Massachusetts Institute of Technology
MIT Video Course

Video Course Study Guide

Finite Element Procedures for Solids and Structures– Nonlinear Analysis

Klaus-Jürgen Bathe
Professor of Mechanical Engineering, MIT

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Preface

This course on the nonlinear analysis of solids and structures can be thought of as a continuation of the course on the linear analysis of solids and structures (see *Finite Element Procedures for Solids and Structures—Linear Analysis*) or as a stand-alone course.

The objective in this course is to summarize modern and effective finite element procedures for the nonlinear analysis of static and dynamic problems. The modeling of geometric and material nonlinear problems is discussed. The basic finite element formulations employed are presented, efficient numerical procedures are discussed, and recommendations on the actual use of the methods in engineering practice are given. The course is intended for practicing engineers and scientists who want to solve problems using modern and efficient finite element methods.

In this study guide, brief descriptions of the lectures are presented. The markerboard presentations and viewgraphs used in the lectures are also given. Below the brief description of each lecture, reference is made to the accompanying textbook of the course: *Finite Element Procedures in Engineering Analysis,* by K. J. Bathe, Prentice-Hall, Englewood Cliffs, N.J., 1982. Reference is also sometimes made to one or more journal papers.

The textbook sections and examples, listed below the brief description of each lecture, provide important reading and study material for the course.

Acknowledgments

August 1986

I was indeed very fortunate to have had the help of some very able and devoted individuals in the production of this video course.

Theodore (Ted) Sussman, my research assistant, was most helpful in the preparation of the viewgraphs and especially in the design of the problem solutions and the computer laboratory sessions.

Patrick Weygint, Assistant Production Manager, aided me with great patience and a keen eye for details during practically every phase of the production. Elizabeth DeRienzo, Production Manager for the Center for Advanced Engineering Study, MIT, showed great skill and cooperation in directing the actual videotaping. Richard Noyes, Director of the MIT Video Course Program, contributed many excellent suggestions throughout the preparation and production of the video course.

The combined efforts of these people plus the professionalism of the video crew and support staff helped me to present what I believe is a very valuable series of video-based lessons in Finite Element Procedures for Solids and Structures–Nonlinear Analysis.

Many thanks to them all!

Klaus-Jürgen Bathe, MIT

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* Topics followed by an asterisk consist of two videotapes

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* Topics followed by an asterisk consist of two videotapes

Topic 1

Introduction to Nonlinear Analysis

Contents:

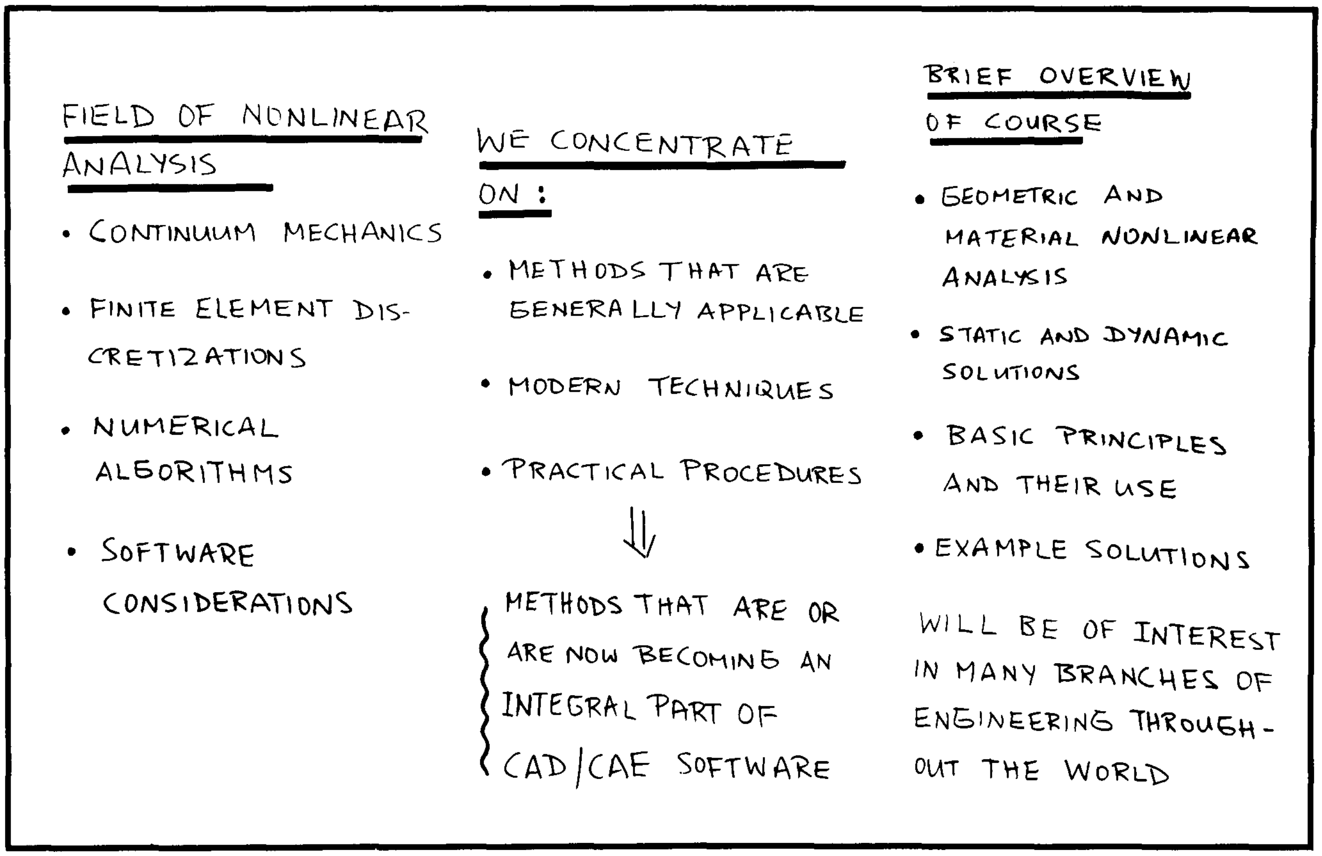
- **Introduction to the course**
- **The importance of nonlinear analysis**
- **Four illustrative films depicting actual and potential nonlinear analysis applications**
- **General recommendations for nonlinear analysis**
- **Modeling of problems**
- **Classification of nonlinear analyses**
- **Example analysis of a bracket, small and large deformations, elasto-plastic response**
- **Two computer-plotted animations**
    - **—elasto-plastic large deformation response of a plate with a hole**
    - **—large displacement response of a diamond-shaped frame**
- **The basic approach of an incremental solution**
- **Time as a variable in static and dynamic solutions**
- **The basic incremental/iterative equations**
- **A demonstrative static and dynamic nonlinear analysis of a shell**

Textbook: Section 6.1

Examples: 6.1, 6.2, 6.3, 6.4

Reference: The shell analysis is reported in

Ishizaki, T., and K. J. Bathe, "On Finite Element Large Displacement and Elastic-Plastic Dynamic Analysis of Shell Structures," *Computers & Structures, 12,* 309–318, 1980.



FIELD OF NONLINEAR ANALYSIS

- CONTINUUM MECHANICS
- FINITE ELEMENT DISCRETIZATIONS
- NUMERICAL ALGORITHMS
- SOFTWARE CONSIDERATIONS

WE CONCENTRATE ON :

- METHODS THAT ARE GENERALLY APPLICABLE
- MODERN TECHNIQUES
- PRACTICAL PROCEDURES

$\Downarrow$

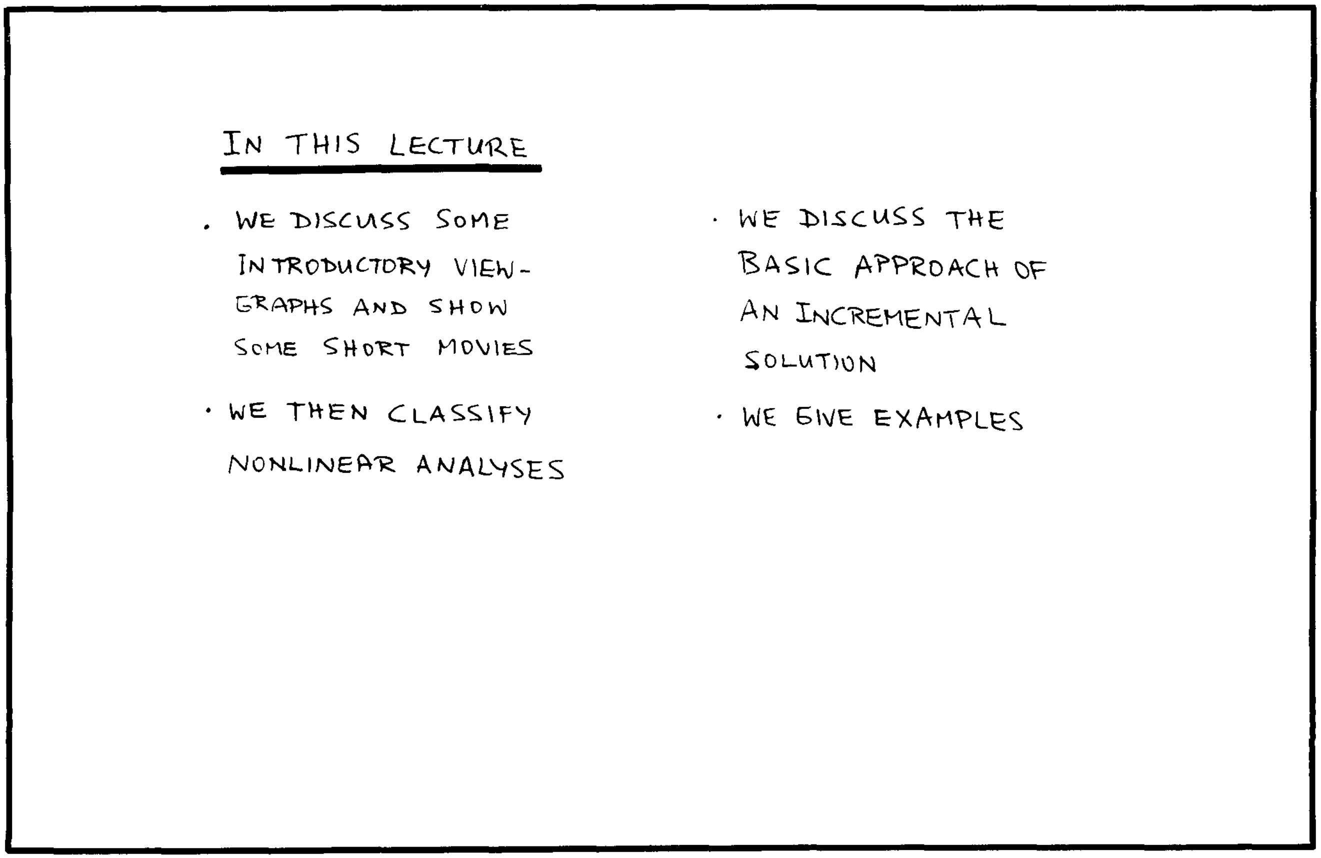
METHODS THAT ARE OR ARE NOW BECOMING AN INTEGRAL PART OF CAD/CAE SOFTWARE

BRIEF OVERVIEW OF COURSE

- GEOMETRIC AND MATERIAL NONLINEAR ANALYSIS
- STATIC AND DYNAMIC SOLUTIONS
- BASIC PRINCIPLES AND THEIR USE
- EXAMPLE SOLUTIONS

WILL BE OF INTEREST IN MANY BRANCHES OF ENGINEERING THROUGHOUT THE WORLD

Markerboard 1-1



IN THIS LECTURE

- WE DISCUSS SOME INTRODUCTORY VIEWGRAPHS AND SHOW SOME SHORT MOVIES
- WE THEN CLASSIFY NONLINEAR ANALYSES
- WE DISCUSS THE BASIC APPROACH OF AN INCREMENTAL SOLUTION
- WE GIVE EXAMPLES

Markerboard 1-2

Transparency 1-1

FINITE ELEMENT NONLINEAR ANALYSIS

- Nonlinear analysis in engineering mechanics can be an art.
- Nonlinear analysis can be a frustration.
- It always is a great challenge.

Transparency 1-2

Some important engineering phenomena can only be assessed on the basis of a nonlinear analysis:

- Collapse or buckling of structures due to sudden overloads
- Progressive damage behavior due to long lasting severe loads
- For certain structures (e.g. cables), nonlinear phenomena need be included in the analysis even for service load calculations.

The need for nonlinear analysis has increased in recent years due to the need for

— use of optimized structures

— use of new materials

— addressing safety-related issues of structures more rigorously

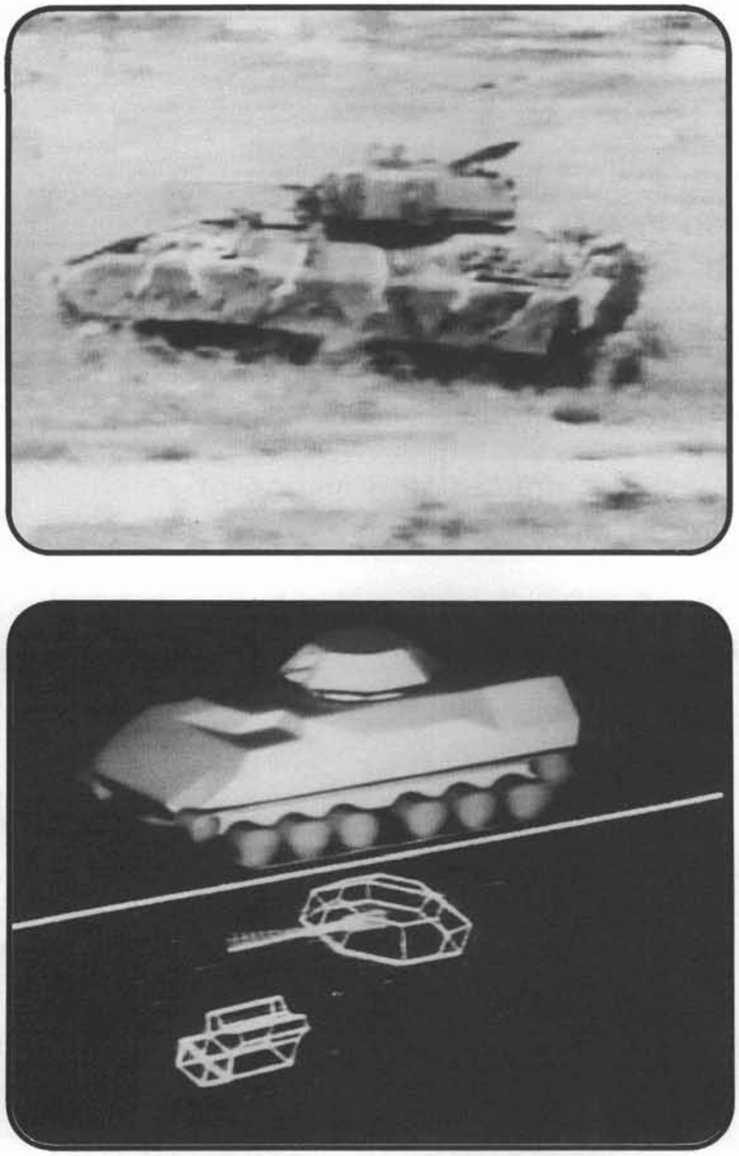
The corresponding benefits can be most important.

Transparency 1-3

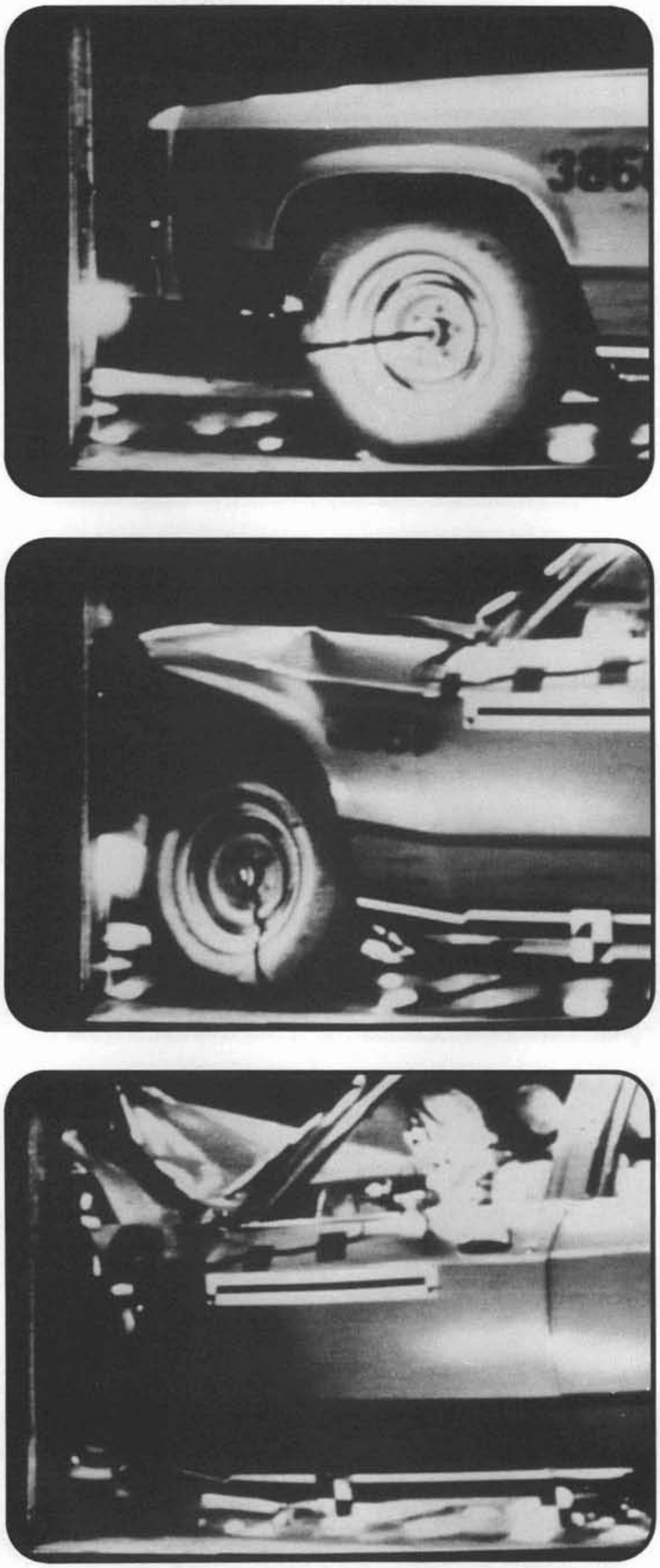
Problems to be addressed by a nonlinear finite element analysis are found in almost all branches of engineering, most notably in,

Nuclear Engineering
Earthquake Engineering
Automobile Industries
Defense Industries
Aeronautical Engineering
Mining Industries
Offshore Engineering
and so on

Transparency 1-4

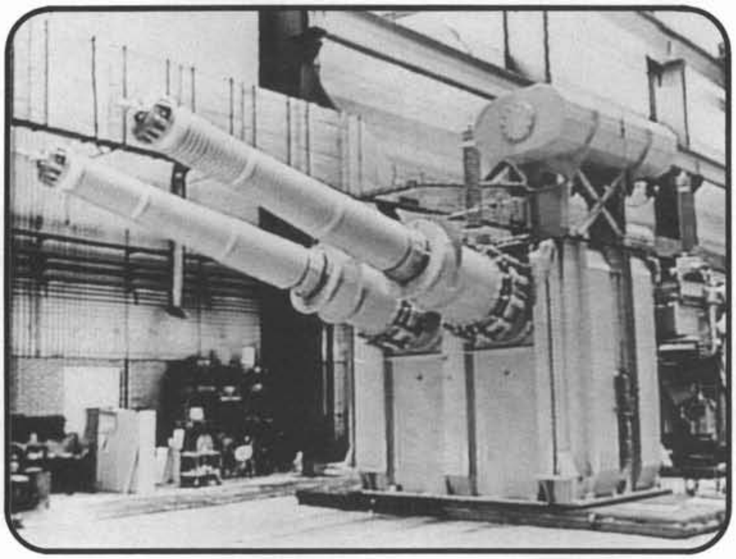


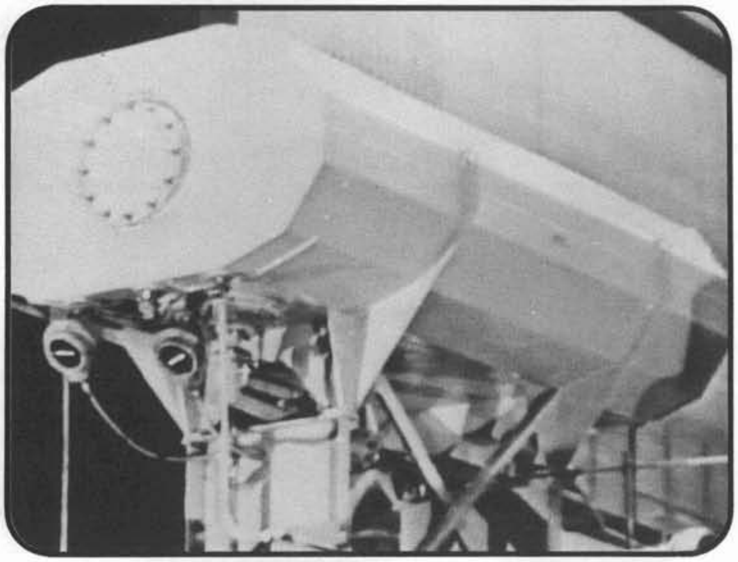
Film Insert Armored Fighting Vehicle
Courtesy of General Electric
CAE International Inc.

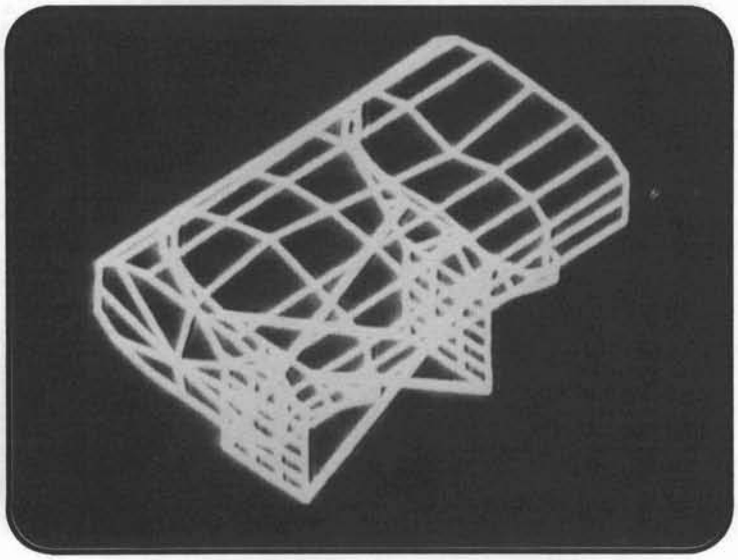


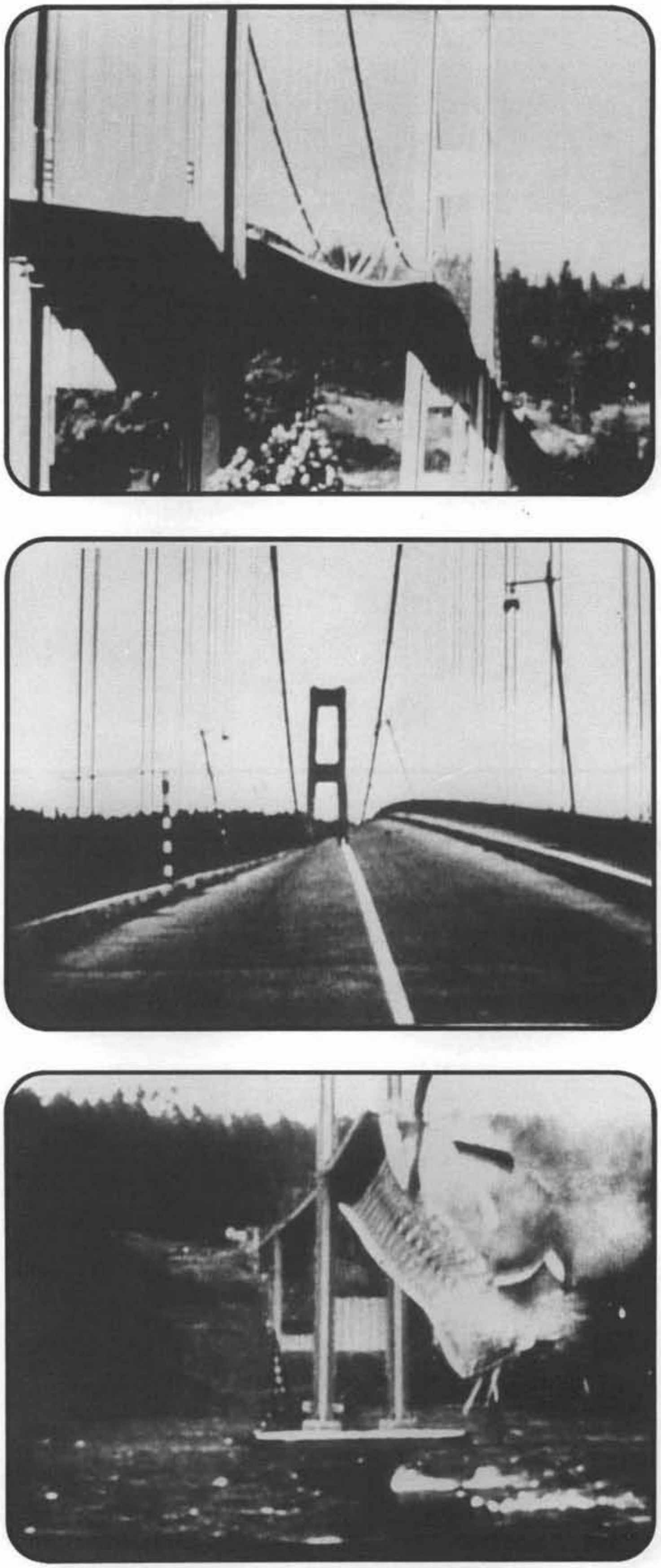
Film Insert Automobile Crash Test
Courtesy of
Ford Occupant
Protection Systems

Film Insert Earthquake Analysis
Courtesy of
ASEA Research
and Innovation-
Transformers
Division









Film Insert Tacoma Narrows Bridge Collapse
Courtesy of Barney D. Elliot

Transparency 1-5

For effective nonlinear analysis, a good physical and theoretical understanding is most important.

$$\underbrace{\begin{array}{c}\text{PHYSICAL}\\ \text{INSIGHT}\end{array} \longleftrightarrow \begin{array}{c}\text{MATHEMATICAL}\\ \text{FORMULATION}\end{array}}_{\left(\begin{array}{c}\text{INTERACTION AND}\\ \text{MUTUAL ENRICHMENT}\end{array}\right)}$$

Transparency 1-6

BEST APPROACH

- Use reliable and generally applicable finite elements.
- With such methods, we can establish models that we understand.
- Start with simple models (of nature) and refine these as need arises.

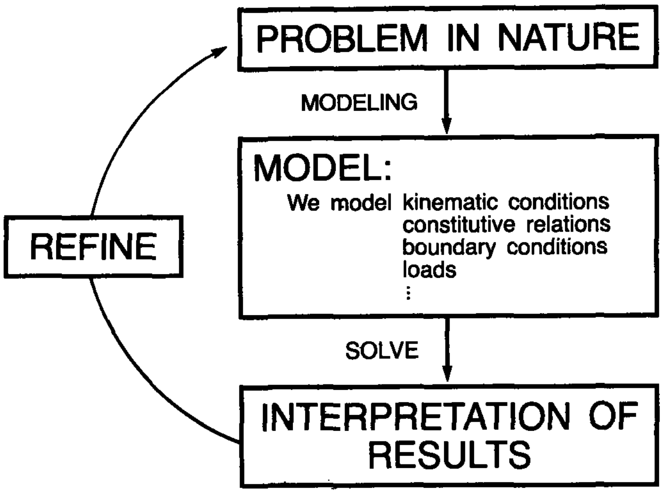
A "PHILOSOPHY" FOR PERFORMING
A NONLINEAR ANALYSIS

TO PERFORM A NONLINEAR ANALYSIS

- Stay with relatively small and reliable models.
- Perform a linear analysis first.
- Refine the model by introducing nonlinearities as desired.
- Important:
  - — Use reliable and well-understood models.
  - — Obtain accurate solutions of the models.

NECESSARY FOR THE INTERPRETATION
OF RESULTS

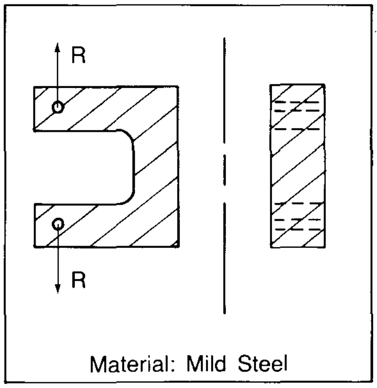
Transparency
1-7



Transparency
1-8

Transparency 1-9

A TYPICAL NONLINEAR PROBLEM



POSSIBLE QUESTIONS:

Yield Load?

Limit Load?

Plastic Zones?

Residual Stresses?

Yielding where Loads are Applied?

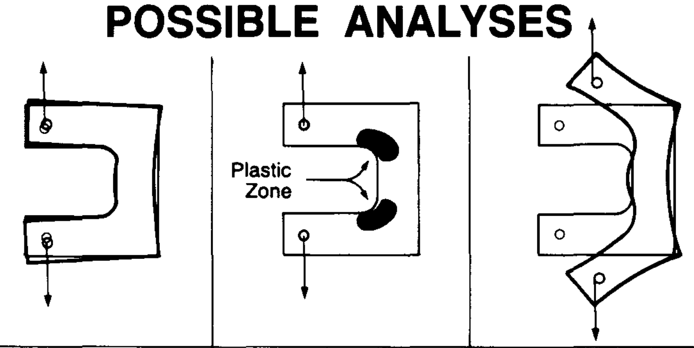
Creep Response?

Permanent Deflections?

⋮

Transparency 1-10

POSSIBLE ANALYSES



Linear elastic analysis	Plastic analysis	Plastic analysis
	(Small deformations)	(Large deformations)
Determine:	Determine:	Determine:
Total Stiffness; Yield Load	Sizes and Shapes of Plastic Zones	Ultimate Load Capacity

Transparency 1-11

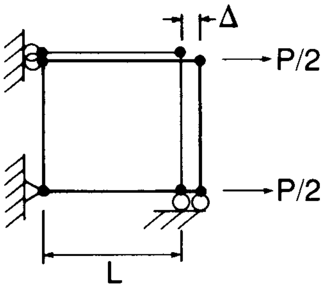
CLASSIFICATION OF NONLINEAR ANALYSES

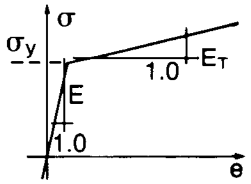
1) Materially-Nonlinear-Only (M.N.O.) analysis:

   - Displacements are infinitesimal.
   - Strains are infinitesimal.
   - The stress-strain relationship is nonlinear.

Transparency 1-12

Example:





Material is elasto-plastic.

$\frac{\Delta}{L} < 0.04$

- As long as the yield point has not been reached, we have a linear analysis.

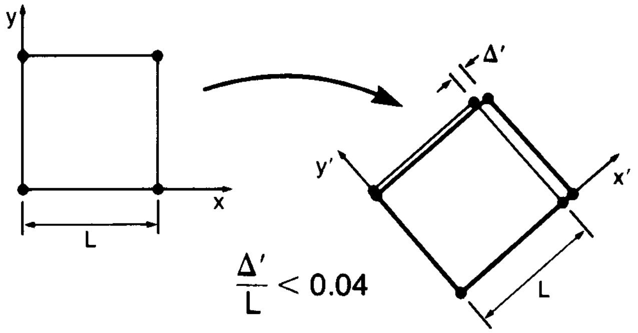
Transparency 1-13

2) Large displacements / large rotations but small strains:

- Displacements and rotations are large.
- Strains are small.
- Stress-strain relations are linear or nonlinear.

Transparency 1-14

Example:



$\frac{\Delta'}{L} < 0.04$

- As long as the displacements are very small, we have an M.N.O. analysis.

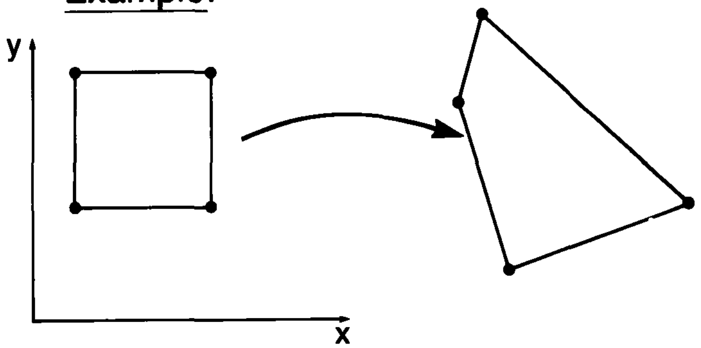
Transparency 1-15

3) Large displacements, large rotations, large strains:

- Displacements are large.
- Rotations are large.
- Strains are large.
- The stress-strain relation is probably nonlinear.

Transparency 1-16

Example:

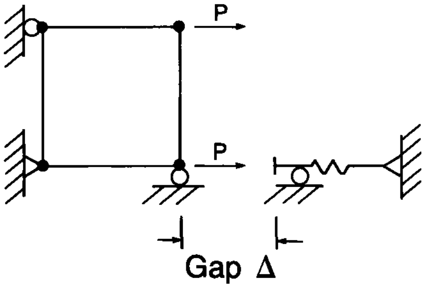


- This is the most general formulation of a problem, considering no nonlinearities in the boundary conditions.

Transparency 1-17

4) Nonlinearities in boundary conditions

Contact problems:

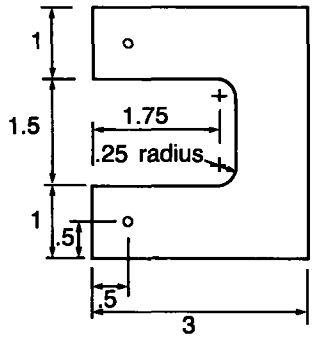


- Contact problems can arise with large displacements, large rotations, materially nonlinear conditions, . . .

Transparency 1-18

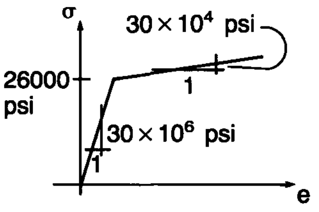
Example: Bracket analysis

All dimensions in inches



thickness = 1 in.

Elasto-plastic material model:

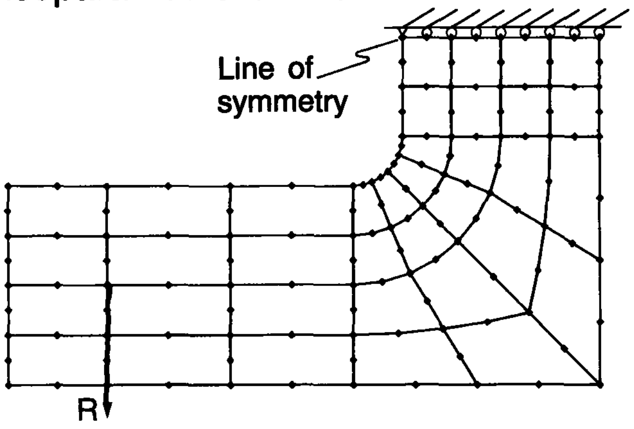


Isotropic hardening

Transparency 1-19

Finite element model: 36 element mesh

- All elements are 8-node isoparametric elements



Transparency 1-20

Three *kinematic* formulations are used:

- Materially-nonlinear-only analysis (small displacements/small rotations and small strains)
- Total Lagrangian formulation (large displacements/large rotations and large strains)
- Updated Lagrangian formulation (large displacements/large rotations and large strains)

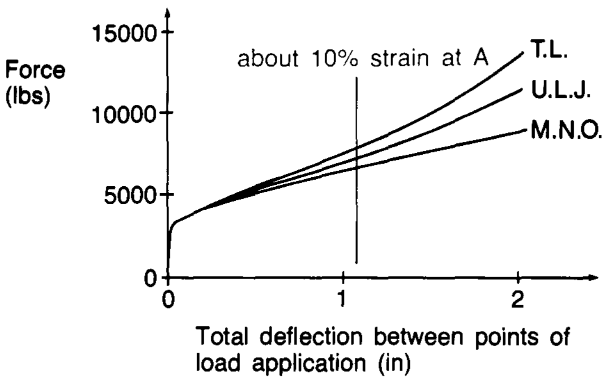
Transparency
1-21

However, different stress-strain laws are used with the total and updated Lagrangian formulations. In this case,

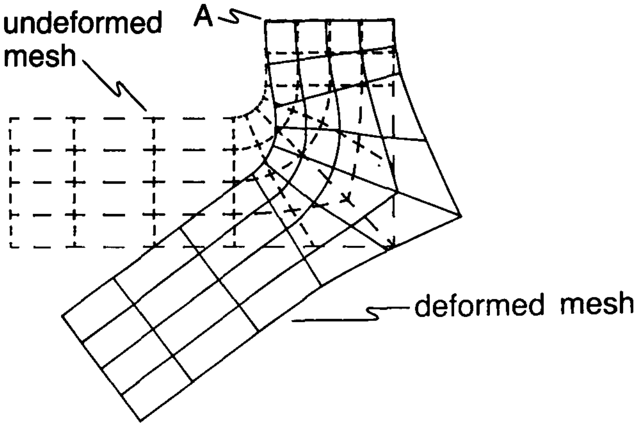
- The material law used in conjunction with the total Lagrangian formulation is actually not applicable to large strain situations (but only to large displ., rotation/ small strain conditions).
- The material law used in conjunction with the updated Lagrangian formulation is applicable to large strain situations.

Transparency
1-22

We present force-deflection curves computed using each of the three kinematic formulations and associated material laws:

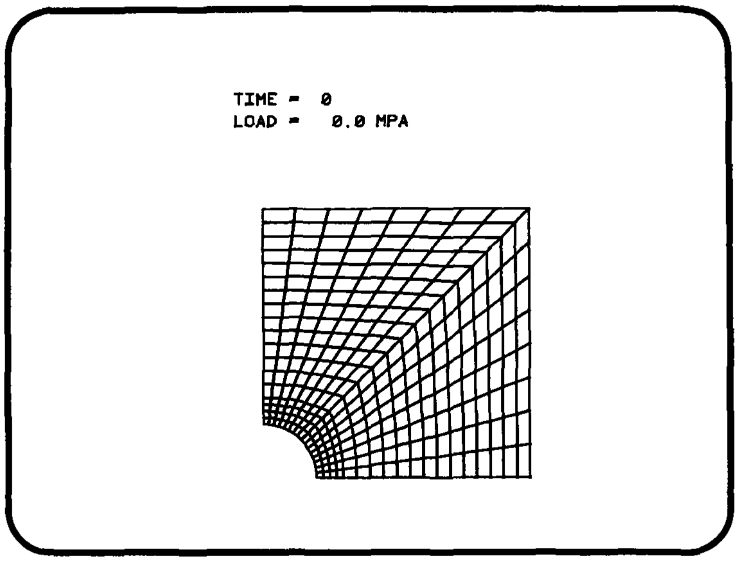


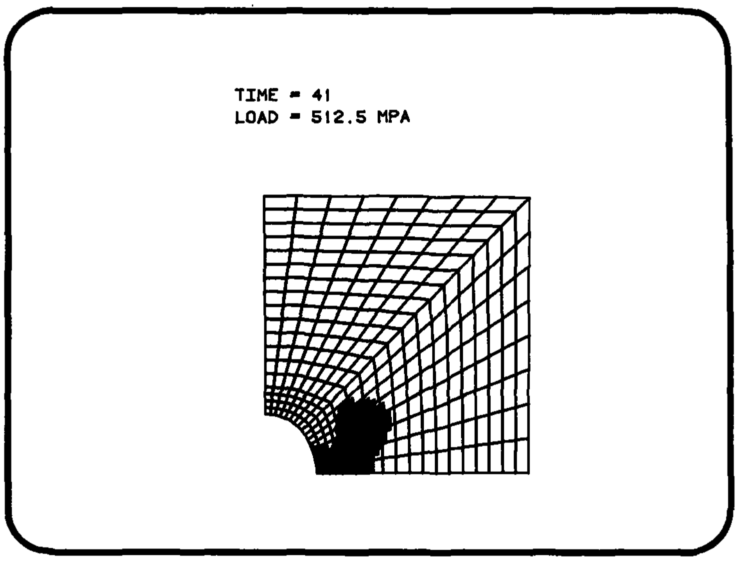
The deformed mesh corresponding to a load level of 12000 lbs is shown below (the U.L.J. formulation is used).

