Topic 15

Use of Elastic Constitutive Relations in Total Lagrangian Formulation

Contents:

- **Basic considerations in modeling material response**
- **Linear and nonlinear elasticity**
- **Isotropic and orthotropic materials**
- **One-dimensional example, large strain conditions**
- **The case of large displacement/small strain analysis, discussion of effectiveness using the total Lagrangian formulation**
- **Hyperelastic material model (Mooney-Rivlin) for analysis of rubber-type materials**
- **Example analysis: Solution of a rubber tensile test specimen**
- **Example analysis: Solution of a rubber sheet with a hole**

Textbook: 6.4, 6.4.1

Reference: The solution of the rubber sheet with a hole is given in

Bathe, K. J., E. Ramm, and E. L. Wilson, "Finite Element Formulations for Large Deformation Dynamic Analysis," *International Journal for Numerical Methods in Engineering*, *9*, 353–386, 1975.

USE OF CONSTITUTIVE RELATIONS

Transparency 15-1

- We developed quite general kinematic relations and finite element discretizations, applicable to small or large deformations.
- To use these finite element formulations, appropriate constitutive relations must be employed.
- Schematically

$$\underline{K} = \int_V \underline{B}^T \underline{C}\, \underline{B}\, dV, \qquad \underline{F} = \int_V \underline{B}^T \underline{\tau}\, dV$$

constitutive relations enter here

Transparency 15-2

For analysis, it is convenient to use the classifications regarding the magnitude of deformations introduced earlier:

- Infinitesimally small displacements
- Large displacements / large rotations, but small strains
- Large displacements / large rotations, and large strains

The applicability of material descriptions generally falls also into these categories.

Transparency 15-3

Recall:

- Materially-nonlinear-only (M.N.O.) analysis assumes (models only) infinitesimally small displacements.
- The total Lagrangian (T.L.) and updated Lagrangian (U.L.) formulations can be employed for analysis of infinitesimally small displacements, of large displacements and of large strains (considering the analysis of 2-D and 3-D solids).

→ All kinematic nonlinearities are fully included.

Transparency 15-4

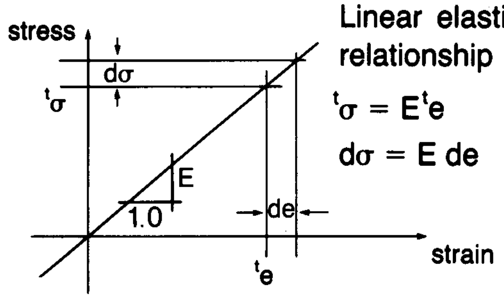
We may use various material descriptions:

Material Model	Examples
Elastic	Almost all materials, for small enough stresses
Hyperelastic	Rubber
Hypoelastic	Concrete
Elastic-plastic	Metals, soils, rocks under high stresses
Creep	Metals at high temperatures
Viscoplastic	Polymers, metals

Transparency 15-5

ELASTIC MATERIAL BEHAVIOR:

In linear, infinitesimal displacement, small strain analysis, we are used to employing



Linear elastic stress-strain relationship

$^t\sigma = E\,^te$

$d\sigma = E\,de$

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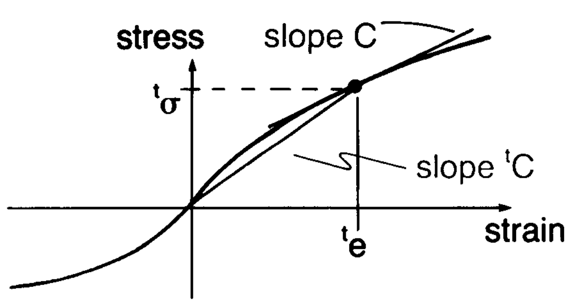
For 1-D nonlinear analysis we can use

