar{v} - \bar{v}\nabla\cdot\bar{P} + \bar{v}\cdot\nabla\bar{P} - \bar{P}\cdot\nabla\bar{v} \tag{6}$$

In this example, the first and last terms vanish because the motion is rigid body, while (because there is no y variation), the next to last term $\bar{v}\cdot\nabla\bar{P} = U\partial\bar{P}/\partial y = 0$. The remaining term is simply $\rho_p\bar{v}$.

Prob. 2.9.1 a) With $\bar{M}$ the only source of $\bar{H}$, it is reasonable to presume that $\bar{H}$ only depends on x and it follows from Gauss' law for $\bar{H}$ that

$$\nabla\cdot\bar{H} = -\nabla\cdot\bar{M} \Rightarrow \frac{\partial H_x}{\partial x} = \frac{\rho_o}{\mu_o}\cos(\pi x/a) \Rightarrow H_x = \frac{\rho_o a}{\mu_o\pi}\sin\left(\frac{\pi x}{a}\right) \tag{1}$$

b) A solution to Faraday's law that also satisfies the boundary conditions follows by simply setting the arguments of the curls equal.

$$\bar{E} = -\mu_o\bar{M}\times\bar{v} = \frac{\rho_o a}{\pi} U \sin\left(\frac{\pi x}{a}\right)\bar{i}_z \tag{2}$$

c) The current is zero because $\bar{E}'=0$. To see this, use the results of Eqs. 1 and 2 to evaluate

$$\bar{E}' = \bar{E} + \bar{v}\times\mu_o\bar{H} = \bar{i}_z\left[\frac{\rho_o a}{\pi}U\sin\left(\frac{\pi x}{a}\right) - \frac{\rho_o a}{\pi}U\sin\left(\frac{\pi x}{a}\right)\right] = 0 \tag{3}$$

Prob. 2.11.1 With regions to the left, above and below the movable electrode denoted by (a), (b) and (c) respectively, the electric fields there (with up defined as positive) are

$$E_a = (v_2 - v_1)/b \ ; \ E_b = -v_1/(b-\xi_2) \ ; \ E_c = v_2/\xi_2 \tag{1}$$

On the upper electrode, the total charge is the area $d(a-\xi_1)$ times the charge per unit area on the left section of the electrode, $-\epsilon_o E_a$, plus the area $d\xi_1$ times the charge per unit area on the right section, $-\epsilon_o E_b$. The charge on the lower electrode follows similarly so that the capacitance matrix is

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = d\epsilon_o \begin{bmatrix} \frac{a-\xi_1}{b} + \frac{\xi_1}{b-\xi_2} & -\frac{(a-\xi_1)}{b} \\ -\frac{(a-\xi_1)}{b} & \frac{a-\xi_1}{b} + \frac{\xi_1}{\xi_2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{2}$$

Prob. 2.12.1 Define regions (a) and (b) as between the two coils and inside the inner one respectively and it follows that the magnetic fields are uniform in each region and given by

$$H_a = \frac{i_1}{d} \ ; \ H_b = H_a + \frac{i_2}{d} = \frac{i_1}{d} + \frac{i_2}{d} \tag{1}$$

These fields are defined as positive into the paper. Note that they satisfy Ampere's law and the divergence condition in the volume and the jump and boundary conditions at the boundaries. For the contours as defined, the normal to the surface defining λ_1 is into the paper. The fields are uniform, so the surface integral is carried out by multiplying the flux density, $\mu_o H$, by the appropriate area. For example, λ_1 is found as

$$\lambda_1 = \mu_o \frac{i_1}{d}\pi(a^2 - \xi^2) + \mu_o\left(\frac{i_1}{d} + \frac{i_2}{d}\right)\pi\xi^2 \tag{2}$$

Thus, the flux linkages are

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \frac{\mu_o \pi a^2}{d} & \frac{\mu_o \pi \xi^2}{d} \\ \frac{\mu_o \pi \xi^2}{d} & \frac{\mu_o \pi \xi^2}{d} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \tag{3}$$

Prob. 2.13.1 It is a line integration in the state-space (v_1, v_2, ξ_1, ξ_2) that is called for. The system has already been assembled mechanically, so the displacements (ξ_1, ξ_2) are fixed. The remaining path of integration in the space (v_1, v_2) is carried out by raising v_1 to its final value with $v_2=0$ and then raising v_2 with v_1 fixed (so that $\delta v_1=0$) at its final value. Thus,

$$w' = \int q_1 \delta v_1 + q_2 \delta v_2 = \int_0^{v_1} q_1(v_1', 0, \xi_1, \xi_2)\delta v_1' + \int_0^{v_2} q_2(v_1, v_2', \xi_1, \xi_2)\delta v_2' \qquad (1)$$

and with the introduction of the capacitance matrix,

$$w' = \frac{1}{2} C_{11} v_1^2 + C_{21} v_1 v_2 + \frac{1}{2} C_{22} v_2^2 \qquad (2)$$

Note that $C_{21}=C_{12}$.

Prob. 2.13.2 Even with the nonlinear dielectric, the electric field between the electrodes is simply v/b. Thus, the surface charge on the lower electrode, where there is free space, is $D=\epsilon_o E=\epsilon_o v/b$, while that adjacent to the dielectric is

$$D = \frac{\epsilon_o v}{b} + v/b\, \alpha_1 \sqrt{\alpha_2^2 + \left(\frac{v}{b}\right)^2} \qquad (1)$$

It follows that the net charge is

$$q = d a \epsilon_o \frac{v}{b} + \frac{d \xi v}{\alpha_1 \sqrt{\alpha_2^2 b^2 + v^2}} \qquad (2)$$

so that

$$w' = \int_0^v q\,\delta v = \frac{\epsilon_o}{2}\frac{d}{b}(a) v^2 + \frac{d\xi}{\alpha_1}\left[(\alpha_2^2 b^2 + v^2)^{\frac{1}{2}} - \alpha_2 b\right] \qquad (3)$$

Prob. 2.14.1 a) To find the energy, it is first necessary to invert the terminal relations found in Prob. 2.14.1. Cramer's rule yields

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} \frac{d}{\mu_o \pi (a^2-\xi^2)} & \frac{-d}{\mu_o \pi (a^2-\xi^2)} \\ \frac{-d}{\mu_o \pi (a^2-\xi^2)} & \frac{d}{\mu_o \pi (a^2-\xi^2)}\left(\frac{a}{\xi}\right)^2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \qquad (1)$$

Integration of Eq. 2.14.11 in (λ_1, λ_2) space can be carried out along any path. But, in particular, integrate on λ_1 with $\lambda_2=0$. Then, with λ_1 at its final value, integrate on λ with $\delta\lambda_1=0$.