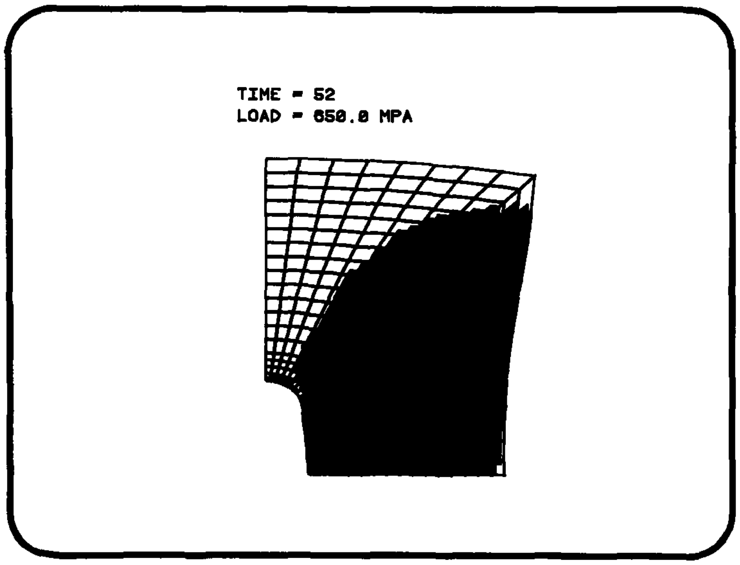


Transparency 1-23

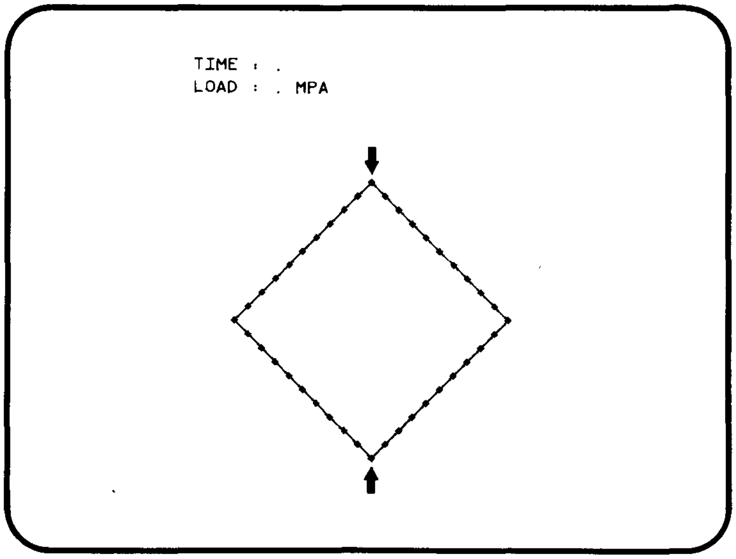
Computer Animation
Plate with hole



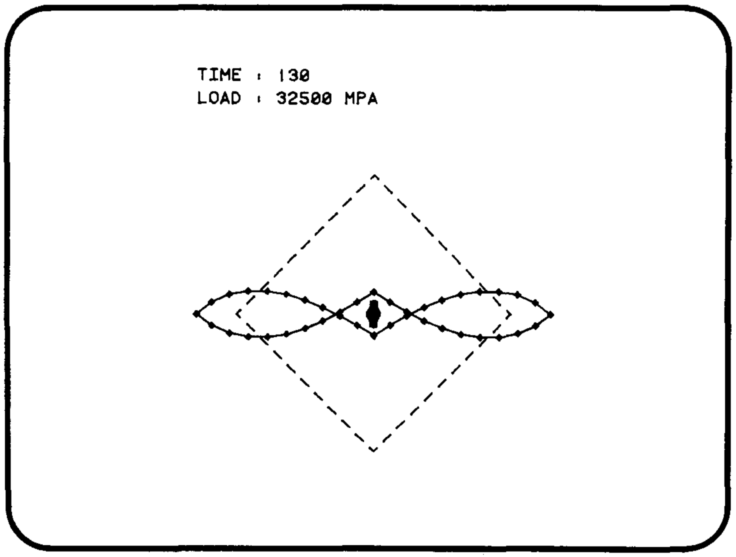




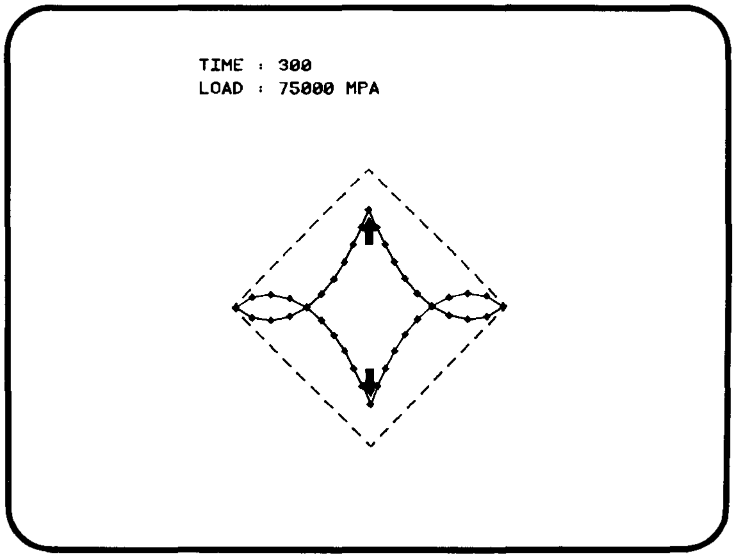
Computer Animation
Diamond shaped frame



```
TIME : .
LOAD : . MPA
```



```
TIME : 130
LOAD : 32500 MPA
```



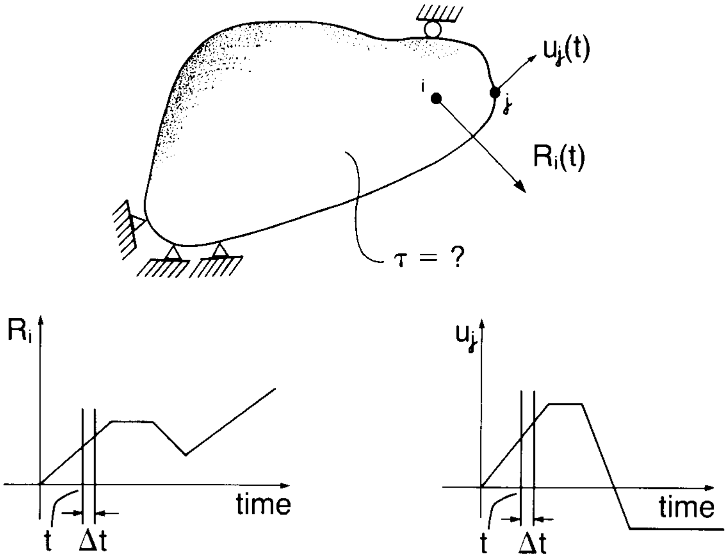
```
TIME : 300
LOAD : 75000 MPA
```

Transparency 1-24

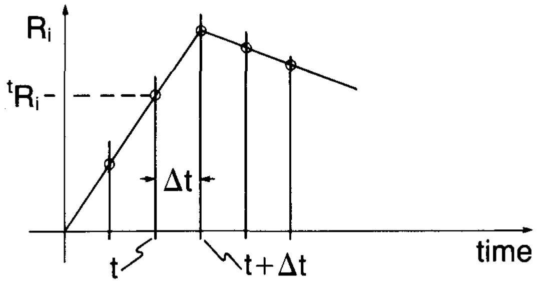
THE BASIC APPROACH OF AN INCREMENTAL SOLUTION

- We consider a body (a structure or solid) subjected to force and displacement boundary conditions that are changing.
- We describe the externally applied forces and the displacement boundary conditions as functions of time.

Transparency 1-25



Since we anticipate nonlinearities, we use an incremental approach, measured in load steps or time steps



Transparency 1-26

When the applied forces and displacements vary

- slowly, meaning that the frequencies of the loads are much smaller than the natural frequencies of the structure, we have a static analysis;
- fast, meaning that the frequencies of the loads are in the range of the natural frequencies of the structure, we have a dynamic analysis.

Transparency 1-27

Transparency 1-28

Meaning of time variable

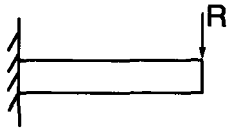
- Time is a pseudo-variable, only denoting the load level

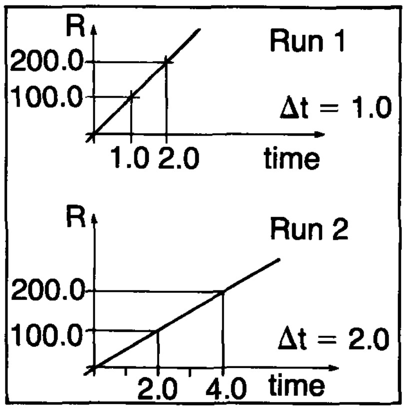
  in

  Nonlinear static analysis with time-independent material properties

Transparency 1-29

Example:





Identically the same results are obtained in Run 1 and Run 2

Transparency 1-30

Time is an actual variable

— in dynamic analysis

— in nonlinear static analysis with time-dependent material properties (creep)

Now Δt must be chosen carefully with respect to the physics of the problem, the numerical technique used and the costs involved.

Transparency 1-31

At the end of each load (or time) step, we need to satisfy the three basic requirements of mechanics:

- Equilibrium
- Compatibility
- The stress-strain law

This is achieved – in an approximate manner using finite elements – by the application of the principle of virtual work.

Transparency 1-32

We idealize the body as an assemblage of finite elements and apply the principle of virtual work to the unknown state at time t+Δt.

$$ {}^{t+\Delta t}\underline{R} = {}^{t+\Delta t}\underline{F} $$

vector of externally applied nodal point forces (these include the inertia forces in dynamic analysis)

vector of nodal point forces equivalent to the internal element stresses

Transparency 1-33

- Now assume that the solution at time t is known. Hence ${}^{t}\tau_{ij}$, ${}^{t}V$, ... are known.
- We want to obtain the solution corresponding to time t+Δt (i.e., for the loads applied at time t+Δt).
- For this purpose, we solve in static analysis

$$ {}^{t}\underline{K}\;\Delta\underline{U} = {}^{t+\Delta t}\underline{R} - {}^{t}\underline{F} $$

$$ {}^{t+\Delta t}\underline{U} \doteq {}^{t}\underline{U} + \Delta\underline{U} $$

Transparency 1-34

More generally, we solve

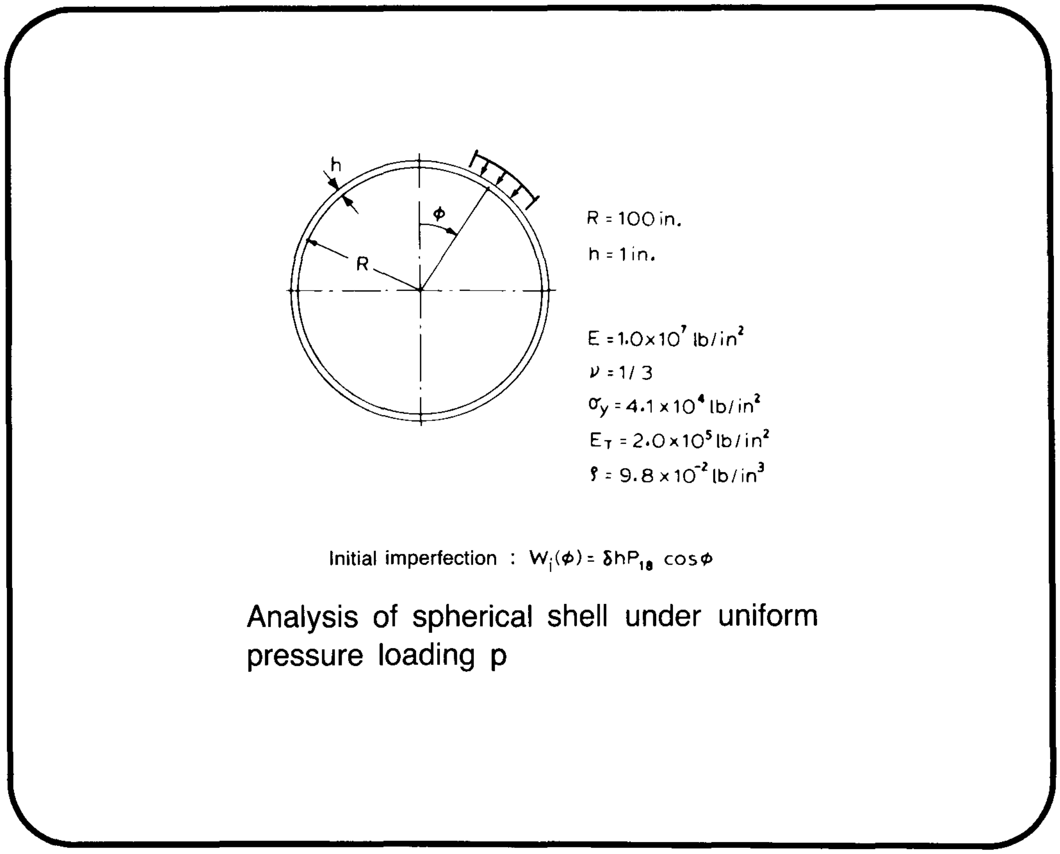
$$ {}^{t}\underline{K}\,\Delta\underline{U}^{(i)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)} $$

$$ {}^{t+\Delta t}\underline{U}^{(i)} = {}^{t+\Delta t}\underline{U}^{(i-1)} + \Delta\underline{U}^{(i)} $$

using

$$ {}^{t+\Delta t}\underline{F}^{(0)} = {}^{t}\underline{F}\,, \qquad {}^{t+\Delta t}\underline{U}^{(0)} = {}^{t}\underline{U} $$

Slide 1-1



R = 100 in.

h = 1 in.

$E = 1.0 \times 10^7$ lb/in^2

$\nu = 1/3$

$\sigma_y = 4.1 \times 10^4$ lb/in^2

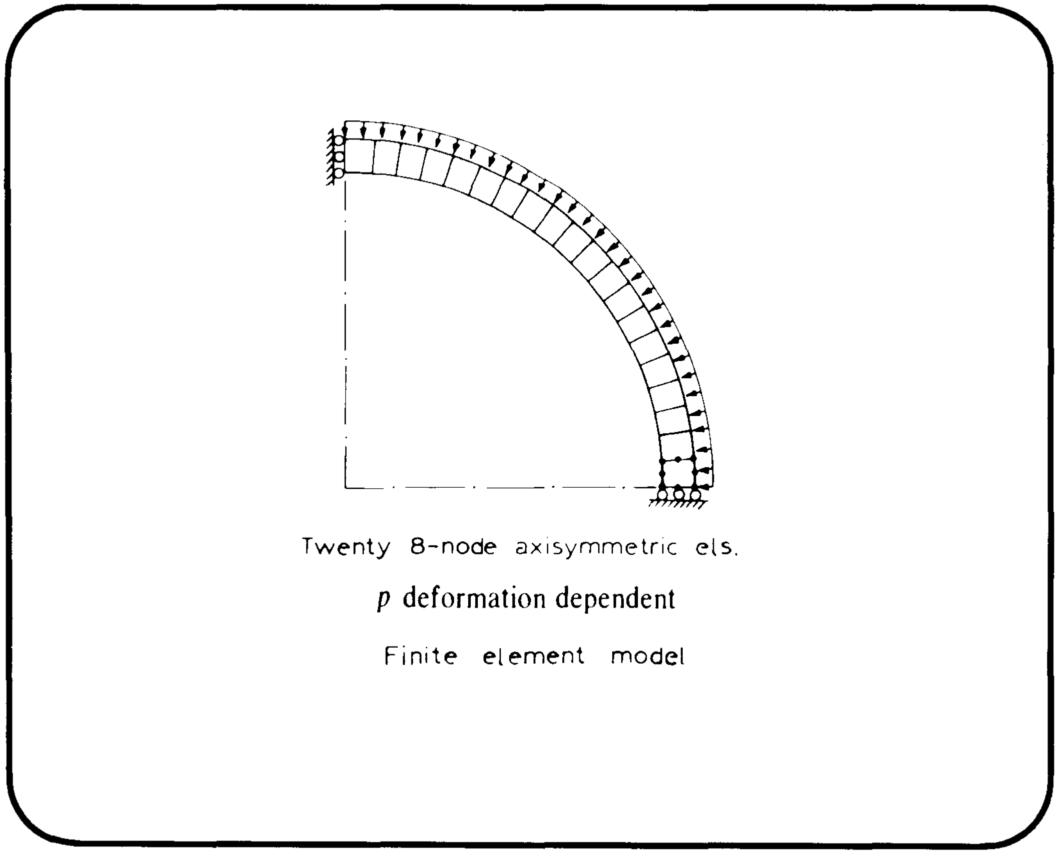
$E_T = 2.0 \times 10^5$ lb/in^2

$\rho = 9.8 \times 10^{-2}$ lb/in^3

Initial imperfection : $W_i(\phi) = \delta h P_{18} \cos\phi$

Analysis of spherical shell under uniform pressure loading p

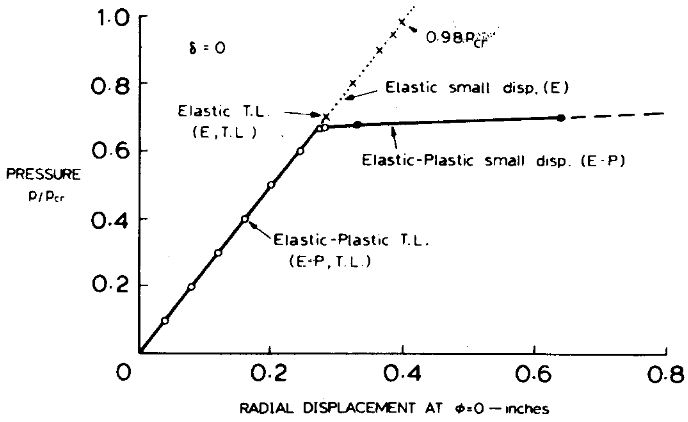
Slide 1-2



Twenty 8-node axisymmetric els.

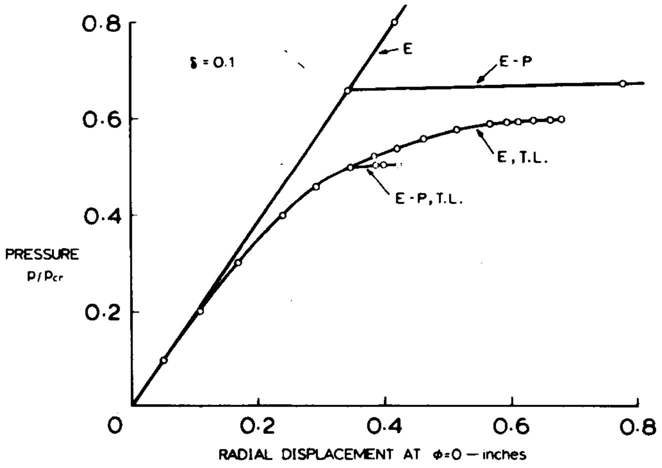
p deformation dependent

Finite element model



Static response of perfect ($\delta = 0$) shell

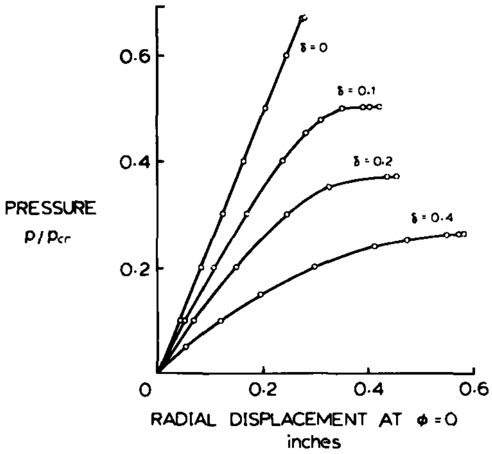
Slide
1-3



Static response of imperfect ($\delta = 0.1$) shell

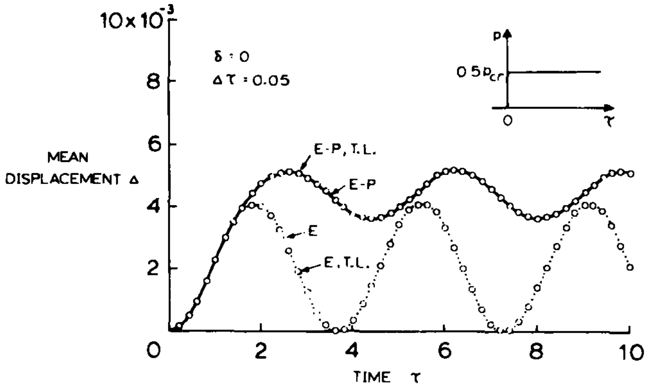
Slide
1-4

Slide 1-5

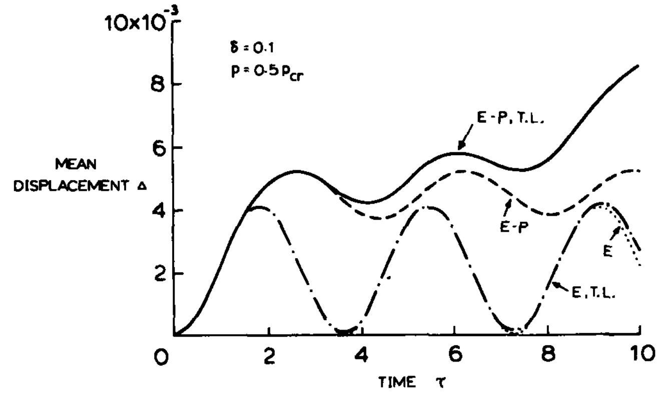


Elastic-plastic static buckling behavior of the shell with various levels of initial imperfection

Slide 1-6

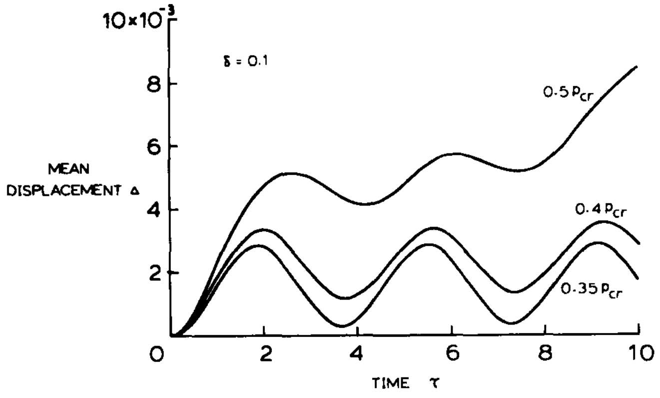


Dynamic response of perfect ($\delta = 0$) shell under step external pressure.



Dynamic response of imperfect ($\delta = 0.1$) shell under step external pressure.

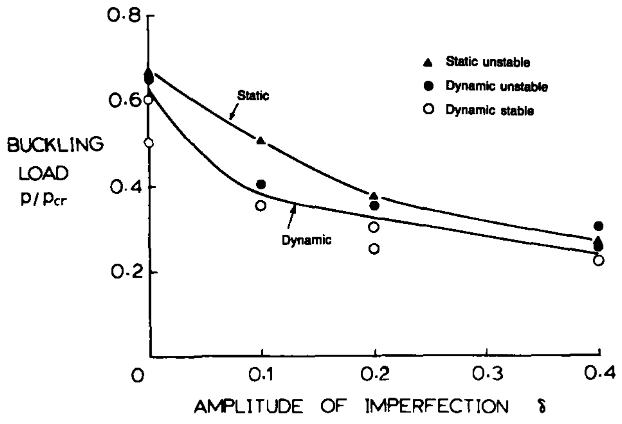
Slide 1-7



Elastic-plastic dynamic response of imperfect ($\delta = 0.1$) shell

Slide 1-8

Slide
1-9



Effect of initial imperfections on the elastic–plastic buckling load of the shell

Topic 2

Basic Considerations in Nonlinear Analysis

Contents:

- **The principle of virtual work in general nonlinear analysis, including all material and geometric nonlinearities**
- **A simple instructive example**
- **Introduction to the finite element incremental solution, statement and physical explanation of governing finite element equations**
- **Requirements of equilibrium, compatibility, and the stress-strain law**
- **Nodal point equilibrium versus local equilibrium**
- **Assessment of accuracy of a solution**
- **Example analysis: Stress concentration factor calculation for a plate with a hole in tension**
- **Example analysis: Fracture mechanics stress intensity factor calculation for a plate with an eccentric crack in tension**
- **Discussion of mesh evaluation by studying stress jumps along element boundaries and pressure band plots**

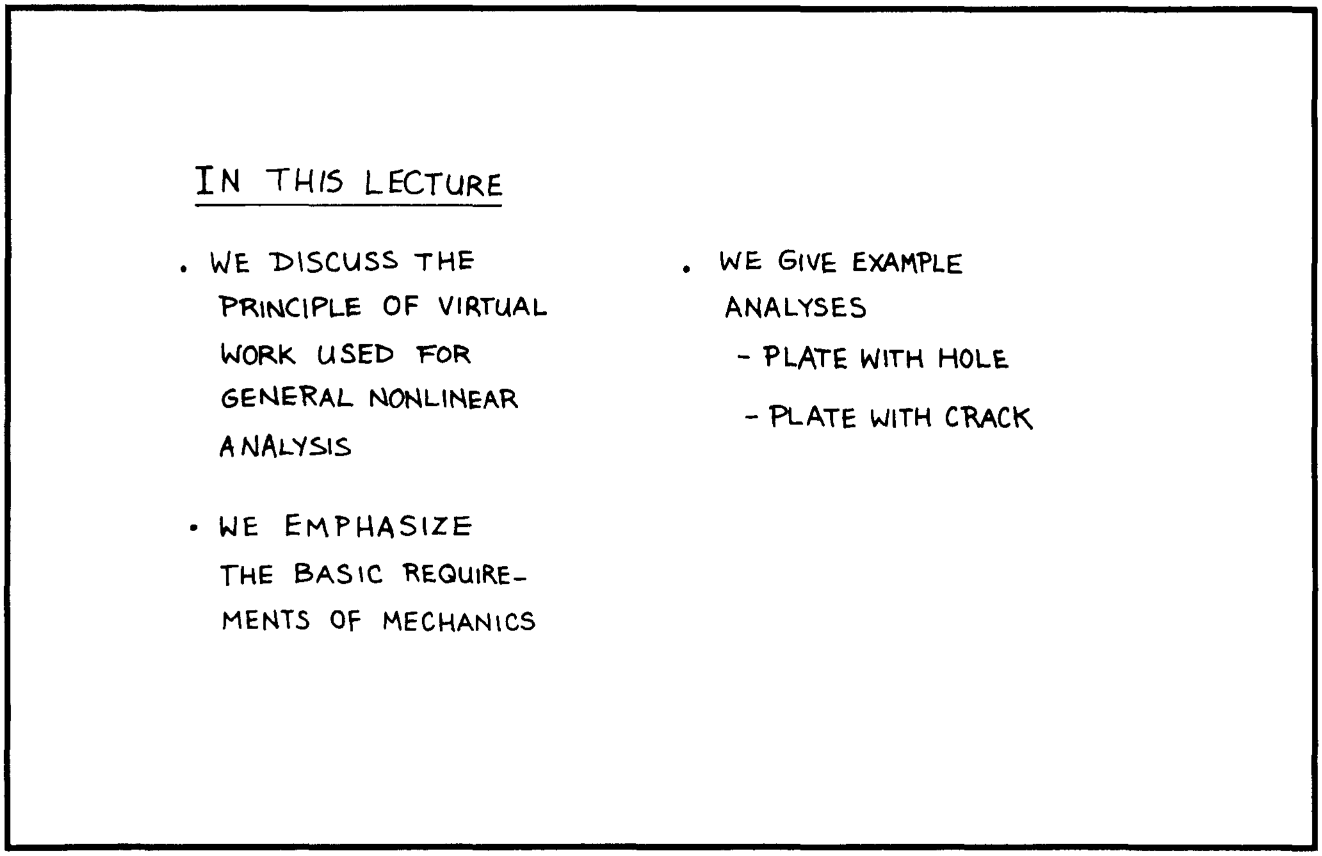
Textbook: Section 6.1

Examples: 6.1, 6.2, 6.3, 6.4

References: The evaluation of finite element solutions is studied in

Sussman, T., and K. J. Bathe, "Studies of Finite Element Procedures—On Mesh Selection," *Computers & Structures, 21*, 257–264, 1985.

Sussman, T., and K. J. Bathe, "Studies of Finite Element Procedures—Stress Band Plots and the Evaluation of Finite Element Meshes," *Engineering Computations*, to appear.



IN THIS LECTURE

- WE DISCUSS THE PRINCIPLE OF VIRTUAL WORK USED FOR GENERAL NONLINEAR ANALYSIS
- WE EMPHASIZE THE BASIC REQUIREMENTS OF MECHANICS
- WE GIVE EXAMPLE ANALYSES
  - PLATE WITH HOLE
  - PLATE WITH CRACK

Markerboard
2-1

Transparency 2-1

THE PRINCIPLE OF VIRTUAL WORK

$$\int_{{}^tV} {}^t\tau_{ij}\, \delta_t e_{ij}\, {}^t dV = {}^t\mathcal{R}$$

where

$${}^t\mathcal{R} = \int_{{}^tV} {}^t f_i^B\, \delta u_i\, {}^t dV + \int_{{}^tS} {}^t f_i^S\, \delta u_i^S\, {}^t dS$$

${}^t\tau_{ij}$ = forces per unit area at time t (Cauchy stresses)

$$\delta_t e_{ij} = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial {}^t x_j} + \frac{\partial \delta u_j}{\partial {}^t x_i}\right)$$

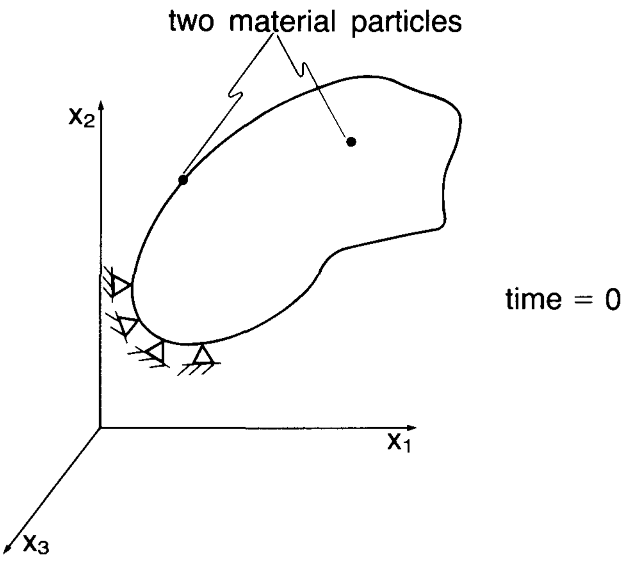
Transparency 2-2

and

δu_i, $\delta_t e_{ij}$ = virtual displacements and corresponding virtual strains

${}^tV$, ${}^tS$ = volume and surface area at time t

${}^t f_i^B$, ${}^t f_i^S$ = externally applied forces per unit current volume and unit current area



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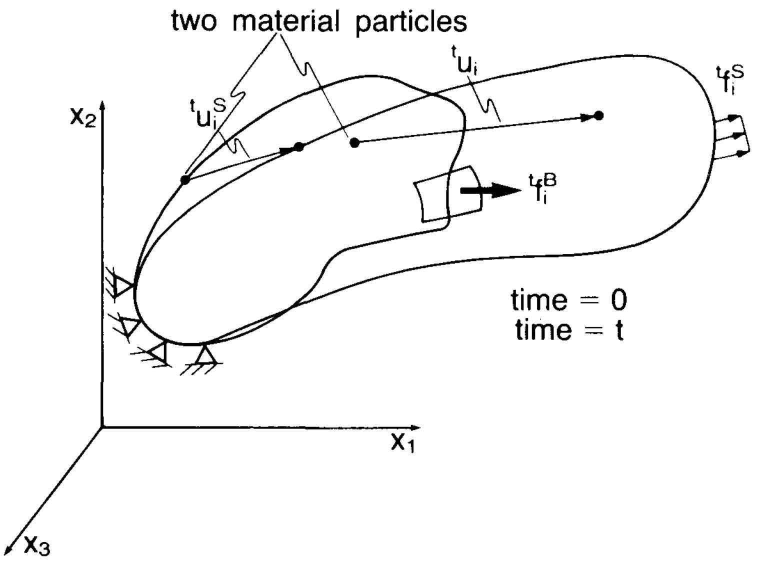

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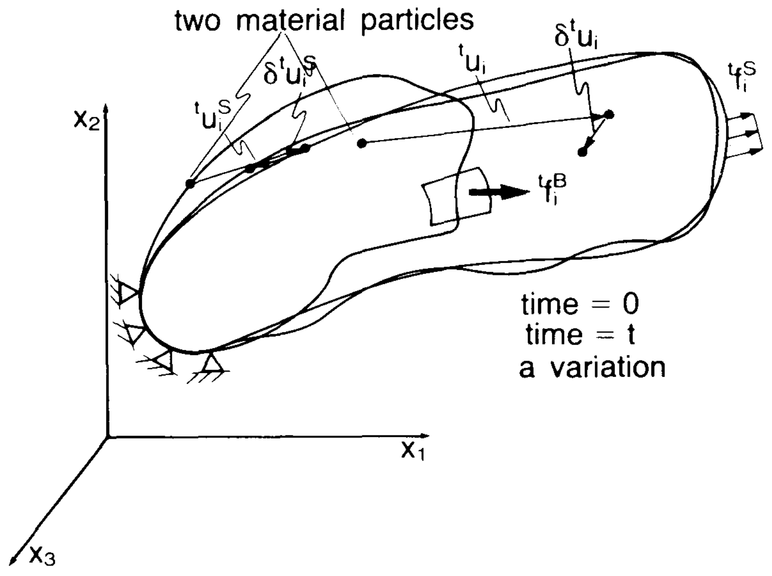
Transparency 2-5



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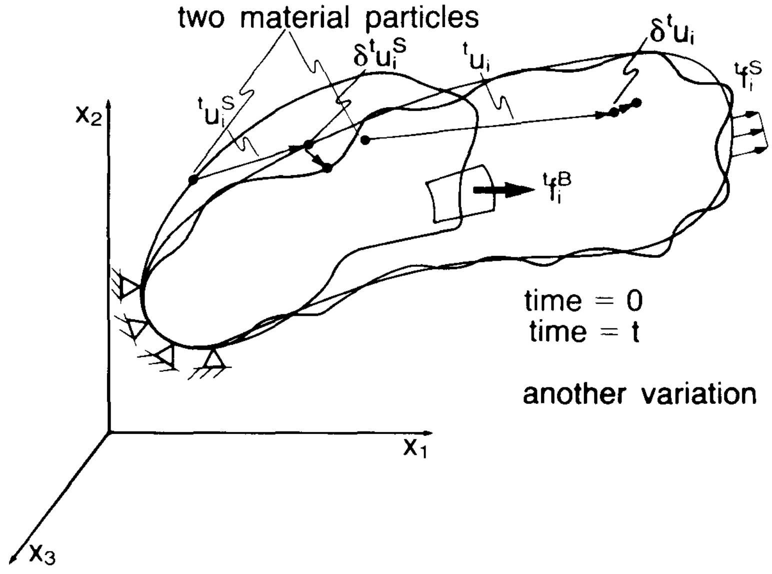


Transparency 2-7

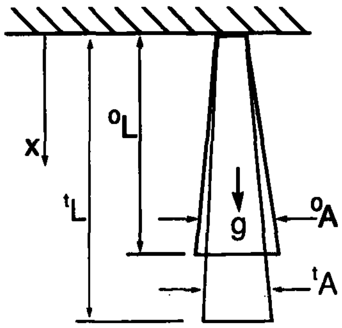
Note: Integrating the principle of virtual work by parts gives

- Governing differential equations of motion
- Plus force (natural) boundary conditions

just like in infinitesimal displacement analysis.

Transparency 2-8

Example: Truss stretching under its own weight



Assume:

- Plane cross-sections remain plane
- Constant uniaxial stress on each cross-section

We then have a one-dimensional analysis.

Transparency 2-9

Using these assumptions,

$$\int_{^tV} {}^t\tau_{ij}\,\delta_t e_{ij}\,{}^tdV = \int_{^tL} {}^t\tau\,\delta_t e\,{}^tA\,{}^tdx\,,$$

$${}^t\mathcal{R} = \int_{^tL} {}^t\rho g\,\delta u\,{}^tA\,{}^tdx$$

Hence the principle of virtual work is now

$$\int_{^tL} {}^t\tau\,{}^tA\,\delta_t e\,{}^tdx = \int_{^tL} {}^t\rho g\,{}^tA\,\delta u\,{}^tdx$$

where

$$\delta_t e = \frac{\partial \delta u}{\partial {}^tx}$$

Transparency 2-10

We now recover the differential equation of equilibrium using integration by parts:

$$\int_{^tL}\left[\frac{\partial}{\partial {}^tx}({}^t\tau\,{}^tA) + {}^t\rho g\,{}^tA\right]\delta u\,{}^tdx - [({}^t\tau\,{}^tA)\delta u]\Big|_0^{^tL} = 0$$

Since the variations δu are arbitrary (except at $x = 0$), we obtain

$$\frac{\partial}{\partial {}^tx}({}^t\tau\,{}^tA) + {}^t\rho g\,{}^tA = 0\,, \qquad ({}^t\tau\,{}^tA)\Big|_{^tL} = 0$$

THE GOVERNING DIFFERENTIAL EQUATION	THE FORCE (NATURAL) BOUNDARY CONDITION

Transparency
2-11

FINITE ELEMENT APPLICATION OF THE PRINCIPLE OF VIRTUAL WORK

$$\int_{^tV} {}^t\tau_{ij}\, \delta_t e_{ij}\, {}^t dV = \int_{^tV} {}^t f_i^B\, \delta u_i\, {}^t dV + \int_{^tS} {}^t f_i^S\, \delta u_i^S\, {}^t dS$$

$$\downarrow$$

BY THE FINITE ELEMENT METHOD

$$\downarrow$$

$$\delta \underline{U}^T\, {}^t\underline{F} = \delta \underline{U}^T\, {}^t\underline{R}$$

Transparency
2-12

- Now assume that the solution at time t is known. Hence ${}^t\tau_{ij}$, ${}^tV$, . . . are known.
- We want to obtain the solution corresponding to time $t+\Delta t$ (i.e., for the loads applied at time $t+\Delta t$).
- The principle of virtual work gives for time $t+\Delta t$

$${}^{t+\Delta t}\underline{F} = {}^{t+\Delta t}\underline{R}$$

Transparency 2-13

To solve for the unknown state at time t+Δt, we assume

$$ {}^{t+\Delta t}\underline{F} = {}^{t}\underline{F} + {}^{t}\underline{K}\,\Delta\underline{U} $$

Hence we solve

$$ {}^{t}\underline{K}\,\Delta\underline{U} = {}^{t+\Delta t}\underline{R} - {}^{t}\underline{F} $$

and obtain

$$ {}^{t+\Delta t}\underline{U} \doteq {}^{t}\underline{U} + \Delta\underline{U} $$

Transparency 2-14

More generally, we solve

$$ {}^{t}\underline{K}\,\Delta\underline{U}^{(i)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)} $$

$$ {}^{t+\Delta t}\underline{U}^{(i)} = {}^{t+\Delta t}\underline{U}^{(i-1)} + \Delta\underline{U}^{(i)} $$

using

$$ {}^{t+\Delta t}\underline{F}^{(0)} = {}^{t}\underline{F}\,, \qquad {}^{t+\Delta t}\underline{U}^{(0)} = {}^{t}\underline{U} $$

Transparency 2-15

- Nodal point equilibrium is satisfied when the equation

  $${}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)} = \underline{0}$$

  is satisfied.

- Compatibility is satisfied provided a compatible element layout is used.

- The stress-strain law enters in the calculation of ${}^{t}\underline{K}$ and ${}^{t+\Delta t}\underline{F}^{(i-1)}$.

Transparency 2-16

Most important is the appropriate calculation of ${}^{t+\Delta t}\underline{F}^{(i-1)}$ from ${}^{t+\Delta t}\underline{U}^{(i-1)}$.

The general procedure is:

$${}^{t+\Delta t}\underline{U}^{(i-1)} \xrightarrow{\text{gives}} \text{strains} \underset{\substack{\uparrow \\ \left(\text{CONSTITUTIVE RELATIONS ENTER}\right)}}{\xrightarrow{\text{gives}}} \text{stresses} \xrightarrow{\text{gives}} {}^{t+\Delta t}\underline{F}^{(i-1)}$$

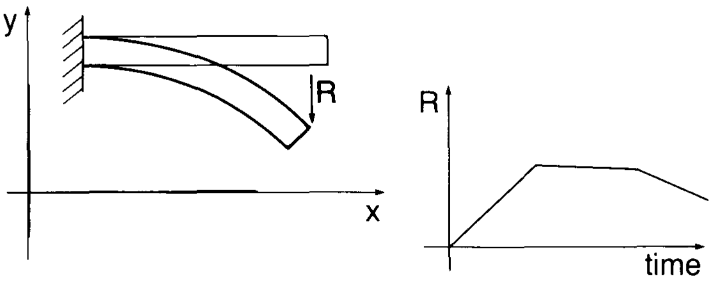
Note:

$${}^{t+\Delta t}\underline{\sigma}^{(i-1)} = {}^{t}\underline{\sigma} + \int_{{}^{t}\underline{e}}^{{}^{t+\Delta t}\underline{e}^{(i-1)}} \underline{C}\, d\underline{e}$$

Transparency 2-17

Here we assumed that the nodal point loads are independent of the structural deformations. The loads are given as functions of time only.

Example:



Transparency 2-18

WE SATISFY THE BASIC REQUIREMENTS OF MECHANICS:

Stress-strain law

Need to evaluate the stresses correctly from the strains.

Compatibility

Need to use compatible element meshes and satisfy displacement boundary conditions.

Transparency 2-19

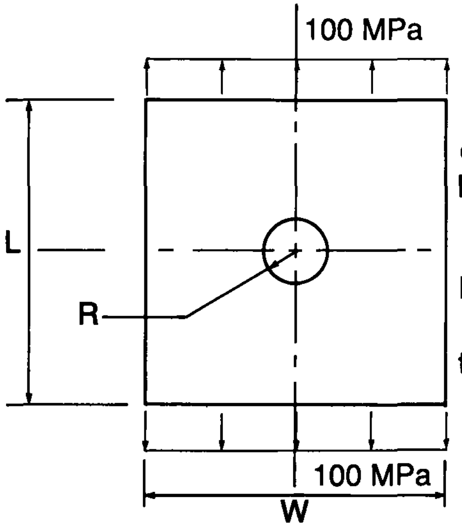
Equilibrium

- Corresponding to the finite element nodal point degrees of freedom (global equilibrium)
- Locally if a fine enough finite element discretization is used

Check:
- Whether the stress boundary conditions are satisfied
- Whether there are no unduly large stress jumps between elements

Transparency 2-20

Example: Plate with hole in tension



$E = 207000$ MPa
$\nu = 0.3$
$\sigma_y = 740$ MPa
$E_T = 2070$ MPa

$R = 0.01$ m
$L = W = 0.1$ m
thickness = 0.01 m

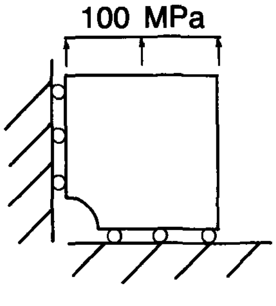
Transparency 2-21

Purpose of analysis:

To accurately determine the stresses in the plate, assuming that the load is small enough so that a linear elastic analysis may be performed.

Transparency 2-22

Using symmetry, we only need to model one quarter of the plate:



Transparency 2-23

Accuracy considerations:

Recall, in a displacement-based finite element solution,

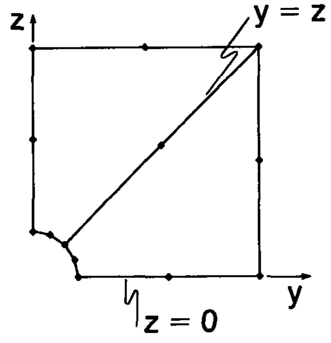
- Compatibility is satisfied.
- The material law is satisfied.
- Equilibrium (locally) is only approximately satisfied.