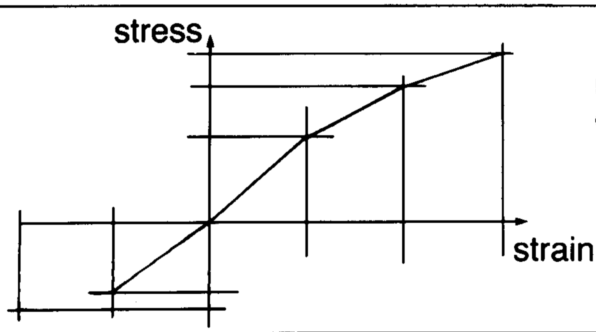


Nonlinear elastic stress-strain relationship

$^t\sigma = {}^tC\,^te$

not constant

$d\sigma = C\,de$



In practice, a piecewise linear description is used

Transparency 15-7

We can generalize the elastic material behavior using:

$$ {}_0^tS_{ij} = {}_0^tC_{ijrs}\ {}_0^t\varepsilon_{rs} $$

$$ d_0S_{ij} = {}_0C_{ijrs}\ d_0\varepsilon_{rs} $$

This material description is frequently employed with

- the usual constant material moduli used in infinitesimal displacement analysis
- rubber-type materials

Transparency 15-8

Use of constant material moduli, for an isotropic material:

$$ {}_0^tC_{ijrs} = {}_0C_{ijrs} = \lambda\, \delta_{ij}\, \delta_{rs} + \mu(\delta_{ir}\, \delta_{js} + \delta_{is}\, \delta_{jr}) $$

Lamé constants:

$$ \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}\ ,\quad \mu = \frac{E}{2(1+\nu)} $$

Kronecker delta:

$$ \delta_{ij} = \begin{cases} 0; & i \neq j \\ 1; & i = j \end{cases} $$

Transparency 15-9

Examples:

2-D plane stress analysis:

$$
{}_0\underline{C} = \frac{E}{1-\nu^2}\left[\begin{array}{cc|c} 1 & \nu & 0 \\ \nu & 1 & 0 \\ \hline 0 & 0 & \dfrac{1-\nu}{2} \end{array}\right]
$$

corresponds to ${}_0^tS_{12} = \mu\,({}_0^t\varepsilon_{12} + {}_0^t\varepsilon_{21})$

Transparency 15-10

2-D axisymmetric analysis:

$$
\underline{C} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \dfrac{\nu}{1-\nu} & 0 & \dfrac{\nu}{1-\nu} \\ \dfrac{\nu}{1-\nu} & 1 & 0 & \dfrac{\nu}{1-\nu} \\ 0 & 0 & \dfrac{1-2\nu}{2(1-\nu)} & 0 \\ \dfrac{\nu}{1-\nu} & \dfrac{\nu}{1-\nu} & 0 & 1 \end{bmatrix}
$$

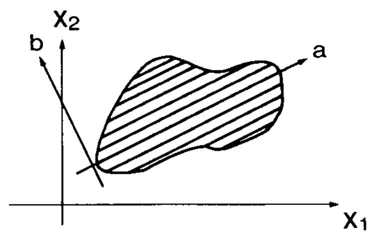
Transparency 15-11

For an orthotropic material, we also use the usual constant material moduli:

Example: 2-D plane stress analysis

local coordinate system a-b

$$ {}_{0}C_{\ell}^{-1} = \begin{bmatrix} \frac{1}{E_a} & -\frac{\nu_{ab}}{E_b} & 0 \\ & \frac{1}{E_b} & 0 \\ \text{sym.} & & \frac{1}{G_{ab}} \end{bmatrix} $$



$E_a \neq E_b$

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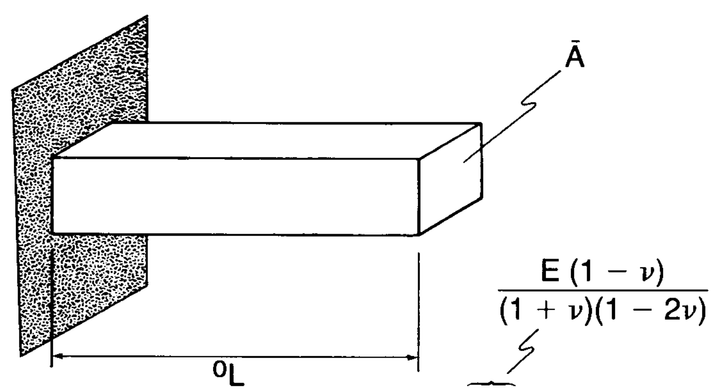
Sample analysis: One-dimensional problem:

Material constants E, ν



Constitutive relation: ${}_{0}^{t}S_{11} = \tilde{E}\; {}_{0}^{t}\varepsilon_{11}$

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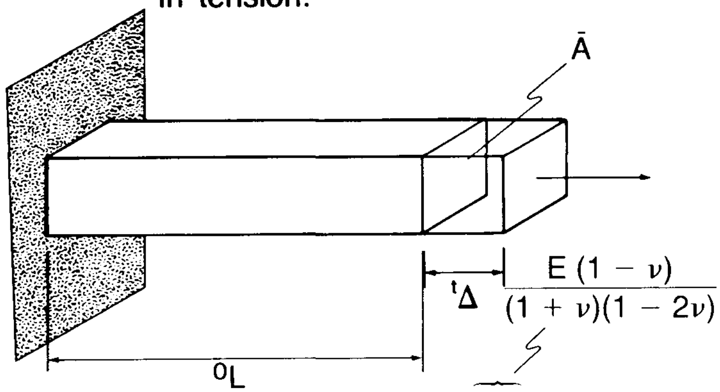
Sample analysis: One-dimensional problem:

Material constants E, ν

In tension:



Constitutive relation: ${}_0^tS_{11} = \tilde{E}\ {}_0^t\varepsilon_{11}$

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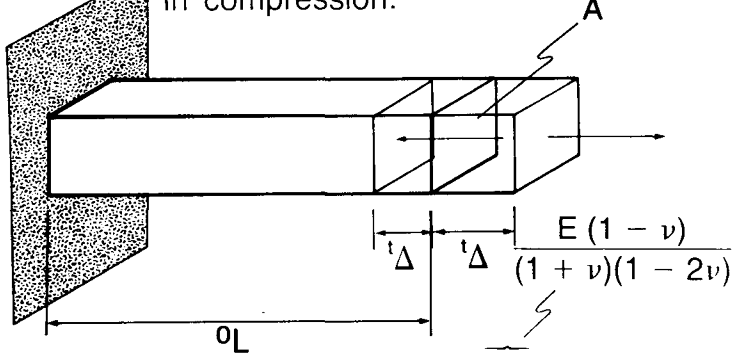
Sample analysis: One-dimensional problem:

Material constants E, ν

In tension:

In compression:



Constitutive relation: ${}_0^tS_{11} = \tilde{E}\ {}_0^t\varepsilon_{11}$

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We establish the force-displacement relationship:

$$ {}^t_0\varepsilon_{11} = \underbrace{{}^t_0u_{1,1}}_{\frac{{}^tL - {}^0L}{{}^0L}} + \frac{1}{2}\left({}^t_0u_{1,1}\right)^2 $$

$$ = \frac{1}{2}\left[\left(\frac{{}^tL}{{}^0L}\right)^2 - 1\right], $$

$$ {}^t_0S_{11} = \frac{{}^0\rho}{{}^t\rho}\,{}^0_tx_{1,1}\,{}^t\tau_{11}\,{}^0_tx_{1,1} $$

$$ = \frac{{}^tL}{{}^0L}\left(\frac{{}^0L}{{}^tL}\right)\frac{{}^tP}{\bar{A}}\left(\frac{{}^0L}{{}^tL}\right) = \frac{{}^0L}{{}^tL}\,\frac{{}^tP}{\bar{A}} $$

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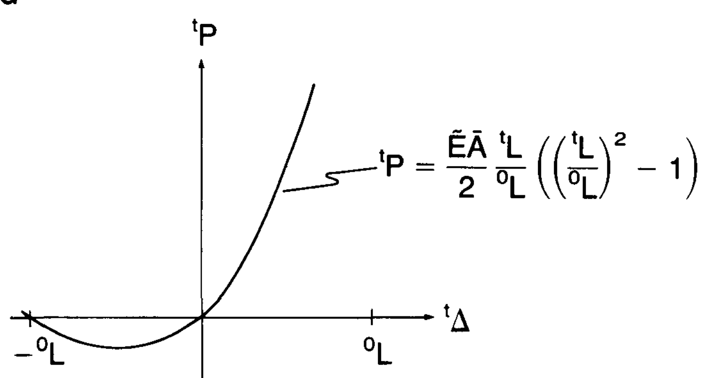
Using ${}^tL = {}^0L + {}^t\Delta$, ${}^t_0S_{11} = \tilde{E}\,{}^t_0\varepsilon_{11}$, we find



This is not a realistic material description for large strains.

