Topic 11

Solution of the Nonlinear Finite Element Equations in Static Analysis—Part II

Contents:

- **Automatic load step incrementation for collapse and post-buckling analysis**
- **Constant arc-length and constant increment of work constraints**
- **Geometrical interpretations**
- **An algorithm for automatic load incrementation**
- **Linearized buckling analysis, solution of eigenproblem**
- **Value of linearized buckling analysis**
- **Example analysis: Collapse of an arch—linearized buckling analysis and automatic load step incrementation, effect of initial geometric imperfections**

Textbook: Sections 6.1, 6.5.2

Reference: The automatic load stepping scheme is presented in

Bathe, K. J., and E. Dvorkin, "On the Automatic Solution of Nonlinear Finite Element Equations," *Computers & Structures, 17,* 871–879, 1983.

- WE DISCUSSED IN THE PREVIOUS LECTURE SOLUTION SCHEMES TO SOLVE

$$ {}^{t+\Delta t}\underline{R} = {}^{t+\Delta t}\underline{F} $$

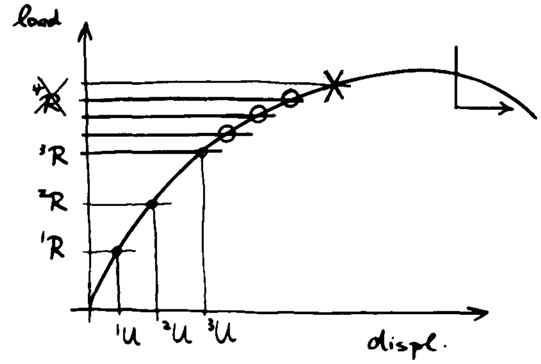
WITH ${}^{t+\Delta t}\underline{R}$ PRESCRIBED FOR EACH LOAD LEVEL

EXAMPLE:

$$ {}^{t}\underline{K}\,\Delta\underline{U}^{(i)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)} $$

$$ {}^{t+\Delta t}\underline{U}^{(i)} = {}^{t+\Delta t}\underline{U}^{(i-1)} + \Delta\underline{U}^{(i)} $$

SCHEMATICALLY:



- DIFFICULTIES ARE ENCOUNTERED TO CALCULATE COLLAPSE LOADS

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AUTOMATIC LOAD STEP INCREMENTATION

- To obtain more rapid convergence in each load step
- To have the program select load increments automatically
- To solve for post-buckling response

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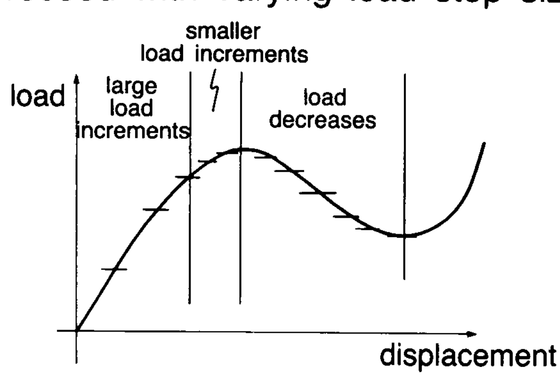
An effective solution procedure would proceed with varying load step sizes:



- Load increment for each step is to be adjusted in magnitude for rapid convergence.

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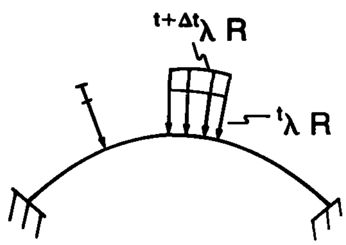
We compute ${}^{t+\Delta t}\underline{R}$ using

$${}^{t+\Delta t}\underline{R} = {}^{t+\Delta t}\lambda\, \underbrace{\underline{R}} \leftarrow \text{a constant vector}$$

Hence we assume: Deformation-independent loading.
All loads are identically scaled.