Prob. 2.14.1 (cont.)

$$w = \int_0^{\lambda_1} i_1(\lambda_1',0)\,d\lambda_1' + \int_0^{\lambda_2} i_2(\lambda_1,\lambda_2')\,d\lambda_2' \tag{2}$$

$$= \frac{1}{2}\left[\frac{d}{\mu_0\pi(a^2-\xi^2)}\right]\lambda_1^2 - \left[\frac{d}{\mu_0\pi(a^2-\xi^2)}\right]\lambda_1\lambda_2 + \frac{1}{2}\left[\frac{d}{\mu_0\pi(a^2-\xi^2)}\left(\frac{a}{\xi}\right)^2\right]\lambda_2^2$$

b) The coenergy is found from Eq. 2.14.12 where the flux linkages as given in the solution to Prob. 2.12.1 can be used directly. Now, the integration is in (i_1,i_2) space, and is carried out as in part (a), but with the i's playing the role of the λ's.

$$w' = \int_0^{i_1} \lambda_1(i_1',0)\,di_1' + \int_0^{i_2} \lambda_2(i_1,i_2')\,di_2' \tag{3}$$

$$= \frac{1}{2}\left(\frac{\mu_0\pi a^2}{d}\right)i_1^2 + \frac{\mu_0\pi\xi^2}{d}i_1 i_2 + \frac{1}{2}\left(\frac{\mu_0\pi\xi^2}{d}\right)i_2^2$$

Prob. 2.15.1 Following the outlined procedure,

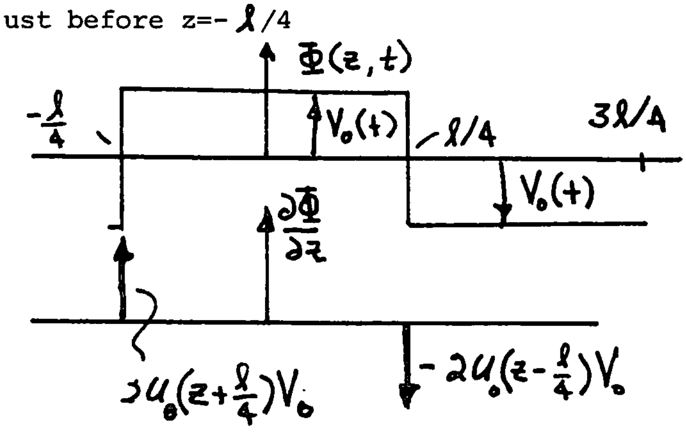
$$\int_z^{z+\ell} \Phi(z,t)\,e^{jk_m z}\,dz = \int_z^{z+\ell} \sum_{n=-\infty}^{+\infty} \tilde{\Phi}_n\, e^{j(k_m - k_n)z}\,dz \tag{1}$$

Each term in the series is integrated to give

$$= \sum_{n=-\infty}^{+\infty} \frac{\tilde{\Phi}_n}{j} e^{j\frac{2\pi}{\ell}(m-n)z}\left[e^{j2\pi(m-n)} - 1\right] \Big/ \frac{2\pi}{\ell}(m-n) \tag{2}$$

Thus, for $m \neq n$, all terms vanish. The term $m=n$ is evaluated by either taking the limit $m \to n$ of Eq. 2 or returning to Eq. 1 to see that the right hand side is simply $\tilde{\Phi}_m \ell$. Thus, solution for $\tilde{\Phi}_m$ gives Eq. 8.

Prob. 2.15.2 One period of the distribution is sketched as a function of z as shown. Note that the function starts just before $z=-\ell/4$ and terminates just before $z=3\ell/4$. The coefficients follow directly from Eq. 8. Especially for ramp functions, it is often convenient to make use of the fact that



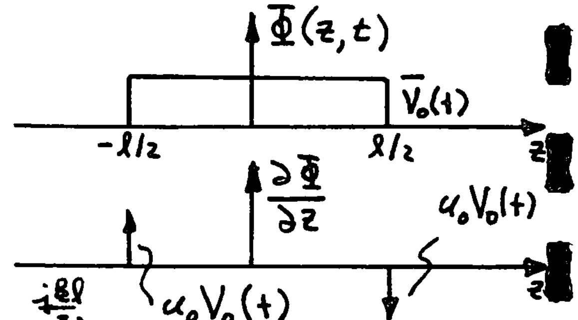
$$\frac{\partial\Phi}{\partial z} \longleftrightarrow -jk_m\tilde{\Phi}_m \tag{1}$$

Prob. 2.15.2 (cont.)

and find the coefficients of the derivative of $\Phi(z,t)$, as shown in the sketch. Thus,

$$-jk_m\tilde{\Phi}_m = \frac{1}{l}\int_{-l/4}^{l/4}\frac{\partial\Phi}{\partial z}e^{jk_m z}dz = \frac{2V_o}{l}\left(e^{-j\frac{k_m l}{4}} - e^{j\frac{k_m l}{4}}\right) \tag{2}$$

and it follows that the coefficients are as given. Note that m=0 must give $\tilde{\Phi}_m = 0$ because there is no space average to the potential. That the other even components vanish is implicite to Eq. 2.