We can observe the equilibrium error by plotting stress discontinuities.

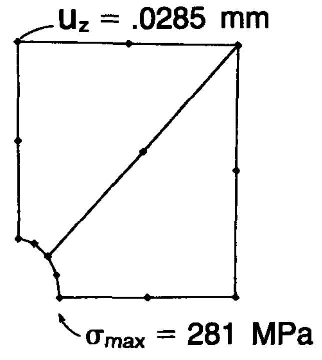
Transparency 2-24

Two element mesh: All elements are two-dimensional 8-node isoparametric elements.

Undeformed mesh:

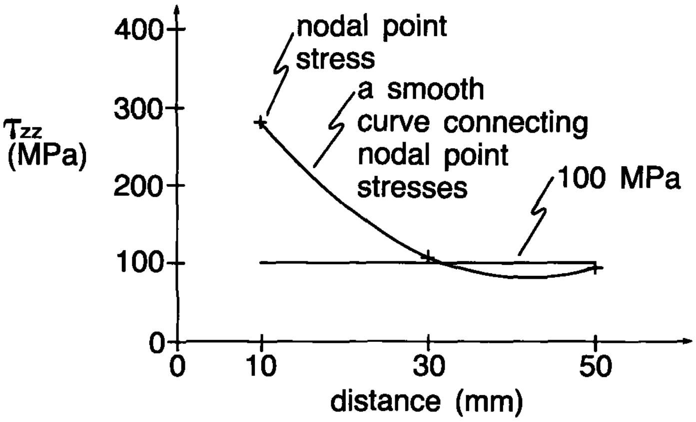


Deformed mesh (displacements amplified):



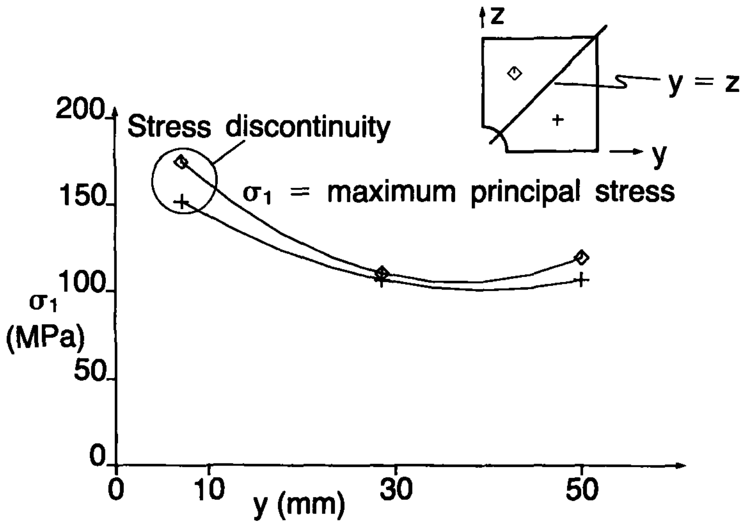
Transparency 2-25

Plot stresses (evaluated at the nodal points) along the line z=0:

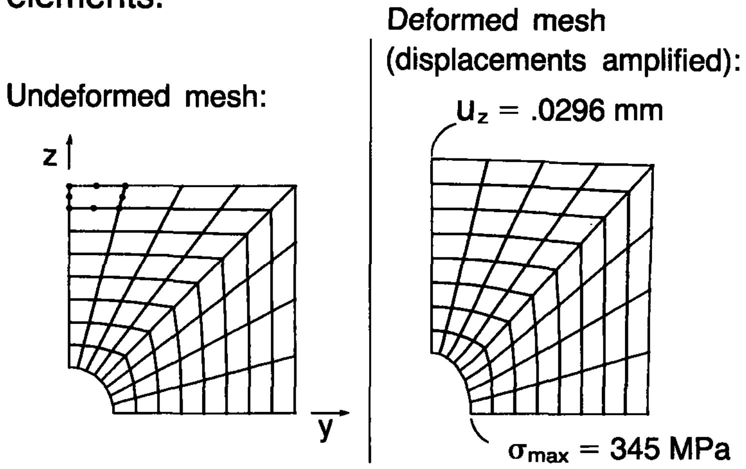


Transparency 2-26

Plot stresses along the line y = z:

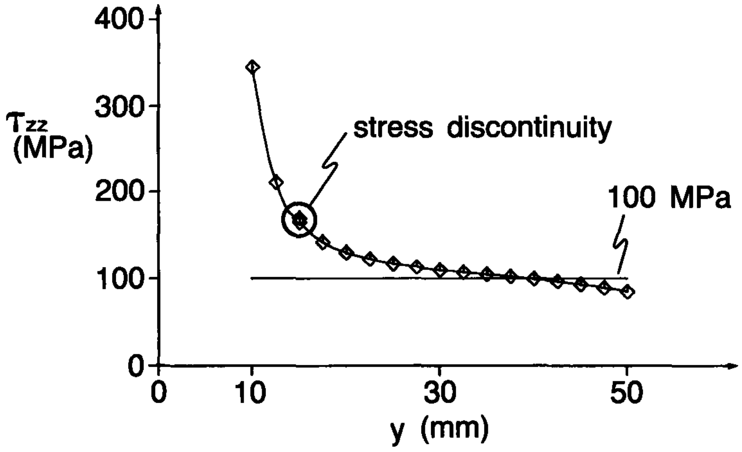


Sixty-four element mesh: All elements are two-dimensional 8-node isoparametric elements.



Transparency 2-27

Plot stresses along the line z=0:

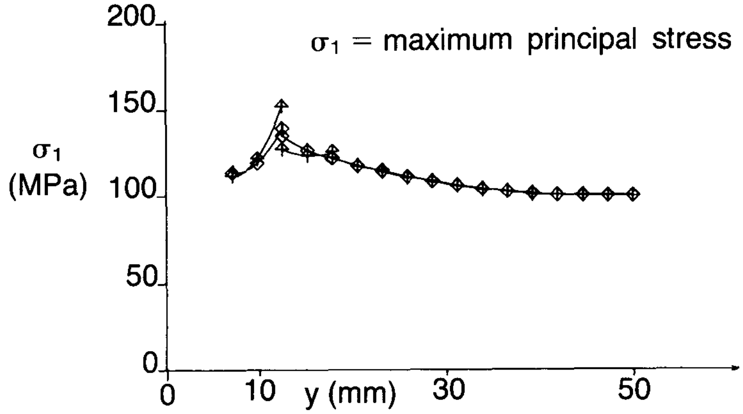


Transparency 2-28

Transparency 2-29

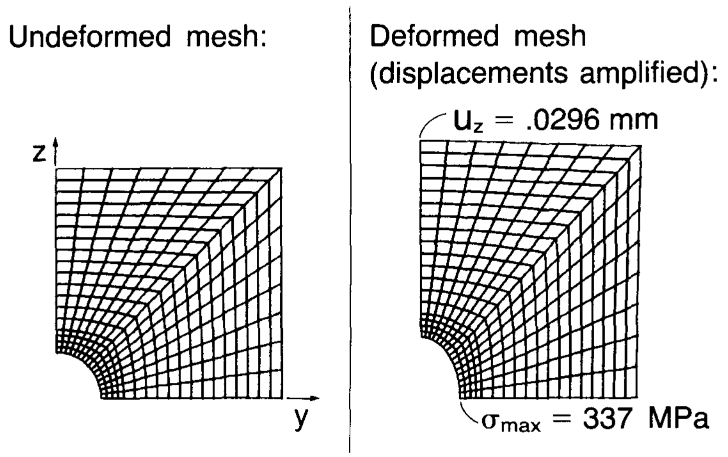
Plot stresses along the line y = z:

The stress discontinuities are negligible for y > 20 mm.

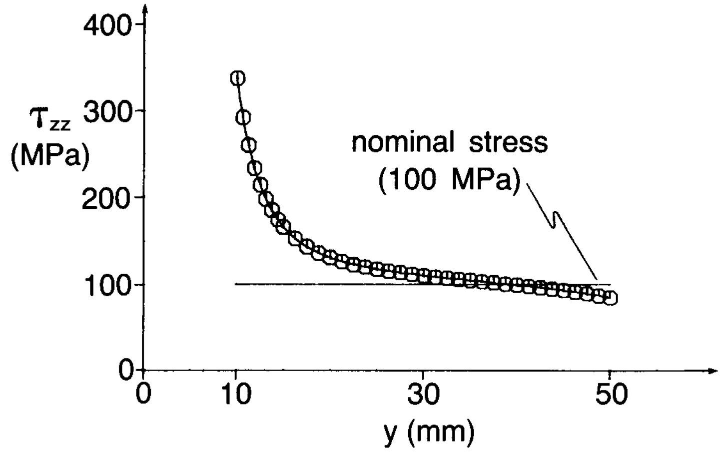


Transparency 2-30

288 element mesh: All elements are two-dimensional 8-node elements.



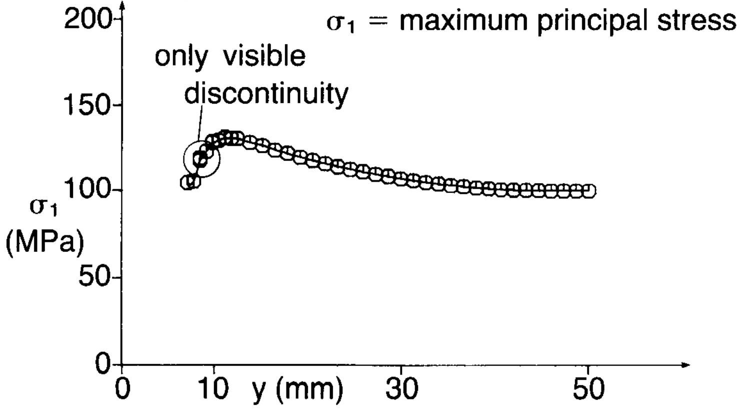
Plot stresses along the line z = 0:



Transparency 2-31

Plot stresses along the line y = z:

- There are no visible stress discontinuities between elements on opposite sides of the line y = z.



Transparency 2-32

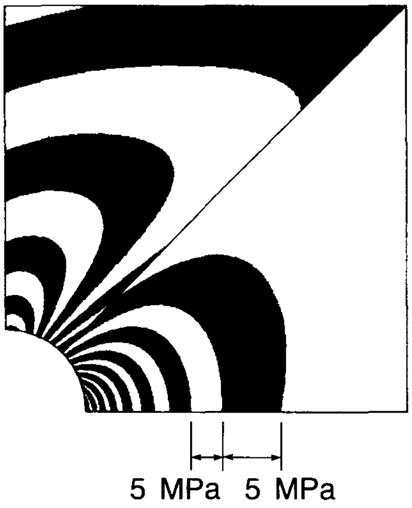
Transparency 2-33

- To be confident that the stress discontinuities are small everywhere, we should plot stress jumps along each line in the mesh.
- An alternative way of presenting stress discontinuities is by means of a pressure band plot:
  - — Plot bands of constant pressure where

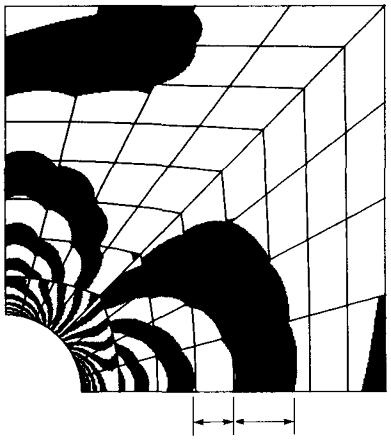
$$\text{pressure} = \frac{-(\tau_{xx} + \tau_{yy} + \tau_{zz})}{3}$$

Transparency 2-34

Two element mesh: Pressure band plot

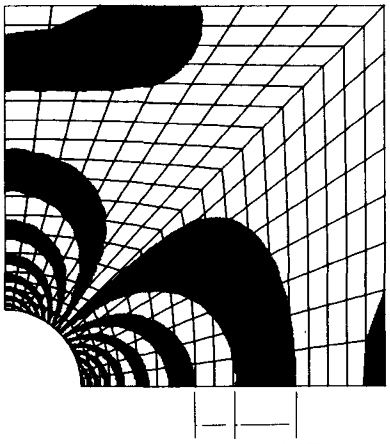


Sixty-four element mesh: Pressure band plot



Transparency 2-35

288 element mesh: Pressure band plot



Transparency 2-36

Transparency 2-37

We see that stress discontinuities are represented by breaks in the pressure bands. As the mesh is refined, the pressure bands become smoother.

- The stress state everywhere in the mesh is represented by one picture.
- The pressure band plot may be drawn by a computer program.
- However, actual magnitudes of pressures are not directly displayed.

Transparency 2-38

Summary of results for plate with hole meshes:

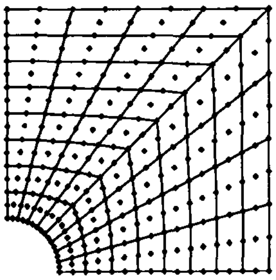
Number of elements	Degrees of freedom	Relative cost	Displacement at top (mm)	Stress concentration factor
2	20	0.08	.0285	2.81
64	416	1.0	.0296	3.45
288	1792	7.2	.0296	3.37

Transparency 2-39

- Two element mesh cannot be used for stress predictions.
- Sixty-four element mesh gives reasonably accurate stresses. However, further refinement at the hole is probably desirable.
- 288 element mesh is overrefined for linear elastic stress analysis. However, this refinement may be necessary for other types of analyses.

Transparency 2-40

Now consider the effect of using 9-node isoparametric elements. Consider the 64 element mesh discussed earlier, where each element is a 9-node element:



Will the solution improve significantly?

Transparency 2-41

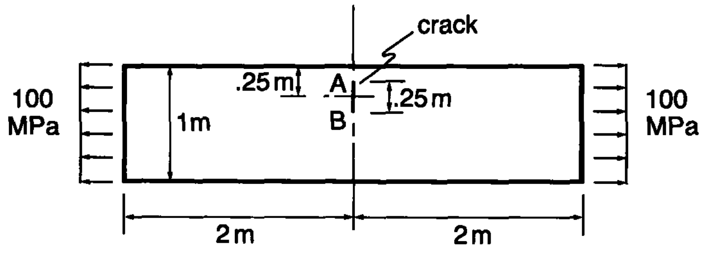
No, the answers do not improve significantly:

	Sixty-four 8-node elements	Sixty-four 9-node elements
Number of degrees of freedom	416	544
Displacement at top (mm)	.029576	.029577
Stress concentration factor	3.452	3.451

The stress jump and pressure band plots do not change significantly.

Transparency 2-42

Example: Plate with eccentric crack in tension



E = 207000 MPa
ν = 0.3
K_c = 110 MPa $\sqrt{m}$

thickness = 0.01 m
plane stress

- Will the crack propagate?

Background:

Assuming that the theory of linear elastic fracture mechanics is applicable, we have

K_I = stress intensity factor for a mode I crack

K_I determines the "strength" of the $1/\sqrt{r}$ stress singularity at the crack tip.

$K_I > K_C$ – crack will propagate

(K_C is a property of the material)

Transparency 2-43

Computation of K_I: From energy considerations, we have for plane stress situations

$$K_I = \sqrt{EG} \ , \ G = -\frac{\partial \Pi}{\partial A}$$

where Π = total potential energy

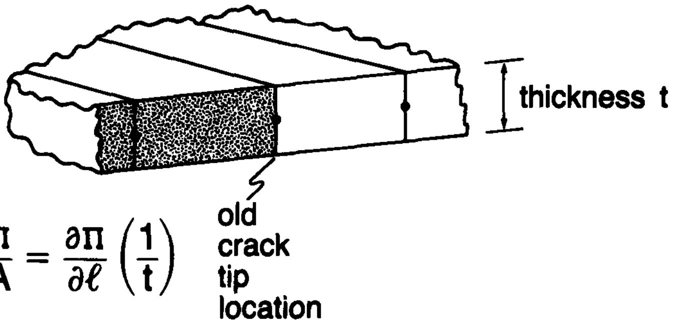
A = area of the crack surface

G is known as the "energy release rate" for the crack.

Transparency 2-44

Transparency 2-45

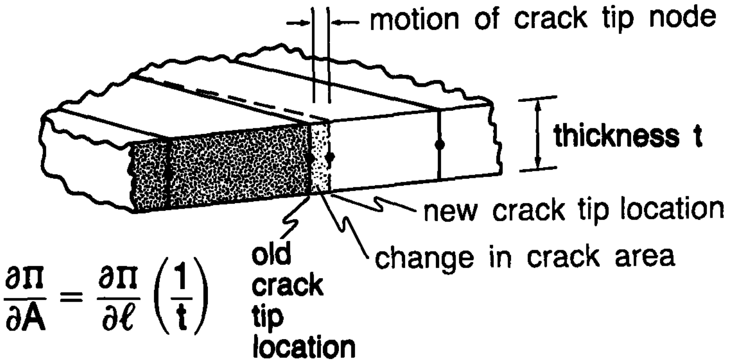
In this finite element analysis, each crack tip is represented by a node. Hence the change in the area of the crack may be written in terms of the motion of the node at the crack tip.



$$\frac{\partial \Pi}{\partial A} = \frac{\partial \Pi}{\partial \ell}\left(\frac{1}{t}\right)$$

Transparency 2-46

In this finite element analysis, each crack tip is represented by a node. Hence the change in the area of the crack may be written in terms of the motion of the node at the crack tip.



$$\frac{\partial \Pi}{\partial A} = \frac{\partial \Pi}{\partial \ell}\left(\frac{1}{t}\right)$$

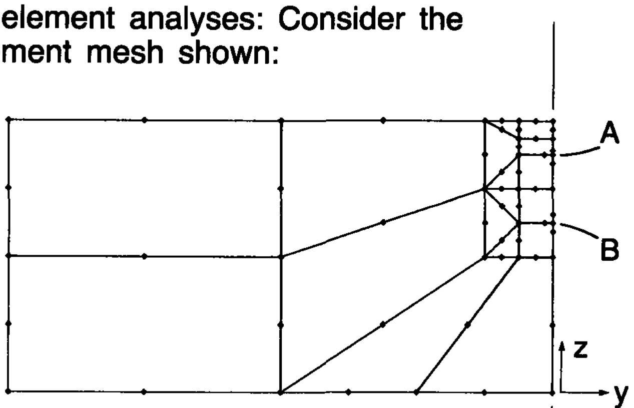
Transparency 2-47

The quantities $\frac{\partial \Pi}{\partial \ell}$ may be efficiently computed using equations based on the chain differentiation of the total potential with respect to the nodal coordinates describing the crack tip. This computation is performed at the end of (but as part of) the finite element analysis.

See T. Sussman and K. J. Bathe, "The Gradient of the Finite Element Variational Indicator with Respect to Nodal Point Coordinates . . . ", Int. J. Num. Meth. Engng. Vol. 21, 763–774 (1985).

Transparency 2-48

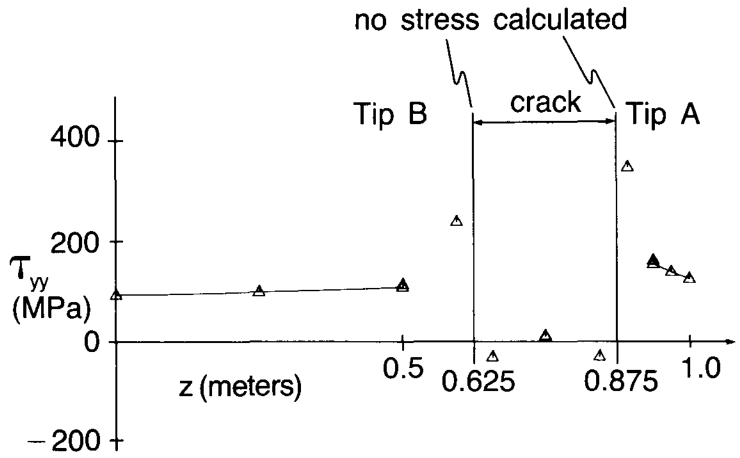
Finite element analyses: Consider the 17 element mesh shown:



- The mid-side nodes nearest the crack tip are located at the quarter-points.

Transparency 2-49

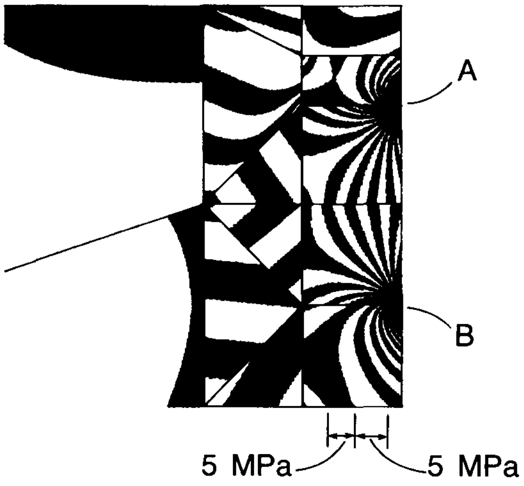
Results: Plot of stresses on line of symmetry for 17 element mesh.



Transparency 2-50

Pressure band plot (detail):

- The pressure jumps are larger than 5 MPa.



Based on the pressure band plot, we conclude that the mesh is too coarse for accurate stress prediction.

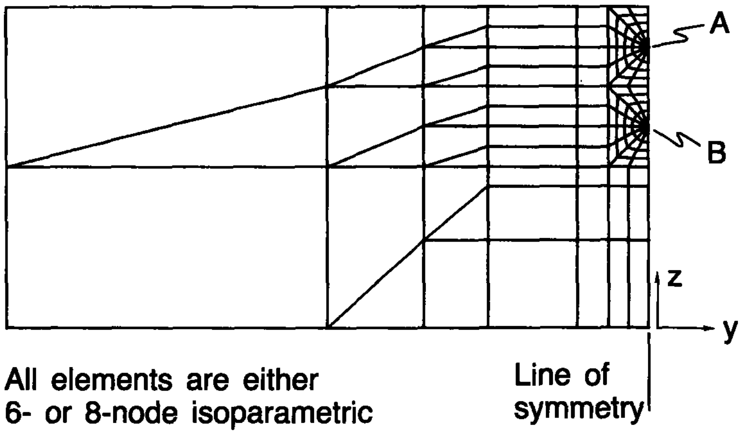
However, good results are obtained for the stress intensity factors (when they are calculated as described earlier):

$K_A = 72.6$ MPa$\sqrt{m}$ (analytical solution $= 72.7$ MPa$\sqrt{m}$)

$K_B = 64.5$ MPa$\sqrt{m}$ (analytical solution $= 68.9$ MPa$\sqrt{m}$)

Transparency 2-51

Now consider the 128 element mesh shown:

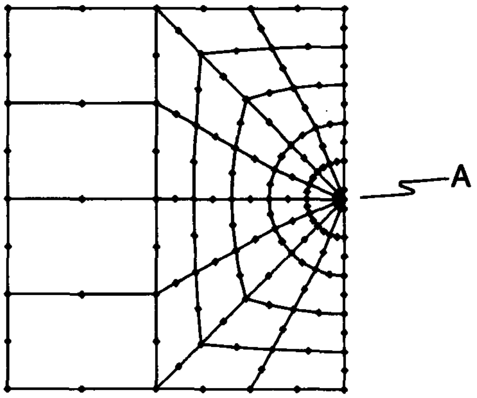


All elements are either 6- or 8-node isoparametric elements.

Transparency 2-52

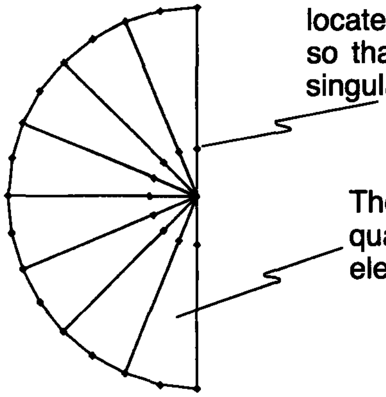
Transparency 2-53

Detail of 128 element mesh:



Transparency 2-54

Close-up of crack tip A:

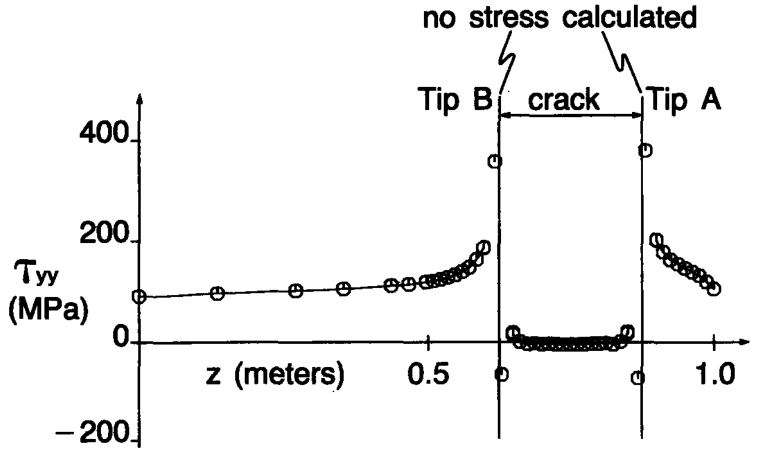


mid-side nodes nearest the crack tip are located at the "quarter-points" so that the $1/\sqrt{r}$ stress singularity is properly modeled.

These elements are 6-node quadratic isoparametric elements (degenerated).

Transparency 2-55

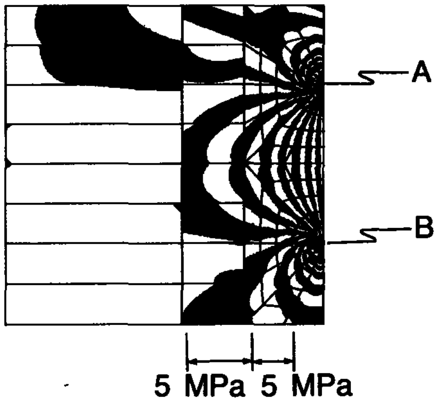
Results: Stress plot on line of symmetry for 128 element mesh.



Transparency 2-56

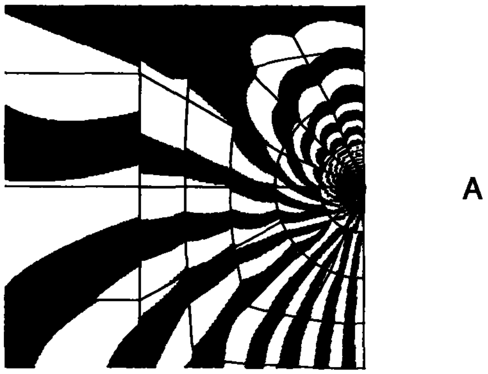
Pressure band plot (detail) for 128 element mesh:

- The pressure jumps are smaller than 5 MPa for all elements far from the crack tips.



Transparency 2-57

A close-up shows that the stress jumps are larger than 5 MPa in the first and second rings of elements surrounding crack tip A.



Transparency 2-58

Based on the pressure band plot, we conclude that the mesh is fine enough for accurate stress calculation (except for the elements near the crack tip nodes).

We also obtain good results for the stress intensity factors:

$K_A = 72.5$ MPa $\sqrt{m}$ (analytical solution $= 72.7$ MPa $\sqrt{m}$)
$K_B = 68.8$ MPa $\sqrt{m}$ (analytical solution $= 68.9$ MPa $\sqrt{m}$)

We see that the degree of refinement needed for a mesh in linear elastic analysis is dependent upon the type of result desired.

- Displacements — coarse mesh
- Stress intensity factors — coarse mesh
- Lowest natural frequencies and associated mode shapes — coarse mesh
- Stresses — fine mesh

General nonlinear analysis — usually fine mesh

Transparency 2-59

Topic 3

Lagrangian Continuum Mechanics Variables for General Nonlinear Analysis

Contents:

- **The principle of virtual work in terms of the 2nd Piola-Kirchhoff stress and Green-Lagrange strain tensors**
- **Deformation gradient tensor**
- **Physical interpretation of the deformation gradient**
- **Change of mass density**
- **Polar decomposition of deformation gradient**
- **Green-Lagrange strain tensor**
- **Second Piola-Kirchhoff stress tensor**
- **Important properties of the Green-Lagrange strain and 2nd Piola-Kirchhoff stress tensors**
- **Physical explanations of continuum mechanics variables**
- **Examples demonstrating the properties of the continuum mechanics variables**

Textbook: Sections 6.2.1, 6.2.2

Examples: 6.5, 6.6, 6.7, 6.8, 6.10, 6.11, 6.12, 6.13, 6.14

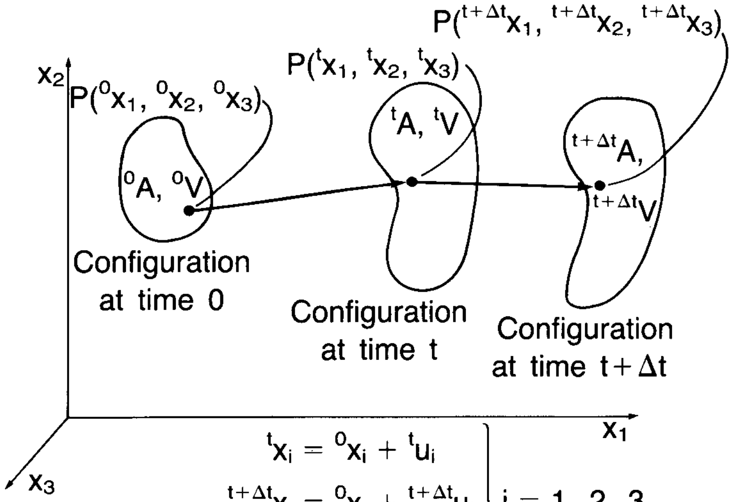
CONTINUUM MECHANICS FORMULATION

For
- Large displacements
- Large rotations
- Large strains

Hence we consider a body subjected to arbitrary large motions,

We use a Lagrangian description.

Transparency 3-1



$$\left.\begin{aligned} {}^t x_i &= {}^0 x_i + {}^t u_i \\ {}^{t+\Delta t} x_i &= {}^0 x_i + {}^{t+\Delta t} u_i \\ u_i &= {}^{t+\Delta t} u_i - {}^t u_i \end{aligned}\right\} i = 1,\ 2,\ 3$$

Transparency 3-2

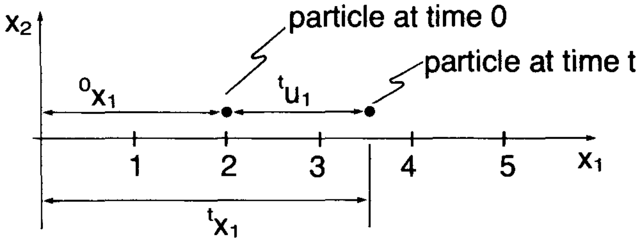
Transparency 3-3

Regarding the notation we need to keep firmly in mind that

— the Cartesian axes are stationary.

— the unit distances along the x_i-axes are the same for ${}^{0}x_i$, ${}^{t}x_i$, ${}^{t+\Delta t}x_i$.

Example:



Transparency 3-4

PRINCIPLE OF VIRTUAL WORK

Corresponding to time t+Δt:

$$\int_{^{t+\Delta t}V} {}^{t+\Delta t}\tau_{ij}\ \delta_{t+\Delta t}e_{ij}\ {}^{t+\Delta t}dV = {}^{t+\Delta t}\mathcal{R}$$

where

$${}^{t+\Delta t}\mathcal{R} = \int_{^{t+\Delta t}V} {}^{t+\Delta t}f_i^B\ \delta u_i\ {}^{t+\Delta t}dV + \int_{^{t+\Delta t}S} {}^{t+\Delta t}f_i^S\ \delta u_i^S\ {}^{t+\Delta t}dS$$

Transparency 3-5

and

$^{t+\Delta t}\tau_{ij}$ = Cauchy stresses (forces/unit area at time $t+\Delta t$)

$$\delta_{t+\Delta t}e_{ij} = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial^{t+\Delta t}x_j} + \frac{\partial \delta u_j}{\partial^{t+\Delta t}x_i}\right)$$

= variation in the small strains referred to the configuration at time $t+\Delta t$

Transparency 3-6

We need to rewrite the principle of virtual work, using new stress and strain measures:

- We cannot integrate over an unknown volume.
- We cannot directly work with increments in the Cauchy stresses.

We introduce:

$^{t}_{0}\underline{S}$ = 2nd Piola-Kirchhoff stress tensor

$^{t}_{0}\underline{\varepsilon}$ = Green-Lagrange strain tensor

Transparency 3-7

The 2nd Piola-Kirchhoff stress tensor:

$${}_0^tS_{ij} = \frac{{}^0\rho}{{}^t\rho}\; {}_t^0x_{i,m}\; {}^t\tau_{mn}\; {}_t^0x_{j,n}$$

The Green-Lagrange strain tensor:

$${}_0^t\varepsilon_{ij} = \frac{1}{2}\left({}_0^tu_{i,j} + {}_0^tu_{j,i} + {}_0^tu_{k,i}\;{}_0^tu_{k,j}\right)$$

where $${}_t^0x_{i,m} = \frac{\partial\,{}^0x_i}{\partial\,{}^tx_m}\ ,\quad {}_0^tu_{i,j} = \frac{\partial\,{}^tu_i}{\partial\,{}^0x_j}$$

Transparency 3-8

Note: We are using the indicial notation with the summation convention.

For example,

$$\begin{aligned}{}_0^tS_{11} = \frac{{}^0\rho}{{}^t\rho}\,\big[&{}_t^0x_{1,1}\,{}^t\tau_{11}\;{}_t^0x_{1,1}\\ &+ {}_t^0x_{1,1}\;{}^t\tau_{12}\;{}_t^0x_{1,2}\\ &+ \ldots\\ &+ {}_t^0x_{1,3}\;{}^t\tau_{33}\;{}_t^0x_{1,3}\big]\end{aligned}$$

Transparency 3-9

Using the 2nd Piola-Kirchhoff stress and Green-Lagrange strain tensors, we have

$$\int_{{}^tV} {}^t\tau_{ij}\, \delta_t e_{ij}\, {}^t dV = \int_{{}^0V} {}^t_0S_{ij}\, \delta\, {}^t_0\varepsilon_{ij}\, {}^0dV$$

This relation holds for all times

$\Delta t, 2\Delta t, \ldots, t, t+\Delta t, \ldots$

Transparency 3-10

To develop the incremental finite element equations we will use

$$\int_{{}^0V} {}^{t+\Delta t}_{\ \ \ 0}S_{ij}\, \delta\, {}^{t+\Delta t}_{\ \ \ 0}\varepsilon_{ij}\, {}^0dV = {}^{t+\Delta t}\mathcal{R}$$

- We now integrate over a known volume, ${}^0V$.
- We can incrementally decompose ${}^{t+\Delta t}_{\ \ \ 0}S_{ij}$ and ${}^{t+\Delta t}_{\ \ \ 0}\varepsilon_{ij}$, i.e.

$$ {}^{t+\Delta t}_{\ \ \ 0}S_{ij} = {}^t_0S_{ij} + {}_0S_{ij}$$
$$ {}^{t+\Delta t}_{\ \ \ 0}\varepsilon_{ij} = {}^t_0\varepsilon_{ij} + {}_0\varepsilon_{ij}$$

Transparency 3-11

Before developing the incremental continuum mechanics and finite element equations, we want to discuss

- some important kinematic relationships used in geometric nonlinear analysis
- some properties of the 2nd Piola-Kirchhoff stress and Green-Lagrange strain tensors

Transparency 3-12

To explain some important properties of the 2nd Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor, we consider the

Deformation Gradient Tensor

- This tensor captures the straining and the rigid body rotations of the material fibers.
- It is a very fundamental quantity used in continuum mechanics.

Transparency 3-13

The deformation gradient is defined as

$$
{}^t_0\underline{X} = \begin{bmatrix} \frac{\partial\,{}^tx_1}{\partial\,{}^0x_1} & \frac{\partial\,{}^tx_1}{\partial\,{}^0x_2} & \frac{\partial\,{}^tx_1}{\partial\,{}^0x_3} \\ \frac{\partial\,{}^tx_2}{\partial\,{}^0x_1} & \frac{\partial\,{}^tx_2}{\partial\,{}^0x_2} & \frac{\partial\,{}^tx_2}{\partial\,{}^0x_3} \\ \frac{\partial\,{}^tx_3}{\partial\,{}^0x_1} & \frac{\partial\,{}^tx_3}{\partial\,{}^0x_2} & \frac{\partial\,{}^tx_3}{\partial\,{}^0x_3} \end{bmatrix}
$$

in a Cartesian coordinate system

Using indicial notation,

$$
{}^t_0X_{ij} = \frac{\partial\,{}^tx_i}{\partial\,{}^0x_j} = {}^t_0x_{i,j}
$$

Transparency 3-14

Another way to write the deformation gradient:

$$
{}^t_0\underline{X} = ({}_0\underline{\nabla}\,{}^t\underline{x}^T)^T
$$

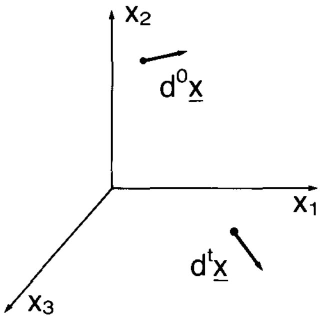
where

$$
{}_0\underline{\nabla} = \begin{bmatrix} \frac{\partial}{\partial\,{}^0x_1} \\ \frac{\partial}{\partial\,{}^0x_2} \\ \frac{\partial}{\partial\,{}^0x_3} \end{bmatrix}, \quad {}^t\underline{x}^T = [{}^tx_1 \quad {}^tx_2 \quad {}^tx_3]
$$

the gradient operator

Transparency 3-15

The deformation gradient describes the deformations (rotations and stretches) of material fibers:

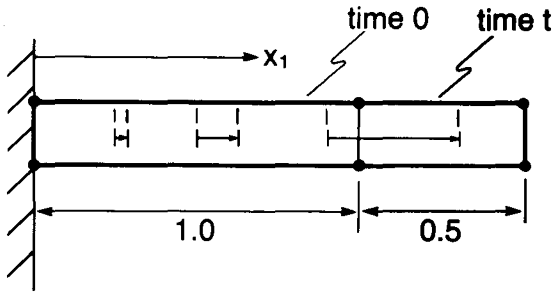


The vectors $d^0\underline{x}$ and $d^t\underline{x}$ represent the orientation and length of a material fiber at times 0 and t. They are related by

$$d^t\underline{x} = {}^t_0\underline{X}\; d^0\underline{x}$$

Transparency 3-16

Example: One-dimensional deformation



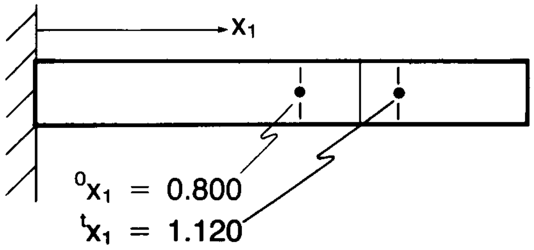
Deformation field: ${}^t x_1 = {}^0 x_1 + 0.5({}^0 x_1)^2$

$$\downarrow$$

$${}^t_0 X_{11} = \frac{\partial\, {}^t x_1}{\partial\, {}^0 x_1} = 1 + {}^0 x_1$$

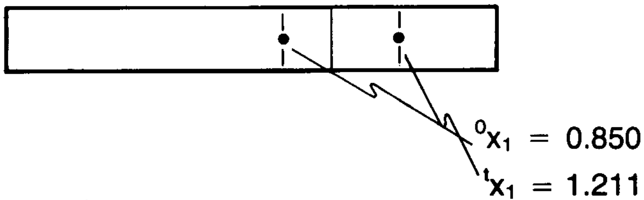
Transparency 3-17

Consider a material particle initially at $x_1 = 0.8$:



Transparency 3-18

Consider an adjacent material particle:



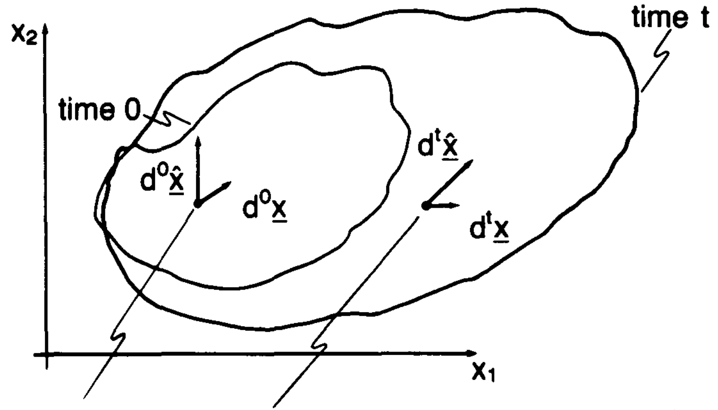
Compute ${}^t_0X_{11}$:

$$\frac{\Delta\,{}^tx_1}{\Delta\,{}^0x_1} = \frac{1.211 - 1.120}{.850 - .800} = 1.82 \leftarrow \text{Estimate}$$

$$\left.{}^t_0X_{11}\right|_{{}^0x_1=0.8} = 1.80$$

Transparency 3-19

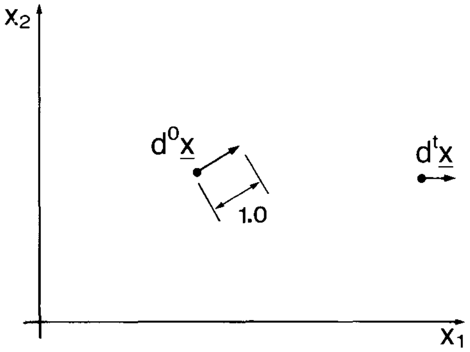
Example: Two-dimensional deformation



$$\overline{({}^0x_1,\ {}^0x_2)} \rightarrow \overline{({}^tx_1,\ {}^tx_2)}:\ {}^t_0\underline{X} = \begin{bmatrix} .481 & .667 \\ -.385 & .667 \end{bmatrix}$$

Transparency 3-20

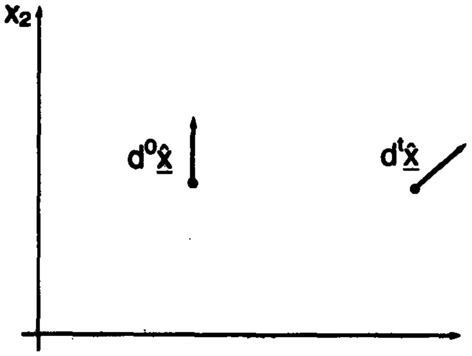
Considering $d^0\underline{x}$,



$$\begin{array}{ccc} d^t\underline{x} & = & {}^t_0\underline{X} \quad d^0\underline{x} \end{array}$$

$$\begin{bmatrix} .75 \\ 0 \end{bmatrix} = \begin{bmatrix} .481 & .667 \\ -.385 & .667 \end{bmatrix} \begin{bmatrix} .866 \\ .5 \end{bmatrix}$$

Considering $d\,^0\hat{\underline{x}}$,

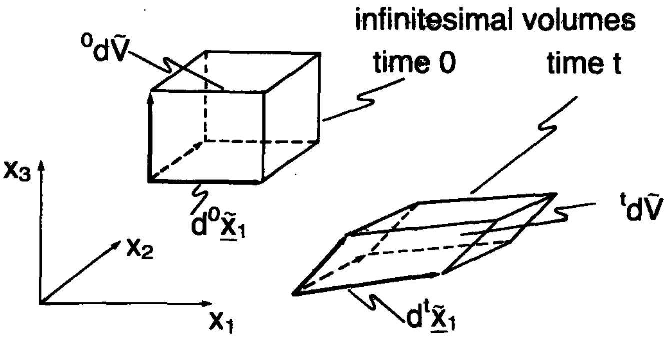


$$d\,^t\hat{\underline{x}} = {}^t_0\underline{X}\; d\,^0\hat{\underline{x}}$$

$$\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}.481 & .667\\ -.385 & .667\end{bmatrix}\begin{bmatrix}0\\1.5\end{bmatrix}$$

Transparency 3-21

The mass densities $^0\rho$ and $^t\rho$ may be related using the deformation gradient:



Three material fibers describe each volume.