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- The usual isotropic and orthotropic material relationships (constant E, ν, E_a, etc.) are mostly employed in large displacement / large rotation, but small strain analysis.
- Recall that the components of the 2nd Piola-Kirchhoff stress tensor and of the Green-Lagrange strain tensor are invariant under a rigid body motion (rotation) of the material.

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— Hence only the actual straining increases the components of the Green-Lagrange strain tensor and, through the material relationship, the components of the 2nd Piola-Kirchhoff stress tensor.

— The effect of rotating the material is included in the T.L. formulation,

$$ {}_0^t\underline{F} = \int_{{}^0V} \underbrace{{}_0^t\underline{B}_L^T}_{\text{includes rotation}} \; \underbrace{{}_0^t\hat{\underline{S}}}_{\text{invariant under a rigid body rotation}} \; {}^0dV $$

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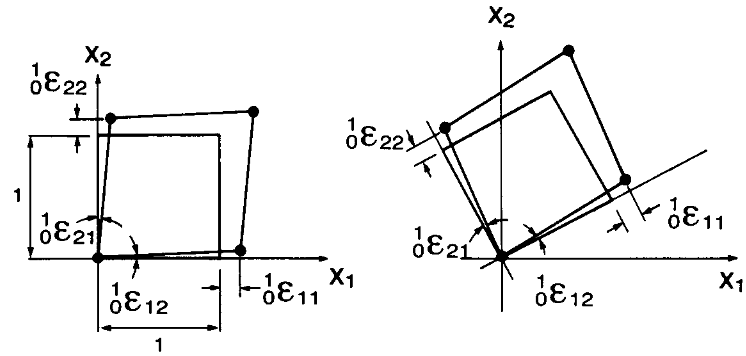
Pictorially:



Deformation to state 1 (small strain situation)

Rigid rotation from state 1 to state 2

Transparency 15-20

For small strains,

$${}^1_0\varepsilon_{11}\ ,\ {}^1_0\varepsilon_{22}\ ,\ {}^1_0\varepsilon_{12} = {}^1_0\varepsilon_{21} << 1,$$

$${}^1_0S_{ij} = \underbrace{{}^1_0C_{ijrs}}_{\text{a function of } E,\ \nu}\ {}^1_0\varepsilon_{rs},$$

$${}^1_0S_{ij} \doteq {}^1\tau_{ij}$$

Also, since state 2 is reached by a rigid body rotation,

$${}^2_0\varepsilon_{ij} = {}^1_0\varepsilon_{ij}\ ,\ {}^2_0S_{ij} = {}^1_0S_{ij},$$

$${}^2\underline{\tau} = \underbrace{\underline{R}}_{\text{rotation matrix}}\ {}^1\underline{\tau}\ \underline{R}^T$$

Transparency 15-21

Applications:

- Large displacement / large rotation but small strain analysis of beams, plates and shells. These can frequently be modeled using 2-D or 3-D elements. Actual beam and shell elements will be discussed later.
- Linearized buckling analysis of structures.

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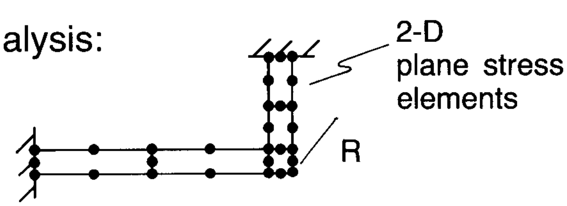
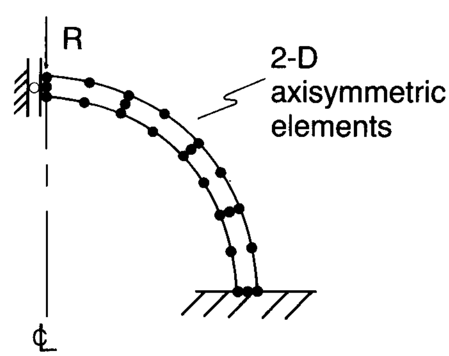
Frame analysis:



Axisymmetric shell:



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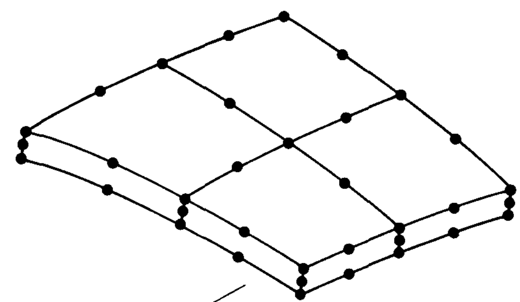
General shell:



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Hyperelastic material model: formulation of rubber-type materials

$$ {}_0^tS_{ij} = \frac{\partial\, {}_0^tW}{\partial\, {}_0^t\varepsilon_{ij}} \neq {}_0^tC_{ijrs}\, {}_0^t\varepsilon_{rs} $$

$$ d_0S_{ij} = {}_0C_{ijrs}\, d_0\varepsilon_{rs}, \qquad {}_0C_{ijrs} = \frac{\partial^2\, {}_0^tW}{\partial\, {}_0^t\varepsilon_{ij}\, \partial\, {}_0^t\varepsilon_{rs}} $$

where

${}_0^tW$ = strain energy density function (per unit original volume)

Rubber is assumed to be an isotropic material, hence

$$ {}_{0}^{t}W = \text{function of } (I_1, I_2, I_3) $$

where the I_i's are the invariants of the Cauchy-Green deformation tensor (with components ${}_{0}^{t}C_{ij}$):

$$ I_1 = {}_{0}^{t}C_{ii} $$

$$ I_2 = \frac{1}{2}\left(I_1^2 - {}_{0}^{t}C_{ij}\,{}_{0}^{t}C_{ij}\right) $$

$$ I_3 = \det\left({}_{0}^{t}\underline{C}\right) $$

Transparency 15-25

Example: Mooney-Rivlin material law

$$ {}_{0}^{t}W = \underbrace{C_1}(I_1 - 3) + \underbrace{C_2}(I_2 - 3) $$

material constants

with

$I_3 = 1$ ← incompressibility constraint

Note, in general, the displacement-based finite element formulations presented above should be extended to include the incompressibility constraint effectively. A special case, however, is the analysis of plane stress problems.

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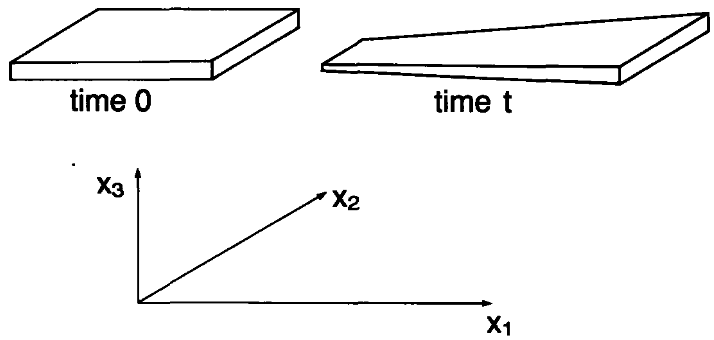
Special case of Mooney-Rivlin law: plane stress analysis



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For this (two-dimensional) problem,

$$ {}_0^t\underline{C} = \begin{bmatrix} {}_0^tC_{11} & {}_0^tC_{12} & 0 \\ {}_0^tC_{21} & {}_0^tC_{22} & 0 \\ 0 & 0 & {}_0^tC_{33} \end{bmatrix} $$

Since the rubber is assumed to be incompressible, we set det $({}_0^t\underline{C})$ to 1 by choosing

$$ {}_0^tC_{33} = \frac{1}{({}_0^tC_{11}\,{}_0^tC_{22} - {}_0^tC_{12}\,{}_0^tC_{21})} $$