Prob. 2.15.3 The dependence on z of Φ and its spatial derivative are as sketched. Because the transform of $\partial\Phi/\partial z \leftrightarrow -jk\tilde{\Phi}$, the integration over the two impulse functions gives simply

$$-jk\tilde{\Phi} = \int_{-\infty}^{+\infty}\frac{\partial\Phi}{\partial z}e^{jkz}dz = 2V_o\frac{\left(e^{-j\frac{kl}{2}} - e^{j\frac{kl}{2}}\right)}{2} \tag{1}$$

Solution of this expression for $\tilde{\Phi}$ results in the given transform. More direct, but less convenient, is the direct evaluation of Eq. 2.15.10.

Prob. 2.15.4 Evaluation of the required space average is carried out by fixing attention on one value of n in the infinite series on n and considering the terms of the infinite series on m. Thus,

$$\langle AB\rangle_z = \sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}\tilde{A}_n\tilde{B}_m\frac{1}{l}\int_z^{z+l}\exp -j(k_n+k_m)dz \tag{1}$$

$$= \sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}\tilde{A}_n\tilde{B}_m\frac{j}{(n+m)}\exp -2\pi j(n+m)$$

Thus, all terms are zero except the one having n=-m. That term is best evaluated using the original expression to carry out the integration. Thus,

$$\langle AB\rangle_z = \sum_{n=-\infty}^{+\infty}\tilde{A}_n\tilde{B}_{-n} \tag{2}$$

Because the Fourier series is required to be real, $\tilde{B}_{-n} = \tilde{B}_n^*$ and hence the given expression of Eq. 2.15.17 follows.

Prob. 2.16.1 To be formal about deriving transfer relations of Table 2.16.1, start with Eq. 2.16.14

$$\tilde{\Phi} = \tilde{\Phi}_1 \sinh \gamma x + \tilde{\Phi}_2 \cosh \gamma x \tag{1}$$

and require that $\tilde{\Phi}(x=\Delta) \equiv \tilde{\Phi}^\alpha,\ \tilde{\Phi}(x=0) \equiv \tilde{\Phi}^\beta$. Thus,

$$\begin{bmatrix} \sinh \gamma\Delta & \cosh \gamma\Delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\Phi}_1 \\ \tilde{\Phi}_2 \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^\alpha \\ \tilde{\Phi}^\beta \end{bmatrix} \tag{2}$$

Inversion gives (by Cramer's rule)

$$\begin{bmatrix} \tilde{\Phi}_1 \\ \tilde{\Phi}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sinh \gamma\Delta} & -\coth \gamma\Delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^\alpha \\ \tilde{\Phi}^\beta \end{bmatrix} \tag{3}$$

Because $\tilde{D}_x = -\epsilon\, \partial\tilde{\Phi}/\partial x$, it follows for Eq. 1 that

$$\tilde{D}_x = -\epsilon\gamma \left(\tilde{\Phi}_1 \cosh \gamma x + \tilde{\Phi}_2 \sinh \gamma x\right) \tag{4}$$

Evaluation at the respective boundaries gives

$$\begin{bmatrix} \tilde{D}_x^\alpha \\ \tilde{D}_x^\beta \end{bmatrix} = -\epsilon\gamma \begin{bmatrix} \cosh \gamma\Delta & \sinh \gamma\Delta \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\Phi}_1 \\ \tilde{\Phi}_2 \end{bmatrix} \tag{5}$$

Finally, substitution of Eq. 3 for the column matrix on the right in Eq. 5 gives

$$\begin{aligned} \begin{bmatrix} \tilde{D}_x^\alpha \\ \tilde{D}_x^\beta \end{bmatrix} &= -\epsilon\gamma \begin{bmatrix} \cosh \gamma\Delta & \sinh \gamma\Delta \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sinh \gamma\Delta} & -\coth \gamma\Delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^\alpha \\ \tilde{\Phi}^\beta \end{bmatrix} \\ &= -\epsilon\gamma \begin{bmatrix} \coth \gamma\Delta & -\frac{1}{\sinh \gamma\Delta} \\ \frac{1}{\sinh \gamma\Delta} & -\coth \gamma\Delta \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^\alpha \\ \tilde{\Phi}^\beta \end{bmatrix} \end{aligned} \tag{6}$$

which is Eq. (a) of Table 2.16.1.

Prob. 2.16.1 (cont.)

The second form, Eq. (b), is obtained by applying Cramer's rule to the inversion of Eq. 8. Note that the determinant of the coefficients is

$$\text{Det} = -\coth^2\gamma\Delta + \frac{1}{\sinh^2\gamma\Delta} = \frac{1-\cosh^2\gamma\Delta}{\sinh^2\gamma\Delta} = -1 \qquad (7)$$

so

$$\begin{bmatrix} \tilde{\Phi}^\alpha \\ \tilde{\Phi}^\beta \end{bmatrix} = \frac{1}{\epsilon\gamma}\begin{bmatrix} -\coth\gamma\Delta & \dfrac{1}{\sinh\gamma\Delta} \\ \dfrac{-1}{\sinh\gamma\Delta} & \coth\gamma\Delta \end{bmatrix}\begin{bmatrix} \tilde{D}_x^\alpha \\ \tilde{D}_x^\beta \end{bmatrix} \qquad (8)$$

Prob. 2.16.2 For the limit m=0,k=0, solutions are combined to satisfy the potential constraints by Eq. 2.16.20, and it follows that the electric displacement is

$$\tilde{D}_r = -\epsilon\frac{\partial\tilde{\Phi}}{\partial r} = -\epsilon\,\tilde{\Phi}^\alpha\frac{\left(\frac{1}{r}\right)}{\ln\left(\frac{\alpha}{\beta}\right)} + \epsilon\,\tilde{\Phi}^\beta\frac{\left(\frac{1}{r}\right)}{\ln\left(\frac{\alpha}{\beta}\right)} \qquad (1)$$

This is evaluated at the respective boundaries to give Eq. (a) of Table 2.16.2 with f_m and g_m as defined for k=0,m=0.