Transparency 3-22

Transparency 3-23

For an infinitesimal volume, we note that mass is conserved:

$$ {}^{t}\rho\ {}^{t}d\tilde{V} = {}^{0}\rho\ {}^{0}d\tilde{V} $$

volume at time t

volume at time 0

However, we can show that

$$ {}^{t}d\tilde{V} = \det {}_{0}^{t}\underline{X}\ {}^{0}d\tilde{V} $$

Hence

$$ {}^{0}\rho = {}^{t}\rho\ \det {}_{0}^{t}\underline{X} $$

Transparency 3-24

Proof that ${}^{t}d\tilde{V} = \det {}_{0}^{t}\underline{X}\ {}^{0}d\tilde{V}$:

$$ d^{0}\underline{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ds_1 \ ; \ d^{0}\underline{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} ds_2 $$

$$ d^{0}\underline{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} ds_3 $$

Hence ${}^{0}d\tilde{V} = ds_1\, ds_2\, ds_3$.

Transparency 3-25

But $d\,{}^t\underline{x}_i = {}^t_0\underline{X}\, d\,{}^0\underline{x}_i$; i = 1, 2, 3

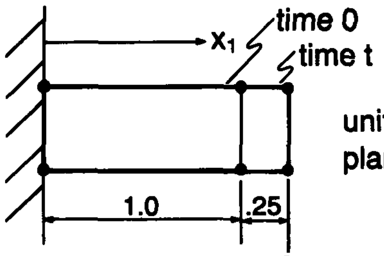
$$\text{and } {}^t d\tilde{V} = (d\,{}^t\underline{x}_1 \times d\,{}^t\underline{x}_2) \cdot d\,{}^t\underline{x}_3$$

$$= \det {}^t_0\underline{X}\, ds_1\, ds_2\, ds_3$$

$$= \det {}^t_0\underline{X}\, {}^0 d\tilde{V}$$

Transparency 3-26

Example: One-dimensional stretching



uniform stretching
plane strain conditions

Deformation field: ${}^t x_1 = {}^0 x_1 + 0.25\,{}^0 x_1$

$$\text{Deformation gradient: } {}^t_0\underline{X} = \begin{bmatrix} 1.25 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \det {}^t_0\underline{X} = 1.25$$

Hence ${}^0\rho = 1.25\,{}^t\rho$ (${}^t\rho < {}^0\rho$ makes physical sense)

Transparency 3-27

We also use the inverse deformation gradient:

$$d\,{}^0\underline{x} = {}^0_t\underline{X}\; d\,{}^t\underline{x}$$

MATERIAL FIBER AT TIME 0

MATERIAL FIBER AT TIME t

Mathematically, ${}^0_t\underline{X} = ({}^t_0\underline{X})^{-1}$

Proof:
$$\begin{aligned} d\,{}^0\underline{x} &= {}^0_t\underline{X}\,({}^t_0\underline{X}\; d\,{}^0\underline{x}) \\ &= ({}^0_t\underline{X}\;{}^t_0\underline{X})\; d\,{}^0\underline{x} \\ &= \underline{I}\; d\,{}^0\underline{x} \end{aligned}$$

Transparency 3-28

An important point is:

$${}^t_0\underline{X} = {}^t_0\underline{R}\;{}^t_0\underline{U}$$

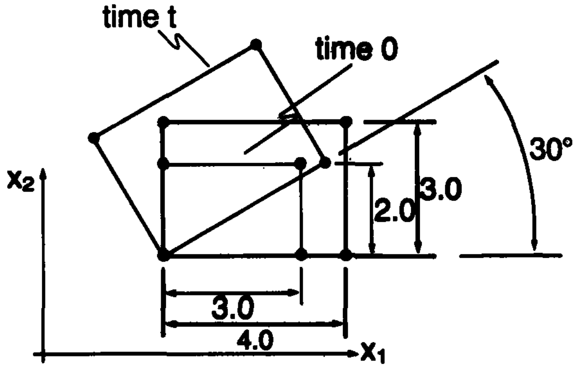
Polar decomposition of ${}^t_0\underline{X}$:

${}^t_0\underline{R}$ = orthogonal (rotation) matrix

${}^t_0\underline{U}$ = symmetric (stretch) matrix

We can always decompose ${}^t_0\underline{X}$ in the above form.

Example: Uniform stretch and rotation



$$\begin{array}{ccc} {}^t_0\underline{X} & = & {}^t_0\underline{R} \quad {}^t_0\underline{U} \end{array}$$

$$\begin{bmatrix} 1.154 & -0.750 \\ 0.667 & 1.299 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.500 \\ 0.500 & 0.866 \end{bmatrix} \begin{bmatrix} 1.333 & 0 \\ 0 & 1.500 \end{bmatrix}$$

Transparency 3-29

Using the deformation gradient, we can describe the (right) Cauchy-Green deformation tensor

$$ {}^t_0\underline{C} = {}^t_0\underline{X}^T \, {}^t_0\underline{X} $$

This tensor depends only on the stretch tensor ${}^t_0\underline{U}$:

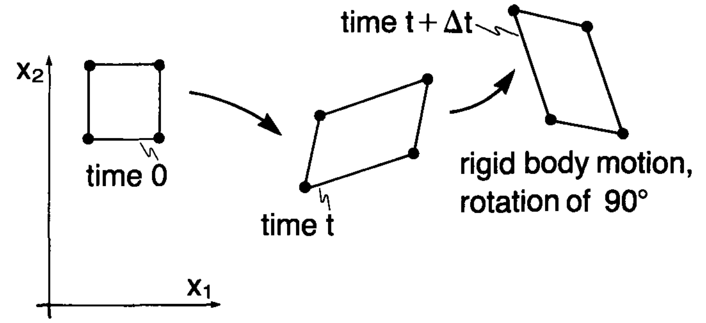
$$ \begin{aligned} {}^t_0\underline{C} &= ({}^t_0\underline{U}^T \, {}^t_0\underline{R}^T)\,({}^t_0\underline{R}\, {}^t_0\underline{U}) \\ &= ({}^t_0\underline{U})^2 \quad \text{(since } {}^t_0\underline{R} \text{ is orthogonal)} \end{aligned} $$

Hence ${}^t_0\underline{C}$ is invariant under a rigid body rotation.

Transparency 3-30

Transparency 3-31

Example: Two-dimensional motion



rigid body motion, rotation of 90°

$${}^{t}_{0}\underline{X} = \begin{bmatrix} 1.5 & .2 \\ .5 & 1 \end{bmatrix} \qquad {}^{t+\Delta t}_{\ 0}\underline{X} = \begin{bmatrix} -.5 & -1 \\ 1.5 & .2 \end{bmatrix}$$

$${}^{t}_{0}\underline{C} = \begin{bmatrix} 2.5 & .8 \\ .8 & 1.04 \end{bmatrix} \qquad {}^{t+\Delta t}_{\ 0}\underline{C} = \begin{bmatrix} 2.5 & .8 \\ .8 & 1.04 \end{bmatrix}$$

Transparency 3-32

The Green-Lagrange strain tensor measures the stretching deformations. It can be written in several equivalent forms:

1) ${}^{t}_{0}\underline{\varepsilon} = \frac{1}{2}\left({}^{t}_{0}\underline{C} - \underline{I}\right)$

From this,

- ${}^{t}_{0}\underline{\varepsilon}$ is symmetric.
- For a rigid body motion between times t and t+Δt, ${}^{t+\Delta t}_{\ 0}\underline{\varepsilon} = {}^{t}_{0}\underline{\varepsilon}$.
- For a rigid body motion between times 0 and t, ${}^{t}_{0}\underline{\varepsilon} = \underline{0}$.

Transparency 3-33

- ${}_0^t\underline{\varepsilon}$ is symmetric because ${}_0^t\underline{C}$ is symmetric

$$ {}_0^t\underline{\varepsilon} = \frac{1}{2}({}_0^t\underline{C} - \underline{I}) $$

- For a rigid body motion from t to t+Δt, we have

$$ {}_0^{t+\Delta t}\underline{X} = \underline{R}\;{}_0^t\underline{X} $$

$$ {}_0^{t+\Delta t}\underline{C} = {}_0^t\underline{C} \Rightarrow {}_0^{t+\Delta t}\underline{\varepsilon} = {}_0^t\underline{\varepsilon} $$

- For a rigid body motion

$$ {}_0^t\underline{C} = \underline{I} \Rightarrow {}_0^t\underline{\varepsilon} = \underline{0} $$

Transparency 3-34

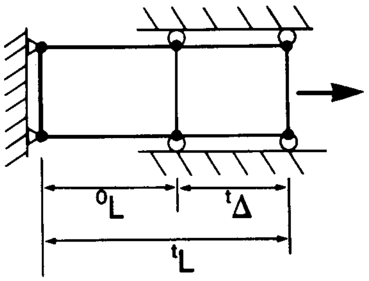
$$ 2)\quad {}_0^t\varepsilon_{ij} = \frac{1}{2}\,(\underbrace{{}_0^tu_{i,j} + {}_0^tu_{j,i}}_{\text{LINEAR IN DISPLACEMENTS}} + \underbrace{{}_0^tu_{k,i}\;{}_0^tu_{k,j}}_{\text{NONLINEAR IN DISPLACEMENTS}}) $$

where $\;{}_0^tu_{i,j} = \dfrac{\partial\,{}^tu_i}{\partial\,{}^0x_j}$

Important point: This strain tensor is exact and holds for any amount of stretching.

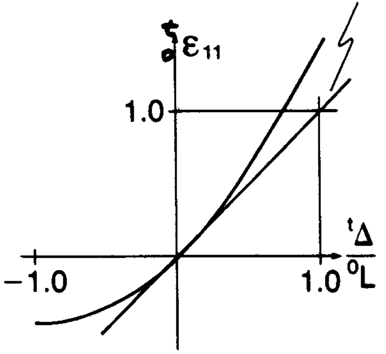
Transparency 3-35

Example: Uniaxial strain



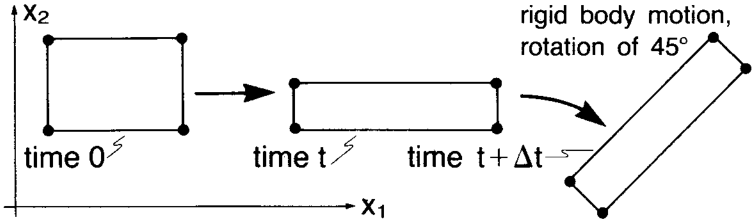
$$ {}^t_0\varepsilon_{11} = \frac{{}^t\Delta}{{}^0L} + \frac{1}{2}\left(\frac{{}^t\Delta}{{}^0L}\right)^2 $$

engineering strain



Transparency 3-36

Example: Biaxial straining and rotation



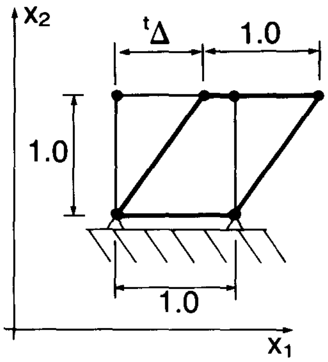
$$ {}^t_0\underline{X} = \begin{bmatrix} 1.5 & 0 \\ 0 & .5 \end{bmatrix} \qquad {}^{t+\Delta t}_{\ 0}\underline{X} = \begin{bmatrix} 1.06 & -.354 \\ 1.06 & .354 \end{bmatrix} $$

$$ {}^t_0\underline{C} = \begin{bmatrix} 2.25 & 0 \\ 0 & .25 \end{bmatrix} \qquad {}^{t+\Delta t}_{\ 0}\underline{C} = \begin{bmatrix} 2.25 & 0 \\ 0 & .25 \end{bmatrix} $$

$$ {}^t_0\underline{\varepsilon} = \begin{bmatrix} .625 & 0 \\ 0 & -.375 \end{bmatrix} \qquad {}^{t+\Delta t}_{\ 0}\underline{\varepsilon} = \begin{bmatrix} .625 & 0 \\ 0 & -.375 \end{bmatrix} $$

Transparency 3-37

Example: Simple shear



$$ {}^t_0\underline{X} = \begin{bmatrix} 1.0 & {}^t\Delta \\ 0 & 1.0 \end{bmatrix} $$

$$ {}^t_0\underline{C} = \begin{bmatrix} 1.0 & {}^t\Delta \\ {}^t\Delta & 1.0 + ({}^t\Delta)^2 \end{bmatrix} $$

$$ {}^t_0\underline{\varepsilon} = \begin{bmatrix} 0 & {}^t\Delta/2 \\ {}^t\Delta/2 & ({}^t\Delta)^2/2 \end{bmatrix} $$

For small displacements, ${}^t_0\underline{\varepsilon}$ is approximately equal to the small strain tensor.

Transparency 3-38

The 2nd Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor are energetically conjugate:

${}^t\tau_{ij}\,\delta_t e_{ij}$ = Virtual work at time t per unit current volume

${}^t_0S_{ij}\,\delta\,{}^t_0\varepsilon_{ij}$ = Virtual work at time t per unit original volume

where ${}^t_0S_{ij}$ is the 2nd Piola-Kirchhoff stress tensor.

Transparency 3-39

The 2nd Piola-Kirchhoff stress tensor:

$${}^t_0S_{ij} = \frac{{}^0\rho}{{}^t\rho}\, {}^0_tx_{i,m}\, {}^tT_{mn}\, {}^0_tx_{j,n}$$ — INDICIAL NOTATION

$${}^t_0\underline{S} = \frac{{}^0\rho}{{}^t\rho}\, {}^0_t\underline{X}\, {}^t\underline{T}\, {}^0_t\underline{X}^T$$ — MATRIX NOTATION

Solving for the Cauchy stresses gives

$${}^tT_{ij} = \frac{{}^t\rho}{{}^0\rho}\, {}^t_0x_{i,m}\, {}^t_0S_{mn}\, {}^t_0x_{j,n}$$ — INDICIAL NOTATION

$${}^t\underline{T} = \frac{{}^t\rho}{{}^0\rho}\, {}^t_0\underline{X}\, {}^t_0\underline{S}\, {}^t_0\underline{X}^T$$ — MATRIX NOTATION

Transparency 3-40

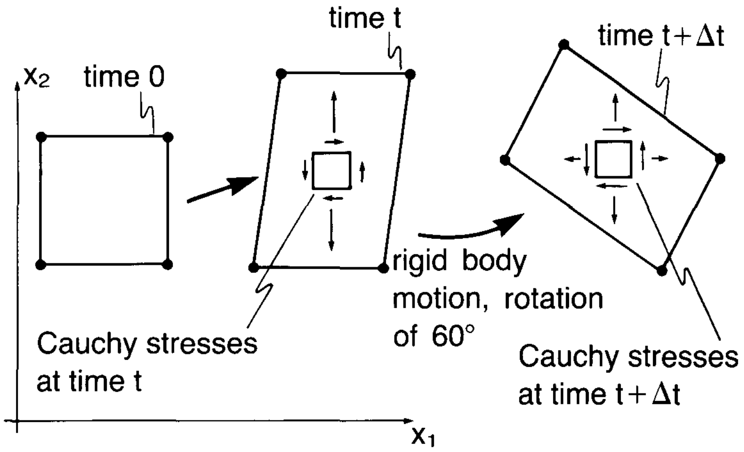
Properties of the 2nd Piola-Kirchhoff stress tensor:

- ${}^t_0\underline{S}$ is symmetric.
- ${}^t_0\underline{S}$ is invariant under a rigid-body motion (translation and/or rotation).

  Hence ${}^t_0\underline{S}$ changes only when the material is deformed.
- ${}^t_0\underline{S}$ has no direct physical interpretation.

Transparency 3-41

Example: Two-dimensional motion



Transparency 3-42

At time t,	At time t+Δt,
${}^t_0\underline{X} = \begin{bmatrix} 1 & .2 \\ 0 & 1.5 \end{bmatrix}$	${}^{t+\Delta t}_{\ 0}\underline{X} = \begin{bmatrix} .5 & -1.20 \\ .866 & .923 \end{bmatrix}$
${}^t\underline{\tau} = \begin{bmatrix} 0 & 1000 \\ 1000 & 2000 \end{bmatrix}$	${}^{t+\Delta t}\underline{\tau} = \begin{bmatrix} 634 & -1370 \\ -1370 & 1370 \end{bmatrix}$
${}^t_0\underline{S} = \begin{bmatrix} -346 & 733 \\ 733 & 1330 \end{bmatrix}$	${}^{t+\Delta t}_{\ 0}\underline{S} = \begin{bmatrix} -346 & 733 \\ 733 & 1330 \end{bmatrix}$

Topic 4

Total Lagrangian Formulation for Incremental General Nonlinear Analysis

Contents:

- **Review of basic principle of virtual work equation, objective in incremental solution**
- **Incremental stress and strain decompositions in the total Lagrangian form of the principle of virtual work**
- **Linear and nonlinear strain increments**
- **Initial displacement effect**
- **Considerations for finite element discretization with continuum elements (isoparametric solids with translational degrees of freedom only) and structural elements (with translational and rotational degrees of freedom)**
- **Consistent linearization of terms in the principle of virtual work for the incremental solution**
- **The "out-of-balance" virtual work term**
- **Derivation of iterative equations**
- **The modified Newton-Raphson iteration, flow chart of complete solution**

Textbook:

Sections 6.2.3, 8.6, 8.6.1

Transparency 4-1

TOTAL LAGRANGIAN FORMULATION

We have so far established that

$$\int_{{}^{0}V} {}^{t+\Delta t}_{\;\;0}S_{ij}\,\delta\,{}^{t+\Delta t}_{\;\;0}\varepsilon_{ij}\,{}^{0}dV = {}^{t+\Delta t}\mathcal{R}$$

is totally equivalent to

$$\int_{{}^{t+\Delta t}V} {}^{t+\Delta t}\tau_{ij}\,\delta_{t+\Delta t}e_{ij}\,{}^{t+\Delta t}dV = {}^{t+\Delta t}\mathcal{R}$$

Transparency 4-2

Recall:

▶ $$\int_{{}^{t+\Delta t}V} {}^{t+\Delta t}\tau_{ij}\,\delta_{t+\Delta t}e_{ij}\,{}^{t+\Delta t}dV = {}^{t+\Delta t}\mathcal{R}$$

is an expression of

- Equilibrium
- Compatibility
- The stress-strain law

all at time $t+\Delta t$.

Transparency 4-3

- We employ an incremental solution procedure:

  Given the solution at time t, we seek the displacement increments u_i to obtain the displacements at time $t+\Delta t$

  $$ {}^{t+\Delta t}u_i = {}^{t}u_i + u_i $$

  We can then evaluate, from the total displacements, the Cauchy stresses at time $t+\Delta t$. These stresses will satisfy the principle of virtual work at time $t+\Delta t$.

Transparency 4-4

- Our goal is, for the finite element solution, to linearize the equation of the principle of virtual work, so as to finally obtain

  $$ \underbrace{{}^{t}\underline{K}}_{\substack{\text{tangent}\\ \text{stiffness}\\ \text{matrix}}}\;\underbrace{\Delta\underline{U}^{(1)}}_{\substack{\text{nodal point}\\ \text{displacement}\\ \text{increments}}} = \underbrace{{}^{t+\Delta t}\underline{R}}_{\substack{\text{externally applied}\\ \text{loads at}\\ \text{time } t+\Delta t}} - \underbrace{{}^{t}\underline{F}}_{\substack{\text{vector of}\\ \text{nodal point forces}\\ \text{corresponding to}\\ \text{the element}\\ \text{internal stresses}\\ \text{at time } t}} $$

  The vector $\Delta\underline{U}^{(1)}$ now gives an approximation to the displacement increment $\underline{U} = {}^{t+\Delta t}\underline{U} - {}^{t}\underline{U}$.

Transparency 4-5

The equation

$$\begin{array}{ccccccc} {}^{t}\underline{K} & \Delta\underline{U}^{(1)} & = & {}^{t+\Delta t}\underline{R} & - & {}^{t}\underline{F} \\ \left[\qquad\right] & [\,] & = & [\,] & - & [\,] \\ n\times n & n\times 1 & & n\times 1 & & n\times 1 \end{array}$$

is valid

- for a single finite element (n = number of element degrees of freedom)
- for an assemblage of elements (n = total number of degrees of freedom)

Transparency 4-6

▸ We cannot "simply" linearize the principle of virtual work when it is written in the form

$$\int_{{}^{t+\Delta t}V} {}^{t+\Delta t}\tau_{ij}\, \delta_{t+\Delta t}e_{ij}\, {}^{t+\Delta t}dV = {}^{t+\Delta t}\mathcal{R}$$

- We cannot integrate over an unknown volume.
- We cannot directly increment the Cauchy stresses.

Transparency 4-7

- To linearize, we choose a known reference configuration and use 2nd Piola-Kirchhoff stresses and Green-Lagrange strains as described below.

  Two practical choices for the reference configuration:

  - time = 0 → total Lagrangian formulation
  - time = t → updated Lagrangian formulation

Transparency 4-8

TOTAL LAGRANGIAN FORMULATION

Because ${}^{t+\Delta t}_{0}S_{ij}$ and ${}^{t+\Delta t}_{0}\varepsilon_{ij}$ are energetically conjugate,

the principle of virtual work

$$\int_{{}^{t+\Delta t}V} {}^{t+\Delta t}\tau_{ij}\, \delta_{t+\Delta t}e_{ij}\, {}^{t+\Delta t}dV = {}^{t+\Delta t}\mathscr{R}$$

can be written as

$$\int_{{}^{0}V} {}^{t+\Delta t}_{0}S_{ij}\, \delta\, {}^{t+\Delta t}_{0}\varepsilon_{ij}\, {}^{0}dV = {}^{t+\Delta t}\mathscr{R}$$

Transparency 4-9

We already know the solution at time t (${}^t_0S_{ij}$, ${}^t_0u_{i,j}$, etc.). Therefore we decompose the unknown stresses and strains as

$$ {}^{t+\Delta t}_{0}S_{ij} = \underbrace{{}^t_0S_{ij}} + \underbrace{{}_0S_{ij}} $$

known unknown increments

$$ {}^{t+\Delta t}_0\varepsilon_{ij} = \overbrace{{}^t_0\varepsilon_{ij}} + \overbrace{{}_0\varepsilon_{ij}} $$

Transparency 4-10

In terms of displacements, using

$$ {}^t_0\varepsilon_{ij} = \frac{1}{2}\left({}^t_0u_{i,j} + {}^t_0u_{j,i} + {}^t_0u_{k,i}\,{}^t_0u_{k,j}\right) $$

and

$$ {}^{t+\Delta t}_0\varepsilon_{ij} = \frac{1}{2}\left({}^{t+\Delta t}_0u_{i,j} + {}^{t+\Delta t}_0u_{j,i} + {}^{t+\Delta t}_0u_{k,i}\,{}^{t+\Delta t}_0u_{k,j}\right) $$

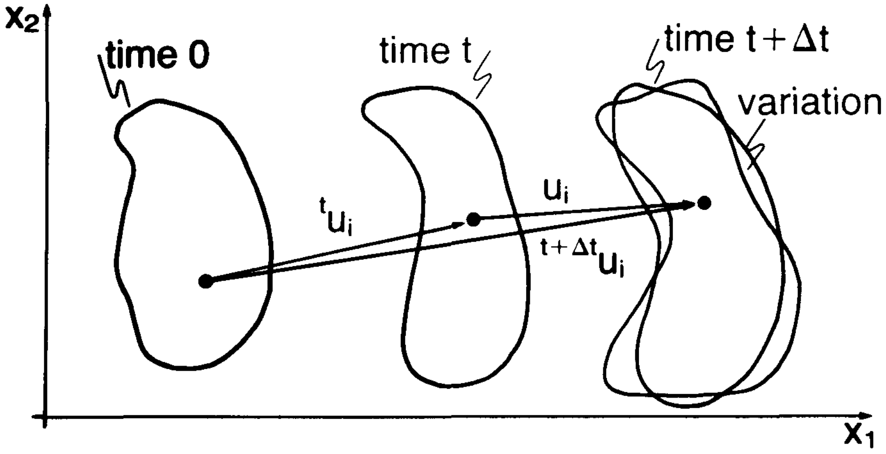
we find

$$ {}_0\varepsilon_{ij} = \frac{1}{2}\underbrace{\left({}_0u_{i,j} + {}_0u_{j,i} + \underbrace{{}^t_0u_{k,i}\,{}_0u_{k,j} + {}_0u_{k,i}\,{}^t_0u_{k,j}}_{\text{initial displacement effect}}\right)}_{\text{linear in } u_i} + \underbrace{\frac{1}{2}\,{}_0u_{k,i}\,{}_0u_{k,j}}_{\text{nonlinear in } u_i} $$

Transparency
4-11

We note $\delta\, {}^{t+\Delta t}_{\ 0}\varepsilon_{ij} = \delta\, {}_0\varepsilon_{ij}$

- Makes sense physically, because each variation is taken on the displacements at time $t+\Delta t$, with ${}^t u_i$ fixed.



Transparency
4-12

We define

$$ {}_0e_{ij} = \underbrace{\frac{1}{2}\left({}_0u_{i,j} + {}_0u_{j,i} + {}^t_0u_{k,i}\,{}_0u_{k,j} + {}_0u_{k,i}\,{}^t_0u_{k,j}\right)}_{\text{LINEAR STRAIN INCREMENT}} $$

$$ {}_0\eta_{ij} = \underbrace{\frac{1}{2}\,{}_0u_{k,i}\,{}_0u_{k,j}}_{\text{NONLINEAR STRAIN INCREMENT}} $$

Hence

$$ {}_0\varepsilon_{ij} = {}_0e_{ij} + {}_0\eta_{ij}\,, \qquad \delta_0\varepsilon_{ij} = \delta_0 e_{ij} + \delta_0\eta_{ij} $$

Transparency 4-13

An interesting observation:

- We have identified above, from continuum mechanics considerations, incremental strain terms

  $_0e_{ij}$ — linear in the displacement increments u_i
  $_0\eta_{ij}$ — nonlinear (second = order) in the displacement increments u_i

- In finite element analysis, the displacements are interpolated in terms of nodal point variables.

Transparency 4-14

- In isoparametric finite element analysis of solids, the finite element internal displacements depend linearly on the nodal point displacements.

$$^t u_i = \sum_{k=1}^{N} h_k \, ^t u^k_i$$

Hence, the exact linear strain increment and nonlinear strain increment are given by $_0e_{ij}$ and $_0\eta_{ij}$.

Transparency 4-15

- However, in the formulation of degenerate isoparametric beam and shell elements, the finite element internal displacements are expressed in terms of nodal point displacements and rotations.

${}^{t}u_i$ = f (linear in nodal point displacements but nonlinear in nodal point rotations)

Transparency 4-16

- For isoparametric beam and shell elements
  - the exact linear strain increment is given by ${}_0e_{ij}$, linear in the incremental nodal point variables
  - only an approximation to the second-order nonlinear strain increment is given by $\frac{1}{2}\,{}_0u_{k,i}\,{}_0u_{k,j}$, second-order in the incremental nodal point displacements and rotations

Transparency 4-17

The equation of the principle of virtual work becomes

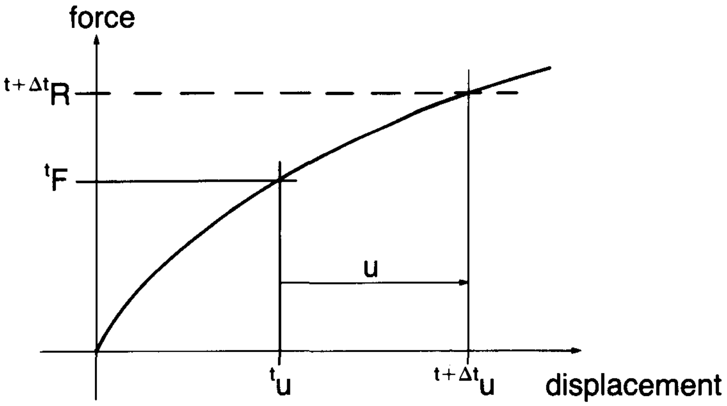
$$\int_{^0V} {}_0S_{ij}\, \delta_0\varepsilon_{ij}\, {}^0dV + \int_{^0V} {}^t_0S_{ij}\, \delta_0\eta_{ij}\, {}^0dV$$

$$= {}^{t+\Delta t}\mathcal{R} - \int_{^0V} {}^t_0S_{ij}\, \delta_0 e_{ij}\, {}^0dV$$

Given a variation δu_i, the right-hand-side is known. The left-hand-side contains unknown displacement increments.

Important: So far, no approximations have been made.

Transparency 4-18



All we have done so far is to write the principle of virtual work in terms of ^tu_i and u_i.

Transparency 4-19

- The equation of the principle of virtual work is in general a complicated nonlinear function in the unknown displacement increment.
- We obtain an approximate equation by neglecting all higher-order terms in u_i (so that only linear terms in u_i remain). This leads to

$$ {}^{t}_{0}\underline{K}\,\Delta\underline{U} = {}^{t+\Delta t}\underline{R} - {}^{t}_{0}\underline{F} $$

The process of neglecting higher-order terms is called linearization.

Transparency 4-20

Now we begin to linearize the terms that contain the unknown displacement increments.

1) The term $\int_{{}^0V} {}^{t}_{0}S_{ij}\,\delta_0\eta_{ij}\,{}^{0}dV$

is linear in u_i:

- ${}^{t}_{0}S_{ij}$ does not contain u_i.
- $\delta_0\eta_{ij} = \frac{1}{2}\,{}_0u_{k,i}\,\delta_0u_{k,j} + \frac{1}{2}\,\delta_0u_{k,i}\,{}_0u_{k,j}$

  is linear in u_i.

Transparency 4-21

2) The term $\int_{^0V} {_0S_{ij}}\, \delta_0\varepsilon_{ij}\, {^0dV}$ contains linear and higher-order terms in u_i:

- ${_0S_{ij}}$ is a nonlinear function (in general) of ${_0\varepsilon_{ij}}$.
- $\delta_0\varepsilon_{ij} = \delta_0 e_{ij} + \delta_0\eta_{ij}$ is a linear function of u_i.

We need to neglect all higher-order terms in u_i.

Transparency 4-22

Linearization of ${_0S_{ij}}\,\delta_0\varepsilon_{ij}$:

Our objective is to express (by approximation) ${_0S_{ij}}$ as a linear function of u_i (noting that ${_0S_{ij}}$ equals zero if u_i equals zero).

We also recognize that $\delta_0\varepsilon_{ij}$ contains only constant and linear terms in u_i. We will see that only the constant term $\delta_0 e_{ij}$ should be included.

Transparency 4-23

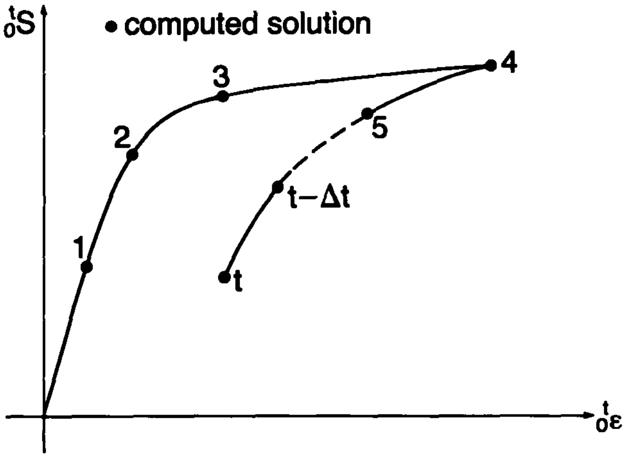
${}_0S_{ij}$ can be written as a Taylor series in ${}_0\varepsilon_{ij}$:

$$ {}_0S_{ij} = \underbrace{\left.\frac{\partial\, {}^t_0S_{ij}}{\partial\, {}^t_0\varepsilon_{rs}}\right|_t}_{\text{known}} \underbrace{{}_0\varepsilon_{rs}}_{\text{linear and quadratic in } u_i} + \text{higher-order terms} $$

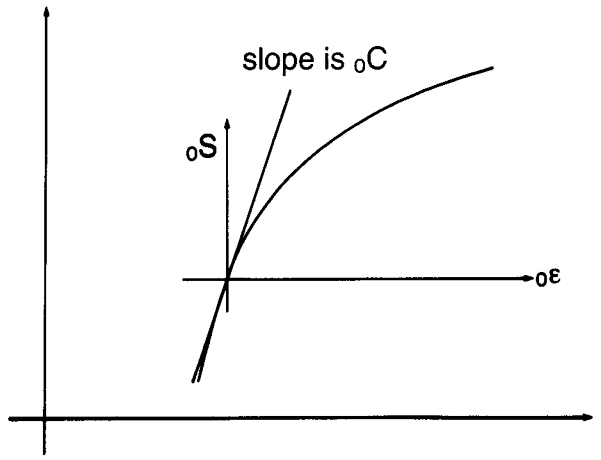
$$ \doteq \left.\frac{\partial\, {}^t_0S_{ij}}{\partial\, {}^t_0\varepsilon_{rs}}\right|_t (\underbrace{{}_0e_{rs}}_{\text{linear in } u_i} + \underbrace{{}_0\eta_{rs}}_{\text{quadratic in } u_i}) \doteq \underbrace{{}_0C_{ijrs}\, {}_0e_{rs}}_{\text{linearized term}} $$

Transparency 4-24

Example: A one-dimensional stress- strain law



At time t,



Transparency 4-25

Hence we obtain

$${}_0S_{ij}\,\delta_0\varepsilon_{ij} \doteq \underbrace{{}_0C_{ijrs}\,{}_0e_{rs}}_{}\,(\delta_0 e_{ij} + \delta_0\eta_{ij})$$

$$= \underbrace{{}_0C_{ijrs}\,{}_0e_{rs}\,\underbrace{\delta_0 e_{ij}}_{\substack{\text{does not}\\ \text{contain } u_i}}}_{\text{linear in } u_i} + \underbrace{{}_0C_{ijrs}\,{}_0e_{rs}\,\underbrace{\delta_0\eta_{ij}}_{\text{linear in } u_i}}_{\text{quadratic in } u_i}$$

$$\doteq \underbrace{{}_0C_{ijrs}\,{}_0e_{rs}\,\delta_0 e_{ij}}_{\text{linearized result}}$$

Transparency 4-26

Transparency 4-27

The final linearized equation is

$$\underbrace{\int_{^0V} {}_0C_{ijrs}\, {}_0e_{rs}\, \delta_0 e_{ij}\, {}^0dV + \int_{^0V} {}_0^tS_{ij}\, \delta_0\eta_{ij}\, {}^0dV}_{\delta \underline{U}^T\, {}_0^t\underline{K}\, \Delta\underline{U}}$$

$$= \underbrace{{}^{t+\Delta t}\mathcal{R} - \int_{^0V} {}_0^tS_{ij}\, \delta_0 e_{ij}\, {}^0dV}_{\delta\underline{U}^T\, ({}^{t+\Delta t}\underline{R} - {}_0^t\underline{F})}$$

when discretized using the finite element method

Transparency 4-28

- An important point is that

$$\int_{^0V} {}_0^tS_{ij}\, \delta_0 e_{ij}\, {}^0dV = \underbrace{\int_{^0V} {}_0^tS_{ij}\, \delta_0^t\varepsilon_{ij}\, {}^0dV}_{\text{the virtual work due to the element internal stresses at time t}}$$

because

$$\delta_0 e_{ij} = \delta_0^t\varepsilon_{ij}$$

- We interpret

$$ {}^{t+\Delta t}\mathcal{R} - \int_{^0V} {}_0^tS_{ij}\, \delta_0 e_{ij}\, {}^0dV$$

as an “out-of-balance” virtual work term.

Transparency 4-29

Mathematical explanation that $\delta_0 e_{ij} = \delta_0^{\,t}\varepsilon_{ij}$:

We had $\delta\,{}^{t+\Delta t}_{\ \ 0}\varepsilon_{ij} = \delta_0 e_{ij} + \delta_0 \eta_{ij}$.

If $u_i = 0$, then the configuration at time $t+\Delta t$ is identical to the configuration at time t. Hence $\delta\,{}^{t+\Delta t}_{\ \ 0}\varepsilon_{ij}\Big|_{u_i=0} = \delta_0^{\,t}\varepsilon_{ij}$.

It follows that

$$\delta\,{}^{t+\Delta t}_{\ \ 0}\varepsilon_{ij}\Big|_{u_i=0} = \overset{\delta_0 e_{ij}}{\overset{\nearrow}{\delta_0 e_{ij}\Big|_{u_i=0}}} + \overset{0}{\overset{\nearrow}{\delta_0 \eta_{ij}\Big|_{u_i=0}}} = \delta_0^{\,t}\varepsilon_{ij}$$

Transparency 4-30

This result makes physical sense because equilibrium was assumed to be satisfied at time t. Hence we can write

$$\int_{{}^0V} {}_0C_{ijrs}\ {}_0e_{rs}\ \delta_0 e_{ij}\ {}^0dV + \int_{{}^0V} {}^{t}_{0}S_{ij}\ \delta_0 \eta_{ij}\ {}^0dV$$

$$= {}^{t+\Delta t}\mathcal{R} - {}^{t}\mathcal{R}$$

Check: Suppose that ${}^{t+\Delta t}\mathcal{R} = {}^{t}\mathcal{R}$ and that the material is elastic. Then ${}^{t+\Delta t}u_i$ must equal ${}^{t}u_i$, hence $u_i = 0$. This is satisfied by the above equation.

Transparency 4-31

We may rewrite the linearized governing equation as follows:

$$\int_{{}^0V} {}_0C_{ijrs}\, \Delta_0 e_{rs}^{(1)}\, \delta_0 e_{ij}\, {}^0dV + \int_{{}^0V} {}^t_0S_{ij}\, \delta\Delta_0\eta_{ij}^{(1)}\, {}^0dV$$

$$= {}^{t+\Delta t}\mathcal{R} - \int_{{}^0V} \underbrace{{}^{t+\Delta t}_{\quad 0}S_{ij}^{(0)}}_{{}^t_0S_{ij}}\ \underbrace{\delta\, {}^{t+\Delta t}_{\quad 0}\varepsilon_{ij}^{(0)}}_{\delta\,{}^t_0\varepsilon_{ij}}\ {}^0dV$$

Transparency 4-32

When the linearized governing equation is discretized, we obtain

$${}^t_0\underline{K}\,\Delta\underline{U}^{(1)} = {}^{t+\Delta t}\underline{R} - \underbrace{{}^{t+\Delta t}_{\quad 0}\underline{F}^{(0)}}_{{}^t_0\underline{F}}$$

We then use

$${}^{t+\Delta t}\underline{U}^{(1)} = \underbrace{{}^{t+\Delta t}\underline{U}^{(0)}}_{{}^t\underline{U}} + \Delta\underline{U}^{(1)}$$

Transparency 4-33

Having obtained an approximate solution ${}^{t+\Delta t}\underline{U}^{(1)}$, we can compute an improved solution:

$$\int_{^0V} {}_0C_{ijrs}\, \Delta_0 e_{rs}^{(2)}\, \delta_0 e_{ij}\, {}^0dV + \int_{^0V} {}^t_0S_{ij}\, \delta\Delta_0\eta_{ij}^{(2)}\, {}^0dV$$

$$= {}^{t+\Delta t}\mathcal{R} - \int_{^0V} {}^{t+\Delta t}_{\ \ 0}S_{ij}^{(1)}\, \delta\, {}^{t+\Delta t}_{\ \ 0}\varepsilon_{ij}^{(1)}\, {}^0dV$$

which, when discretized, gives

$${}^t_0\underline{K}\, \Delta\underline{U}^{(2)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}_{\ \ 0}\underline{F}^{(1)}$$

We then use

$${}^{t+\Delta t}\underline{U}^{(2)} = {}^{t+\Delta t}\underline{U}^{(1)} + \Delta\underline{U}^{(2)}$$

Transparency 4-34

In general,

$$\int_{^0V} {}_0C_{ijrs}\, \Delta_0 e_{rs}^{(k)}\, \delta_0 e_{ij}\, {}^0dV + \int_{^0V} {}^t_0S_{ij}\, \delta\Delta_0\eta_{ij}^{(k)}\, {}^0dV$$

$$= {}^{t+\Delta t}\mathcal{R} - \int_{^0V} {}^{t+\Delta t}_{\ \ 0}S_{ij}^{(k-1)}\, \delta\, {}^{t+\Delta t}_{\ \ 0}\varepsilon_{ij}^{(k-1)}\, {}^0dV$$

which, when discretized, gives

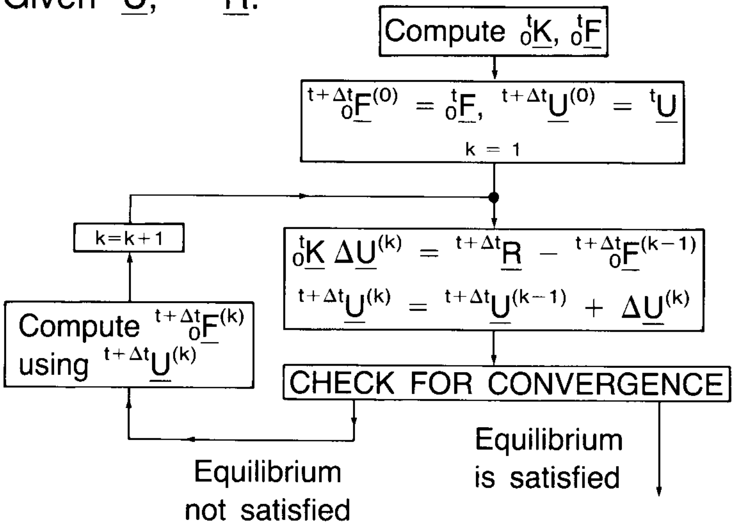
$${}^t_0\underline{K}\, \Delta\underline{U}^{(k)} = {}^{t+\Delta t}\underline{R} - \underbrace{{}^{t+\Delta t}_{\ \ 0}\underline{F}^{(k-1)}}_{\text{computed from } {}^{t+\Delta t}u_i^{(k-1)}}$$

(for k = 1, 2, 3, ...)