Transparency 15-29

We can now evaluate I_1, I_2:

$$I_1 = {}_0^tC_{11} + {}_0^tC_{22} + \frac{1}{({}_0^tC_{11}\,{}_0^tC_{22} - {}_0^tC_{12}\,{}_0^tC_{21})}$$

$$I_2 = {}_0^tC_{11}\,{}_0^tC_{22} + \frac{{}_0^tC_{11} + {}_0^tC_{22}}{({}_0^tC_{11}\,{}_0^tC_{22} - {}_0^tC_{12}\,{}_0^tC_{21})} - \frac{1}{2}({}_0^tC_{12})^2 - \frac{1}{2}({}_0^tC_{21})^2$$

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The 2nd Piola-Kirchhoff stresses are

$${}_0^tS_{ij} = \frac{\partial\,{}_0^tW}{\partial\,{}_0^t\varepsilon_{ij}} = 2\,\frac{\partial\,{}_0^tW}{\partial\,{}_0^tC_{ij}} \qquad \left(\begin{matrix}\text{remember} \\ {}_0^tC_{ij} = 2\,{}_0^t\varepsilon_{ij} + \delta_{ij}\end{matrix}\right)$$

$$= 2\,\frac{\partial}{\partial\,{}_0^tC_{ij}}\Big[C_1\,(I_1 - 3) + C_2\,(I_2 - 3)\Big]$$

$$= 2\,C_1\,\frac{\partial I_1}{\partial\,{}_0^tC_{ij}} + 2\,C_2\,\frac{\partial I_2}{\partial\,{}_0^tC_{ij}}$$

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Performing the indicated differentiations gives

$$\begin{bmatrix} {}_0^tS_{11} \\ {}_0^tS_{22} \\ {}_0^tS_{12} \end{bmatrix} = 2\,C_1 \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - ({}_0^tC_{33})^2 \begin{bmatrix} {}_0^tC_{22} \\ {}_0^tC_{11} \\ -{}_0^tC_{12} \end{bmatrix} \right\}$$

$$+\,2\,C_2 \left\{ {}_0^tC_{33} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \left[1 - ({}_0^tC_{33})^2\,({}_0^tC_{11} + {}_0^tC_{22})\right] \begin{bmatrix} {}_0^tC_{22} \\ {}_0^tC_{11} \\ -{}_0^tC_{12} \end{bmatrix} \right\}$$

This is the stress-strain relationship.

Transparency 15-32

We can also evaluate the tangent constitutive tensor ${}_0C_{ijrs}$ using

$${}_0C_{ijrs} = \frac{\partial^2\,{}_0^tW}{\partial\,{}_0^t\varepsilon_{ij}\,\partial\,{}_0^t\varepsilon_{rs}}$$

$$= \underbrace{4\,C_1 \frac{\partial^2 I_1}{\partial\,{}_0^tC_{ij}\,\partial\,{}_0^tC_{rs}} + 4\,C_2 \frac{\partial^2 I_2}{\partial\,{}_0^tC_{ij}\,\partial\,{}_0^tC_{rs}}}_{\text{For the Mooney-Rivlin law}}$$

etc.

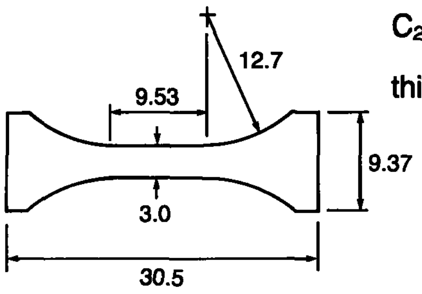
Example: Analysis of a tensile test specimen:

Mooney-Rivlin constants:

$C_1 = .234\ N/mm^2$

$C_2 = .117\ N/mm^2$

thickness = 1 mm



All dimensions in millimeters

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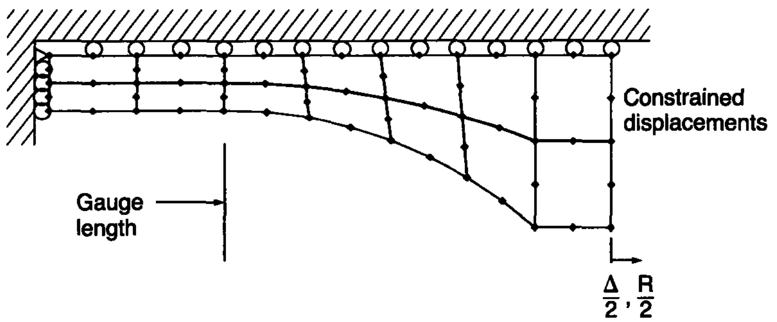
Finite element mesh: Fourteen 8-node elements



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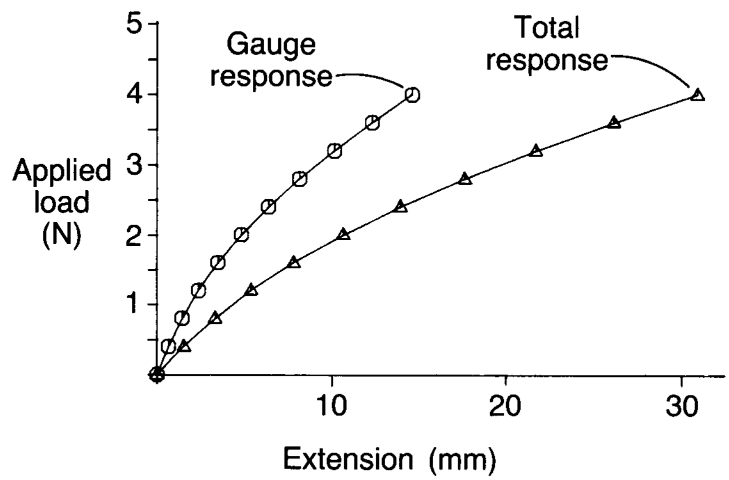
Results: Force–deflection curves



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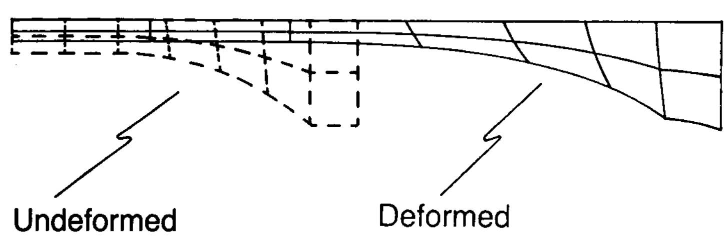
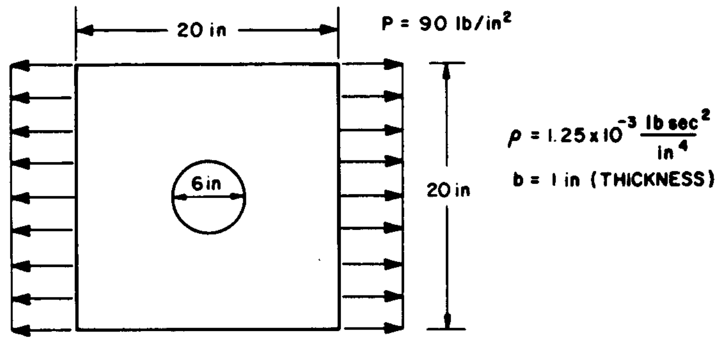
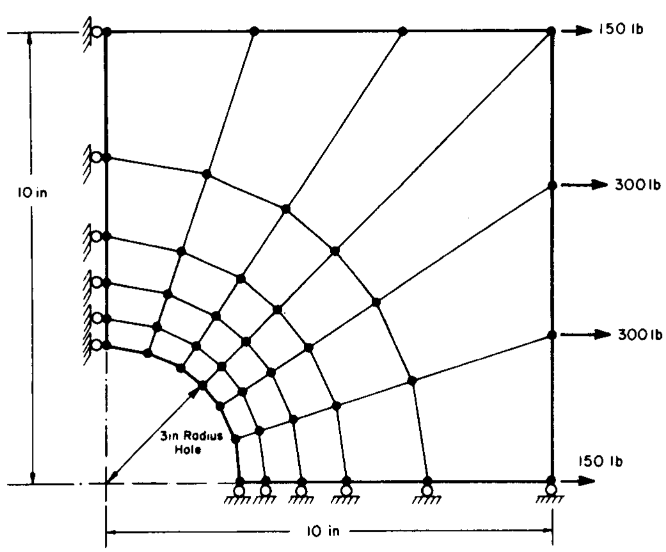
Final deformed mesh (force = 4 N):