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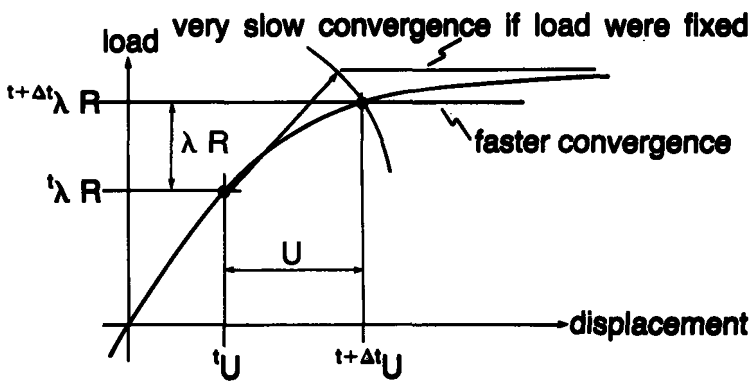
The basic approach:



$${}^{t+\Delta t}\lambda^{(i)} = {}^{t}\lambda + \underbrace{\lambda^{(i)}} \leftarrow \Sigma\, \Delta\lambda^{(k)}$$

$${}^{t+\Delta t}U^{(i)} = {}^{t}U + \underbrace{U^{(i)}} \leftarrow \Sigma\, \Delta U^{(k)}$$

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The governing equations are now:

$$ {}^{\tau}\underline{K}\,\Delta\underline{U}^{(i)} = \underbrace{({}^{t+\Delta t}\lambda^{(i-1)} + \Delta\lambda^{(i)})}_{{}^{t+\Delta t}\lambda^{(i)}}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)} $$

with a constraint equation

$$ f(\Delta\lambda^{(i)}, \Delta\underline{U}^{(i)}) = 0 $$

The unknowns are $\Delta\underline{U}^{(i)}$, $\Delta\lambda^{(i)}$.

$\tau = t$ in the modified Newton-Raphson iteration.

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We may rewrite the equilibrium equations to obtain

$$ {}^{\tau}\underline{K}\,\Delta\underline{\bar{U}}^{(i)} = {}^{t+\Delta t}\lambda^{(i-1)}\,\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)} $$

$$ {}^{\tau}\underline{K}\,\Delta\underline{\bar{\bar{U}}} = \underline{R} $$

only solve this once for each load step.

Hence, we can add these to obtain

$$ \Delta\underline{U}^{(i)} = \Delta\underline{\bar{U}}^{(i)} + \Delta\lambda^{(i)}\,\Delta\underline{\bar{\bar{U}}} $$

Constraint equations:

1 Spherical constant arc-length criterion

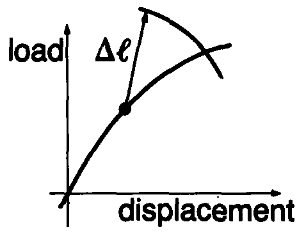
$$(\lambda^{(i)})^2 + (\underline{U}^{(i)})^T (\underline{U}^{(i)})/\beta = (\Delta\ell)^2$$

where

$\lambda^{(i)} = {}^{t+\Delta t}\lambda^{(i)} - {}^{t}\lambda$

$\underline{U}^{(i)} = {}^{t+\Delta t}\underline{U}^{(i)} - {}^{t}\underline{U}$

β = A normalizing factor applied to displacement components (to make all terms dimensionless)



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This equation may be solved for $\Delta\lambda^{(i)}$ as follows:

Using $\lambda^{(i)} = \lambda^{(i-1)} + \Delta\lambda^{(i)}$

and $\underline{U}^{(i)} = \underline{U}^{(i-1)} + \Delta\underline{U}^{(i)}$

$$= \underline{U}^{(i-1)} + \Delta\bar{\underline{U}}^{(i)} + \Delta\lambda^{(i)}\, \Delta\bar{\bar{\underline{U}}}$$

we obtain a quadratic equation in $\Delta\lambda^{(i)}$ ($\Delta\bar{\underline{U}}^{(i)}$ and $\Delta\bar{\bar{\underline{U}}}$ are known vectors).