Note that ${}^{t+\Delta t}\underline{U}^{(k)} = {}^t\underline{U} + \sum_{j=1}^{k} \Delta\underline{U}^{(j)}$.

Transparency 4-35

Given ${}^{t}\underline{U}$, ${}^{t+\Delta t}\underline{R}$:



Compute ${}^{t}_{0}\underline{K}$, ${}^{t}_{0}\underline{F}$

$${}^{t+\Delta t}_{0}\underline{F}^{(0)} = {}^{t}_{0}\underline{F}, \quad {}^{t+\Delta t}\underline{U}^{(0)} = {}^{t}\underline{U}$$

k = 1

$${}^{t}_{0}\underline{K}\, \Delta\underline{U}^{(k)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}_{0}\underline{F}^{(k-1)}$$

$${}^{t+\Delta t}\underline{U}^{(k)} = {}^{t+\Delta t}\underline{U}^{(k-1)} + \Delta\underline{U}^{(k)}$$

CHECK FOR CONVERGENCE

Equilibrium not satisfied

Compute ${}^{t+\Delta t}_{0}\underline{F}^{(k)}$ using ${}^{t+\Delta t}\underline{U}^{(k)}$

k=k+1

Equilibrium is satisfied

Topic 5

Updated Lagrangian Formulation for Incremental General Nonlinear Analysis

Contents:

- **Principle of virtual work in terms of 2nd Piola-Kirchhoff stresses and Green-Lagrange strains referred to the configuration at time *t***
- **Incremental stress and strain decompositions in the updated Lagrangian form of the principle of virtual work**
- **Linear and nonlinear strain increments**
- **Consistent linearization of terms in the principle of virtual work**
- **The "out-of-balance" virtual work term**
- **Iterative equations for modified Newton-Raphson solution**
- **Flow chart of complete solution**
- **Comparison to total Lagrangian formulation**

Textbook: Section 6.2.3

A SUMMARY OF THE T.L.F.

- THE BASIC EQN. WE USE IS

$$\int_{^{t+\Delta t}V} {}^{t+\Delta t}\tau_{ij}\,\delta_{t+\Delta t}e_{ij}\,{}^{t+\Delta t}dV = {}^{t+\Delta t}R$$

- WE INTRODUCE

$${}^{t+\Delta t}_{0}S_{ij} \qquad {}^{t+\Delta t}_{t}S_{ij}$$

$${}^{t+\Delta t}_{0}\varepsilon_{ij} \qquad {}^{t+\Delta t}_{t}\varepsilon_{ij}$$

- WE DECOMPOSE

$${}^{t+\Delta t}_{0}S_{ij} = \overset{\checkmark}{{}^{t}_{0}S_{ij}} + \overset{?}{{}_{0}S_{ij}}$$

$${}^{t+\Delta t}_{0}\varepsilon_{ij} = \overset{\checkmark}{{}^{t}_{0}\varepsilon_{ij}} + \overset{?}{{}_{0}\varepsilon_{ij}}$$

- WE NOTE

$${}_{0}\varepsilon_{ij} = \underset{\text{LINEAR}}{{}_{0}e_{ij}} + \underset{\text{NONLINEAR}}{{}_{0}\eta_{ij}}$$

IN u_i ← particle displ.

- WE OBTAIN

$$\int_{^{0}V} {}^{t+\Delta t}_{0}S_{ij}\,\delta\,{}^{t+\Delta t}_{0}\varepsilon_{ij}\,{}^{0}dV = {}^{t+\Delta t}R$$

- SUBSTITUTION AND LINEARIZATION GIVES

$$\int_{^{0}V} {}_{0}C_{ijrs}\,{}_{0}e_{rs}\,\delta_{0}e_{ij}\,{}^{0}dV + \int_{^{0}V} {}^{t}_{0}S_{ij}\,\delta_{0}\eta_{ij}\,{}^{0}dV = {}^{t+\Delta t}R - \int_{^{0}V} {}^{t}_{0}S_{ij}\,\delta_{0}e_{ij}\,{}^{0}dV$$

Markerboard 5-1

- IN THE ITERATION WE HAVE

$$\ldots = {}^{t+\Delta t}R - \int_{^{0}V} {}^{t+\Delta t}_{0}S_{ij}^{(k-1)}\,\delta\,{}^{t+\Delta t}_{0}\varepsilon_{ij}^{(k-1)}\,{}^{0}dV$$

$k = 1, 2, 3, \ldots$

- THE F.E. DISCRETIZATION GIVES

$${}^{t}_{0}\underline{K}\,\Delta\underline{U}^{(i)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}_{0}\underline{F}^{(i-1)}$$

$i = 1, 2, 3, \ldots$

- AT CONVERGENCE

$${}^{t+\Delta t}\underline{R} = {}^{t+\Delta t}_{0}\underline{F}$$

WE SATISFY:

- COMPATIBILITY
- STRESS-STRAIN LAW
- EQUILIBRIUM
  - NODAL POINT EQUILIBRIUM
  - LOCAL EQUILIBRIUM IF MESH IS FINE ENOUGH

Markerboard 5-2

Transparency 5-1

UPDATED LAGRANGIAN FORMULATION

Because ${}^{t+\Delta t}_{\ t}S_{ij}$ and ${}^{t+\Delta t}_{\ t}\varepsilon_{ij}$ are energetically conjugate,

the principle of virtual work

$$\int_{{}^{t+\Delta t}V} {}^{t+\Delta t}\tau_{ij}\,\delta_{t+\Delta t}e_{ij}\,{}^{t+\Delta t}dV = {}^{t+\Delta t}\mathcal{R}$$

can be written as

$$\int_{{}^{t}V} {}^{t+\Delta t}_{\ t}S_{ij}\,\delta\,{}^{t+\Delta t}_{\ t}\varepsilon_{ij}\,{}^{t}dV = {}^{t+\Delta t}\mathcal{R}$$

Transparency 5-2

We already know the solution at time t (${}^t_tS_{ij}$, ${}^t_tu_{i,j}$, etc.). Therefore we decompose the unknown stresses and strains as

$${}^{t+\Delta t}_{\ t}S_{ij} = \underbrace{{}^t_tS_{ij}} + \underbrace{{}_tS_{ij}} = {}^t\tau_{ij} + {}_tS_{ij}$$

known unknown increments

$${}^{t+\Delta t}_{\ t}\varepsilon_{ij} = \overbrace{\cancelto{0}{{}^t_t\varepsilon_{ij}}} + \overbrace{{}_t\varepsilon_{ij}} = {}_t\varepsilon_{ij}$$

Transparency 5-3

In terms of displacements, using

$$ {}^{t+\Delta t}_{\ \ \ t}\varepsilon_{ij} = \frac{1}{2}\left({}^{t+\Delta t}_{\ \ \ t}u_{i,j} + {}^{t+\Delta t}_{\ \ \ t}u_{j,i} + {}^{t+\Delta t}_{\ \ \ t}u_{k,i}\;{}^{t+\Delta t}_{\ \ \ t}u_{k,j}\right) $$

we find

$$ {}_t\varepsilon_{ij} = \underbrace{\frac{1}{2}\left({}_tu_{i,j} + {}_tu_{j,i}\right)}_{\text{linear in } u_i} + \underbrace{\frac{1}{2}\,{}_tu_{k,i}\;{}_tu_{k,j}}_{\text{nonlinear in } u_i} $$

(No initial displacement effect)

Transparency 5-4

We define

$$ {}_te_{ij} = \frac{1}{2}\left({}_tu_{i,j} + {}_tu_{j,i}\right) \quad \text{linear strain increment} $$

$$ {}_t\eta_{ij} = \frac{1}{2}\,{}_tu_{k,i}\;{}_tu_{k,j} \quad \text{nonlinear strain increment} $$

Hence

$$ {}_t\varepsilon_{ij} = {}_te_{ij} + {}_t\eta_{ij} $$

$$ \delta_t\varepsilon_{ij} = \delta_te_{ij} + \delta_t\eta_{ij} $$

Transparency 5-5

The equation of the principle of virtual work becomes

$$\int_{{}^tV} {}_tS_{ij}\, \delta_t\varepsilon_{ij}\, {}^tdV + \int_{{}^tV} {}^t\tau_{ij}\, \delta_t\eta_{ij}\, {}^tdV$$

$$= {}^{t+\Delta t}\mathcal{R} - \int_{{}^tV} {}^t\tau_{ij}\, \delta_t e_{ij}\, {}^tdV$$

Given a variation δu_i, the right-hand-side is known. The left-hand-side contains unknown displacement increments.

Important: So far, no approximations have been made.

Transparency 5-6

Just as in the total Lagrangian formulation,

- The equation of the principle of virtual work is in general a complicated nonlinear function in the unknown displacement increment.
- Therefore we linearize this equation to obtain the approximate equation

$$ {}^t_t\underline{K}\, \Delta\underline{U} = {}^{t+\Delta t}\underline{R} - {}^t_t\underline{F} $$

We begin to linearize the terms containing the unknown displacement increments.

1) The term $\int_{{}^tV} {}^t\tau_{ij}\,\delta_t\eta_{ij}\,{}^tdV$

is linear in u_i.

- ${}^t\tau_{ij}$ does not contain u_i.
- $\delta_t\eta_{ij} = \frac{1}{2}\,{}_tu_{k,i}\,\delta_tu_{k,j} + \frac{1}{2}\,\delta_tu_{k,i}\,{}_tu_{k,j}$

  is linear in u_i.

Transparency 5-7

2) The term $\int_{{}^tV} {}_tS_{ij}\,\delta_t\varepsilon_{ij}\,{}^tdV$ contains linear and higher-order terms in u_i.

- ${}_tS_{ij}$ is a nonlinear function (in general) of ${}_t\varepsilon_{ij}$.
- $\delta_t\varepsilon_{ij} = \delta_te_{ij} + \delta_t\eta_{ij}$ is a linear function of u_i.

We need to neglect all higher-order terms in u_i.

Transparency 5-8

Transparency 5-9

${}_tS_{ij}$ can be written as a Taylor series in ${}_t\varepsilon_{ij}$:

$$ {}_tS_{ij} = \underbrace{\left.\frac{\partial\,{}_tS_{ij}}{\partial\,{}_t\varepsilon_{rs}}\right|_t}_{\text{known}} \underbrace{{}_t\varepsilon_{rs}}_{\text{linear and quadratic in } u_i} + \text{higher-order terms} $$

$$ \doteq \left.\frac{\partial\,{}_tS_{ij}}{\partial\,{}_t\varepsilon_{rs}}\right|_t \Big(\underbrace{{}_te_{rs}}_{\substack{\text{linear}\\ \text{in } u_i}} + \underbrace{{}_t\eta_{rs}}_{\substack{\text{quadratic}\\ \text{in } u_i}}\Big) \doteq \underbrace{{}_tC_{ijrs}\,{}_te_{rs}}_{\text{linearized term}} $$

Transparency 5-10

Hence we obtain

$$ \underbrace{{}_tS_{ij}}\,\delta\,{}_t\varepsilon_{ij} \doteq \underbrace{{}_tC_{ijrs}\,{}_te_{rs}}\,(\delta\,{}_te_{ij} + \delta\,{}_t\eta_{ij}) $$

$$ = \underbrace{{}_tC_{ijrs}\,{}_te_{rs}\,\underbrace{\delta\,{}_te_{ij}}_{\substack{\text{does not}\\ \text{contain } u_i}}}_{\text{linear in } u_i} + \underbrace{{}_tC_{ijrs}\,{}_te_{rs}\,\underbrace{\delta\,{}_t\eta_{ij}}_{\text{linear in } u_i}}_{\text{quadratic in } u_i} $$

$$ \doteq \underbrace{{}_tC_{ijrs}\;{}_te_{rs}\;\delta\,{}_te_{ij}}_{\text{linearized result}} $$

The final linearized equation is

$$\underbrace{\int_{^tV} {}_tC_{ijrs}\, {}_te_{rs}\, \delta_t e_{ij}\, {}^tdV + \int_{^tV} {}^t\tau_{ij}\, \delta_t \eta_{ij}\, {}^tdV}_{\delta \underline{U}^T\, {}^t_t\underline{K}\, \Delta \underline{U}}$$

$$= \underbrace{{}^{t+\Delta t}\mathcal{R} - \int_{^tV} {}^t\tau_{ij}\, \delta_t e_{ij}\, {}^tdV}_{\delta \underline{U}^T\, ({}^{t+\Delta t}\underline{R} - {}^t_t\underline{F})}$$

when
discretized
using the
finite element
method

Transparency 5-11

An important point is that

$$\int_{^tV} {}^t\tau_{ij}\, \delta_t e_{ij}\, {}^tdV$$

is the virtual work due to element internal stresses at time t. We interpret

$${}^{t+\Delta t}\mathcal{R} - \int_{^tV} {}^t\tau_{ij}\, \delta_t e_{ij}\, {}^tdV$$

as an “out-of-balance” virtual work term.

Transparency 5-12

Transparency 5-13

Solution using updated Lagrangian formulation

Displacement iteration:

$$ {}^{t+\Delta t}u_i^{(k)} = {}^{t+\Delta t}u_i^{(k-1)} + \Delta u_i^{(k)}, \quad {}^{t+\Delta t}u_i^{(0)} = {}^{t}u_i $$

Modified Newton iteration:

$$ \int_{{}^{t}V} {}_{t}C_{ijrs}\, \Delta_t e_{rs}^{(k)}\, \delta_t e_{ij}\, {}^{t}dV + \int_{{}^{t}V} {}^{t}\tau_{ij}\, \delta \Delta_t \eta_{ij}^{(k)}\, {}^{t}dV $$

$$ = {}^{t+\Delta t}\mathscr{R} - \int_{{}^{t+\Delta t}V^{(k-1)}} {}^{t+\Delta t}\tau_{ij}^{(k-1)}\, \delta_{t+\Delta t} e_{ij}^{(k-1)}\, {}^{t+\Delta t}dV $$

$k = 1, 2, \ldots$

Transparency 5-14

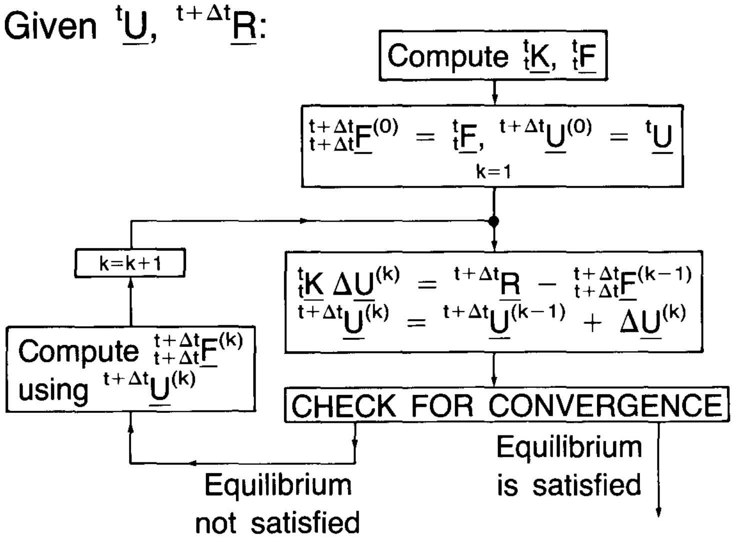
which, when discretized, gives

$$ {}_{t}^{t}\underline{K}\, \Delta \underline{U}^{(k)} = {}^{t+\Delta t}\underline{R} - \underbrace{{}_{t+\Delta t}^{t+\Delta t}\underline{F}^{(k-1)}}_{\text{computed from } {}^{t+\Delta t}u_i^{(k-1)}} $$

(for $k = 1, 2, 3, \ldots$)

Note that $ {}^{t+\Delta t}\underline{U}^{(k)} = {}^{t}\underline{U} + \sum_{j=1}^{k} \Delta \underline{U}^{(j)}$.

Given ${}^t\underline{U}$, ${}^{t+\Delta t}\underline{R}$:



Compute ${}^t_t\underline{K}$, ${}^t_t\underline{F}$

${}^{t+\Delta t}_{t+\Delta t}\underline{F}^{(0)} = {}^t_t\underline{F}$, ${}^{t+\Delta t}\underline{U}^{(0)} = {}^t\underline{U}$

k=1

k=k+1

${}^t_t\underline{K}\,\Delta\underline{U}^{(k)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}_{t+\Delta t}\underline{F}^{(k-1)}$

${}^{t+\Delta t}\underline{U}^{(k)} = {}^{t+\Delta t}\underline{U}^{(k-1)} + \Delta\underline{U}^{(k)}$

Compute ${}^{t+\Delta t}_{t+\Delta t}\underline{F}^{(k)}$ using ${}^{t+\Delta t}\underline{U}^{(k)}$

CHECK FOR CONVERGENCE

Equilibrium not satisfied

Equilibrium is satisfied

Transparency 5-15

Comparison of T.L. and U.L. formulations

- In the T.L. formulation, all derivatives are with respect to the initial coordinates whereas in the U.L. formulation, all derivatives are with respect to the current coordinates.
- In the U.L. formulation we work with the actual physical stresses (Cauchy stress).

Transparency 5-16

Transparency 5-17

The same assumptions are made in the linearization and indeed the same finite element stiffness and force vectors are calculated (when certain transformation rules are followed).

Topic 6

Formulation of Finite Element Matrices

Contents:

- **Summary of principle of virtual work equations in total and updated Lagrangian formulations**
- **Deformation-independent and deformation-dependent loading**
- **Materially-nonlinear-only analysis**
- **Dynamic analysis, implicit and explicit time integration**
- **Derivations of finite element matrices for total and updated Lagrangian formulations, materially-nonlinear-only analysis**
- **Displacement and strain-displacement interpolation matrices**
- **Stress matrices**
- **Numerical integration and application of Gauss and Newton-Cotes formulas**
- **Example analysis: Elasto-plastic beam in bending**
- **Example analysis: A numerical experiment to test for correct element rigid body behavior**

Textbook: Sections 6.3, 6.5.4

- WE HAVE DEVELOPED THE GENERAL INCREMENTAL CONTINUUM MECHANICS EQUATIONS IN THE PREVIOUS LECTURES
- IN THIS LECTURE
    - WE DISCUSS THE F.E. MATRICES USED IN STATIC AND DYNAMIC ANALYSIS, IN GENERAL MATRIX TERMS
- THE F.E. MATRICES ARE FORMULATED, AND WE DISCUSS THEIR EVALUATION BY NUMERICAL INTEGRATION

Markerboard 6-1

Transparency
6-1

DERIVATION OF ELEMENT MATRICES

The governing continuum mechanics equation for the total Lagrangian (T.L.) formulation is

$$\int_{{}^0V} {}_0C_{ijrs}\,{}_0e_{rs}\,\delta_0 e_{ij}\,{}^0dV + \int_{{}^0V} {}_0^tS_{ij}\,\delta_0\eta_{ij}\,{}^0dV$$

$$= {}^{t+\Delta t}\mathcal{R} - \int_{{}^0V} {}_0^tS_{ij}\,\delta_0 e_{ij}\,{}^0dV$$

Transparency
6-2

The governing continuum mechanics equation for the updated Lagrangian (U.L.) formulation is

$$\int_{{}^tV} {}_tC_{ijrs}\,{}_te_{rs}\,\delta_t e_{ij}\,{}^tdV + \int_{{}^tV} {}^t\tau_{ij}\,\delta_t\eta_{ij}\,{}^tdV$$

$$= {}^{t+\Delta t}\mathcal{R} - \int_{{}^tV} {}^t\tau_{ij}\,\delta_t e_{ij}\,{}^tdV$$

Transparency 6-3

For the T.L. formulation, the modified Newton iteration procedure is

(for k = 1, 2, 3, ...)

$$\int_{{}^0V} {}_0C_{ijrs}\, \Delta_0 e_{rs}^{(k)}\, \delta_0 e_{ij}\, {}^0dV + \int_{{}^0V} {}_0^tS_{ij}\, \delta\Delta_0\eta_{ij}^{(k)}\, {}^0dV$$

$$= {}^{t+\Delta t}\mathcal{R} - \int_{{}^0V} {}^{t+\Delta t}_{\ \ \ 0}S_{ij}^{(k-1)}\, \delta\, {}^{t+\Delta t}_{\ \ \ 0}\epsilon_{ij}^{(k-1)}\, {}^0dV$$

where we use

$${}^{t+\Delta t}u_i^{(k)} = {}^{t+\Delta t}u_i^{(k-1)} + \Delta u_i^{(k)}$$

with initial conditions

$${}^{t+\Delta t}u_i^{(0)} = {}^tu_i, \quad {}^{t+\Delta t}_{\ \ \ 0}S_{ij}^{(0)} = {}_0^tS_{ij}, \quad {}^{t+\Delta t}_{\ \ \ 0}\epsilon_{ij}^{(0)} = {}_0^t\epsilon_{ij}$$

Transparency 6-4

For the U. L. formulation, the modified Newton iteration procedure is

(for k = 1, 2, 3, ...)

$$\int_{{}^tV} {}_tC_{ijrs}\, \Delta_t e_{rs}^{(k)}\, \delta_t e_{ij}\, {}^tdV + \int_{{}^tV} {}^t\tau_{ij}\, \delta\Delta_t\eta_{ij}^{(k)}\, {}^tdV$$

$$= {}^{t+\Delta t}\mathcal{R} - \int_{{}^{t+\Delta t}V^{(k-1)}} {}^{t+\Delta t}\tau_{ij}^{(k-1)}\, \delta_{t+\Delta t}e_{ij}^{(k-1)}\, {}^{t+\Delta t}dV$$

where we use

$${}^{t+\Delta t}u_i^{(k)} = {}^{t+\Delta t}u_i^{(k-1)} + \Delta u_i^{(k)}$$

with initial conditions

$${}^{t+\Delta t}u_i^{(0)} = {}^tu_i, \quad {}^{t+\Delta t}\tau_{ij}^{(0)} = {}^t\tau_{ij}, \quad {}_{t+\Delta t}e_{ij}^{(0)} = {}_te_{ij}$$

Transparency 6-5

Assuming that the loading is deformation-independent,

$${}^{t+\Delta t}\mathcal{R} = \int_{{}^{0}V} {}^{t+\Delta t}_{\;\;\;0}f_i^B \, \delta u_i \, {}^{0}dV + \int_{{}^{0}S} {}^{t+\Delta t}_{\;\;\;0}f_i^S \, \delta u_i^S \, {}^{0}dS$$

For a dynamic analysis, the inertia force loading term is

$$\int_{{}^{t+\Delta t}V} {}^{t+\Delta t}\rho \; {}^{t+\Delta t}\ddot{u}_i \, \delta u_i \; {}^{t+\Delta t}dV = \underbrace{\int_{{}^{0}V} {}^{0}\rho \; {}^{t+\Delta t}\ddot{u}_i \, \delta u_i \; {}^{0}dV}_{\text{may be evaluated at time 0}}$$

Transparency 6-6

If the external loads are deformation-dependent,

$$\int_{{}^{t+\Delta t}V} {}^{t+\Delta t}f_i^B \, \delta u_i \; {}^{t+\Delta t}dV \doteq \int_{{}^{t+\Delta t}V^{(k-1)}} {}^{t+\Delta t}f_i^{B(k-1)} \, \delta u_i \; {}^{t+\Delta t}dV$$

and

$$\int_{{}^{t+\Delta t}S} {}^{t+\Delta t}f_i^S \, \delta u_i^S \; {}^{t+\Delta t}dS \doteq \int_{{}^{t+\Delta t}S^{(k-1)}} {}^{t+\Delta t}f_i^{S(k-1)} \, \delta u_i^S \; {}^{t+\Delta t}dS$$

Transparency 6-7

Materially-nonlinear-only analysis:

$$\int_V C_{ijrs}\, \Delta e_{rs}^{(k)}\, \delta e_{ij}\, dV = {}^{t+\Delta t}\mathcal{R} - \int_V {}^{t+\Delta t}\sigma_{ij}^{(k-1)}\, \delta e_{ij}\, dV$$

This equation is obtained from the governing T.L. and U.L. equations by realizing that, neglecting geometric nonlinearities,

$$ {}^{t+\Delta t}_{0}S_{ij} \equiv {}^{t+\Delta t}\tau_{ij} \equiv \underbrace{{}^{t+\Delta t}\sigma_{ij}}_{\text{physical stress}}$$

Transparency 6-8

Dynamic analysis:

Implicit time integration:

$$ {}^{t+\Delta t}\mathcal{R} = {}^{t+\Delta t}\mathcal{R}_{\substack{\text{external}\\ \text{loads}}} - \int_{{}^0V} {}^0\rho\; {}^{t+\Delta t}\ddot{u}_i\, \delta u_i\, {}^0dV$$

Explicit time integration:

T.L. $\int_{{}^0V} {}^t_0S_{ij}\, \delta\, {}^t_0\epsilon_{ij}\, {}^0dV = {}^t\mathcal{R}$

U.L. $\int_{{}^tV} {}^t\tau_{ij}\, \delta_t e_{ij}\, {}^tdV = {}^t\mathcal{R}$

M.N.O. $\int_V {}^t\sigma_{ij}\, \delta e_{ij}\, dV = {}^t\mathcal{R}$

Transparency 6-9

The finite element equations corresponding to the continuum mechanics equations are

Materially-nonlinear-only analysis:

Static analysis:

$${}^{t}\underline{K}\,\Delta\underline{U}^{(i)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)} \qquad (6.55)$$

Dynamic analysis, implicit time integration:

$$\underline{M}\,{}^{t+\Delta t}\ddot{\underline{U}}^{(i)} + {}^{t}\underline{K}\,\Delta\underline{U}^{(i)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)} \qquad (6.56)$$

Dynamic analysis, explicit time integration:

$$\underline{M}\,{}^{t}\ddot{\underline{U}} = {}^{t}\underline{R} - {}^{t}\underline{F} \qquad (6.57)$$

Transparency 6-10

Total Lagrangian formulation:

Static analysis:

$$({}^{t}_{0}\underline{K}_{L} + {}^{t}_{0}\underline{K}_{NL})\,\Delta\underline{U}^{(i)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}_{0}\underline{F}^{(i-1)}$$

Dynamic analysis, implicit time integration:

$$\underline{M}\,{}^{t+\Delta t}\ddot{\underline{U}}^{(i)} + ({}^{t}_{0}\underline{K}_{L} + {}^{t}_{0}\underline{K}_{NL})\,\Delta\underline{U}^{(i)}$$
$$= {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}_{0}\underline{F}^{(i-1)}$$

Dynamic analysis, explicit time integration:

$$\underline{M}\,{}^{t}\ddot{\underline{U}} = {}^{t}\underline{R} - {}^{t}_{0}\underline{F}$$

Transparency 6-11

Updated Lagrangian formulation:

Static analysis:

$$({}^{t}_{t}\underline{K}_{L} + {}^{t}_{t}\underline{K}_{NL})\,\Delta\underline{U}^{(i)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}_{t+\Delta t}\underline{F}^{(i-1)}$$

Dynamic analysis, implicit time integration:

$$\underline{M}\,{}^{t+\Delta t}\underline{\ddot{U}}^{(i)} + ({}^{t}_{t}\underline{K}_{L} + {}^{t}_{t}\underline{K}_{NL})\,\Delta\underline{U}^{(i)}$$
$$= {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}_{t+\Delta t}\underline{F}^{(i-1)}$$

Dynamic analysis, explicit time integration:

$$\underline{M}\,{}^{t}\underline{\ddot{U}} = {}^{t}\underline{R} - {}^{t}_{t}\underline{F}$$

Transparency 6-12

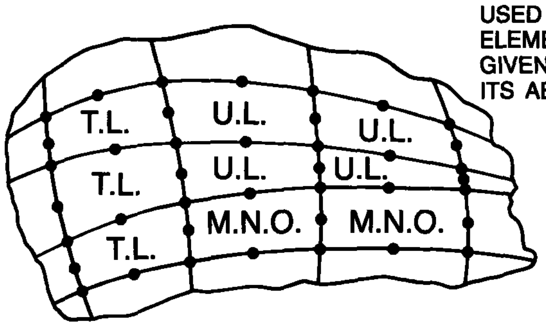
The above expressions are valid for

- a single finite element ($\underline{U}$ contains the element nodal point displacements)
- an assemblage of elements ($\underline{U}$ contains all nodal point displacements)

In practice, element matrices are calculated and then assembled into the global matrices using the direct stiffness method.

Transparency 6-13

Considering an assemblage of elements, we will see that different formulations may be used in the same analysis:



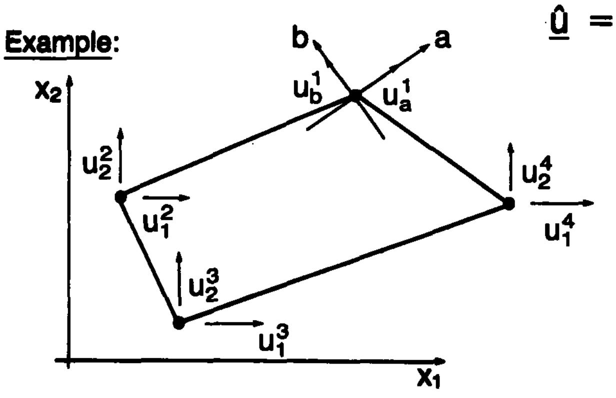
THE FORMULATION
USED FOR EACH
ELEMENT IS
GIVEN BY
ITS ABBREVIATION

Transparency 6-14

We now concentrate on a single element.

The vector $\hat{\underline{u}}$ contains the element incremental nodal point displacements

Example:

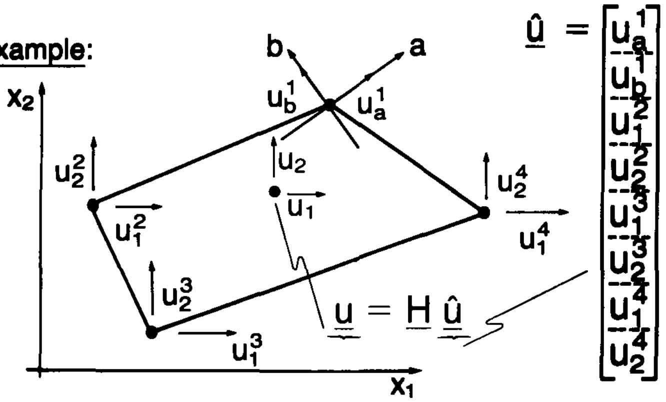


$$\hat{\underline{u}} = \begin{bmatrix} u_a^1 \\ \hdashline u_b^1 \\ \hdashline u_1^2 \\ \hdashline u_2^2 \\ \hdashline u_1^3 \\ \hdashline u_2^3 \\ \hdashline u_1^4 \\ \hdashline u_2^4 \end{bmatrix}$$

Transparency 6-15

We may write the displacements at any point in the element in terms of the element nodal displacements:

Example:



Transparency 6-16

Finite element discretization of governing continuum mechanics equations:

For all analysis types:

$$\int_V {}^0\rho\ {}^{t+\Delta t}\ddot{u}_i\ \delta u_i\ {}^0dV \rightarrow \delta\underline{\hat{u}}^T \left(\underbrace{{}^0\rho \int_{{}^0V} \underline{H}^T\, \underline{H}\ {}^0dV}_{\underline{M}}\right) {}^{t+\Delta t}\ddot{\underline{\hat{u}}}$$

where we used $\underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{\text{displacements at a point within the element}} = \underline{H}\,\underline{\hat{u}}$

Transparency 6-17

and

$${}^{t+\Delta t}\mathcal{R} = \int_{{}^0V} {}^{t+\Delta t}_{\ \ 0}f_i^B \, \delta u_i \, {}^0dV + \int_{{}^0S} {}^{t+\Delta t}_{\ \ 0}f_i^S \, \delta u_i^S \, {}^0dS$$

$$\Big\downarrow$$

$$\delta \underline{\hat{u}}^T \underbrace{\left(\int_{{}^0V} \underline{H}^T \, {}^{t+\Delta t}_{\ \ 0}\underline{f}^B \, {}^0dV + \int_{{}^0S} \underline{H}^{ST} \, {}^{t+\Delta t}_{\ \ 0}\underline{f}^S \, {}^0dS \right)}_{{}^{t+\Delta t}\underline{R}}$$

where $\left.\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right|_{\text{on } S} = \underline{H}^S \, \underline{\hat{u}}$

Transparency 6-18

Materially-nonlinear-only analysis:

Considering an incremental displacement u_i,

$$\int_V C_{ijrs} \, e_{rs} \, \delta e_{ij} \, dV \rightarrow \delta \underline{\hat{u}}^T \underbrace{\left(\int_V \underline{B}_L^T \, \underline{C} \, \underline{B}_L \, dV \right)}_{{}^t\underline{K}} \underline{\hat{u}}$$

where

$$\underline{e} = \underline{B}_L \underline{\hat{u}}$$

a vector containing components of e_{ij}

Example: Two-dimensional plane stress element:

$$\underline{e} = \begin{bmatrix} e_{11} \\ e_{22} \\ 2\,e_{12} \end{bmatrix}$$

Transparency 6-19

and

$$\int_V {}^t\sigma_{ij}\,\delta e_{ij}\,dV \rightarrow \delta\hat{\underline{u}}^T \underbrace{\left(\int_V \underline{B}_L^T\,{}^t\hat{\underline{\Sigma}}\,dV\right)}_{{}^t\underline{F}}$$

where ${}^t\hat{\underline{\Sigma}}$ is a vector containing components of ${}^t\sigma_{ij}$.

Example: Two-dimensional plane stress element:

$${}^t\hat{\underline{\Sigma}} = \begin{bmatrix} {}^t\sigma_{11} \\ {}^t\sigma_{22} \\ {}^t\sigma_{12} \end{bmatrix}$$

Transparency 6-20

Total Lagrangian formulation:

Considering an incremental displacement u_i,

$$\int_{{}^0V} {}_0C_{ijrs}\,{}_0e_{rs}\,\delta_0 e_{ij}\,{}^0dV \rightarrow \delta\hat{\underline{u}}^T \underbrace{\left(\int_{{}^0V} {}^t_0\underline{B}_L^T\,{}_0\underline{C}\,{}^t_0\underline{B}_L\,{}^0dV\right)}_{{}^t_0\underline{K}_L} \hat{\underline{u}}$$

where

$$\underbrace{{}_0\underline{e}} = {}^t_0\underline{B}_L\,\hat{\underline{u}}$$

a vector containing
components of ${}_0e_{ij}$

Transparency 6-21

$$\int_{^0V} {}_0^tS_{ij}\,\delta_0\eta_{ij}\,{}^0dV \rightarrow \delta\underline{\hat{u}}^T \underbrace{\left(\int_{^0V} {}_0^t\underline{B}_{NL}^T\,{}_0^t\underline{S}\,{}_0^t\underline{B}_{NL}\,{}^0dV\right)}_{{}_0^t\underline{K}_{NL}} \underline{\hat{u}}$$

where

${}_0^t\underline{S}$ is a matrix containing components of ${}_0^tS_{ij}$

${}_0^t\underline{B}_{NL}\,\underline{\hat{u}}$ contains components of ${}_0u_{i,j}$

Transparency 6-22

and

$$\int_{^0V} {}_0^tS_{ij}\,\delta_0e_{ij}\,{}^0dV \rightarrow \delta\underline{\hat{u}}^T \underbrace{\left(\int_{^0V} {}_0^t\underline{B}_L^T\,{}_0^t\underline{\hat{S}}\,{}^0dV\right)}_{{}_0^t\underline{F}}$$

where ${}_0^t\underline{\hat{S}}$ is a vector containing components of ${}_0^tS_{ij}$.

Transparency 6-23

Updated Lagrangian formulation:

Considering an incremental displacement u_i,

$$\int_{^tV} {}_tC_{ijrs}\; {}_te_{rs}\; \delta_t e_{ij}\; {}^tdV \rightarrow \delta\underline{\hat{u}}^T \underbrace{\left(\int_{^tV} {}^t_t\underline{B}_L^T\; {}_t\underline{C}\; {}^t_t\underline{B}_L\; {}^tdV\right)}_{{}^t_t\underline{K}_L} \underline{\hat{u}}$$

where

$$\underbrace{{}_t\underline{e}}_{} = {}^t_t\underline{B}_L\; \underline{\hat{u}}$$

a vector containing components of ${}_te_{ij}$

Transparency 6-24

$$\int_{^tV} {}^t\tau_{ij}\; \delta_t\eta_{ij}\; {}^tdV \rightarrow \delta\underline{\hat{u}}^T \underbrace{\left(\int_{^tV} {}^t_t\underline{B}_{NL}^T\; {}^t\underline{\tau}\; {}^t_t\underline{B}_{NL}\; {}^tdV\right)}_{{}^t_t\underline{K}_{NL}} \underline{\hat{u}}$$

where

${}^t\underline{\tau}$ is a matrix containing components of ${}^t\tau_{ij}$

${}^t_t\underline{B}_{NL}\; \underline{\hat{u}}$ contains components of ${}_tu_{i,j}$

Transparency 6-25

and

$$\int_{{}^tV} {}^t\tau_{ij}\, \delta_t e_{ij}\, {}^t dV \rightarrow \delta \hat{\underline{u}}^T \underbrace{\left(\int_{{}^tV} {}^t_t\underline{B}_L^T\, {}^t\hat{\underline{\tau}}\, {}^t dV\right)}_{{}^t_t\underline{F}}$$

where ${}^t\hat{\underline{\tau}}$ is a vector containing components of ${}^t\tau_{ij}$

Transparency 6-26

- The finite element stiffness and mass matrices and force vectors are evaluated using numerical integration (as in linear analysis).
- In isoparametric finite element analysis we have, schematically, in 2-D analysis

$$\underline{K} = \int_{-1}^{+1} \int_{-1}^{+1} \underbrace{\underline{B}^T\, \underline{C}\, \underline{B} \det \underline{J}}_{\underline{G}}\, dr\, ds$$

$$\underline{K} \doteq \sum_i \sum_j \alpha_{ij} \underline{G}_{ij}$$

Transparency 6-27

And similarly

$$\underline{F} = \int_{-1}^{+1}\int_{-1}^{+1} \underbrace{\underline{B}^T \hat{\underline{T}} \det \underline{J}}_{\underline{G}}\, dr\, ds$$

$$\underline{F} \doteq \sum_i \sum_j \alpha_{ij} \underline{G}_{ij}$$

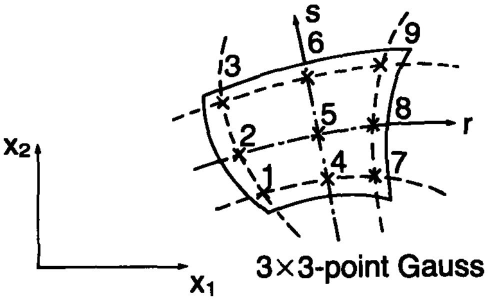
$$\underline{M} = \int_{-1}^{+1}\int_{-1}^{+1} \underbrace{\rho \underline{H}^T\, \underline{H} \det \underline{J}}_{G}\, dr\, ds$$

$$\underline{M} \doteq \sum_i \sum_j \alpha_{ij} \underline{G}_{ij}$$

Transparency 6-28

Frequently used is Gauss integration:

Example: 2-D analysis



r, s values:
±0.7745...
0.0

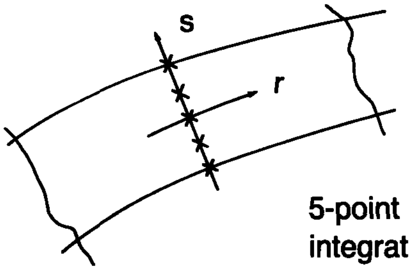
3×3-point Gauss integration

All integration points are in the interior of the element.

Transparency 6-29

Also used is Newton-Cotes integration:

Example: shell element



5-point Newton-Cotes integration in s-direction

Integration points are on the boundary and the interior of the element.

Transparency 6-30

Gauss versus Newton-Cotes Integration:

- Use of n Gauss points integrates a polynomial of order 2n-1 exactly, whereas use of n Newton-Cotes points integrates only a polynomial of n-1 exactly.
  Hence, for analysis of solids we generally use Gauss integration.
- Newton-Cotes integration involves points on the boundaries.
  Hence, Newton-Cotes integration may be effective for structural elements.

Transparency 6-31

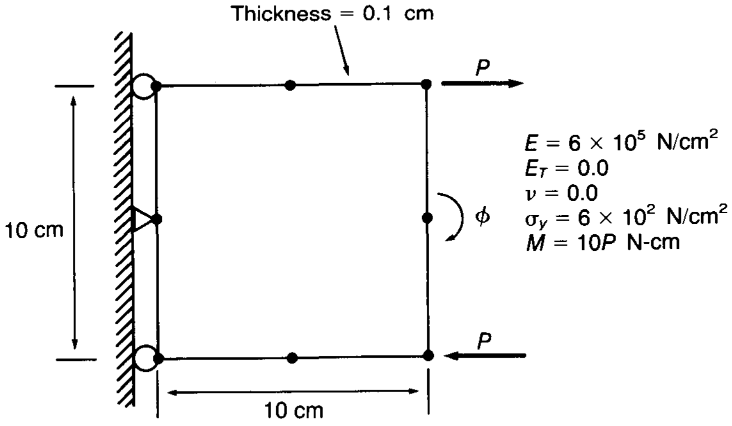
In principle, the integration schemes are employed as in linear analysis:

- The integration order must be high enough not to have spurious zero energy modes in the elements.
- The appropriate integration order may, in nonlinear analysis, be higher than in linear analysis (for example, to model more accurately the spread of plasticity). On the other hand, too high an order of integration is also not effective; instead, more elements should be used.