For k=0,m$\neq$0, the correct combination of potentials is given by Eq. 2.16.21. It follows that

$$\tilde{D}_r = \epsilon m\left\{\frac{\tilde{\Phi}^\alpha}{\beta}\left[\frac{\left(\frac{\beta}{r}\right)^{m+1}+\left(\frac{r}{\beta}\right)^{m-1}}{\left(\frac{\beta}{\alpha}\right)^m-\left(\frac{\alpha}{\beta}\right)^m}\right] - \frac{\tilde{\Phi}^\alpha}{\alpha}\left[\frac{\left(\frac{r}{\alpha}\right)^{m-1}+\left(\frac{\alpha}{r}\right)^{m+1}}{\left(\frac{\beta}{\alpha}\right)^m-\left(\frac{\alpha}{\beta}\right)^m}\right]\right\} \qquad (2)$$

Evaluation of this expression at the respective boundaries gives Eqs. (a) of Table 2.16.2 with entries f_m and g_m as defined for the case k=0,m=0.

For k $\neq$ 0,m $\neq$ 0, the potential is given by Eq. 2.16.25. Thus, the electric displacement is

$$\tilde{D}_r = -jk\left\{\tilde{\Phi}^\alpha\frac{[H_m(jk\beta)J_m'(jkr) - J_m(jk\beta)H_m'(jkr)]}{[H_m(jk\beta)J_m(jk\alpha) - J_m(jk\beta)H_m(jk\alpha)]}\right.$$

$$\left. + \tilde{\Phi}^\beta\frac{[J_m(jk\alpha)H_m'(jkr) - H_m(jk\alpha)J_m'(jkr)]}{[H_m(jk\beta)J_m(jk\alpha) - J_m(jk\beta)H_m(jk\alpha)]}\right\} \qquad (3)$$

and evaluation at the respective boundaries gives Eqs. (a) of the table with f_m and g_m as defined in terms of H_m and J_m. To obtain g_m in the form given,

Prob. 2.16.2 (cont.)

use the identity in the footnote to the table. These entries can be written in terms of the modified functions, K_m and I_m by using Eqs. 2.16.22.

In taking the limit where the inside boundary goes to zero, it is necessary to evaluate

$$\tilde{D}_r^\alpha = \epsilon\left[f_m(0,\alpha)\,\tilde{\Phi}^\alpha + g_m(\alpha,0)\,\tilde{\Phi}^\beta \right] \tag{4}$$

Because K_m and H_m approach infinity as their arguments go to zero, $g_m(\alpha,0) \rightarrow 0$. Also, in the expression for f_m in terms of the functions H_m and J_m, the first t erm in the numerator dominates the second while the second term in the denominator dominates the first. Thus, f_m becomes

$$f_m(0,\alpha) \rightarrow jk\frac{H_m(jk\beta)\,J_m'(jk\alpha)}{-J_m(jk\alpha)\,H_m(jk\beta)} \tag{5}$$

and with the use of Eqs. 2.16.22, this expression becomes the one given in the table.

In the opposite extreme, where the outside boundary goes to infinity, the desired relation is

$$\tilde{D}_r^\beta = \epsilon\left[g_m(\beta,\infty)\,\tilde{\Phi}^\alpha + f_m(\infty,\beta)\,\tilde{\Phi}^\beta \right] \tag{6}$$

Here, note that I_m and J_m (and hence I_m' and J_m') go to infinity as their arguments become large. Thus, $g_m(\beta,\infty) \rightarrow 0$ and in the expressions for f_m, the second term in the numerator and first term in the denominator dominate to give

$$f_m(\infty,\beta) \rightarrow -jk\frac{J_m(jkx)\,H_m'(jk\beta)}{J_m(jkx)\,H_m'(jk\beta)} = -jk\frac{H_m'(jk\beta)}{H_m(jk\beta)} = \frac{-k\,K_m'(k\beta)}{K_m(k\beta)} \tag{7}$$

To invert these results and determine relations in the form of Eqs. (b) of the table, note that the first case, k=0,m=0 involves solutions that are not independent. This reflects the physical fact that it is only the potential difference that matters in this limit and that $(\tilde{\Phi}^\alpha, \tilde{\Phi}^\beta)$ are not really independent variables. Mathematically, the inversion process leads to an infinite determinant.

In general, Cramer's rule gives the inversion of Eqs. (a) as

Prob. 2.16.2 (cont.)

$$F_m(\beta,\alpha) = \epsilon f_m(\alpha,\beta)/\mathrm{Det}\ ;\ F_m(\alpha,\beta) = \epsilon f_m(\beta,\alpha)$$
$$G_m(\beta,\alpha) = -\epsilon g_m(\beta,\alpha)/\mathrm{Det}\ ;\ G_m(\alpha,\beta) = -\epsilon g_m(\alpha,\beta)$$

where
$$\mathrm{Det} = \epsilon\{f_m(\beta,\alpha)f_m(\alpha,\beta) - g_m(\beta,\alpha)g_m(\alpha,\beta)$$

Prob. 2.16.3 The outline for solving this problem is the same as for Prob. 2.16.2. The starting point is Eq. 2.16.36 rather than the three potential distributions representing limiting cases and the general case in Prob. 2.16.2.