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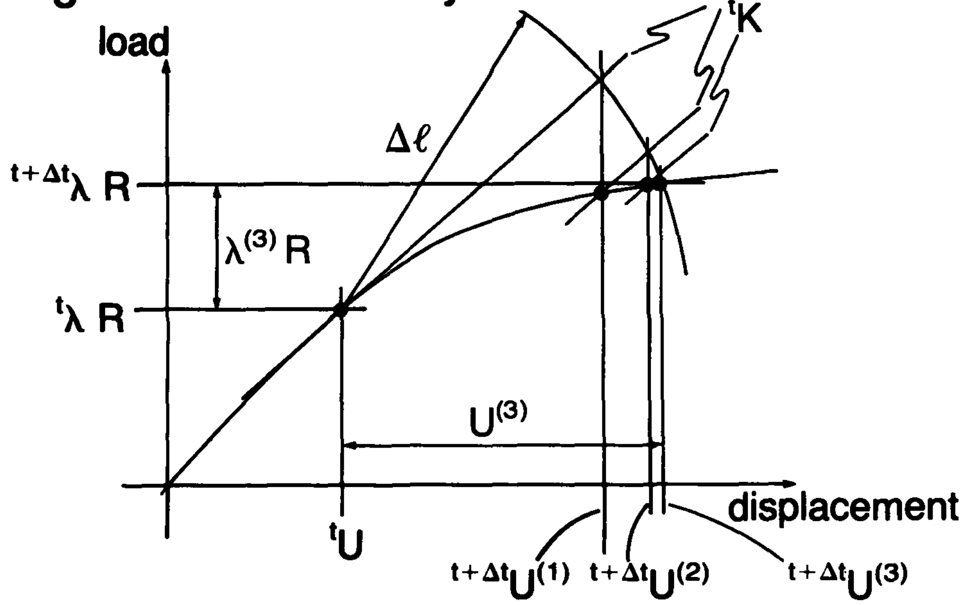
Geometrical interpretation for single degree of freedom system:



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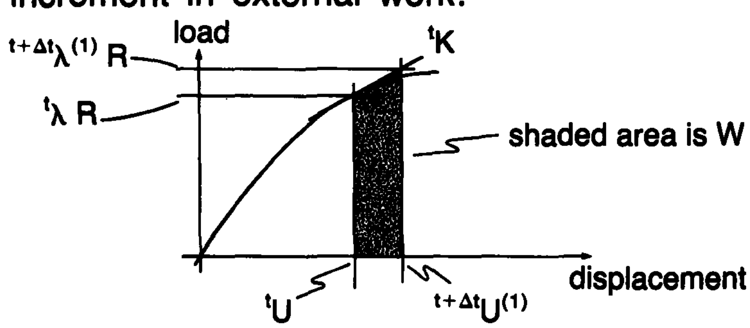
2 "Constant" increment of external work criterion

First iteration: $\left({}^{t}\lambda + \frac{1}{2}\Delta\lambda^{(1)}\right)\underline{R}^{T}\,\Delta\underline{U}^{(1)} = W$

where W is the (preselected) increment in external work:



Successive iterations (i = 2, 3, ...)

$$\left({}^{t+\Delta t}\lambda^{(i-1)} + \frac{1}{2}\Delta\lambda^{(i)}\right) \underline{R}^T \Delta\underline{U}^{(i)} = 0$$

This has solutions:

- $\underline{R}^T \Delta\underline{U}^{(i)} = 0 \qquad \left(\Delta\lambda^{(i)} = -\frac{\underline{R}^T \Delta\bar{\underline{U}}^{(i)}}{\underline{R}^T \Delta\bar{\bar{\underline{U}}}}\right)$
- $\underbrace{{}^{t+\Delta t}\lambda^{(i)} = -{}^{t+\Delta t}\lambda^{(i-1)}}$

  load reverses direction
  (This solution is disregarded)

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Our algorithm:

- Specify $\underline{R}$ and the displacement at one degree of freedom corresponding to ${}^{\Delta t}\lambda$. Solve for ${}^{\Delta t}\underline{U}$.
- Set $\Delta\ell$.
- Use [1] for the next load steps.
- Calculate W for each load step. When W does not change appreciably, or difficulties are encountered with [1], use [2] for the next load step.