Transparency 6-32

Example: Test of effect of integration order

Finite element model considered:



$E = 6 \times 10^5$ N/cm^2

$E_T = 0.0$

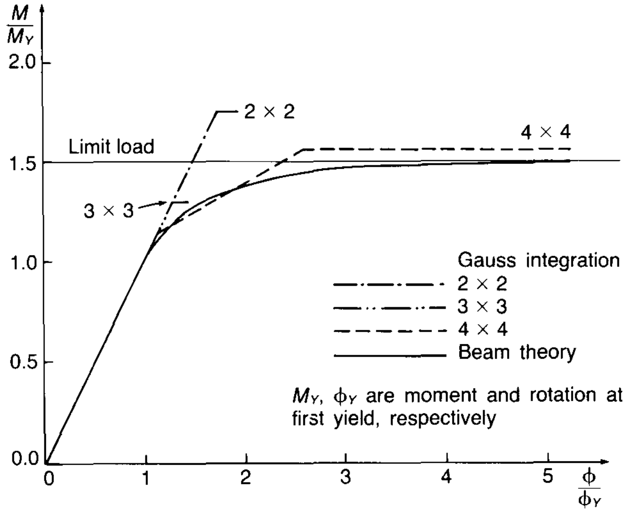
$\nu = 0.0$

$\sigma_y = 6 \times 10^2$ N/cm^2

$M = 10P$ N-cm

Transparency 6-33

Calculated response:

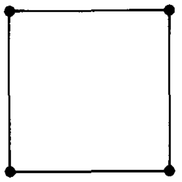


M_Y, ϕ_Y are moment and rotation at first yield, respectively

Transparency 6-34

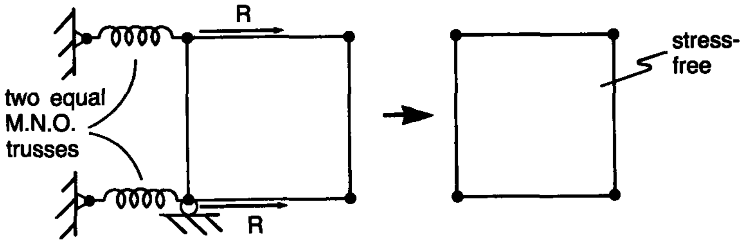
Problem: Design numerical experiments which test the ability of a finite element to correctly model large rigid body translations and large rigid body rotations.

— Consider a single two-dimensional square 4-node finite element:



plane stress
or plane strain

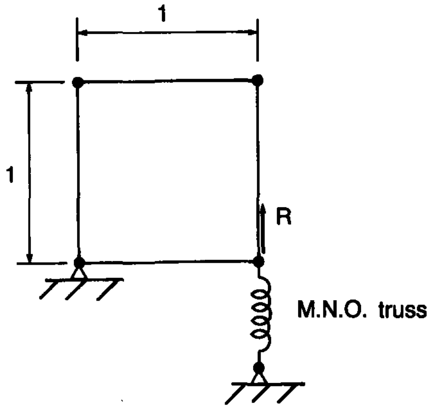
Numerical experiment to test whether a 4-node element can model a large rigid body translation:



This result will be obtained if any of the finite element formulations discussed (T.L., U.L., M.N.O. or linear) is used.

Transparency 6-35

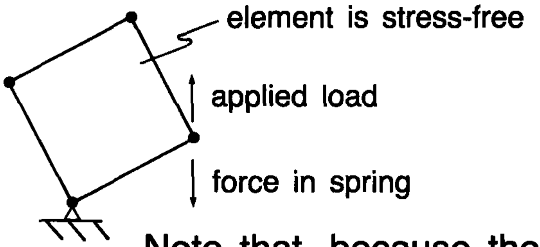
Numerical experiment to test whether a 4-node element can model a large rigid body rotation:



Transparency 6-36

Transparency 6-37

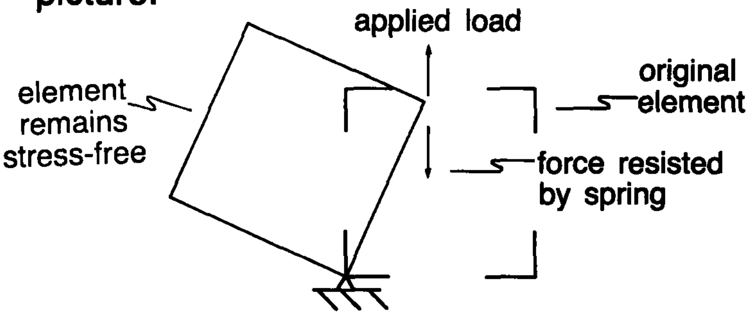
When the load is applied, the element should rotate as a rigid body. The load should be transmitted entirely through the truss.



Note that, because the spring is modeled using an M.N.O. truss element, the force transmitted by the truss is always vertical.

Transparency 6-38

After the load is applied, the element should look as shown in the following picture.



This result will be obtained if the T.L. or U.L. formulations are used to model the 2-D element.

Topic 7

Two- and Three-Dimensional Solid Elements; Plane Stress, Plane Strain, and Axisymmetric Conditions

Contents:

- **Isoparametric interpolations of coordinates and displacements**
- **Consistency between coordinate and displacement interpolations**
- **Meaning of these interpolations in large displacement analysis, motion of a material particle**
- **Evaluation of required derivatives**
- **The Jacobian transformations**
- **Details of strain-displacement matrices for total and updated Lagrangian formulations**
- **Example of 4-node two-dimensional element, details of matrices used**

Textbook: Sections 6.3.2, 6.3.3

Example: 6.17

- FINITE ELEMENTS CAN IN GENERAL BE CATEGORIZED AS

- CONTINUUM ELEMENTS (SOLID)
- STRUCTURAL ELEMENTS

IN THIS LECTURE

- WE CONSIDER THE 2-D CONTINUUM ISOPARAMETRIC ELEMENTS
- THESE ELEMENTS ARE USED VERY WIDELY
- THE ELEMENTS ARE VERY GENERAL ELEMENTS FOR GEOMETRIC AND MATERIAL NONLINEAR CONDITIONS
- WE ALSO POINT OUT HOW GENERAL 3-D ELEMENTS ARE CALCULATED USING THE SAME PROCEDURES

Markerboard
7-1

Transparency 7-1

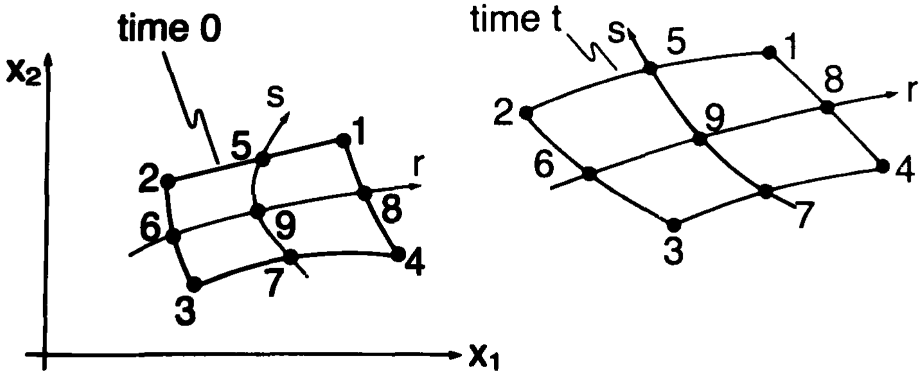
TWO- AND THREE-DIMENSIONAL SOLID ELEMENTS

- Two-dimensional elements comprise
  - — plane stress and plane strain elements
  - — axisymmetric elements
- The derivations used for the two-dimensional elements can be easily extended to the derivation of three-dimensional elements.

  Hence we concentrate our discussion now first on the two-dimensional elements.

Transparency 7-2

TWO-DIMENSIONAL AXISYMMETRIC, PLANE STRAIN AND PLANE STRESS ELEMENTS



Because the elements are isoparametric,

$$ {}^{0}x_1 = \sum_{k=1}^{N} h_k \, {}^{0}x_1^k \;,\quad {}^{0}x_2 = \sum_{k=1}^{N} h_k \, {}^{0}x_2^k $$

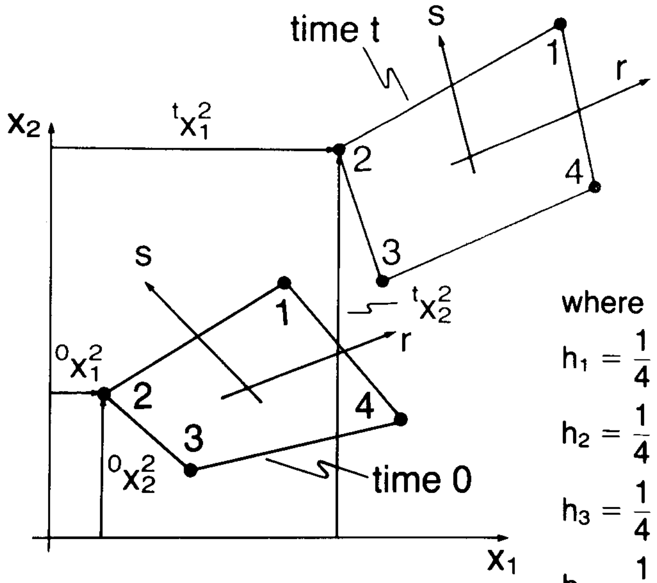
and

$$ {}^{t}x_1 = \sum_{k=1}^{N} h_k \, {}^{t}x_1^k \;,\quad {}^{t}x_2 = \sum_{k=1}^{N} h_k \, {}^{t}x_2^k $$

where the h_k's are the isoparametric interpolation functions.

Transparency 7-3

Example: A four-node element



$$ {}^{t}x_i = \sum_{k=1}^{4} h_k \, {}^{t}x_i^k $$

$$ {}^{0}x_i = \sum_{k=1}^{4} h_k \, {}^{0}x_i^k $$

where

$$ h_1 = \frac{1}{4}(1 + r)(1 + s) $$

$$ h_2 = \frac{1}{4}(1 - r)(1 + s) $$

$$ h_3 = \frac{1}{4}(1 - r)(1 - s) $$

$$ h_4 = \frac{1}{4}(1 + r)(1 - s) $$

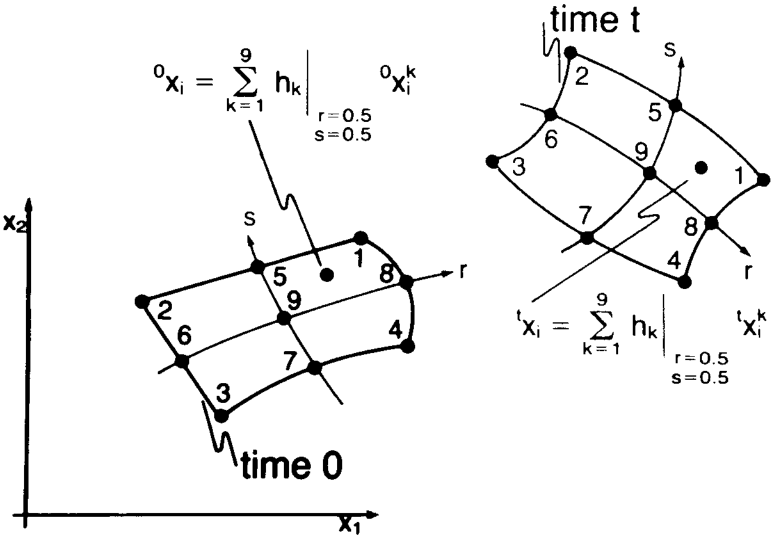
Transparency 7-4

Transparency 7-5

Example: Motion of a material particle

Consider the material particle at r = 0.5, s = 0.5:

Important: The isoparametric coordinates of a material particle never change



$${}^0x_i = \sum_{k=1}^{9} h_k \Big|_{\substack{r=0.5\\ s=0.5}} \; {}^0x_i^k$$

$${}^tx_i = \sum_{k=1}^{9} h_k \Big|_{\substack{r=0.5\\ s=0.5}} \; {}^tx_i^k$$

Transparency 7-6

A major advantage of the isoparametric finite element discretization is that we may directly write

$${}^tu_1 = \sum_{k=1}^{N} h_k \, {}^tu_1^k \quad , \quad {}^tu_2 = \sum_{k=1}^{N} h_k \, {}^tu_2^k$$

and

$$u_1 = \sum_{k=1}^{N} h_k \, u_1^k \quad , \quad u_2 = \sum_{k=1}^{N} h_k \, u_2^k$$

Transparency 7-7

This is easily shown: for example,

$$ {}^{t}x_i = \sum_{k=1}^{N} h_k \, {}^{t}x_i^k $$

$$ {}^{0}x_i = \sum_{k=1}^{N} h_k \, {}^{0}x_i^k $$

Subtracting the second equation from the first equation gives

$$ \underbrace{{}^{t}x_i - {}^{0}x_i}_{{}^{t}u_i} = \sum_{k=1}^{N} h_k \underbrace{({}^{t}x_i^k - {}^{0}x_i^k)}_{{}^{t}u_i^k} $$

Transparency 7-8

The element matrices require the following derivatives:

$$ {}_{0}^{t}u_{i,j} = \frac{\partial\, {}^{t}u_i}{\partial\, {}^{0}x_j} = \sum_{k=1}^{N} \left(\frac{\partial h_k}{\partial\, {}^{0}x_j} \right) {}^{t}u_i^k $$

$$ {}_{0}u_{i,j} = \frac{\partial u_i}{\partial\, {}^{0}x_j} = \sum_{k=1}^{N} \left(\frac{\partial h_k}{\partial\, {}^{0}x_j} \right) u_i^k $$

$$ {}_{t}u_{i,j} = \frac{\partial u_i}{\partial\, {}^{t}x_j} = \sum_{k=1}^{N} \left(\frac{\partial h_k}{\partial\, {}^{t}x_j} \right) u_i^k $$

Transparency 7-9

These derivatives are evaluated using a Jacobian transformation (the chain rule):

$$\frac{\partial h_k}{\partial r} = \boxed{\frac{\partial h_k}{\partial {}^0x_1}} \frac{\partial {}^0x_1}{\partial r} + \boxed{\frac{\partial h_k}{\partial {}^0x_2}} \frac{\partial {}^0x_2}{\partial r}$$

$$\frac{\partial h_k}{\partial s} = \boxed{\frac{\partial h_k}{\partial {}^0x_1}} \frac{\partial {}^0x_1}{\partial s} + \boxed{\frac{\partial h_k}{\partial {}^0x_2}} \frac{\partial {}^0x_2}{\partial s}$$

REQUIRED DERIVATIVES

In matrix form,

$$\begin{bmatrix} \dfrac{\partial h_k}{\partial r} \\ \dfrac{\partial h_k}{\partial s} \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial {}^0x_1}{\partial r} & \dfrac{\partial {}^0x_2}{\partial r} \\ \dfrac{\partial {}^0x_1}{\partial s} & \dfrac{\partial {}^0x_2}{\partial s} \end{bmatrix}}_{{}^0\underline{J}} \begin{bmatrix} \dfrac{\partial h_k}{\partial {}^0x_1} \\ \dfrac{\partial h_k}{\partial {}^0x_2} \end{bmatrix}$$

Transparency 7-10

The required derivatives are computed using a matrix inversion:

$$\begin{bmatrix} \dfrac{\partial h_k}{\partial {}^0x_1} \\ \dfrac{\partial h_k}{\partial {}^0x_2} \end{bmatrix} = {}^0\underline{J}^{-1} \begin{bmatrix} \dfrac{\partial h_k}{\partial r} \\ \dfrac{\partial h_k}{\partial s} \end{bmatrix}$$

The entries in ${}^0\underline{J}$ are computed using the interpolation functions. For example,

$$\frac{\partial {}^0x_1}{\partial r} = \sum_{k=1}^{N} \frac{\partial h_k}{\partial r} \, {}^0x_1^k$$

Transparency 7-11

The derivatives taken with respect to the configuration at time t can also be evaluated using a Jacobian transformation.

$$\begin{bmatrix}\frac{\partial h_k}{\partial r}\\[4pt] \frac{\partial h_k}{\partial s}\end{bmatrix} = \underbrace{\begin{bmatrix}\frac{\partial\,{}^tx_1}{\partial r} & \frac{\partial\,{}^tx_2}{\partial r}\\[4pt] \frac{\partial\,{}^tx_1}{\partial s} & \boxed{\frac{\partial\,{}^tx_2}{\partial s}}\end{bmatrix}}_{{}^t\underline{J}} \begin{bmatrix}\frac{\partial h_k}{\partial\,{}^tx_1}\\[4pt] \frac{\partial h_k}{\partial\,{}^tx_2}\end{bmatrix}$$

$$\sum_{k=1}^{N} \frac{\partial h_k}{\partial s}\,{}^tx_2^k$$

$$\begin{bmatrix}\frac{\partial h_k}{\partial\,{}^tx_1}\\[4pt] \frac{\partial h_k}{\partial\,{}^tx_2}\end{bmatrix} = {}^t\underline{J}^{-1} \begin{bmatrix}\frac{\partial h_k}{\partial r}\\[4pt] \frac{\partial h_k}{\partial s}\end{bmatrix}$$

Transparency 7-12

We can now compute the required element matrices for the total Lagrangian formulation:

Element Matrix	Matrices Required
${}^t_0\underline{K}_L$	${}_0\underline{C}$, ${}^t_0\underline{B}_L$
${}^t_0\underline{K}_{NL}$	${}^t_0\underline{S}$, ${}^t_0\underline{B}_{NL}$
${}^t_0\underline{F}$	${}^t_0\hat{\underline{S}}$, ${}^t_0\underline{B}_L$

Transparency 7-13

We define ${}_0\underline{C}$ so that

$$\begin{bmatrix} {}_0S_{11} \\ {}_0S_{22} \\ {}_0S_{12} \\ {}_0S_{33} \end{bmatrix} = {}_0\underline{C} \begin{bmatrix} {}_0e_{11} \\ {}_0e_{22} \\ 2\,{}_0e_{12} \\ {}_0e_{33} \end{bmatrix}$$

analogous to
${}_0S_{ij} = {}_0C_{ijrs}\,{}_0e_{rs}$

For example, we may choose (axisymmetric analysis),

$$ {}_0\underline{C} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 & \frac{\nu}{1-\nu} \\ \frac{\nu}{1-\nu} & 1 & 0 & \frac{\nu}{1-\nu} \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 1 \end{bmatrix}$$

Transparency 7-14

We note that, in two-dimensional analysis,

$$ {}_0e_{11} = {}_0u_{1,1} + \underbrace{{}^t_0u_{1,1}\;{}_0u_{1,1} + {}^t_0u_{2,1}\;{}_0u_{2,1}} $$

$$ {}_0e_{22} = {}_0u_{2,2} + \underbrace{{}^t_0u_{1,2}\;{}_0u_{1,2} + {}^t_0u_{2,2}\;{}_0u_{2,2}} $$

$$ 2\,{}_0e_{12} = ({}_0u_{1,2} + {}_0u_{2,1}) + \underbrace{({}^t_0u_{1,1}\,{}_0u_{1,2} + {}^t_0u_{2,1}\,{}_0u_{2,2} + {}^t_0u_{1,2}\,{}_0u_{1,1} + {}^t_0u_{2,2}\,{}_0u_{2,1})} $$

$$ {}_0e_{33} = \frac{u_1}{{}^0x_1} + \underbrace{\left(\frac{{}^tu_1}{{}^0x_1}\right)\frac{u_1}{{}^0x_1}} $$

INITIAL DISPLACEMENT EFFECT

Transparency 7-15

and

$${}_0\eta_{11} = \frac{1}{2}\left(({}_0u_{1,1})^2 + ({}_0u_{2,1})^2\right)$$

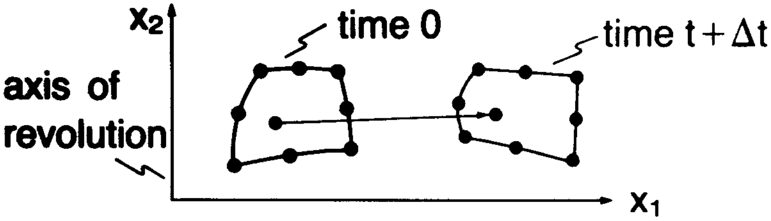
$${}_0\eta_{22} = \frac{1}{2}\left(({}_0u_{1,2})^2 + ({}_0u_{2,2})^2\right)$$

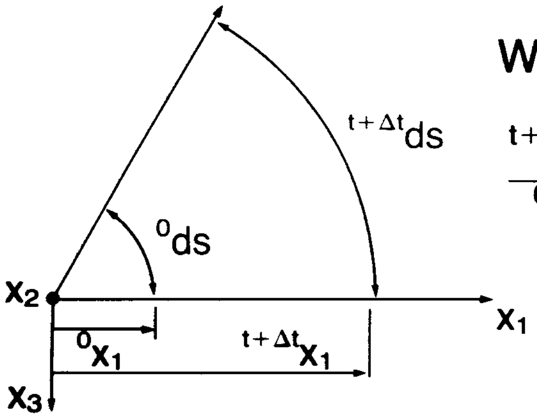
$${}_0\eta_{12} = {}_0\eta_{21} = \frac{1}{2}\left({}_0u_{1,1}\;{}_0u_{1,2} + {}_0u_{2,1}\;{}_0u_{2,2}\right)$$

$${}_0\eta_{33} = \frac{1}{2}\left(\frac{u_1}{{}^0x_1}\right)^2$$

Transparency 7-16

Derivation of ${}_0e_{33}$, ${}_0\eta_{33}$:





We see that

$$\frac{{}^{t+\Delta t}ds}{{}^0ds} = \frac{{}^{t+\Delta t}x_1}{{}^0x_1}$$

Transparency 7-17

Hence $$^{t+\Delta t}_{\ \ 0}\varepsilon_{33} = \frac{1}{2}\left[\left(\frac{^{t+\Delta t}ds}{^{0}ds}\right)^2 - 1\right]$$

$$= \frac{1}{2}\left[\left(\frac{^{t+\Delta t}x_1}{^{0}x_1}\right)^2 - 1\right]$$

$$= \frac{1}{2}\left[\left(\frac{^{0}x_1 + {^{t}u_1} + u_1}{^{0}x_1}\right)^2 - 1\right]$$

$$\vdots$$

$$= \underbrace{\left(\frac{^{t}u_1}{^{0}x_1} + \frac{1}{2}\left(\frac{^{t}u_1}{^{0}x_1}\right)^2\right)}_{^{t}_{0}\varepsilon_{33}} + \underbrace{\left(\underbrace{\frac{u_1}{^{0}x_1} + \left(\frac{^{t}u_1}{^{0}x_1}\right)\frac{u_1}{^{0}x_1}}_{_{0}e_{33}} + \underbrace{\frac{1}{2}\left(\frac{u_1}{^{0}x_1}\right)^2}_{_{0}\eta_{33}}\right)}_{_{0}\varepsilon_{33}}$$

Transparency 7-18

We construct $^{t}_{0}\underline{B}_L$ so that

$$\begin{bmatrix} _{0}e_{11} \\ _{0}e_{22} \\ 2\ _{0}e_{12} \\ \hline _{0}e_{33} \end{bmatrix} = {_{0}\underline{e}} = \underbrace{\left({^{t}_{0}\underline{B}_{L0}} + {^{t}_{0}\underline{B}_{L1}}\right)}_{^{t}_{0}\underline{B}_L}\ \underline{\hat{u}}$$

$_{0}e_{33}$ is only included for axisymmetric analysis

contains initial displacement effect

Transparency 7-19

Entries in ${}^t_0\underline{B}_{L0}$:

$$
\begin{array}{c}
\\ \\
\left[\begin{array}{c|c:c|c}
 & \multicolumn{2}{c|}{\text{node } k} & \\
 & u_1^k & u_2^k & \\
 & {}_0h_{k,1} & 0 & \\
\cdots & 0 & {}_0h_{k,2} & \cdots \\
 & {}_0h_{k,2} & {}_0h_{k,1} & \\
\hline
 & h_k/{}^0x_1 & 0 &
\end{array}\right]
\end{array}
\left[\begin{array}{c}
\vdots \\
\hline
u_1^k \\
\hdashline
u_2^k \\
\hline
\vdots
\end{array}\right]
\begin{array}{c}
\uparrow \\
\text{node } k \\
\downarrow
\end{array}
$$

included only for axisymmetric analysis

This is similar in form to the $\underline{B}$ matrix used in linear analysis.

Transparency 7-20

Entries in ${}^t_0\underline{B}_{L1}$:

$$
\left[\begin{array}{c|l:l|c}
 & \multicolumn{2}{c|}{\text{node } k} & \\
 & \multicolumn{1}{c:}{u_1^k} & \multicolumn{1}{c|}{u_2^k} & \\
 & {}^t_0u_{1,1}\ {}_0h_{k,1} & {}^t_0u_{2,1}\ {}_0h_{k,1} & \\
\cdots & {}^t_0u_{1,2}\ {}_0h_{k,2} & {}^t_0u_{2,2}\ {}_0h_{k,2} & \cdots \\
 & {}^t_0u_{1,1}\ {}_0h_{k,2} & {}^t_0u_{2,1}\ {}_0h_{k,2} & \\
 & \quad + {}^t_0u_{1,2}\ {}_0h_{k,1} & \quad + {}^t_0u_{2,2}\ {}_0h_{k,1} & \\
\hline
 & \frac{{}^tu_1}{{}^0x_1}\ \frac{h_k}{{}^0x_1} & \qquad 0 &
\end{array}\right]
$$

included only for axisymmetric analysis

The initial displacement effect is contained in the terms ${}^t_0u_{i,j}$, $\frac{{}^tu_1}{{}^0x_1}$.

Transparency 7-21

We construct ${}^t_0\underline{B}_{NL}$ and ${}^t_0\underline{S}$ so that

$$\delta\underline{\hat{u}}^T\; {}^t_0\underline{B}_{NL}^T\; {}^t_0\underline{S}\; {}^t_0\underline{B}_{NL}\; \underline{\hat{u}} = {}^t_0S_{ij}\, \delta_0\eta_{ij}$$

Entries in ${}^t_0\underline{S}$:

$$\left[\begin{array}{cccc|c}
{}^t_0S_{11} & {}^t_0S_{12} & 0 & 0 & 0 \\
{}^t_0S_{21} & {}^t_0S_{22} & 0 & 0 & 0 \\
0 & 0 & {}^t_0S_{11} & {}^t_0S_{12} & 0 \\
0 & 0 & {}^t_0S_{21} & {}^t_0S_{22} & 0 \\
\hline
0 & 0 & 0 & 0 & {}^t_0S_{33}
\end{array}\right]$$

included only for axisymmetric analysis

Transparency 7-22

Entries in ${}^t_0\underline{B}_{NL}$:

$$\begin{array}{c|c|c|c}
 & \multicolumn{2}{c|}{\xleftrightarrow{\text{node } k}} & \\
 & \xleftrightarrow{u_1^k} & \xleftrightarrow{u_2^k} & \\
\cdots & {}_0h_{k,1} & 0 & \cdots \\
 & {}_0h_{k,2} & 0 & \\
 & 0 & {}_0h_{k,1} & \\
 & 0 & {}_0h_{k,2} & \\
\hline
 & h_k/{}^0x_1 & 0 & \\
\hline
\end{array}
\quad
\left[\begin{array}{c}
\vdots \\ \hline u_1^k \\ u_2^k \\ \hline \vdots
\end{array}\right]
\quad \updownarrow \text{node } k$$

included only for axisymmetric analysis

Transparency 7-23

${}^t_0\hat{\underline{S}}$ is constructed so that

$$\delta\hat{\underline{u}}^T\ {}^t_0\underline{B}_L^T\ {}^t_0\hat{\underline{S}} = {}^t_0S_{ij}\,\delta_0e_{ij}$$

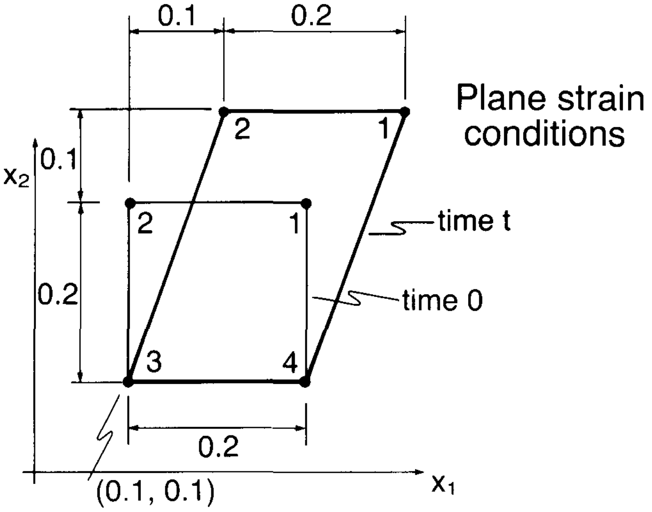
Entries in ${}^t_0\hat{\underline{S}}$:

$$\begin{bmatrix} {}^t_0S_{11} \\ {}^t_0S_{22} \\ {}^t_0S_{12} \\ \hline {}^t_0S_{33} \end{bmatrix}$$

included only for axisymmetric analysis

Transparency 7-24

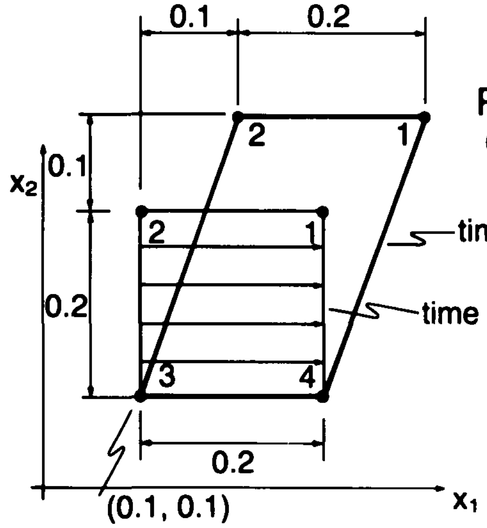
Example: Calculation of ${}^t_0\underline{B}_L$, ${}^t_0\underline{B}_{NL}$



Plane strain conditions

Transparency
7-25

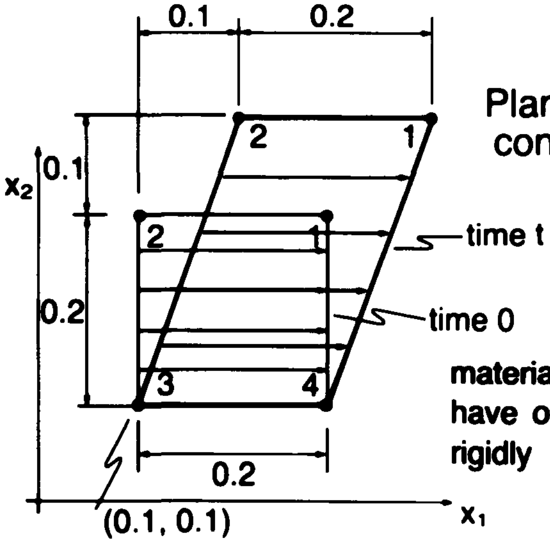
Example: Calculation of ${}^t_0\underline{B}_L$, ${}^t_0\underline{B}_{NL}$



Plane strain
conditions

Transparency
7-26

Example: Calculation of ${}^t_0\underline{B}_L$, ${}^t_0\underline{B}_{NL}$

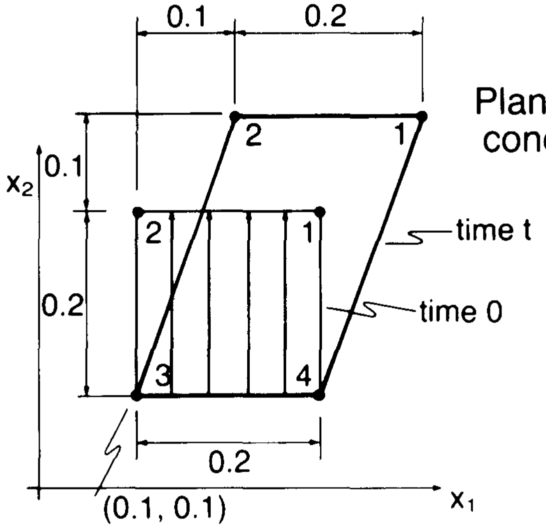


Plane strain
conditions

material fibers
have only translated
rigidly

Transparency 7-27

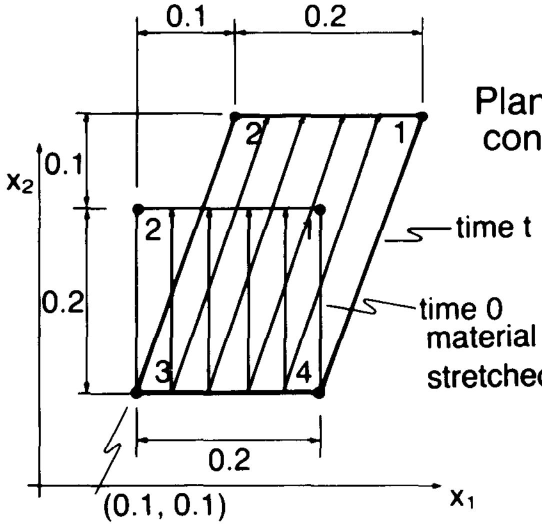
Example: Calculation of ${}^t_0\underline{B}_L$, ${}^t_0\underline{B}_{NL}$



Plane strain conditions

Transparency 7-28

Example: Calculation of ${}^t_0\underline{B}_L$, ${}^t_0\underline{B}_{NL}$

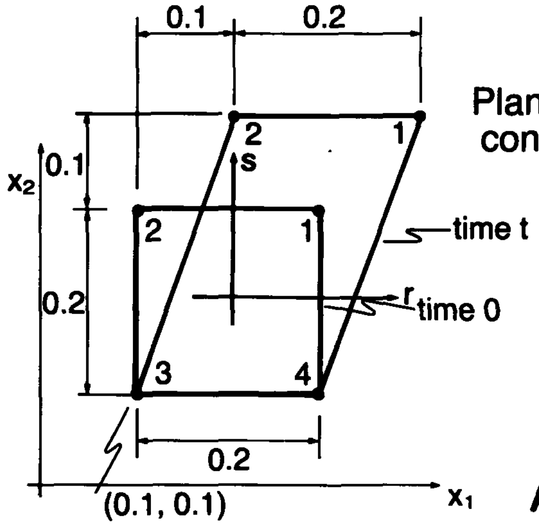


Plane strain conditions

material fibers have stretched and rotated

Transparency 7-29

Example: Calculation of ${}^t_0\underline{B}_L$, ${}^t_0\underline{B}_{NL}$



Plane strain conditions

At time 0,

Transparency 7-30

We can now perform a Jacobian transformation between the (r , s) coordinate system and the (${}^0x_1$, ${}^0x_2$) coordinate system:

By inspection, $\dfrac{\partial\, {}^0x_1}{\partial r} = 0.1 \, , \, \dfrac{\partial\, {}^0x_2}{\partial r} = 0$

$$\frac{\partial\, {}^0x_1}{\partial s} = 0 \quad , \quad \frac{\partial\, {}^0x_2}{\partial s} = 0.1$$

Hence ${}^0\underline{J} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$, $|{}^0\underline{J}| = 0.01$

and

$$\frac{\partial}{\partial\, {}^0x_1} = 10 \frac{\partial}{\partial r} \, , \, \frac{\partial}{\partial\, {}^0x_2} = 10 \frac{\partial}{\partial s}$$

Transparency 7-31

Now we use the interpolation functions to compute ${}_0^tu_{1,1}$, ${}_0^tu_{1,2}$:

node k	$\frac{\partial h_k}{\partial {}^0x_1}$	$\frac{\partial h_k}{\partial {}^0x_2}$	${}^tu_1^k$	$\frac{\partial h_k}{\partial {}^0x_1}\,{}^tu_1^k$	$\frac{\partial h_k}{\partial {}^0x_2}\,{}^tu_1^k$
1	$2.5(1 + s)$	$2.5(1 + r)$	0.1	$0.25(1 + s)$	$0.25(1 + r)$
2	$-2.5(1 + s)$	$2.5(1 - r)$	0.1	$-0.25(1 + s)$	$0.25(1 - r)$
3	$-2.5(1 - s)$	$-2.5(1 - r)$	0.0	0	0
4	$2.5(1 - s)$	$-2.5(1 + r)$	0.0	0	0

Sum: $\underbrace{0.0}_{{}_0^tu_{1,1}}$ $\underbrace{0.5}_{{}_0^tu_{1,2}}$

Transparency 7-32

For this simple problem, we can compute the displacement derivatives by inspection:

From the given dimensions,

$$ {}_0^t\underline{X} = \begin{bmatrix} 1.0 & 0.5 \\ 0.0 & 1.5 \end{bmatrix} $$

Hence

$$ \begin{aligned} {}_0^tu_{1,1} &= {}_0^tX_{11} - 1 = 0 \\ {}_0^tu_{1,2} &= {}_0^tX_{12} \quad\;\; = 0.5 \\ {}_0^tu_{2,1} &= {}_0^tX_{21} \quad\;\; = 0 \\ {}_0^tu_{2,2} &= {}_0^tX_{22} - 1 = 0.5 \end{aligned} $$

Transparency 7-33

We can now construct the columns in ${}^t_0\underline{B}_L$ that correspond to node 3:

$$\begin{array}{l}
\begin{bmatrix} \cdots & \begin{array}{|c:c|} -2.5(1-s) & 0 \\ 0 & -2.5(1-r) \\ -2.5(1-r) & -2.5(1-s) \end{array} & \cdots \end{bmatrix} \quad {}^t_0\underline{B}_{L0} \\
\hline
\begin{bmatrix} \cdots & \begin{array}{|c:c|} 0 & 0 \\ -1.25(1-r) & -1.25(1-r) \\ -1.25(1-s) & -1.25(1-s) \end{array} & \cdots \end{bmatrix} \quad {}^t_0\underline{B}_{L1}
\end{array}$$

Transparency 7-34

Similarly, we construct the columns in ${}^t_0\underline{B}_{NL}$ that correspond to node 3:

$$\begin{bmatrix} \cdots & \begin{array}{|c:c|} -2.5(1-s) & 0 \\ -2.5(1-r) & 0 \\ 0 & -2.5(1-s) \\ 0 & -2.5(1-r) \end{array} & \cdots \end{bmatrix}$$

Transparency 7-35

Consider next the element matrices required for the updated Lagrangian formulation:

Element Matrix	Matrices Required
${}^t_t\underline{K}_L$	${}_t\underline{C}$, ${}^t_t\underline{B}_L$
${}^t_t\underline{K}_{NL}$	${}^t\underline{\tau}$, ${}^t_t\underline{B}_{NL}$
${}^t_t\underline{F}$	${}^t\underline{\hat{\tau}}$, ${}^t_t\underline{B}_L$

Transparency 7-36

We define ${}_t\underline{C}$ so that

$$\begin{bmatrix} {}_tS_{11} \\ {}_tS_{22} \\ {}_tS_{12} \\ {}_tS_{33} \end{bmatrix} = {}_t\underline{C} \begin{bmatrix} {}_te_{11} \\ {}_te_{22} \\ 2\,{}_te_{12} \\ {}_te_{33} \end{bmatrix}$$

analogous to
${}_tS_{ij} = {}_tC_{ijrs}\;{}_te_{rs}$

For example, we may choose (axisymmetric analysis),

$${}_t\underline{C} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \dfrac{\nu}{1-\nu} & 0 & \dfrac{\nu}{1-\nu} \\ \dfrac{\nu}{1-\nu} & 1 & 0 & \dfrac{\nu}{1-\nu} \\ 0 & 0 & \dfrac{1-2\nu}{2(1-\nu)} & 0 \\ \dfrac{\nu}{1-\nu} & \dfrac{\nu}{1-\nu} & 0 & 1 \end{bmatrix}$$

Transparency 7-37

We note that the incremental strain components are, in two-dimensional analysis,

$$
\begin{aligned}
{}_te_{11} &= \frac{\partial u_1}{\partial {}^tx_1} = {}_tu_{1,1} \\
{}_te_{22} &= {}_tu_{2,2} \\
2\,{}_te_{12} &= {}_tu_{1,2} + {}_tu_{2,1} \\
{}_te_{33} &= u_1/{}^tx_1
\end{aligned}
$$

Transparency 7-38

and

$$ {}_t\eta_{11} = \frac{1}{2}\left(({}_tu_{1,1})^2 + ({}_tu_{2,1})^2\right) $$

$$ {}_t\eta_{22} = \frac{1}{2}\left(({}_tu_{1,2})^2 + ({}_tu_{2,2})^2\right) $$

$$ {}_t\eta_{12} = {}_t\eta_{21} = \frac{1}{2}\left({}_tu_{1,1}\,{}_tu_{1,2} + {}_tu_{2,1}\,{}_tu_{2,2}\right) $$

$$ {}_t\eta_{33} = \frac{1}{2}\left(\frac{u_1}{{}^tx_1}\right)^2 $$

Transparency 7-39

We construct ${}^t_t\underline{B}_L$ so that

$$\begin{array}{ll} \left[\begin{array}{c} {}_te_{11} \\ {}_te_{22} \\ 2\,{}_te_{12} \\ \hline {}_te_{33} \end{array}\right] & \begin{array}{l} = {}_t\underline{e} = {}^t_t\underline{B}_L\,\underline{\hat{u}} \\ \text{only included for} \\ \text{axisymmetric analysis} \end{array} \end{array}$$

Transparency 7-40

Entries in ${}^t_t\underline{B}_L$:

$$\begin{array}{cccccc} & \multicolumn{2}{c}{\text{node } k} & & & \\ & u_1^k & u_2^k & & & \\ \left[\begin{array}{c} \\ \cdots \\ \\ \hline \\ \end{array}\right. & \begin{array}{c} {}_th_{k,1} \\ 0 \\ {}_th_{k,2} \\ \hline h_k/{}^tx_1 \end{array} & \begin{array}{c} 0 \\ {}_th_{k,2} \\ {}_th_{k,1} \\ \hline 0 \end{array} & \left.\begin{array}{c} \\ \cdots \\ \\ \hline \\ \end{array}\right] & \left[\begin{array}{c} \vdots \\ \hline u_1^k \\ \hdashline u_2^k \\ \hline \vdots \end{array}\right] & \begin{array}{c} \uparrow \\ \text{node } k \\ \downarrow \end{array} \end{array}$$

only included for axisymmetric analysis

This is similar in form to the $\underline{B}$ matrix used in linear analysis.