Prob. 2.16.4 a) With the z-t dependence $\exp j(\omega t - kz)$, Maxwell's equations become

$$\nabla\cdot\bar{E} = 0 \Rightarrow \frac{\partial\hat{E}_x}{\partial x} = jk\hat{E}_z \tag{1}$$
$$\nabla\cdot\bar{H} = 0 \Rightarrow \frac{\partial\hat{H}_x}{\partial x} = jk\hat{H}_z \tag{2}$$
$$\nabla\times\bar{E} = -\frac{\partial\mu_o\bar{H}}{\partial t} \Rightarrow \begin{cases} jk\hat{E}_y = -j\omega\mu_o\hat{H}_x & (3)\\ -jk\hat{E}_x - \dfrac{\partial\hat{E}_z}{\partial x} = -j\omega\mu_o\hat{H}_y & (4)\\ \dfrac{\partial\hat{E}_y}{\partial x} = -j\omega\mu_o\hat{H}_z & (5)\end{cases}$$
$$\nabla\times\bar{H} = \frac{\partial\epsilon_o\bar{E}}{\partial t} \Rightarrow \begin{cases} jk\hat{H}_y = j\omega\epsilon\hat{E}_x & (6)\\ -jk\hat{H}_x - \dfrac{\partial\hat{H}_z}{\partial x} = j\omega\epsilon\hat{E}_y & (7)\\ \dfrac{\partial\hat{H}_y}{\partial x} = j\omega\epsilon\hat{E}_z & (8)\end{cases}$$

The components $\hat{E}_x$, $\hat{E}_y$, $\hat{H}_x$, $\hat{H}_y$ can be written in terms of $\hat{E}_z$ and $\hat{H}_z$ as follows. Equations 3 and 7 combine to $\left(\gamma^2 \equiv k^2 - (\omega/c)^2\right)$

$$\hat{H}_x = \frac{jk}{\gamma^2}\frac{\partial\hat{H}_z}{\partial x} \tag{9}$$

and Eqs. 4 and 6 give

$$\hat{E}_x = \frac{jk}{\gamma^2}\frac{\partial\hat{E}_z}{\partial x} \tag{10}$$

As a result, Eqs. 6 and 3 give

$$\hat{H}_y = \frac{j\omega\epsilon_o}{\gamma^2}\frac{\partial\hat{E}_z}{\partial x} \tag{11}$$
$$\hat{E}_y = \frac{-j\omega\mu_o}{\gamma^2}\frac{\partial\hat{H}_z}{\partial x} \tag{12}$$

Combining Ampere's and Faraday's laws gives

$$c^2\nabla^2\begin{pmatrix}\bar{H}\\ \bar{E}\end{pmatrix} = \frac{\partial^2}{\partial t^2}\begin{pmatrix}\bar{H}\\ \bar{E}\end{pmatrix} \tag{13}$$

Thus, it follows that

$$\frac{\partial^2}{\partial x^2}\begin{pmatrix}\hat{H}_z\\ \hat{E}_z\end{pmatrix} + \gamma^2\begin{pmatrix}\hat{H}_z\\ \hat{E}_z\end{pmatrix} = 0 \tag{14}$$

Prob. 2.16.4(cont.)

b) Solutions to Eqs. 14 satisfying the boundary conditions are

$$\begin{bmatrix} \hat{H}_z \\ \hat{E}_z \end{bmatrix} = \begin{bmatrix} \hat{H}_z^\alpha \\ \hat{E}_z^\alpha \end{bmatrix} \frac{\sinh \gamma x}{\sinh \gamma \Delta} - \begin{bmatrix} \hat{H}_z^\beta \\ \hat{E}_z^\beta \end{bmatrix} \frac{\sinh \gamma (x-\Delta)}{\sinh \gamma \Delta} \qquad \begin{matrix} (15) \\ (16) \end{matrix}$$

c) Use is now made of Eqs. 9 and 10 to obtain

$$\hat{E}_x = \frac{jk}{\gamma} \left\{ \hat{E}_z^\alpha \frac{\cosh \gamma x}{\sinh \gamma \Delta} - \hat{E}_z^\beta \frac{\cosh \gamma (x-\Delta)}{\sinh \gamma \Delta} \right\} \qquad (17)$$

$$\hat{H}_x = \frac{jk}{\gamma} \left\{ \hat{H}_z^\alpha \frac{\cosh \gamma x}{\sinh \gamma \Delta} - \hat{H}_z^\beta \frac{\cosh \gamma (x-\Delta)}{\sinh \gamma \Delta} \right\} \qquad (18)$$

Also, from Eqs. 3 and 6,

$$\hat{E}_y = -\frac{\omega \mu_o}{k} \hat{H}_x \qquad (19)$$

$$\hat{H}_y = \frac{\omega \epsilon_o}{k} \hat{E}_x \qquad (20)$$

Evaluation of these expressions at the respective boundaries gives the transfer relations summarized in the problem.

d) In the quasistatic limit, times of interest, $1/\omega$, are much longer than the propagation time of an electromagnetic wave in the system. For propagation across the guide, this time is $\Delta/c = \Delta\sqrt{\mu_o \epsilon_o}$. Thus,

$$\Delta\gamma \simeq k\Delta \qquad (21)$$

Note that $k\Delta$ must be larger than τ_{em}/τ, but too large a value of $k\Delta$ means no interaction between the two boundaries. Now, with $\gamma \to k$, $\hat{E}_z = jk\hat{\Phi}$ and $\hat{H}_z = jk\hat{\psi}$, the relations break into the quasi-static transfer relations.

$$\begin{bmatrix} \epsilon_o \hat{E}_x^\alpha \\ \epsilon_o \hat{E}_x^\beta \end{bmatrix} = \epsilon_o k \begin{bmatrix} -\coth k\Delta & \frac{1}{\sinh k\Delta} \\ -\frac{1}{\sinh k\Delta} & \coth k\Delta \end{bmatrix} \begin{bmatrix} \hat{\Phi}^\alpha \\ \hat{\Phi}^\beta \end{bmatrix} \qquad (22)$$

$$\begin{bmatrix} \mu_o \hat{H}_x^\alpha \\ \mu_o \hat{H}_x^\beta \end{bmatrix} = \mu_o k \begin{bmatrix} -\coth k\Delta & \frac{1}{\sinh k\Delta} \\ -\frac{1}{\sinh k\Delta} & \coth k\Delta \end{bmatrix} \begin{bmatrix} \hat{\psi}^\alpha \\ \hat{\psi}^\beta \end{bmatrix} \qquad (23)$$