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— Note that $\Delta\ell$ is adjusted for the next load step based on the number of iterations used in the last load step.

— Also, ${}^{\tau}\underline{K}$ is recalculated when convergence is slow. Full Newton-Raphson iterations are automatically employed when deemed more effective.

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Linearized buckling analysis:

The physical phenomena of buckling or collapse are represented by the mathematical criterion

$$\det({}^{\tau}\underline{K}) = 0$$

where τ denotes the load level associated with buckling or collapse.

The criterion det $({}^{\tau}\underline{K}) = 0$ implies that the equation

$$ {}^{\tau}\underline{K}\,\underline{U}^* = \underline{0} $$

has a non-trivial solution for $\underline{U}^*$ (and α U* is a solution with α being any constant). Hence we can select a small load ε for which very large displacements are obtained.

This means that the structure is unstable.

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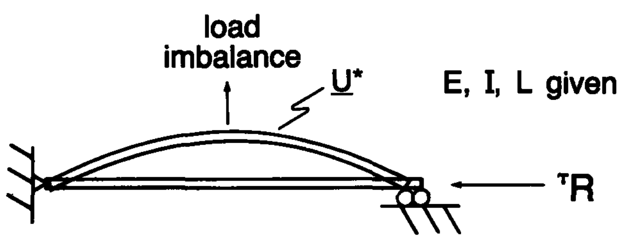
Physically, the smallest load imbalance will trigger the buckling (collapse) displacements:



Pinned-pinned beam

$$ {}^{\tau}R = \frac{\pi^2 EI}{L^2} $$

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We want to predict the load level and mode shape associated with buckling or collapse. Hence we perform a linearized buckling analysis.

We assume

$$^{\tau}\underline{K} = {}^{t-\Delta t}\underline{K} + \lambda\,({}^{t}\underline{K} - {}^{t-\Delta t}\underline{K})$$

$$^{\tau}\underline{R} = {}^{t-\Delta t}\underline{R} + \lambda\,({}^{t}\underline{R} - {}^{t-\Delta t}\underline{R})$$

λ is a scaling factor which we determine below. We assume here that the value λ we require is greater than 1.