Transparency 7-41

We construct ${}^t_t\underline{B}_{NL}$ and ${}^t\underline{\tau}$ so that

$$\delta\hat{\underline{u}}^T\,{}^t_t\underline{B}_{NL}^T\,{}^t\underline{\tau}\,{}^t_t\underline{B}_{NL}\,\hat{\underline{u}} = {}^t\tau_{ij}\,\delta_t\eta_{ij}$$

Entries in ${}^t\underline{\tau}$:

$$\left[\begin{array}{cccc|c}
{}^t\tau_{11} & {}^t\tau_{12} & 0 & 0 & 0 \\
{}^t\tau_{21} & {}^t\tau_{22} & 0 & 0 & 0 \\
0 & 0 & {}^t\tau_{11} & {}^t\tau_{12} & 0 \\
0 & 0 & {}^t\tau_{21} & {}^t\tau_{22} & 0 \\
\hline
0 & 0 & 0 & 0 & {}^t\tau_{33}
\end{array}\right]$$

included only for axisymmetric analysis

Transparency 7-42

Entries in ${}^t_t\underline{B}_{NL}$:

$$\begin{array}{c|c|c|c}
 & \multicolumn{2}{c|}{\text{node } k} & \\
 & u_1^k & u_2^k & \\
\hline
 & {}_th_{k,1} & 0 & \\
 & {}_th_{k,2} & 0 & \\
\cdots & 0 & {}_th_{k,1} & \cdots \\
 & 0 & {}_th_{k,2} & \\
\hline
 & h_k/{}^tx_1 & 0 & \\
\hline
\end{array}
\qquad
\left[\begin{array}{c}
\vdots \\
\hline
u_1^k \\
\hline
u_2^k \\
\hline
\\
\end{array}\right]
\begin{array}{c}
\uparrow \\
\text{node } k \\
\downarrow
\end{array}$$

included only for axisymmetric analysis

Transparency 7-43

${}^{t}\hat{\underline{T}}$ is constructed so that

$$\delta\hat{\underline{u}}^{T}\,{}^{t}_{t}\underline{B}_{L}^{T}\,{}^{t}\hat{\underline{T}} = {}^{t}\tau_{ij}\,\delta_{t}e_{ij}$$

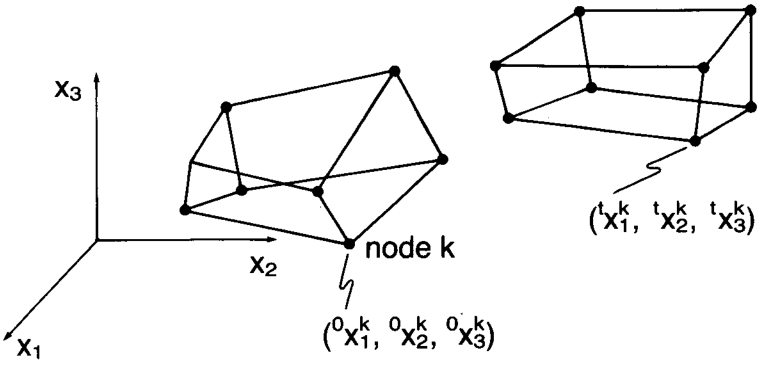
Entries in ${}^{t}\hat{\underline{T}}$:

$$\begin{bmatrix} {}^{t}\tau_{11} \\ {}^{t}\tau_{22} \\ {}^{t}\tau_{12} \\ \hline {}^{t}\tau_{33} \end{bmatrix}$$

included only for axisymmetric analysis

Transparency 7-44

Three-dimensional elements



Transparency 7-45

Here we now use

$$^{0}x_1 = \sum_{k=1}^{N} h_k \, {}^{0}x_1^k \ , \quad {}^{0}x_2 = \sum_{k=1}^{N} h_k \, {}^{0}x_2^k$$

$$^{0}x_3 = \sum_{k=1}^{N} h_k \, {}^{0}x_3^k \ ,$$

where the h_k's are the isoparametric interpolation functions of the three-dimensional element.

Transparency 7-46

Also

$$^{t}x_1 = \sum_{k=1}^{N} h_k \, {}^{t}x_1^k \ , \quad {}^{t}x_2 = \sum_{k=1}^{N} h_k \, {}^{t}x_2^k$$

$$^{t}x_3 = \sum_{k=1}^{N} h_k \, {}^{t}x_3^k$$

and then all the concepts and derivations already discussed are directly applicable to the derivation of the three-dimensional element matrices.

Topic 8

The Two-Noded Truss Element—Updated Lagrangian Formulation

Contents:

- **Derivation of updated Lagrangian truss element displacement and strain-displacement matrices from continuum mechanics equations**
- **Assumption of large displacements and rotations but small strains**
- **Physical explanation of the matrices obtained directly by application of the principle of virtual work**
- **Effect of geometric (nonlinear strain) stiffness matrix**
- **Example analysis: Prestressed cable**

Textbook: Section 6.3.1

Examples: 6.15, 6.16

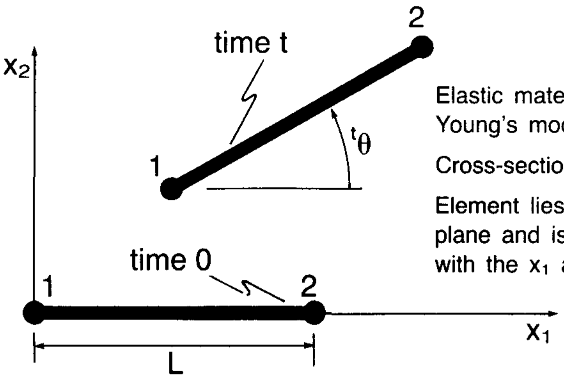
TRUSS ELEMENT DERIVATION

A truss element is a structural member which incorporates the following assumptions:

- Stresses are transmitted only in the direction normal to the cross-section.
- The stress is constant over the cross-section.
- The cross-sectional area remains constant during deformations.

Transparency 8-1

We consider the large rotation–small strain finite element formulation for a straight truss element with constant cross-sectional area.



Elastic material with Young's modulus E

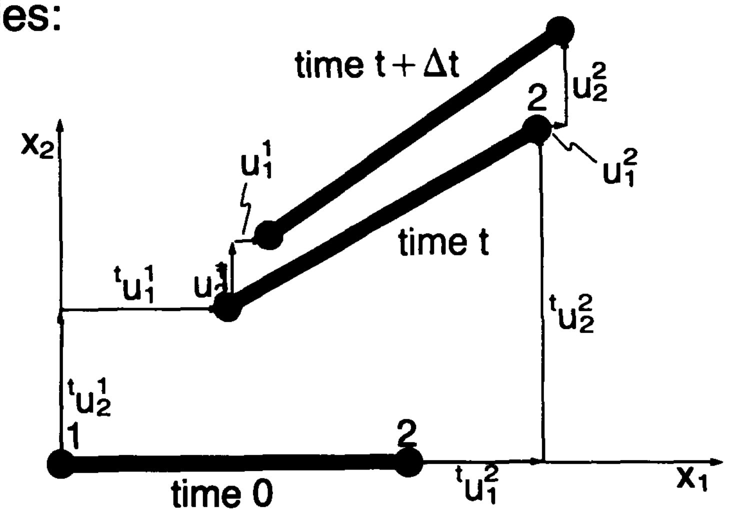
Cross-sectional area A

Element lies in the $x_1 - x_2$ plane and is initially aligned with the x_1 axis.

Transparency 8-2

Transparency 8-3

The deformations of the element are specified by the displacements of its nodes:

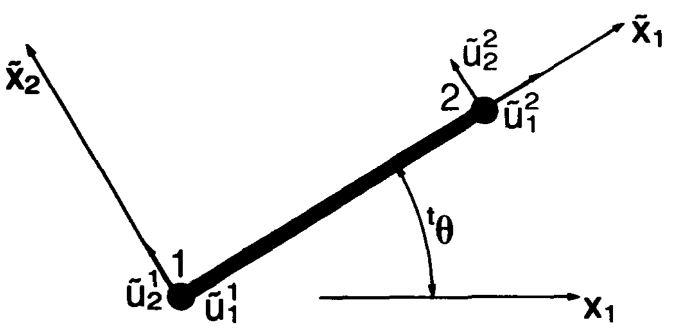


Our goal is to determine the element deformations at time t + Δt.

Transparency 8-4

Updated Lagrangian formulation:

The derivation is simplified if we consider a coordinate system aligned with the truss element at time t.



Transparency 8-5

Written in the rotated coordinate system, the equation of the principle of virtual work is

$$\int_{{}^tV} {}^{t+\Delta t}_{\ \ t}\tilde{S}_{ij}\, \delta\, {}^{t+\Delta t}_{\ \ t}\tilde{\varepsilon}_{ij}\, {}^t dV = {}^{t+\Delta t}\tilde{\mathcal{R}}$$

As we recall, this may be linearized to obtain

$$\int_{{}^tV} {}_t\tilde{C}_{ijrs}\, {}_t\tilde{e}_{rs}\, \delta_t\tilde{e}_{ij}\, {}^t dV + \int_{{}^tV} {}^t\tilde{\tau}_{ij}\, \delta_t\tilde{\eta}_{ij}\, {}^t dV$$
$$= {}^{t+\Delta t}\tilde{\mathcal{R}} - \int_{{}^tV} {}^t\tilde{\tau}_{ij}\, \delta_t\tilde{e}_{ij}\, {}^t dV$$

Transparency 8-6

Because the only non-zero stress component is ${}^t\tilde{\tau}_{11}$, the linearized equation of motion simplifies to

$$\int_{{}^tV} {}_t\tilde{C}_{1111}\, {}_t\tilde{e}_{11}\, \delta_t\tilde{e}_{11}\, {}^t dV + \int_{{}^tV} {}^t\tilde{\tau}_{11}\, \delta_t\tilde{\eta}_{11}\, {}^t dV$$
$$= {}^{t+\Delta t}\tilde{\mathcal{R}} - \int_{{}^tV} {}^t\tilde{\tau}_{11}\, \delta_t\tilde{e}_{11}\, {}^t dV$$

Notice that we need only consider one component of the strain tensor.

Transparency 8-7

We also notice that:

$${}_t\tilde{C}_{1111} = E$$

$${}^t\tilde{\tau}_{11} = \frac{{}^tP}{A}$$

$${}^tV = AL$$

The stress and strain states are constant along the truss.

Hence the equation of motion becomes

$$(EA)\, {}_t\tilde{e}_{11}\, \delta_t\tilde{e}_{11}\, L + {}^tP\, \delta_t\tilde{\eta}_{11}\, L = {}^{t+\Delta t}\tilde{\mathfrak{R}} - {}^tP\, \delta_t\tilde{e}_{11}\, L$$

Transparency 8-8

To proceed, we must express the strain increments in terms of the (rotated) displacement increments:

$${}_t\tilde{e}_{11} = {}^t_t\tilde{\underline{B}}_L\, \hat{\underline{u}}\,,$$

$$\delta_t\tilde{\eta}_{11} = \underbrace{(\delta\hat{\underline{u}}^T\, {}^t_t\underline{B}_{NL}^T)({}^t_t\underline{B}_{NL}\, \hat{\underline{u}})}_{}$$

This form is analogous to the form used in the two-dimensional element formulation.

where

$$\hat{\underline{u}} = \begin{bmatrix} \tilde{u}_1^1 \\ \tilde{u}_2^1 \\ \tilde{u}_1^2 \\ \tilde{u}_2^2 \end{bmatrix}$$

Transparency 8-9

Since ${}_t\tilde{\varepsilon}_{11} = {}_t\tilde{u}_{1,1} + \frac{1}{2}\left(({}_t\tilde{u}_{1,1})^2 + ({}_t\tilde{u}_{2,1})^2\right)$,
we recognize

$$ {}_t\tilde{e}_{11} = {}_t\tilde{u}_{1,1} $$

$$ {}_t\tilde{\eta}_{11} = \frac{1}{2}\left(({}_t\tilde{u}_{1,1})^2 + ({}_t\tilde{u}_{2,1})^2\right) $$

and

$$ \begin{aligned} \delta_t\tilde{\eta}_{11} &= \delta_t\tilde{u}_{1,1}\,{}_t\tilde{u}_{1,1} + \delta_t\tilde{u}_{2,1}\,{}_t\tilde{u}_{2,1} \\ &= \underbrace{[\delta_t\tilde{u}_{1,1}\ \ \delta_t\tilde{u}_{2,1}] \begin{bmatrix} {}_t\tilde{u}_{1,1} \\ {}_t\tilde{u}_{2,1} \end{bmatrix}}_{\text{matrix form}} \end{aligned} $$

Transparency 8-10

We can now write the displacement derivatives in terms of the displacements (this is simple because all quantities are constant along the truss). For example,

$$ {}_t\tilde{u}_{1,1} = \frac{\partial \tilde{u}_1}{\partial\, {}^t\tilde{x}_1} = \frac{\Delta \tilde{u}_1}{\Delta\, {}^t x_1} = \frac{\tilde{u}_1^2 - \tilde{u}_1^1}{L} $$

Hence we obtain

$$ \begin{bmatrix} {}_t\tilde{u}_{1,1} \\ {}_t\tilde{u}_{2,1} \end{bmatrix} = \frac{1}{L}\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}_1^1 \\ \tilde{u}_2^1 \\ \tilde{u}_1^2 \\ \tilde{u}_2^2 \end{bmatrix} \leftarrow \underline{\hat{u}} $$

Transparency 8-11

and

$${}_t\tilde{e}_{11} = \underbrace{\left(\frac{1}{L}\begin{bmatrix}-1 & 0 & 1 & 0\end{bmatrix}\right)}_{{}^t_t\tilde{\underline{B}}_L}\hat{\underline{u}}$$

$$\delta_t\tilde{\eta}_{11} = \underbrace{\delta\hat{\underline{u}}^T\overbrace{\left(\frac{1}{L}\begin{bmatrix}-1 & 0\\ 0 & -1\\ 1 & 0\\ 0 & 1\end{bmatrix}\right)}^{{}^t_t\tilde{\underline{B}}_{NL}^T}}_{\begin{bmatrix}\delta_t\tilde{u}_{1,1} & \delta_t\tilde{u}_{2,1}\end{bmatrix}}\underbrace{\overbrace{\left(\frac{1}{L}\begin{bmatrix}-1 & 0 & 1 & 0\\ 0 & -1 & 0 & 1\end{bmatrix}\right)}^{{}^t_t\tilde{\underline{B}}_{NL}}\hat{\underline{u}}}_{\begin{bmatrix}{}_t\tilde{u}_{1,1}\\ {}_t\tilde{u}_{2,1}\end{bmatrix}}$$

Transparency 8-12

Using these expressions,

$$(EA)\ {}_t\tilde{e}_{11}\ \delta_t\tilde{e}_{11}\ L$$

$$\Big\downarrow$$

$$\delta\hat{\underline{u}}^T\overbrace{\left(\frac{EA}{L}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\right)}^{{}^t_t\tilde{\underline{K}}_L}\hat{\underline{u}}$$

(setting successively each virtual nodal point displacement equal to unity)

Transparency 8-13

${}^tP\ \delta_t\tilde{\eta}_{11}\ L$

$$\delta\hat{\underline{u}}^T \overbrace{\left(\frac{{}^tP}{L}\begin{bmatrix}1&0&-1&0\\0&1&0&-1\\-1&0&1&0\\0&-1&0&1\end{bmatrix}\right)}^{{}^t_t\tilde{\underline{K}}_{NL}}\hat{\underline{u}}$$

Transparency 8-14

and

${}^tP\ \delta_t\tilde{e}_{11}\ L$

$$\delta\hat{\underline{u}}^T \overbrace{\left({}^tP\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}\right)}^{{}^t_t\tilde{\underline{F}}}$$

Transparency 8-15

We have now obtained the required element matrices, expressed in the coordinate system aligned with the truss at time t.

To determine the element matrices in the stationary global coordinate system, we must express the rotated displacement increments $\hat{\underline{u}}$ in terms of the unrotated displacement increments $\hat{\underline{u}}$.

We can show that

$$\begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} = \begin{bmatrix} \cos{}^t\theta & \sin{}^t\theta \\ -\sin{}^t\theta & \cos{}^t\theta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Transparency 8-16

Hence

$$\underbrace{\begin{bmatrix} \tilde{u}_1^1 \\ \tilde{u}_2^1 \\ \tilde{u}_1^2 \\ \tilde{u}_2^2 \end{bmatrix}}_{\hat{\underline{u}}} = \underbrace{\begin{bmatrix} \cos{}^t\theta & \sin{}^t\theta & 0 & 0 \\ -\sin{}^t\theta & \cos{}^t\theta & 0 & 0 \\ 0 & 0 & \cos{}^t\theta & \sin{}^t\theta \\ 0 & 0 & -\sin{}^t\theta & \cos{}^t\theta \end{bmatrix}}_{\underline{T}} \underbrace{\begin{bmatrix} u_1^1 \\ u_2^1 \\ u_1^2 \\ u_2^2 \end{bmatrix}}_{\hat{\underline{u}}}$$

Using this transformation in the equation of motion gives

$$\delta \underline{\hat{u}}^T \, {}^t_t\underline{\tilde{K}}_L \, \underline{\hat{u}} \rightarrow \delta \underline{\hat{u}}^T \underbrace{\underline{T}^T \, {}^t_t\underline{\tilde{K}}_L \, \underline{T}}_{{}^t_t\underline{K}_L} \underline{\hat{u}}$$

$$\delta \underline{\hat{u}}^T \, {}^t_t\underline{\tilde{K}}_{NL} \, \underline{\hat{u}} \rightarrow \delta \underline{\hat{u}}^T \underbrace{\underline{T}^T \, {}^t_t\underline{\tilde{K}}_{NL} \, \underline{T}}_{{}^t_t\underline{K}_{NL}} \underline{\hat{u}}$$

$$\delta \underline{\hat{u}}^T \, {}^t_t\underline{\tilde{F}} \rightarrow \delta \underline{\hat{u}}^T \underbrace{\underline{T}^T \, {}^t_t\underline{\tilde{F}}}_{{}^t_t\underline{F}}$$

Transparency 8-17

Performing the indicated matrix multiplications gives

$${}^t_t\underline{K}_L = \frac{EA}{L}\left[\begin{array}{c|c|c|c} (\cos{}^t\theta)^2 & (\cos{}^t\theta)(\sin{}^t\theta) & -(\cos{}^t\theta)^2 & -(\cos{}^t\theta)(\sin{}^t\theta) \\ \hline & (\sin{}^t\theta)^2 & -(\cos{}^t\theta)(\sin{}^t\theta) & -(\sin{}^t\theta)^2 \\ \hline \text{symmetric} & & (\cos{}^t\theta)^2 & (\cos{}^t\theta)(\sin{}^t\theta) \\ \hline & & & (\sin{}^t\theta)^2 \end{array}\right]$$

Transparency 8-18

Transparency 8-19

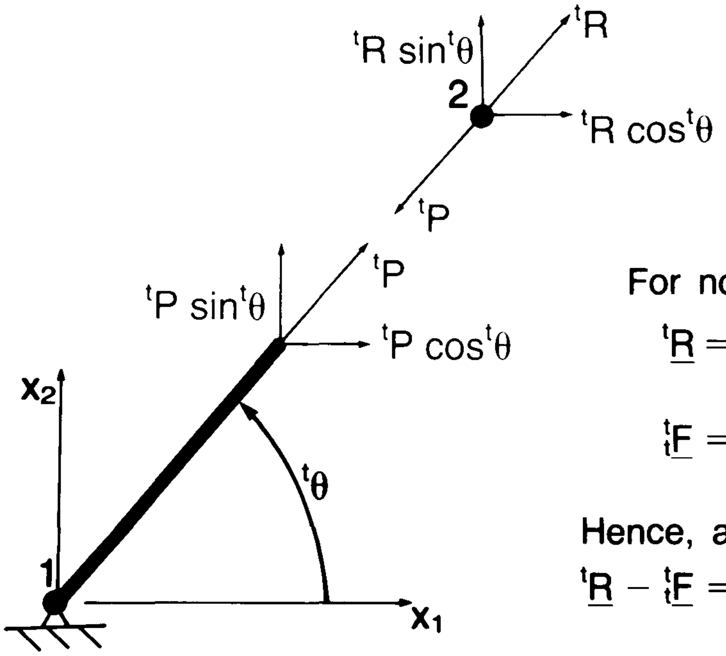
$$ {}^{t}_{t}\underline{K}_{NL} = \frac{{}^{t}P}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ & 1 & 0 & -1 \\ & & 1 & 0 \\ \text{symmetric} & & & 1 \end{bmatrix} $$

and

$$ {}^{t}_{t}\underline{F} = {}^{t}P\begin{bmatrix} -\cos{}^{t}\theta \\ -\sin{}^{t}\theta \\ \cos{}^{t}\theta \\ \sin{}^{t}\theta \end{bmatrix} $$

Transparency 8-20

The vector ${}^{t}_{t}\underline{F}$ makes physical sense:



For node 2,

$$ {}^{t}\underline{R} = \begin{bmatrix} {}^{t}R \cos{}^{t}\theta \\ {}^{t}R \sin{}^{t}\theta \end{bmatrix} $$

$$ {}^{t}_{t}\underline{F} = \begin{bmatrix} {}^{t}P \cos{}^{t}\theta \\ {}^{t}P \sin{}^{t}\theta \end{bmatrix} $$

Hence, at equilibrium,

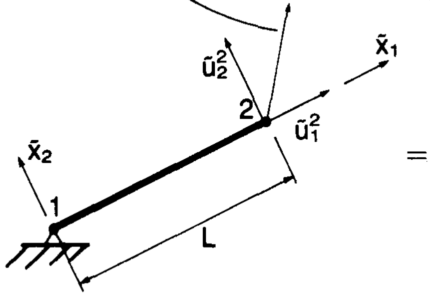
$$ {}^{t}\underline{R} - {}^{t}_{t}\underline{F} = \underline{0} $$

Transparency 8-21

We note that the ${}^{t}_{t}\underline{K}_{NL}$ matrix is unchanged by the coordinate transformation.

- The nonlinear strain increment is related only to the vector magnitude of the displacement increment.

$$\sqrt{(\tilde{u}_1^2)^2 + (\tilde{u}_2^2)^2} = \left(\sqrt{\left(\frac{\partial \tilde{u}_1}{\partial {}^t\tilde{x}_1}\right)^2 + \left(\frac{\partial \tilde{u}_2}{\partial {}^t\tilde{x}_1}\right)^2}\right) L$$



$$= \sqrt{2\,{}_t\eta_{11}}\,L$$

Transparency 8-22

Physically, ${}^{t}_{t}\underline{K}_{NL}$ gives the required change in the externally applied nodal point forces when the truss is rotated.

Consider only $\tilde{u}_2^2$ nonzero.

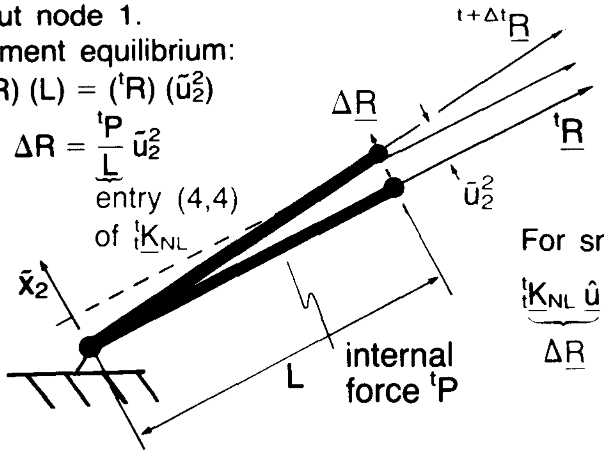
For small $\tilde{u}_2^2$, this gives a rotation about node 1.

Moment equilibrium:

$(\Delta R)\,(L) = ({}^tR)\,(\tilde{u}_2^2)$

or $\Delta R = \underbrace{\frac{{}^tP}{L}}_{}\,\tilde{u}_2^2$

entry (4,4) of ${}^{t}_{t}\underline{K}_{NL}$

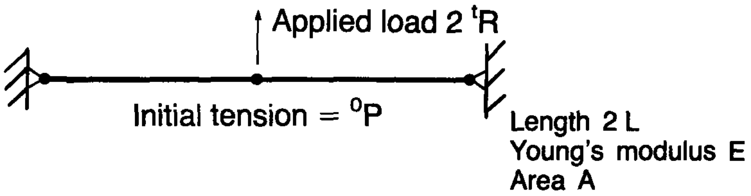


For small $\underline{\hat{u}}$,

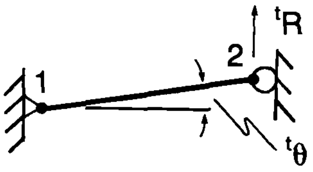
$$\underbrace{{}^{t}_{t}\underline{K}_{NL}\,\underline{\hat{u}}}_{\Delta\underline{R}} = {}^{t+\Delta t}\underline{R} - {}^t\underline{R}$$

Transparency 8-23

Example: Prestressed cable



Finite element model (using symmetry):



Transparency 8-24

Using the U.L. formulation, we obtain

$$\left(\underbrace{\frac{EA}{L}(\sin{}^t\theta)^2}_{{}^t_t\underline{K}_L} + \underbrace{\frac{{}^tP}{L}}_{{}^t_t\underline{K}_{NL}}\right) u_2^2 = {}^{t+\Delta t}R - \underbrace{{}^tP\sin{}^t\theta}_{{}^t_t\underline{F}}$$

Of particular interest is the configuration at time 0, when ${}^t\theta = 0$:

$$\left(\frac{{}^0P}{L}\right) u_2^2 = {}^{\Delta t}R$$

The undeformed cable stiffness is given solely by ${}^t_t\underline{K}_{NL}$.

Transparency 8-25

The cable stiffens as load is applied:

$${}^t_tK = \underbrace{\frac{EA}{L}(\sin{}^t\theta)^2}_{{}^t_tK_L} + \underbrace{\frac{{}^tP}{L}}_{{}^t_tK_{NL}}$$

${}^t_tK_L$ increases as ${}^t\theta$ increases (the truss provides axial stiffness as ${}^t\theta$ increases).

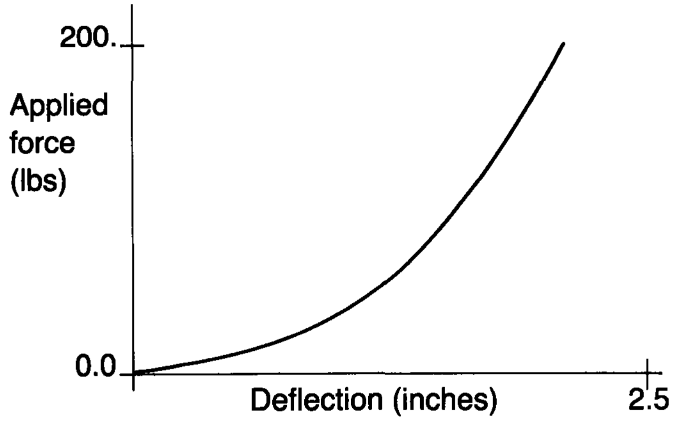
As ${}^t\theta \rightarrow 90°$, the stiffness approaches $\frac{EA}{L}$, but constant L and A means here that only small values of ${}^t\theta$ are permissible.

Transparency 8-26

Using: $L = 120$ in , $A = 1$ in^2,

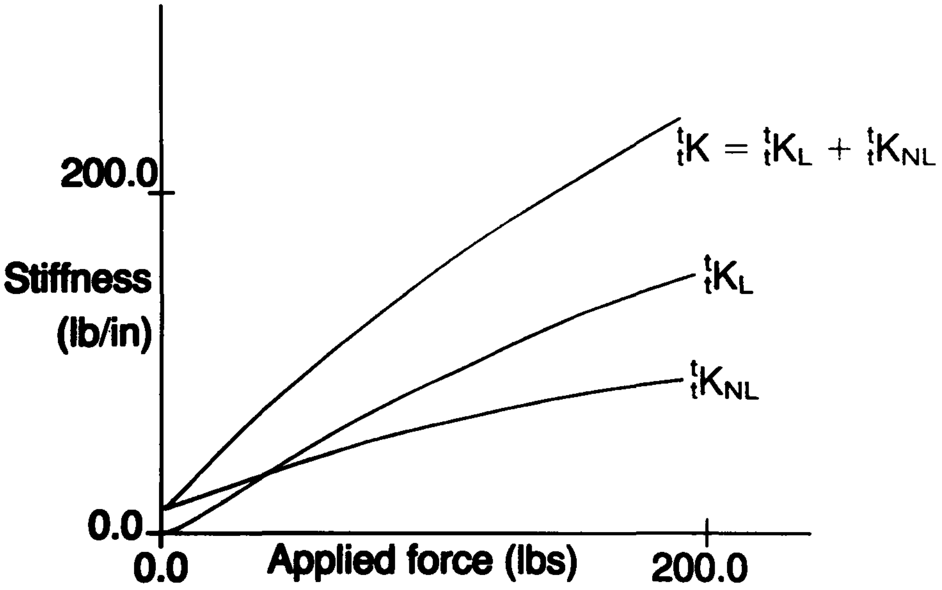
$E = 30 \times 10^6$ psi , ${}^0P = 1000$ lbs

we obtain



Transparency 8-27

We also show the stiffness matrix components as functions of the applied load:



Topic 9

The Two-Noded Truss Element—Total Lagrangian Formulation

Contents:

- **Derivation of total Lagrangian truss element displacement and strain-displacement matrices from continuum mechanics equations**
- **Mathematical and physical explanation that only one component (${}^{t}_{0}S_{11}$) of the 2nd Piola-Kirchhoff stress tensor is nonzero**
- **Physical explanation of the matrices obtained directly by application of the principle of virtual work**
- **Discussion of initial displacement effect**
- **Comparison of updated and total Lagrangian formulations**
- **Example analysis: Collapse of a truss structure**
- **Example analysis: Large displacements of a cable**

Textbook: Section 6.3.1

Examples: 6.15, 6.16

TOTAL LAGRANGIAN FORMULATION OF TRUSS ELEMENT

We directly derive all required matrices in the stationary global coordinate system.

Recall that the linearized equation of the principle of virtual work is

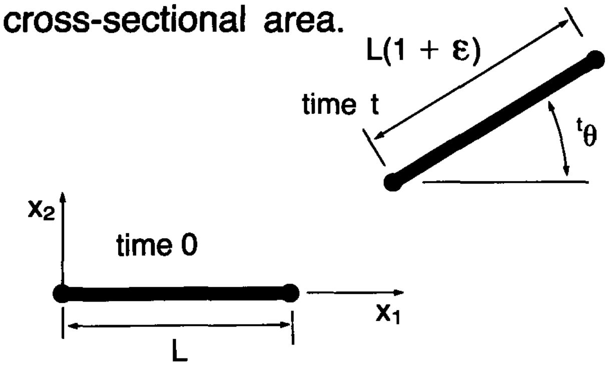
$$\int_{{}^0V} {}_0C_{ijrs}\, {}_0e_{rs}\, \delta_0 e_{ij}\, {}^0dV + \int_{{}^0V} {}_0^tS_{ij}\, \delta_0\eta_{ij}\, {}^0dV$$

$$= {}^{t+\Delta t}\mathcal{R} - \int_{{}^0V} {}_0^tS_{ij}\, \delta_0 e_{ij}\, {}^0dV$$

Transparency 9-1

We will now show that the only non-zero stress component is ${}_0^tS_{11}$.

1) Mathematical explanation:
For simplicity, we assume constant cross-sectional area.



Transparency 9-2

Transparency 9-3

We may show that for the fibers of the truss element

$$ {}^t_0\underline{X} = \begin{bmatrix} (1+\varepsilon)\cos{}^t\theta & -\sin{}^t\theta \\ (1+\varepsilon)\sin{}^t\theta & \cos{}^t\theta \end{bmatrix} $$

Since the truss carries only axial stresses,

$$ {}^t\underline{\tau} = \frac{{}^tP}{A} \underbrace{\begin{bmatrix} (\cos{}^t\theta)^2 & (\cos{}^t\theta)(\sin{}^t\theta) \\ (\cos{}^t\theta)(\sin{}^t\theta) & (\sin{}^t\theta)^2 \end{bmatrix}}_{\text{written in the stationary coordinate frame}} $$

Transparency 9-4

Hence using

$$ {}^t_0\underline{S} = \frac{{}^0\rho}{{}^t\rho}\, {}^0_t\underline{X}\, {}^t\underline{\tau}\, {}^0_t\underline{X}^T $$

we find

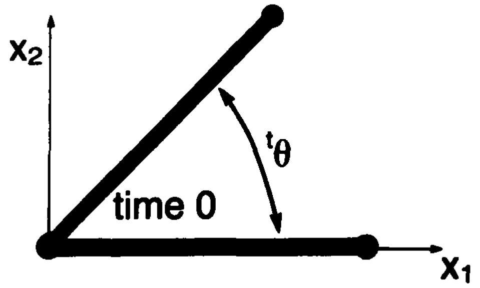
$$ {}^t_0\underline{S} = \begin{bmatrix} \frac{{}^tP}{A}\cancelto{1 \text{ for small } \varepsilon}{\left(\frac{1}{1+\varepsilon}\right)} & 0 \\ 0 & 0 \end{bmatrix} $$

Transparency 9-5

Physical explanation: we utilize an intermediate configuration t*

time t* (conceptual): Element is stretched by ${}^{t}P$.

time t: The element is moved as a rigid body.



$$ {}^{0}_{0}\underline{S} = {}^{0}\underline{\tau} = \begin{bmatrix} {}^{0}P/A & 0 \\ 0 & 0 \end{bmatrix} $$

$$ {}^{t^*}_{0}\underline{S} = {}^{t^*}\underline{\tau} = \begin{bmatrix} {}^{t}P/A & 0 \\ 0 & 0 \end{bmatrix} $$

$$ {}^{t}_{0}S = \begin{bmatrix} {}^{t}P/A & 0 \\ 0 & 0 \end{bmatrix} $$

(the components of the 2nd Piola-Kirchhoff stress tensor do not change during a rigid body motion)

Transparency 9-6

The linearized equation of motion simplifies to

$$ \int_{{}^{0}V} {}_{0}C_{1111}\, {}_{0}e_{11}\, \delta_{0}e_{11}\, {}^{0}dV + \int_{{}^{0}V} {}^{t}_{0}S_{11}\, \delta_{0}\eta_{11}\, {}^{0}dV $$

$$ = {}^{t+\Delta t}\mathcal{R} - \int_{{}^{0}V} {}^{t}_{0}S_{11}\, \delta_{0}e_{11}\, {}^{0}dV $$

Again, we need only consider one component of the strain tensor.

Transparency 9-7

Next we recognize:

$${}_0^tS_{11} = \frac{{}^tP}{A}$$

$${}_0C_{1111} = E \ , \ {}^0V = A\,L$$

The stress and strain states are constant along the truss.

Hence the equation of motion becomes

$$(EA)\ {}_0e_{11}\ \delta_0 e_{11}\ L + {}^tP\ \delta_0\eta_{11}\ L = {}^{t+\Delta t}\mathcal{R} - {}^tP\ \delta_0 e_{11}\ L$$

Transparency 9-8

To proceed, we must express the strain increments in terms of the displacement increments:

$${}_0e_{11} = {}_0^t\underline{B}_L\ \underline{\hat{u}}\ ,$$

$$\delta_0\eta_{11} = (\delta\underline{\hat{u}}^T\ {}_0^t\underline{B}_{NL}^T)\ ({}_0^t\underline{B}_{NL}\ \underline{\hat{u}})$$

where

$$\underline{\hat{u}} = \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_1^2 \\ u_2^2 \end{bmatrix}$$

Transparency 9-9

Since $${}_0\varepsilon_{11} = {}_0u_{1,1} + {}_0^tu_{1,1}\ {}_0u_{1,1} + {}_0^tu_{2,1}\ {}_0u_{2,1} + \frac{1}{2}(({}_0u_{1,1})^2 + ({}_0u_{2,1})^2)$$

we recognize

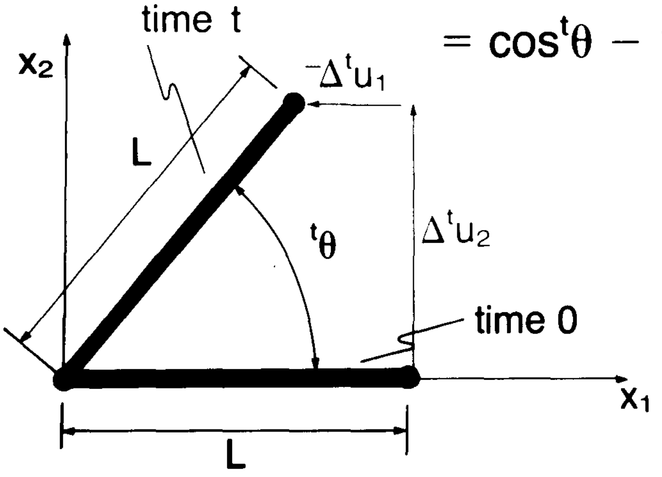
$${}_0e_{11} = {}_0u_{1,1} + {}_0^tu_{1,1}\ {}_0u_{1,1} + {}_0^tu_{2,1}\ {}_0u_{2,1}$$

$$\delta_0\eta_{11} = \delta_0u_{1,1}\ {}_0u_{1,1} + \delta_0u_{2,1}\ {}_0u_{2,1} = [\delta_0u_{1,1}\quad \delta_0u_{2,1}]\begin{bmatrix} {}_0u_{1,1} \\ {}_0u_{2,1} \end{bmatrix}$$

Transparency 9-10

We notice the presence of ${}_0^tu_{1,1}$ and ${}_0^tu_{2,1}$ in ${}_0e_{11}$. These can be evaluated using kinematics:

$${}_0^tu_{1,1} = \frac{\Delta^tu_1}{L} = \cos{}^t\theta - 1 \quad , \quad {}_0^tu_{2,1} = \frac{\Delta\,^tu_2}{L} = \sin{}^t\theta$$



Transparency 9-11

We can now write the displacement derivatives in terms of the displacements (this is simple because all quantities are constant along the truss). For example,

$${}_0u_{1,1} = \frac{\partial u_1}{\partial\, {}^0x_1} = \frac{\Delta u_1}{\Delta\, {}^0x_1} = \frac{u_1^2 - u_1^1}{L}$$

Hence we obtain

$$\begin{bmatrix} {}_0u_{1,1} \\ {}_0u_{2,1} \end{bmatrix} = \frac{1}{L}\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_1^2 \\ u_2^2 \end{bmatrix}$$

Transparency 9-12

Therefore

$${}_0e_{11} = {}_0u_{1,1} + [{}^t_0u_{1,1} \quad {}^t_0u_{2,1}] \begin{bmatrix} {}_0u_{1,1} \\ {}_0u_{2,1} \end{bmatrix}$$

$$= \underbrace{\frac{1}{L}[-1 \quad 0 \quad 1 \quad 0]}_{{}^t_0\underline{B}_{L0}}\, \underline{\hat{u}}$$

$$+ \underbrace{[\cos{}^t\theta - 1 \quad \sin{}^t\theta]\left(\frac{1}{L}\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\right)}_{\text{initial displacement effect } {}^t_0\underline{B}_{L1}}\, \underline{\hat{u}}$$

Transparency 9-13

$$ {}_{0}e_{11} = \underbrace{\frac{1}{L}[\quad -1 \qquad 0 \qquad 1 \qquad 0 \quad]}_{{}^{t}_{0}\underline{B}_{L0}}\ \underline{\hat{u}} $$

$$ + \underbrace{\frac{1}{L}[-(\cos{}^{t}\theta - 1) \ \vdots\ -\sin{}^{t}\theta \ \vdots\ \cos{}^{t}\theta - 1 \ \vdots\ \sin{}^{t}\theta]}_{{}^{t}_{0}\underline{B}_{L1}}\ \underline{\hat{u}} $$

$$ = \underbrace{\frac{1}{L}[\quad -\cos{}^{t}\theta \qquad -\sin{}^{t}\theta \qquad \cos{}^{t}\theta \qquad \sin{}^{t}\theta]}_{{}^{t}_{0}\underline{B}_{L}}\ \underline{\hat{u}} $$

Transparency 9-14

Also

$$ \delta_{0}\eta_{11} = \underbrace{\delta\underline{\hat{u}}^{T} \overbrace{\left(\frac{1}{L}\begin{bmatrix}-1 & 0\\ 0 & -1\\ 1 & 0\\ 0 & 1\end{bmatrix}\right)}^{{}^{t}_{0}\underline{B}^{T}_{NL}}}_{[\delta_{0}u_{1,1} \quad \delta_{0}u_{2,1}]} \underbrace{\overbrace{\left(\frac{1}{L}\begin{bmatrix}-1 & 0 & 1 & 0\\ 0 & -1 & 0 & 1\end{bmatrix}\right)}^{{}^{t}_{0}\underline{B}_{NL}}\ \underline{\hat{u}}}_{\begin{bmatrix}{}_{0}u_{1,1}\\ {}_{0}u_{2,1}\end{bmatrix}} $$

Transparency 9-15

Using these expressions,

$$(EA)\ {}_0e_{11}\ \delta_0e_{11}\ L$$

$$\downarrow$$

$$\delta\underline{\hat{u}}^T \underbrace{\left(\frac{EA}{L}\begin{bmatrix} (\cos{}^t\theta)^2 & (\cos{}^t\theta)(\sin{}^t\theta) & -(\cos{}^t\theta)^2 & -(\cos{}^t\theta)(\sin{}^t\theta) \\ & (\sin{}^t\theta)^2 & -(\cos{}^t\theta)(\sin{}^t\theta) & -(\sin{}^t\theta)^2 \\ & & (\cos{}^t\theta)^2 & (\cos{}^t\theta)(\sin{}^t\theta) \\ \text{symmetric} & & & (\sin{}^t\theta)^2 \end{bmatrix}\right)}_{{}^t_0\underline{K}_L} \underline{\hat{u}}$$

Transparency 9-16

$${}^tP\ \delta_0\eta_{11}\ L$$

$$\searrow$$

$$\delta\underline{\hat{u}}^T \underbrace{\left(\frac{{}^tP}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\right)}_{{}^t_0\underline{K}_{NL}} \underline{\hat{u}}$$

Transparency 9-17

and

$$\delta\underline{\hat{u}}^T \overset{{}^{t}P\,\delta_{0}e_{11}\,L\ \searrow}{\underbrace{\left({}^{t}P \begin{bmatrix} -\cos{}^{t}\theta \\ -\sin{}^{t}\theta \\ \cos{}^{t}\theta \\ \sin{}^{t}\theta \end{bmatrix} \right)}_{{}^{t}_{0}\underline{F}}}$$

Transparency 9-18

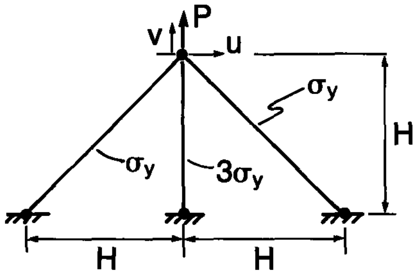
We notice that the element matrices corresponding to the T.L. and U.L. formulations are identical:

- The coordinate transformation used in the U.L. formulation is contained in the "initial displacement effect" matrix used in the T.L. formulation.
- The same can also be shown in detail analytically for a beam element, see K. J. Bathe and S. Bolourchi, Int. J. Num. Meth. in Eng., Vol. 14, pp. 961–986, 1979.