Prob. 2.16.4(cont)

e) Transverse electric(TE) and transverse magnetic(TM) modes between perfectly conducting plates satisfy the boundary conditions

$$\text{(TM)} \quad (H_z = 0) \quad \hat{E}_z^{\alpha,\beta} = 0 \tag{24}$$

$$\text{(TE)} \quad (E_z = 0) \quad \hat{H}_x^{\alpha,\beta} = 0 \tag{25}$$

where the latter condition is expressed in terms of H_z by using Eqs. 12 and 7. Because the modes separate, it is possible to examine them separately. The electric relations are already in the appropriate form for considering the TM modes. The magnetic ones are inverted to obtain

$$\begin{bmatrix} \mu_0 \hat{H}_z^\alpha \\ \mu_0 \hat{H}_z^\beta \end{bmatrix} = -\frac{\gamma\mu}{jk} \begin{bmatrix} -\coth\gamma\Delta & \frac{1}{\sinh\gamma\Delta} \\ -\frac{1}{\sinh\gamma\Delta} & \coth\gamma\Delta \end{bmatrix} \begin{bmatrix} \hat{H}_x^\alpha \\ \hat{H}_x^\beta \end{bmatrix} \tag{26}$$

With the boundary conditions of Eq. 24 in the electric relations and with those of Eq. 25 in these last relations, it is evident that there can be no response unless the determinant of the coefficients vanishes. In each case this requires that

$$-\coth^2\gamma\Delta + \frac{1}{\sinh^2\gamma\Delta} = 0 \tag{27}$$

This has two solutions.

$$\sinh\gamma\Delta = 0 \; ; \; \cosh\gamma\Delta = \pm 1 \tag{28}$$

In either case,

$$\gamma = \frac{jn\pi}{\Delta} \tag{29}$$

It follows from the definition of γ that each mode designated by n must satisfy the dispersion equation

$$\left(\frac{\omega}{c}\right)^2 = k^2 + \left(\frac{n\pi}{\Delta}\right)^2 \tag{30}$$

For propagation of waves through this parallel plate waveguide, k must be real. Thus, all waves attenuate below the cutoff frequency

$$\omega_{cutoff} = \frac{c\pi}{\Delta} \tag{31}$$

because then all have an imagineary wavenumber,k.

Prob. 2.16.5 Gauss' law and $\tilde{E}=-\nabla\Phi$ requires that if there is no free charge

$$\epsilon\nabla^2\Phi+\nabla\epsilon\cdot\nabla\Phi=0 \tag{1}$$

For the given exponential dependence of the permittivity, the x dependence of the coefficients in this expression factors out and it again reduces to a constant coefficient expression

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}+2\gamma\frac{\partial\Phi}{\partial x}=0 \tag{2}$$

In terms of the complex amplitude forms from Table 2.16.1, Eq. 2 requires that

$$\frac{d^2\tilde{\hat{\Phi}}}{dx^2}+2\gamma\frac{d\tilde{\hat{\Phi}}}{dx}-k^2\tilde{\hat{\Phi}}=0 \tag{3}$$

Thus, solutions have the form exp px where $p=-\gamma\pm\lambda,\ \lambda\equiv\sqrt{k^2+\gamma^2}$.

The linear combination of these that satisfies the conditions that $\hat{\Phi}$ be $\hat{\Phi}^\alpha$ and $\tilde{\hat{\Phi}}^\beta$ on the upper and lower surfaces respectively is as given in the problem. The displacement vector is then evaluated as

$$\tilde{D}=-\epsilon_\beta\left\{\tilde{\hat{\Phi}}^\alpha e^{\gamma(x+\Delta)}\frac{[-\gamma\sinh\lambda x+\lambda\cosh\lambda x]}{\sinh\lambda\Delta}\right. \tag{4}$$
$$\left.-\tilde{\hat{\Phi}}^\beta e^{\gamma x}\frac{[-\gamma\sinh\lambda(x-\Delta)+\lambda\cosh\lambda(x-\Delta)]}{\sinh\lambda\Delta}\right.$$

Evaluation of this expression at the respective surfaces then gives the transfer relations summarized in the problem.

Prob. 2.16.6 The fields are governed by

$$\bar{E} = -\nabla \Phi \tag{1}$$

$$\nabla \cdot \bar{D} = 0 \tag{2}$$

Substitution of Eq. 1 and the constitutive law into Eq. 2 gives a generalization of Laplace's equation for the potential.

$$\epsilon_{ij} \frac{\partial^2 \Phi}{\partial x_i \partial x_j} = 0 \tag{3}$$

Substitution of

$$\Phi = \mathrm{Re}\, \tilde{\Phi}(x)\, e^{-j(k_y y + k_z z)} \tag{4}$$

results in

$$\frac{d^2\tilde{\Phi}}{dx^2} - jA\frac{d\tilde{\Phi}}{dx} - B\tilde{\Phi} = 0 \tag{5}$$

where

$$A \equiv k_y \frac{(\epsilon_{xy} + \epsilon_{yx})}{\epsilon_{xx}} + k_z \frac{(\epsilon_{xz} + \epsilon_{zx})}{\epsilon_{xx}} \; ; B = \frac{1}{\epsilon_{xx}}\left[k_y^2 \epsilon_{yy} + k_y k_z (\epsilon_{yz} + \epsilon_{zy}) + k_z^2 \epsilon_{zz}\right]$$

This constant coefficient equation has solutions exp p, where substitution shows that

$$p = j\gamma \pm \lambda \; ; \; \gamma = \frac{A}{2} \; , \; \lambda = \sqrt{B - \frac{A^2}{4}} \tag{6}$$

Thus, solutions take the form

$$\tilde{\Phi} = A_1 e^{j\gamma x} e^{\lambda x} + A_2 e^{j\gamma x} e^{-\lambda x} \tag{7}$$