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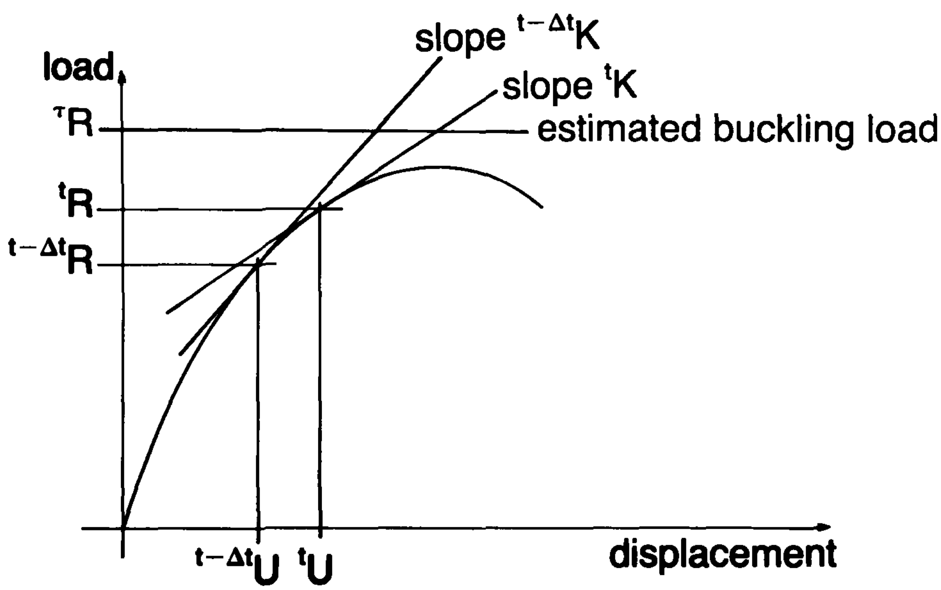
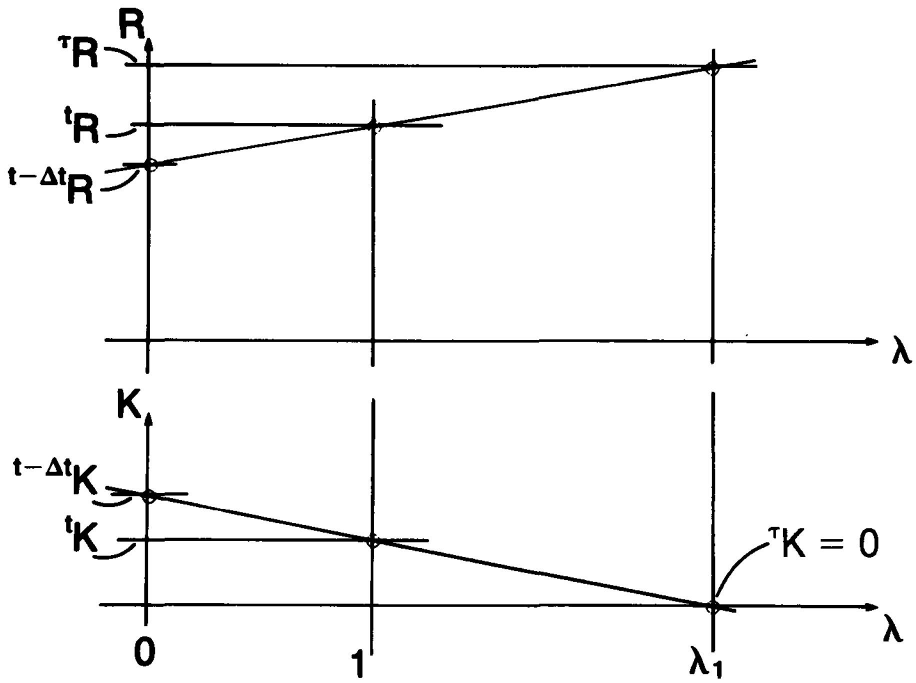
Pictorially, for one degree of freedom:





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The problem of solving for λ such that $\det(^{\tau}\underline{K}) = 0$ is equivalent to the eigenproblem

$$^{t-\Delta t}\underline{K}\,\underline{\phi} = \lambda\,(^{t-\Delta t}\underline{K} - {}^{t}\underline{K})\,\underline{\phi}$$

where $\underline{\phi}$ is the associated eigenvector (buckling mode shape).

In general, $^{t-\Delta t}\underline{K} - {}^{t}\underline{K}$ is indefinite, hence the eigenproblem will have both positive and negative solutions. We want only the smallest positive λ value (and perhaps the next few larger values).

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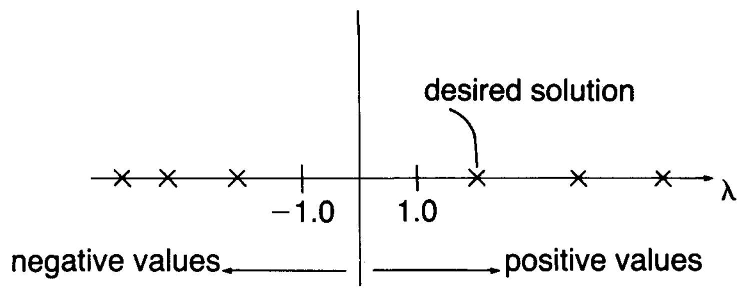
Solution of problem

$$ {}^{t-\Delta t}\underline{K}\,\underline{\phi} = \lambda\,({}^{t-\Delta t}\underline{K} - {}^{t}\underline{K})\,\underline{\phi} $$