Analysis of rubber sheet with hole

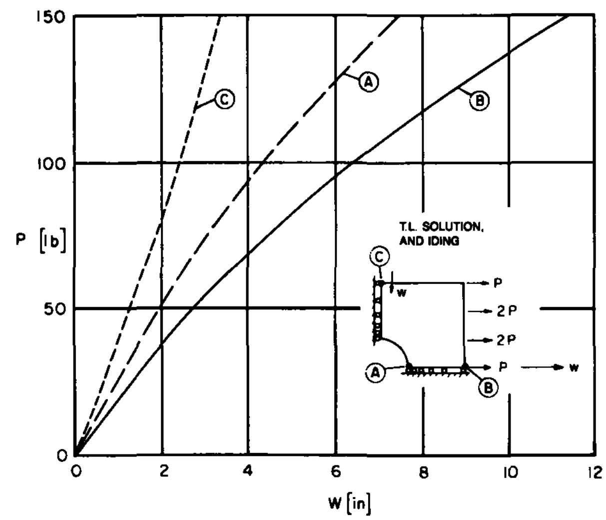
Slide 15-1



Finite element mesh

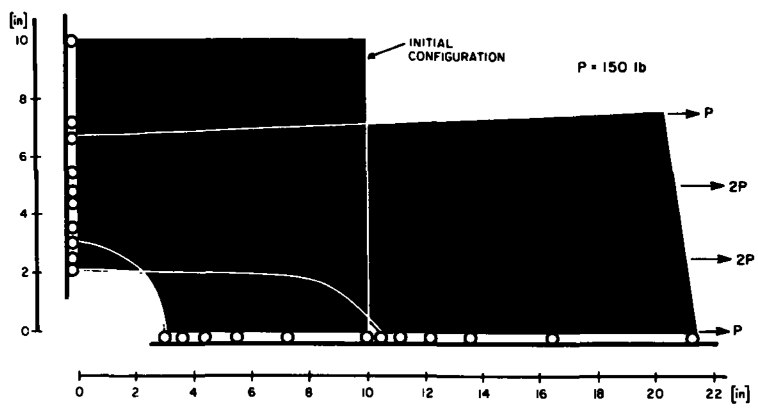
Slide 15-2

Slide 15-3

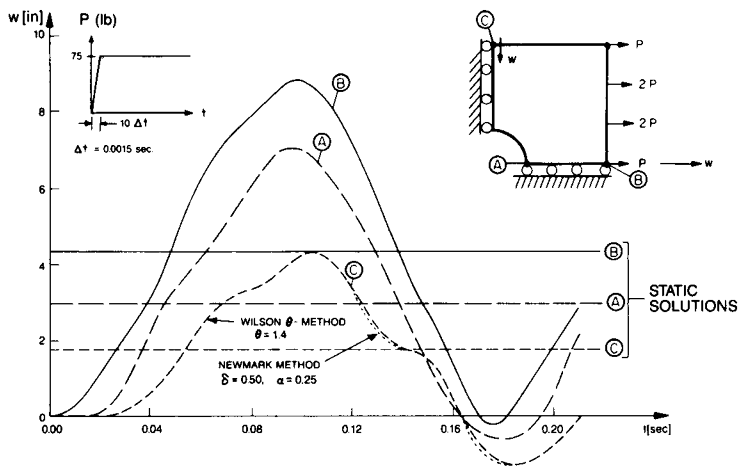


Static load-deflection curve for rubber sheet with hole

Slide 15-4



Deformed configuration drawn to scale of rubber sheet with hole (static analysis)



Displacements versus time for rubber sheet with hole, T.L. solution

Slide 15-5

Resource: Finite Element Procedures for Solids and Structures
Klaus-Jürgen Bathe