The coefficients A_1 and A_2 are determined by requiring that $\tilde{\Phi} = \tilde{\Phi}^\alpha$ and $\tilde{\Phi} = \tilde{\Phi}^\beta$ at $x = \Delta$ and $x=0$ respectively. Thus, in terms of the surface potentials, the potential distribution is given by

$$\tilde{\Phi} = \tilde{\Phi}^\alpha e^{j\gamma(x-\Delta)} \frac{\sinh \lambda x}{\sinh \lambda \Delta} + \tilde{\Phi}^\beta e^{j\gamma x} \frac{\sinh \lambda(\Delta - x)}{\sinh \lambda \Delta} \tag{8}$$

The normal electric displacement follows from the x component of the constitutive law,

$$\tilde{D}_x = \epsilon_{xj} \tilde{E}_j = -\epsilon_{xx} \frac{d\tilde{\Phi}}{dx} + j(\epsilon_{xy} k_y + \epsilon_{xz} k_z)\tilde{\Phi} \tag{9}$$

Evaluation using Eq. 8 then gives

Prob. 2.16.6(cont.)

$$D_x=\left\{-\epsilon_{xx}\left[j\gamma e^{j\gamma(x-\Delta)}\frac{\sinh\lambda x}{\sinh\lambda\Delta}+\lambda e^{j\gamma(x-\Delta)}\frac{\cosh\lambda x}{\sinh\lambda\Delta}\right]+j(\epsilon_{xy}k_y+\epsilon_{xz}k_z)e^{j\gamma(x-\Delta)}\frac{\sinh\lambda x}{\sinh\lambda\Delta}\right\}\tilde{\Phi}^{\alpha} \quad (10)$$

$$+\left\{-\epsilon_{xx}\left[j\gamma e^{j\gamma x}\frac{\sinh\lambda(\Delta-x)}{\sinh\lambda\Delta}-\lambda e^{j\gamma x}\frac{\cosh\lambda(\Delta-x)}{\sinh\lambda\Delta}\right]+j(\epsilon_{xy}k_y+\epsilon_{xz}k_z)e^{j\gamma x}\frac{\sinh\lambda(\Delta-x)}{\sinh\lambda\Delta}\right\}\tilde{\Phi}^{\beta}$$

The required transfer relations follow by evaluating this expression at the respective boundaries.

$$\begin{bmatrix}\tilde{D}_x^{\alpha}\\ \tilde{D}_x^{\beta}\end{bmatrix} \quad \begin{bmatrix}-\epsilon_{xx}[j\gamma+\lambda\coth\lambda\Delta]+j(\epsilon_{xy}k_y+\epsilon_{xz}k_z) & \dfrac{\epsilon_{xx}\lambda e^{j\gamma\Delta}}{\sinh\lambda\Delta}\\ -\dfrac{\epsilon_{xx}\lambda e^{-j\gamma\Delta}}{\sinh\lambda\Delta} & -\epsilon_{xx}[j\gamma-\lambda\coth\lambda\Delta]+j(\epsilon_{xy}k_y+\epsilon_{xz}k_z)\end{bmatrix}\begin{bmatrix}\tilde{\Phi}^{\alpha}\\ \tilde{\Phi}^{\beta}\end{bmatrix} \quad (11)$$

Prob. 2.17.1 In cartesian coordinates, $a^{\alpha}=a^{\beta}$, so that Eq. 2.17.1 requires that $B_{12}=B_{21}$. Comparison of terms in the canonical and particular transfer relations then shows that

$$B_{12}=e^{\lambda\Delta}/\sinh\lambda\Delta=-B_{21}$$

Prob. 2.17.2 Using $\alpha A_{12}=\beta A_{21}$, Table 2.16.2 gives

$$jk\alpha[H_m(jk\alpha)J_m'(jk\alpha)-J_m(jk\alpha)H_m'(jk\alpha)$$

$$=-jk\beta[H_m(jk\beta)J_m'(jk\beta)-J_m(jk\beta)H_m'(jk\beta) \quad (1)$$

These can only be equal for arbitrary α,β if

$$kx[H_m(jkx)J_m'(jkx)-J_m(jkx)H_m'(jkx)]=\text{const.} \quad (2)$$

Limit relations, Eqs. 2.16.22 and 2.16.23, are used to evaluate the constant.

$$ku\left[\left(-\frac{2}{\pi}\right)\sqrt{\frac{\pi}{2u}}\,e^{-u}\left(\frac{e^{u}}{\sqrt{2\pi u}}\left(1-\frac{1}{u}\right)\right)+\frac{1}{\sqrt{2\pi u}}\,e^{u}\left(\frac{2}{\pi}\right)\sqrt{\frac{\pi}{2u}}\,e^{-u}\left(-1-\frac{1}{u}\right)\right]=\text{const.} \quad (3)$$

Thus, as $u\to\infty$ it is clear that $c=-2/\pi$

Prob. 2.17.3 With the assumption that w is a state function, it follows that

$$\delta W = \frac{\partial W}{\partial \tilde{D}^{\alpha}_{nr}} \delta \tilde{D}^{\alpha}_{nr} + \frac{\partial W}{\partial \tilde{D}^{\alpha}_{ni}} \delta \tilde{D}^{\alpha}_{ni} + \frac{\partial W}{\partial \tilde{D}^{\beta}_{nr}} \tilde{D}^{\beta}_{nr} + \frac{\partial W}{\partial \tilde{D}^{\beta}_{ni}} \delta \tilde{D}^{\beta}_{ni}$$