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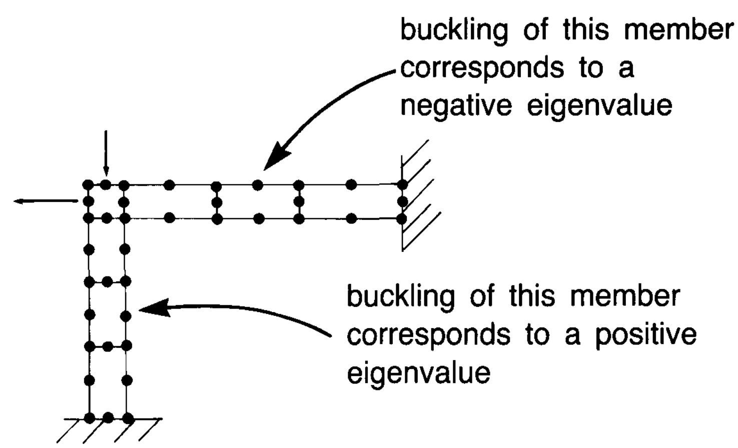
Example of model with both positive and negative eigenvalues:



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We rewrite the eigenvalue problem as follows:

$$ {}^{t}\underline{K}\,\underline{\phi} = \underbrace{\left(\frac{\lambda - 1}{\lambda}\right)}_{\gamma}\ {}^{t-\Delta t}\underline{K}\,\underline{\phi} $$

Now we note that the critical buckling mode of interest is the one for which γ is small and positive.

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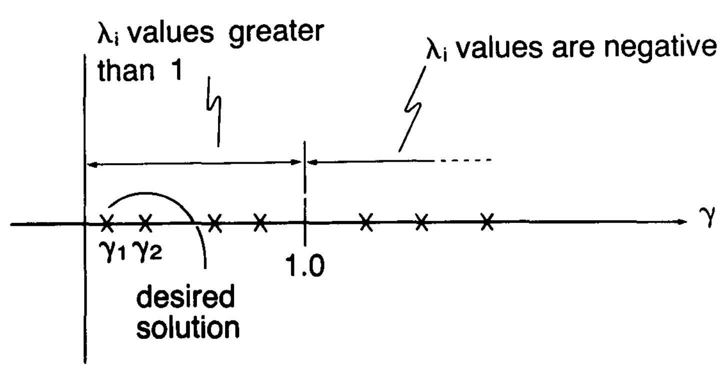
Solution of problem

$$ {}^{t}\underline{K}\,\underline{\phi} = \gamma\ {}^{t-\Delta t}\underline{K}\,\underline{\phi};\quad \gamma = \frac{\lambda - 1}{\lambda} $$



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Value of linearized buckling analysis:

- Not expensive
- Gives insight into possible modes of failure.
- For applicability, important that pre-buckling displacements are small.
- Yields collapse modes that are effectively used to impose imperfections.
  - — To study sensitivity of a structure to imperfections

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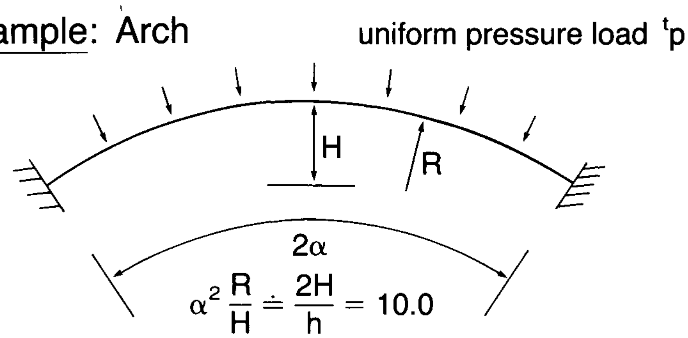
But

— procedure must be employed with great care because the results may be quite misleading.

— procedure only predicts physically realistic buckling or collapse loads when structure buckles "in the Euler column type".

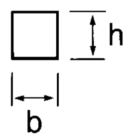
Example: Arch



$$\alpha^2 \frac{R}{H} \doteq \frac{2H}{h} = 10.0$$

$R = 64.85$
$\alpha = 22.5°$
$E = 2.1 \times 10^6$
$\nu = 0.3$
$h = b = 1.0$

Cross-section:



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Finite element model:

- Ten 2-node isoparametric beam elements
- Complete arch is modeled.

Purpose of analysis:

- To determine the collapse mechanism and collapse load level.
- To compute the post-collapse response.