Because the D's are independent variables, the coefficients must agree with those of the expression for δW in the problem statement. Thus, the relations for the Φ's follow. The reciprocity relations follow from taking cross-derivatives of these energy relations

$$-a^{\alpha}\frac{\partial \tilde{\Phi}^{\alpha}_{r}}{\partial \tilde{D}^{\alpha}_{i}} = -a^{\alpha}\frac{\partial \tilde{\Phi}^{\alpha}}{\partial \tilde{D}^{\alpha}_{r}} \quad (1) \qquad -a^{\alpha}\frac{\partial \tilde{\Phi}^{\alpha}_{i}}{\partial \tilde{D}^{\beta}_{r}} = a^{\beta}\frac{\partial \tilde{\Phi}^{\beta}_{r}}{\partial \tilde{D}^{\alpha}_{i}} \quad (4)$$

$$-a^{\alpha}\frac{\partial \tilde{\Phi}^{\alpha}_{r}}{\partial \tilde{D}^{\alpha}_{r}} = a^{\beta}\frac{\partial \tilde{\Phi}^{\beta}_{r}}{\partial \tilde{D}^{\alpha}_{r}} \quad (2) \qquad -a^{\alpha}\frac{\partial \tilde{\Phi}^{\alpha}_{i}}{\partial \tilde{D}^{\beta}_{i}} = a^{\beta}\frac{\partial \tilde{\Phi}^{\beta}_{i}}{\partial \tilde{D}^{\alpha}_{i}} \quad (5)$$

$$-a^{\alpha}\frac{\partial \tilde{\Phi}^{\alpha}_{r}}{\partial \tilde{D}^{\beta}_{r}} = a^{\beta}\frac{\partial \tilde{\Phi}^{\beta}_{i}}{\partial \tilde{D}^{\alpha}_{r}} \quad (3) \qquad a^{\beta}\frac{\partial \tilde{\Phi}^{\beta}_{r}}{\partial \tilde{D}^{\beta}_{i}} = a^{\beta}\frac{\partial \tilde{\Phi}^{\beta}_{i}}{\partial \tilde{D}^{\beta}_{r}} \quad (6)$$

The transfer relation written so as to separate the real and imaginary parts, is equivalent to

$$\begin{bmatrix} \tilde{\Phi}^{\alpha}_{r} \\ \tilde{\Phi}^{\alpha}_{i} \\ \tilde{\Phi}^{\beta}_{r} \\ \tilde{\Phi}^{\beta}_{i} \end{bmatrix} = \begin{bmatrix} -A_{11r} & A_{11i} & A_{12r} & -A_{12i} \\ -A_{11i} & -A_{11r} & A_{12i} & A_{12r} \\ -A_{21r} & A_{21i} & A_{22r} & -A_{22i} \\ -A_{21i} & -A_{21r} & A_{22i} & A_{22r} \end{bmatrix} \begin{bmatrix} \tilde{D}^{\alpha}_{r} \\ \tilde{D}^{\alpha}_{i} \\ \tilde{D}^{\beta}_{r} \\ \tilde{D}^{\beta}_{i} \end{bmatrix}$$

The reciprocity relations (1) and (6) respectively show that these transfer relations require that $A_{11i}=-A_{11i}$ and $A_{22i}=-A_{22i}$, so that the imaginary

Prob. 2.17.3 (cont.)

parts of A_{11} and A_{22} are zero. The other relations show that $a^{\alpha}A_{12r} = a^{\beta}A_{21r}$ and $a^{\alpha}A_{12i} = -a^{\beta}A_{21i}$ so, $a^{\alpha}A_{12} = a^{\beta}A_{21}^{*}$. Of course, A_{12} and hence, A_{21} are actually real.

Prob. 2.17.4 From Problem 2.17.1, for

$$\begin{bmatrix} \tilde{D}_n^{\alpha} \\ \tilde{D}_n^{\beta} \end{bmatrix} = \begin{bmatrix} -B_{11} & B_{12} \\ -B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^{\alpha} \\ \tilde{\Phi}^{\beta} \end{bmatrix} \tag{1}$$

it is shown that

$$-a^{\alpha}\frac{\partial \tilde{D}_n^{\alpha}}{\partial \tilde{\Phi}^{\beta}} = a^{\beta}\frac{\partial \tilde{D}_n^{\beta}}{\partial \tilde{\Phi}^{\alpha}} \tag{2}$$

which requires that

$$B_{12} = B_{21} \tag{3}$$

For this system $B_{12} = B_{21} = \gamma e^{\gamma\Delta}/\sinh \gamma\Delta$.

Prob. 2.18.1 Observe that in cylindrical coordinates (Appendix A) with $\bar{A} = A_{\theta}\bar{i}_{\theta}$

$$\bar{B} = \nabla\times\bar{A} = -\frac{\partial A_{\theta}}{\partial z}\bar{i}_r + \frac{1}{r}\frac{\partial}{\partial r}(rA_{\theta})\bar{i}_z \tag{1}$$

Thus, substitution of $A_{\theta} = \Lambda(r,z)r^{-1}$ gives

$$\bar{B} = -\frac{1}{r}\frac{\partial\Lambda}{\partial z}\bar{i}_r + \frac{1}{r}\frac{\partial\Lambda}{\partial r}\bar{i}_z \tag{2}$$

as in Table 2.18.1.

Prob. 2.18.2 In spherical coordinates with $\bar{A} = A_{\phi}\bar{i}_{\phi}$ (Appendix A),

$$\bar{B} = \nabla\times\bar{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_{\phi}\sin\theta)\bar{i}_r - \frac{1}{r}\frac{\partial}{\partial r}(rA_{\phi})\bar{i}_{\theta} \tag{1}$$

Thus, substitution of $A_{\phi} = \Lambda(r,\theta)(r\sin\theta)^{-1}$ gives

$$\bar{B} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\Lambda}{r}\right)\bar{i}_r - \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\Lambda}{\sin\theta}\right)\bar{i}_{\theta} = \frac{1}{r\sin\theta}\left(\frac{1}{r}\frac{\partial\Lambda}{\partial\theta}\bar{i}_r - \frac{\partial\Lambda}{\partial r}\bar{i}_{\theta}\right) \tag{2}$$

as in Table 2.18.1.

Prob. 2.19.1 The transfer relations are obtained by following the instructions given with Eqs. 2.19.7 through 2.19.12.