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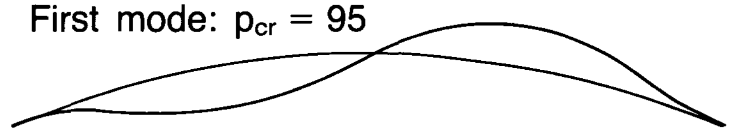
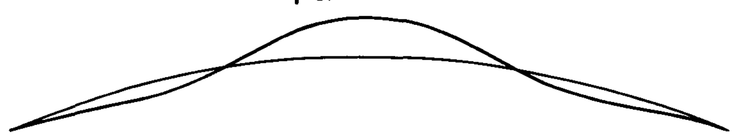
Step 1: Determine collapse mechanisms and collapse loads using a linearized buckling analysis ($^{\Delta t}p = 10$).

First mode: $p_{cr} = 95$



Second mode: $p_{cr} = 150$



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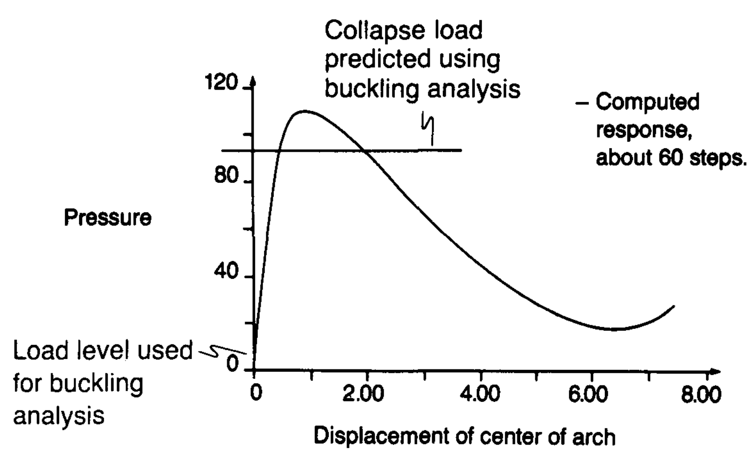
Step 2: Compute the response of the arch using automatic step incrementation.



We have computed the response of a perfect (symmetric) arch. Because the first collapse mode is antisymmetric, that mode is not excited by the pressure loading during the response calculations.

However, a real structure will contain imperfections, and hence will not be symmetric. Therefore, the antisymmetric collapse mode may be excited, resulting in a lower collapse load.

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Hence, we adjust the initial coordinates of the arch to introduce a geometric imperfection. This is done by adding a multiple of the first buckling mode to the geometry of the undeformed arch.

The collapse mode is scaled so that the magnitude of the imperfection is less than 0.01.

The resulting "imperfect" arch is no longer symmetric.

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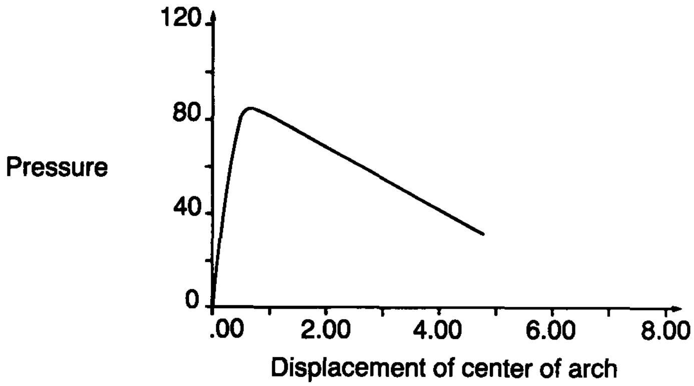
Step 3: Compute the response of the "imperfect" arch using automatic step incrementation.



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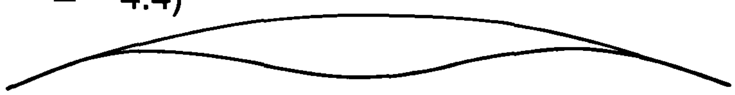
Comparison of post-collapse displacements:

"Perfect" arch: (disp. at center of arch = −4.4)



"Imperfect" arch: (disp. at center of arch = −4.8)