Transparency 9-19

Example: Collapse analysis of a truss structure

H = 5
A = 1
E = 200,000
$E_T = 0$
$\sigma_y = 100$



- Perform collapse analysis using U.L. formulation.
- Test model response when using M.N.O. formulation.

Transparency 9-20

For this structure, we may analytically calculate the elastic limit load and the ultimate limit load. We assume for now that the deflections are infinitesimal.

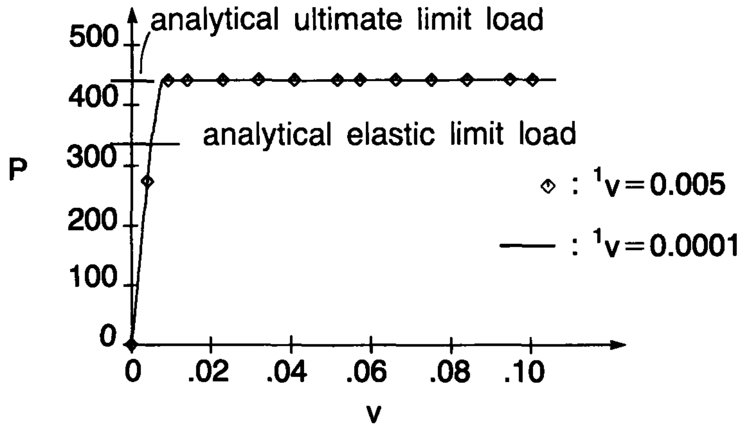
— Elastic limit load
(side trusses just become plastic)

$$P=341.4$$

— Ultimate limit load
(center truss also becomes plastic)

$$P=441.4$$

Using automatic load step incrementation and the U.L. formulation, we obtain the following results:



Transparency 9-21

We now consider an M.N.O. analysis.

We still use the automatic load step incrementation.

- If the stiffness matrix is not reformed, almost identical results are obtained (with reference to the U.L. results).

Transparency 9-22

Transparency 9-23

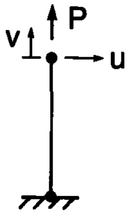
- If the stiffness matrix is reformed for a load level larger than the elastic limit load, the structure is found to be unstable (a zero pivot is found in the stiffness matrix).

  Why?

Transparency 9-24

Explanation:

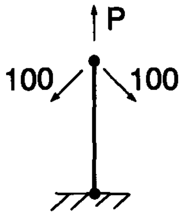
- In the M.N.O. analysis, once the side trusses have become plastic, they no longer contribute stiffness to the structure. Therefore the structure is unstable with respect to a rigid body rotation.



Transparency 9-25

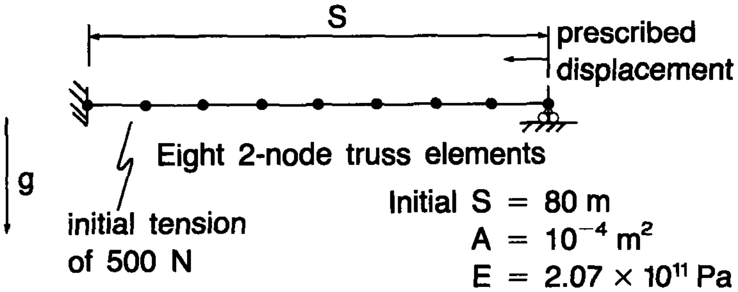
- In the U.L. analysis, once the side trusses have become plastic, they still contribute stiffness because they are transmitting forces (this effect is included in the ${}^t_t\underline{K}_{NL}$ matrix).

  Also, the internal force in the center truss provides stability through the ${}^t_t\underline{K}_{NL}$ matrix.



Transparency 9-26

Example: Large displacements of a uniform cable



Initial $S = 80$ m
$A = 10^{-4}$ m^2
$E = 2.07 \times 10^{11}$ Pa
$\rho = 7750$ kg/m^3

- Determine the deformed shape when $S = 30$ m.

Transparency 9-27

This is a geometrically nonlinear problem (large displacements/large rotations but small strains).

The flexibility of the cable makes the analysis difficult.

- Small perturbations in the nodal coordinates lead to large changes in the out-of-balance loads.
- Use many load steps, with equilibrium iterations, so that the configuration of the cable is never far from an equilibrium configuration.

Transparency 9-28

Solution procedure employed to solve this problem:

- Full Newton iterations without line searches are employed.
- Convergence criteria:

$$\frac{\Delta \underline{U}^{(i)T} \left({}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)}\right)}{\Delta \underline{U}^{(1)T} \left({}^{t+\Delta t}\underline{R} - {}^{t}\underline{F}\right)} \leq 0.001$$

$$\|{}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)}\|_2 \leq 0.01 \text{ N}$$

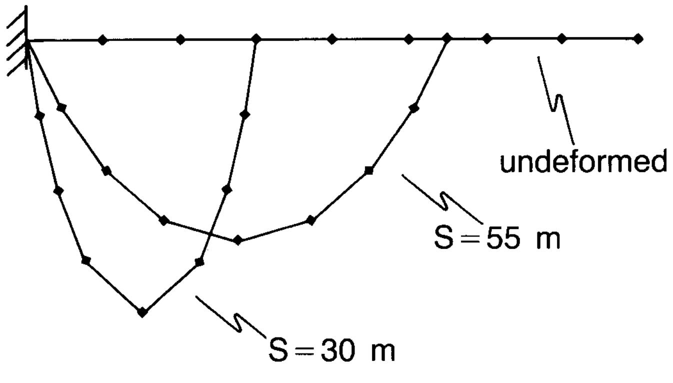
Transparency 9-29

- The gravity loading and the prescribed displacement are applied as follows:

Time step	Comment	Number of equilibrium iterations required per time step
1	The gravity loading is applied.	14
2-1001	The prescribed displacement is applied in 1000 equal steps.	≤5

Transparency 9-30

Pictorially, the results are



Topic 10

Solution of the Nonlinear Finite Element Equations in Static Analysis—Part I

Contents:

- **Short review of Newton-Raphson iteration for the root of a single equation**
- **Newton-Raphson iteration for multiple degree of freedom systems**
- **Derivation of governing equations by Taylor series expansion**
- **Initial stress, modified Newton-Raphson and full Newton-Raphson methods**
- **Demonstrative simple example**
- **Line searches**
- **The Broyden-Fletcher-Goldfarb-Shanno (BFGS) method**
- **Computations in the BFGS method as an effective scheme**
- **Flow charts of modified Newton-Raphson, BFGS, and full Newton-Raphson methods**
- **Convergence criteria and tolerances**

Textbook: Sections 6.1, 8.6, 8.6.1, 8.6.2, 8.6.3

Examples: 6.4, 8.25, 8.26

- WE DERIVED IN THE PREVIOUS LECTURES THE F.E. EQUATIONS

$$ {}^{t}\underline{K}\,\Delta\underline{U}^{(i)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)} $$

$$ {}^{t+\Delta t}\underline{U}^{(i)} = {}^{t+\Delta t}\underline{U}^{(i-1)} + \Delta\underline{U}^{(i)} $$

$$ i = 1, 2, 3, \ldots $$

- IN THIS LECTURE WE CONSIDER VARIOUS TECHNIQUES OF ITERATION AND CONVERGENCE CRITERIA

Markerboard
10-1

Transparency 10-1

SOLUTION OF NONLINEAR EQUATIONS

We want to solve

$$\underbrace{{}^{t+\Delta t}\underline{R}}_{\substack{\text{externally applied}\\ \text{loads}}} - \underbrace{{}^{t+\Delta t}\underline{F}}_{\substack{\text{nodal point forces}\\ \text{corresponding to internal}\\ \text{element stresses}}} = \underline{0}$$

- Loading is deformation-independent
- $${}^{t+\Delta t}\underline{F} = \underbrace{\int_{{}^{0}V} {}^{t+\Delta t}_{\ 0}\underline{B}_L^T \; {}^{t+\Delta t}_{\ 0}\hat{\underline{S}} \; {}^{0}dV}_{\text{T.L. formulation}} = \underbrace{\int_{{}^{t+\Delta t}V} {}^{t+\Delta t}_{t+\Delta t}\underline{B}_L^T \; {}^{t+\Delta t}\hat{\underline{\tau}} \; {}^{t+\Delta t}dV}_{\text{U.L. formulation}}$$

Transparency 10-2

The procedures used are based on the Newton-Raphson method (commonly used to find the roots of an equation).

A historical note:

- Newton gave a version of the method in 1669.
- Raphson generalized and presented the method in 1690.

Both mathematicians used the same concept, and both algorithms gave the same numerical results.

Transparency 10-3

Consider a single Newton-Raphson iteration. We seek a root of f(x), given an estimate to the root, say x_{i-1}, by

$$x_i = x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})}$$

Once x_i is obtained, x_{i+1} may be computed using

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

The process is repeated until the root is obtained.

Transparency 10-4

The formula used for a Newton-Raphson iteration may be derived using a Taylor series expansion.

We can write, for any point x_i and neighboring point x_{i-1},

$$\begin{aligned} f(x_i) &= f(x_{i-1}) + f'(x_{i-1})(x_i - x_{i-1}) \\ &\quad + \text{higher order terms} \\ &\doteq f(x_{i-1}) + f'(x_{i-1})(x_i - x_{i-1}) \end{aligned}$$

Transparency 10-5

Since we want a root of f(x), we set the Taylor series approximation of $f(x_i)$ to zero, and solve for x_i:

$$0 = f(x_{i-1}) + f'(x_{i-1})(x_i - x_{i-1})$$

$$\downarrow$$

$$x_i = x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})}$$

Transparency 10-6

Mathematical example, given merely to demonstrate the Newton-Raphson iteration algorithm:

Let $f(x) = \sin x$, $x_0 = 2$

Using Newton-Raphson iterations, we obtain

i	x_i	error = $\lvert \pi - x_i \rvert$	
0	2.0	1.14	
1	4.185039863	1.04	
2	2.467893675	.67	
3	3.266186277	.12	
4	3.140943912	6.5×10^{-4}	quadratic convergence is observed
5	3.141592654	$< 10^{-9}$	

Transparency 10-7

The approximations obtained using Newton-Raphson iterations exhibit quadratic convergence, if the approximations are "close" to the root.

Mathematically, if $|E_{i-1}| \doteq 10^{-m}$

then $|E_i| \doteq 10^{-2m}$

where E_i is the error in the approximation x_i.

The convergence rate is seen to be quite rapid, once quadratic convergence is obtained.

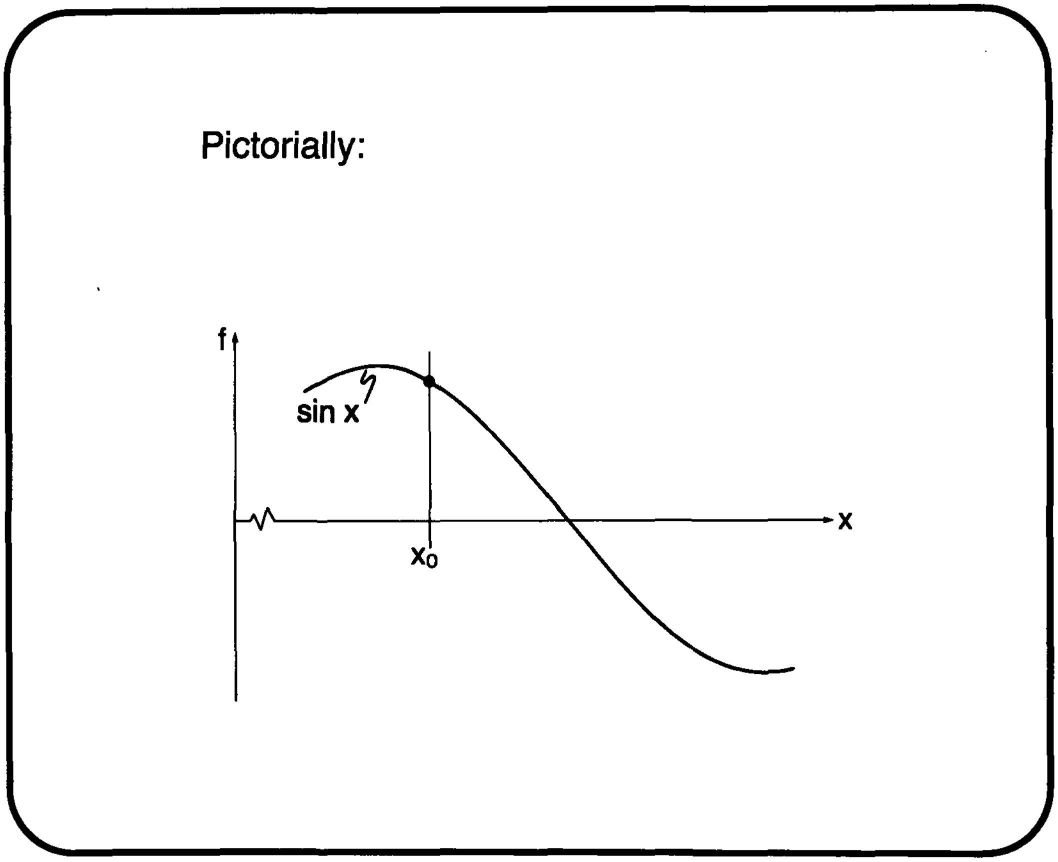
Transparency 10-8

However, if the first approximation x_0 is "far" from the root, Newton-Raphson iterations may not converge to the desired value.

Example: $f(x) = \sin x$, $x_0 = 1.58$

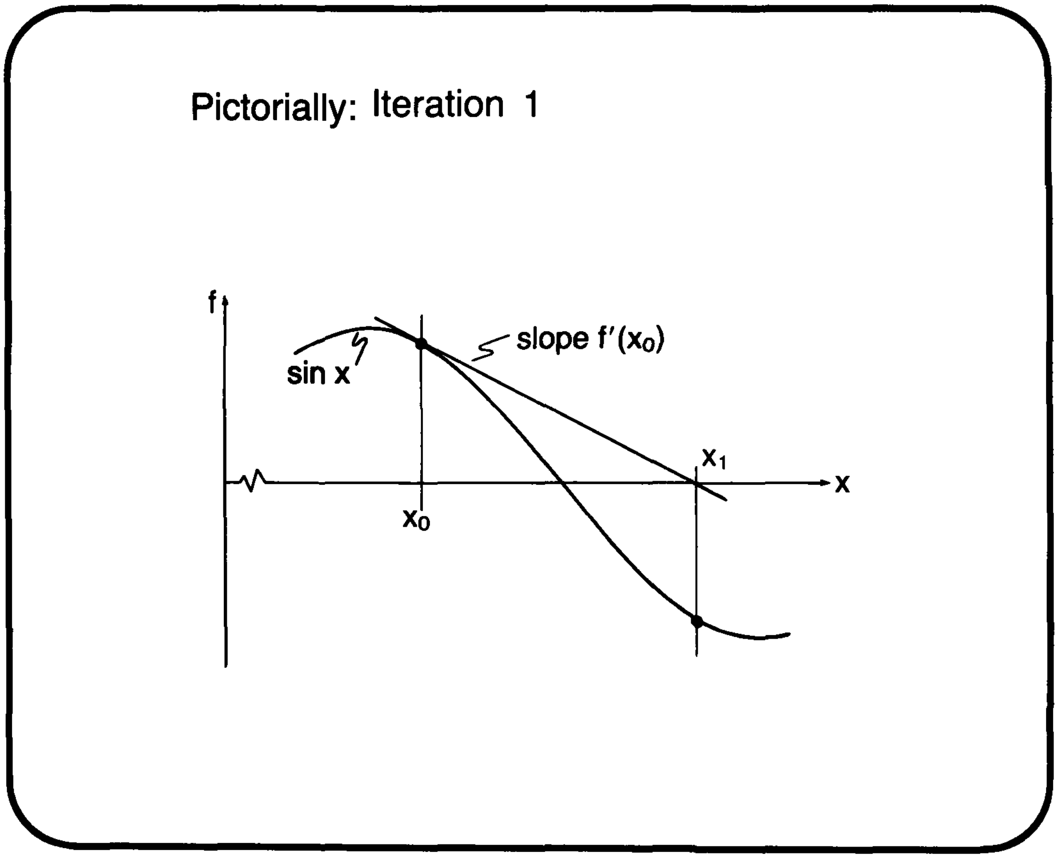
i	x_i	
0	1.58	
1	110.2292036	
2	109.9487161	
3	109.9557430	
4	109.9557429]	not the desired root

Transparency 10-9



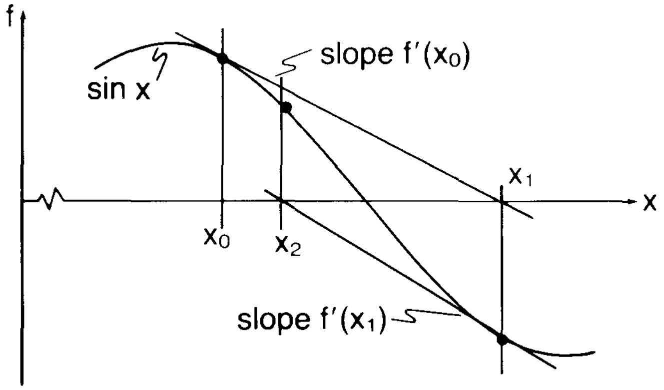
Pictorially:

Transparency 10-10



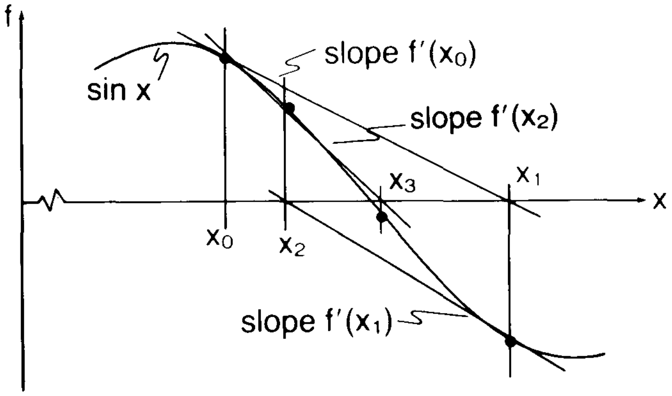
Pictorially: Iteration 1

Pictorially: Iteration 1
Iteration 2



Transparency 10-11

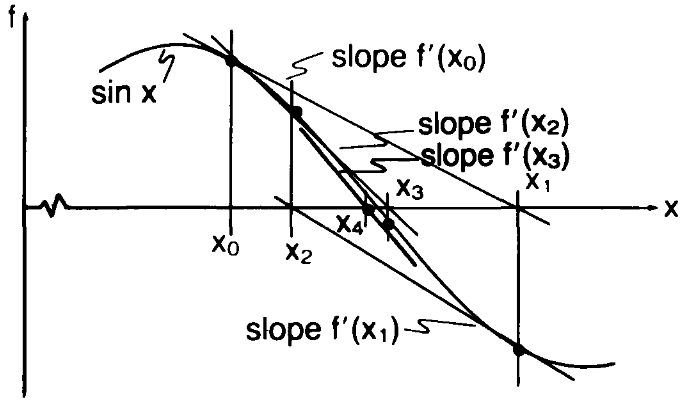
Pictorially: Iteration 1
Iteration 2
Iteration 3



Transparency 10-12

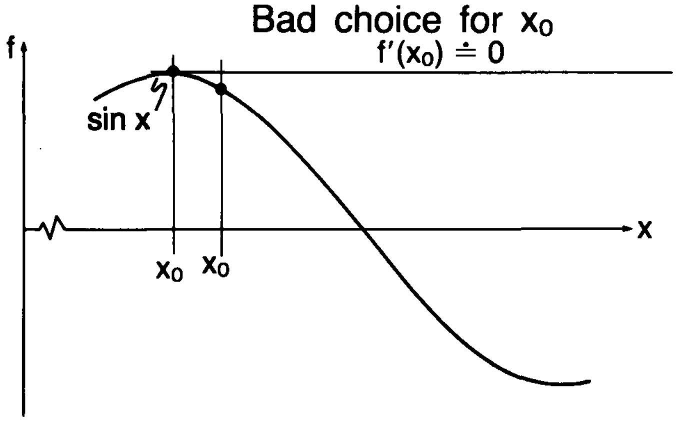
Transparency 10-13

Pictorially: Iteration 1
Iteration 2
Iteration 3
Iteration 4



Transparency 10-14

Pictorially:



Transparency 10-15

Newton-Raphson iterations for multiple degrees of freedom

We would like to solve

$$\underline{f}(\underline{U}) = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F} = \underline{0}$$

where now $\underline{f}$ is a vector (one row for each degree of freedom). For equilibrium, each row in $\underline{f}$ must equal zero.

Transparency 10-16

To derive the iteration formula, we generalize our earlier derivation.

We write

$$\underline{f}({}^{t+\Delta t}\underline{U}^{(i)}) = \underline{f}({}^{t+\Delta t}\underline{U}^{(i-1)}) + \left[\frac{\partial \underline{f}}{\partial \underline{U}}\right]\Bigg|_{{}^{t+\Delta t}\underline{U}^{(i-1)}} ({}^{t+\Delta t}\underline{U}^{(i)} - {}^{t+\Delta t}\underline{U}^{(i-1)}) + \underbrace{\text{higher order terms}}_{\substack{\text{neglected to obtain a} \\ \text{Taylor series approximation}}}$$

Transparency 10-17

Since we want a root of $\underline{f}(\underline{U})$, we set the Taylor series approximation of $\underline{f}({}^{t+\Delta t}\underline{U}^{(i)})$ to zero.

$$\underline{0} = \underline{f}({}^{t+\Delta t}\underline{U}^{(i-1)}) + \left[\frac{\partial \underline{f}}{\partial \underline{U}}\right]\Bigg|_{{}^{t+\Delta t}\underline{U}^{(i-1)}} \overbrace{\Delta \underline{U}^{(i)}}^{{}^{t+\Delta t}\underline{U}^{(i)} - {}^{t+\Delta t}\underline{U}^{(i-1)}}$$

Transparency 10-18

or

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}\Bigg|_{{}^{t+\Delta t}\underline{U}^{(i-1)}} + \underbrace{\begin{bmatrix} \dfrac{\partial f_1}{\partial U_1} & \cdots & \dfrac{\partial f_1}{\partial U_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial U_1} & \cdots & \dfrac{\partial f_n}{\partial U_n} \end{bmatrix}}_{\text{a square matrix}}\Bigg|_{{}^{t+\Delta t}\underline{U}^{(i-1)}} \begin{bmatrix} \Delta U_1^{(i)} \\ \vdots \\ \Delta U_n^{(i)} \end{bmatrix}$$

We now use

$$\underline{f}({}^{t+\Delta t}\underline{U}^{(i-1)}) = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)},$$

$$\left.\frac{\partial \underline{f}}{\partial \underline{U}}\right|_{{}^{t+\Delta t}\underline{U}^{(i-1)}} = \underbrace{\left.\cancelto{0}{\left[\frac{\partial\,{}^{t+\Delta t}\underline{R}}{\partial \underline{U}}\right]}\right|_{{}^{t+\Delta t}\underline{U}^{(i-1)}}}_{\text{because the loads are deformation-independent}} - \underbrace{\left.\left[\frac{\partial\,{}^{t+\Delta t}\underline{F}^{(i-1)}}{\partial \underline{U}}\right]\right|_{{}^{t+\Delta t}\underline{U}^{(i-1)}}}_{}$$

$$= -{}^{t+\Delta t}\underbrace{\underline{K}}_{\text{the } \underline{\text{tangent}} \text{ stiffness matrix}}{}^{(i-1)}$$

Transparency 10-19

<u></u>

Important: ${}^{t+\Delta t}\underline{K}^{(i-1)}$ is symmetric because

- We used symmetric stress and strain measures in our governing equation.
- We interpolated the real displacements and the virtual displacements with exactly the same functions.
- We assumed that the loading was deformation-independent.

Transparency 10-20

Transparency 10-21

Our final result is

$${}^{t+\Delta t}\underline{K}^{(i-1)}\,\Delta\underline{U}^{(i)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)}$$

This is a set of simultaneous linear equations, which can be solved for $\Delta\underline{U}^{(i)}$. Then

$${}^{t+\Delta t}\underline{U}^{(i)} = {}^{t+\Delta t}\underline{U}^{(i-1)} + \Delta\underline{U}^{(i)}$$

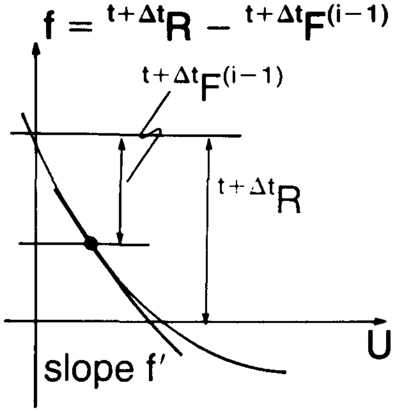
Transparency 10-22

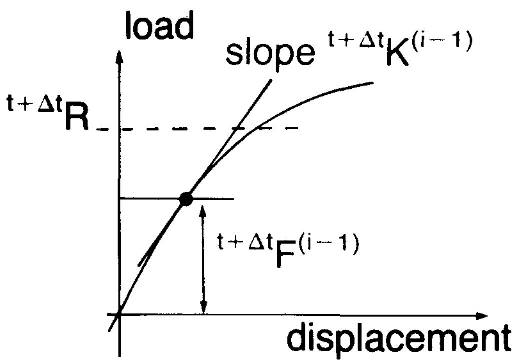
This iteration scheme is referred to as the full Newton-Raphson method (we update the stiffness matrix in each iteration).

The full Newton-Raphson iteration shows mathematically quadratic convergence when solving for the root of an algebraic equation. In finite element analysis, a number of requirements must be fulfilled (for example, the updating of stresses, rotations need careful attention) to actually achieve quadratic convergence.

Transparency 10-23

We can depict the iteration process in two equivalent ways:





This is like a force-deflection curve. We use this representation henceforth.

Transparency 10-24

Modifications:

$${}^{\tau}\underline{K}\,\Delta\underline{U}^{(i)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)}$$

- $\tau = 0$: Initial stress method
- $\tau = t$: Modified Newton method
- Or, more effectively, we update the stiffness matrix at certain times only.

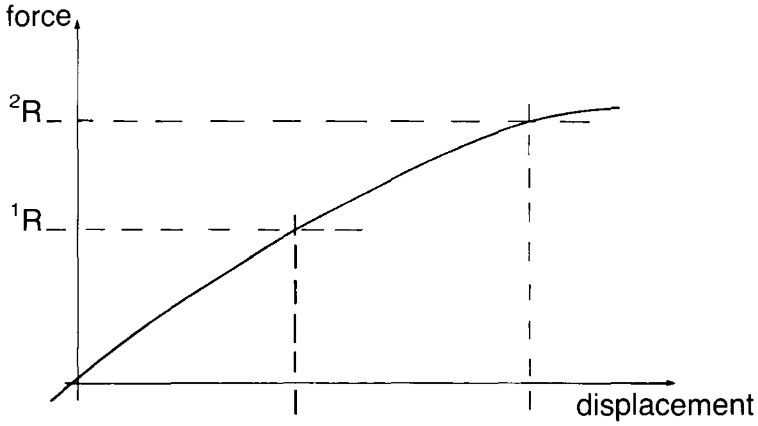
Transparency 10-25

We note:

- The initial stress method and the modified Newton method are much less expensive than the full Newton method per iteration.
- However, many more iterations are necessary to achieve the same accuracy.
- The initial stress method and the modified Newton method "cannot" exhibit quadratic convergence.

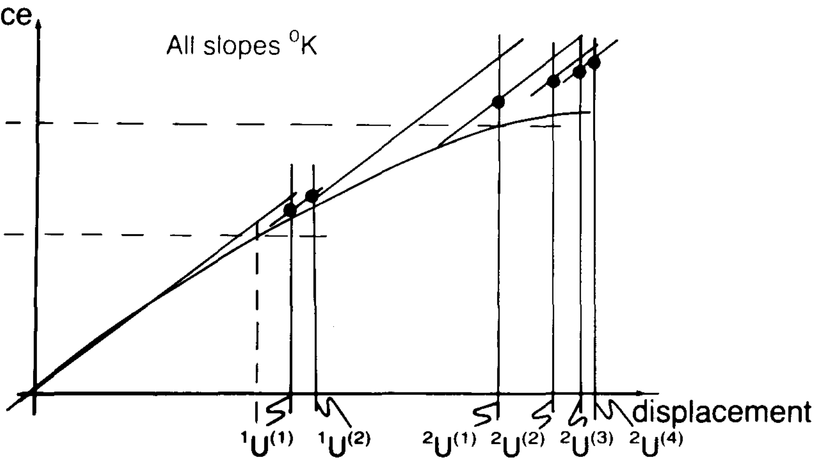
Transparency 10-26

Example: One degree of freedom, two load steps



Initial stress method: $\tau = 0$

Example: One degree of freedom, two load steps



Transparency 10-27

Line searches:

We solve

$$^{\tau}\underline{K}\,\Delta\underline{\bar{U}} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)}$$

and consider forming ${}^{t+\Delta t}\underline{F}^{(i)}$ using

$${}^{t+\Delta t}\underline{U}^{(i)} = {}^{t+\Delta t}\underline{U}^{(i-1)} + \beta\,\Delta\underline{\bar{U}}$$

where we choose β so as to make ${}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i)}$ small “in some sense”.

Transparency 10-28

Transparency 10-29

Aside:

If, for all possible $\underline{U}$, the number

$$\underline{U}^T\,({}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i)}) = 0$$

then $\quad {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i)} = \underline{0}$

Reason: consider

$$\underline{U}^T = [0 \quad 0 \quad 0 \quad \cdots \quad \overset{\text{any row of } \underline{U}}{\overline{1}} \quad \cdots \quad 0 \quad 0]$$

This isolates one row of ${}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i)}$

Transparency 10-30

During the line search, we choose $\underline{U} = \Delta\bar{\underline{U}}$ and seek β such that

$$\Delta\bar{\underline{U}}^T\,({}^{t+\Delta t}\underline{R} - \underbrace{{}^{t+\Delta t}\underline{F}^{(i)}}_{\text{a function of } \beta}) = 0$$

since ${}^{t+\Delta t}\underline{U}^{(i)} = {}^{t+\Delta t}\underline{U}^{(i-1)} + \beta\,\Delta\bar{\underline{U}}$

In practice, we use

$$\frac{\Delta\bar{\underline{U}}^T\,({}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i)})}{\Delta\bar{\underline{U}}^T\,({}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)})} \le \underset{\text{a convergence tolerance}}{\underline{STOL}}$$

Transparency 10-31

BFGS (Broyden-Fletcher-Goldfarb-Shanno) method:

We define

$$\underline{\delta}^{(i)} = {}^{t+\Delta t}\underline{U}^{(i)} - {}^{t+\Delta t}\underline{U}^{(i-1)}$$

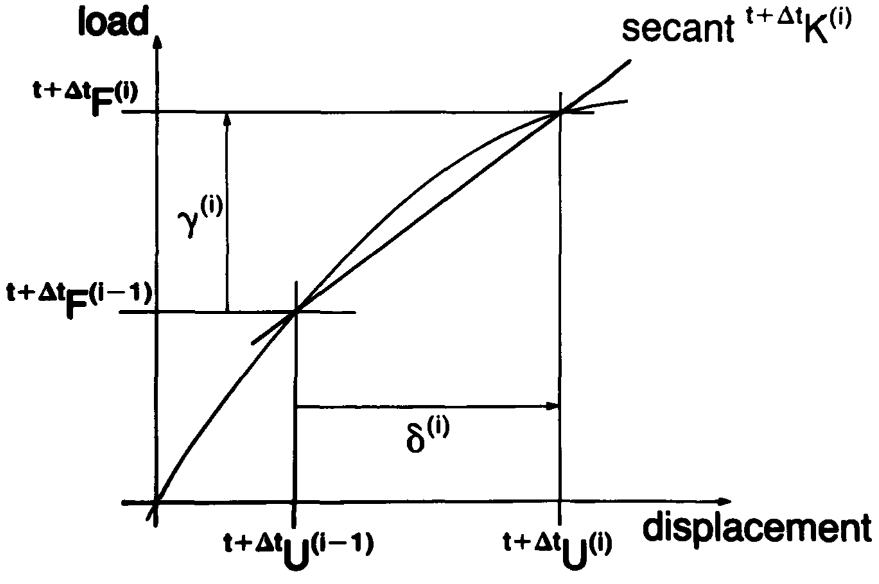
$$\underline{\gamma}^{(i)} = {}^{t+\Delta t}\underline{F}^{(i)} - {}^{t+\Delta t}\underline{F}^{(i-1)}$$

and want a coefficient matrix such that

$$({}^{t+\Delta t}\underline{K}^{(i)})\ \underline{\delta}^{(i)} = \underline{\gamma}^{(i)}$$

Transparency 10-32

Pictorially, for one degree of freedom,



Transparency 10-33

- The BFGS method is an iterative algorithm which produces successive approximations to an effective stiffness matrix (actually, to its inverse).
- A compromise between the full Newton method and the modified Newton method

Transparency 10-34

Step 1: Calculate direction of displacement increment

$$\Delta\bar{\underline{U}}^{(i)} = ({}^{t+\Delta t}\underline{K}^{-1})^{(i-1)}\,({}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)})$$

(Note: We do not calculate the inverse of the coefficient matrix; we use the usual $\underline{L}\,\underline{D}\,\underline{L}^T$ factorization)

Transparency 10-35

Step 2: Line search

$${}^{t+\Delta t}\underline{U}^{(i)} = {}^{t+\Delta t}\underline{U}^{(i-1)} + \beta\, \Delta\bar{\underline{U}}^{(i)}$$

$$\frac{\Delta\bar{\underline{U}}^{(i)T}\left({}^{t+\Delta t}\underline{R} - \overbrace{{}^{t+\Delta t}\underline{F}^{(i)}}^{\text{a function of } \beta}\right)}{\Delta\bar{\underline{U}}^{(i)T}\left({}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)}\right)} \le \text{STOL}$$

Hence we can now calculate $\underline{\delta}^{(i)}$ and $\underline{\gamma}^{(i)}$.

Transparency 10-36

Step 3: Calculation of the new "secant" matrix

$$({}^{t+\Delta t}\underline{K}^{-1})^{(i)} = \underline{A}^{(i)T}\, ({}^{t+\Delta t}\underline{K}^{-1})^{(i-1)}\, \underline{A}^{(i)}$$

where

$\underline{A}^{(i)} = \underline{I} + \underline{v}^{(i)}\ \underline{w}^{(i)T}$

$\underline{v}^{(i)}$ = vector, function of $\underline{\delta}^{(i)}, \underline{\gamma}^{(i)}, {}^{t+\Delta t}\underline{K}^{(i-1)}$

$\underline{w}^{(i)}$ = vector, function of $\underline{\delta}^{(i)}, \underline{\gamma}^{(i)}$

See the textbook.

Transparency 10-37

Important:

- Only vector products are needed to obtain $\underline{v}^{(i)}$ and $\underline{w}^{(i)}$.
- Only vector products are used to calculate $\Delta\bar{\underline{U}}^{(i)}$.

Transparency 10-38

Reason:

$$\begin{aligned}\Delta\bar{\underline{U}}^{(i)} = \{&(\underline{I} + \underline{w}^{(i-1)}\,\underline{v}^{(i-1)T}) \cdots \\ &(\underline{I} + \underline{w}^{(1)}\,\underline{v}^{(1)T})\ {}^{\tau}\underline{K}^{-1}\ (\underline{I} + \underline{v}^{(1)}\,\underline{w}^{(1)T}) \\ &\cdots (\underline{I} + \underline{v}^{(i-1)}\,\underline{w}^{(i-1)T})\} \times \\ &[{}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)}]\end{aligned}$$

Transparency 10-39

In summary

The following solution procedures are most effective, depending on the application.

1) Modified Newton-Raphson iteration with line searches

$$ {}^{t}\underline{K}\,\Delta\bar{\underline{U}}^{(i)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)} $$

$$ {}^{t+\Delta t}\underline{U}^{(i)} = {}^{t+\Delta t}\underline{U}^{(i-1)} + \underbrace{\beta}_{\substack{\text{determined by the}\\ \text{line search}}}\,\Delta\bar{\underline{U}}^{(i)} $$

Transparency 10-40

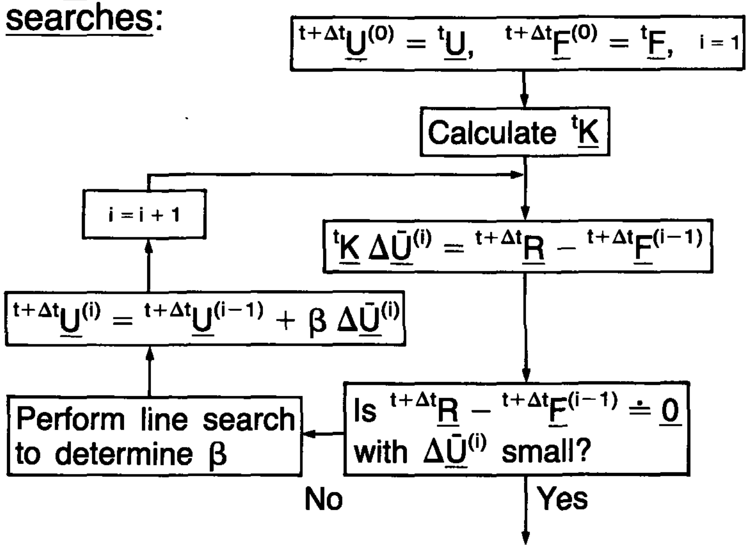
2) BFGS method with line searches

3) Full Newton-Raphson iteration with or without line searches
(full Newton-Raphson iteration with line searches is most powerful)

But, these methods cannot directly be used for post-buckling analyses.

Transparency 10-41

Modified Newton iteration with line searches:



$^{t+\Delta t}\underline{U}^{(0)} = {}^{t}\underline{U}, \quad {}^{t+\Delta t}\underline{F}^{(0)} = {}^{t}\underline{F}, \quad i = 1$

Calculate $^{t}\underline{K}$

$^{t}\underline{K}\,\Delta\bar{\underline{U}}^{(i)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)}$

Is $^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)} \doteq \underline{0}$ with $\Delta\bar{\underline{U}}^{(i)}$ small?

Yes

No

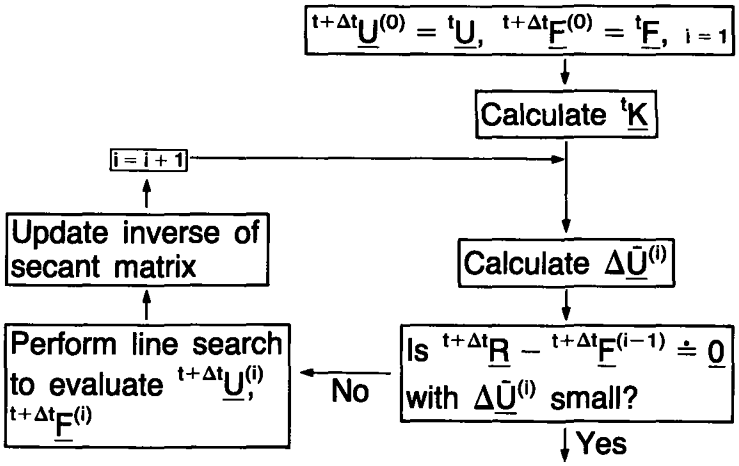
Perform line search to determine β

$^{t+\Delta t}\underline{U}^{(i)} = {}^{t+\Delta t}\underline{U}^{(i-1)} + \beta\,\Delta\bar{\underline{U}}^{(i)}$

$i = i + 1$

Transparency 10-42

BFGS method:



$^{t+\Delta t}\underline{U}^{(0)} = {}^{t}\underline{U}, \quad {}^{t+\Delta t}\underline{F}^{(0)} = {}^{t}\underline{F}, \quad i = 1$

Calculate $^{t}\underline{K}$

Calculate $\Delta\bar{\underline{U}}^{(i)}$

Is $^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)} \doteq \underline{0}$ with $\Delta\bar{\underline{U}}^{(i)}$ small?

Yes

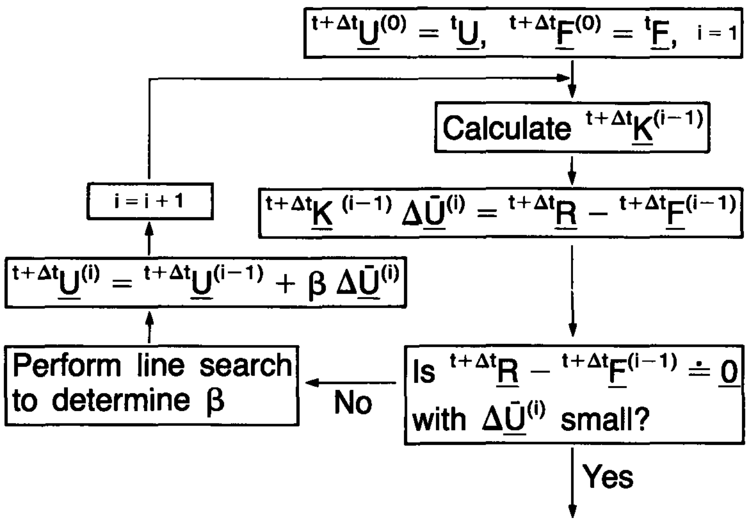
No

Perform line search to evaluate $^{t+\Delta t}\underline{U}^{(i)}$, $^{t+\Delta t}\underline{F}^{(i)}$

Update inverse of secant matrix

$i = i + 1$

Full Newton iteration with line searches:



$^{t+\Delta t}\underline{U}^{(0)} = {}^{t}\underline{U}, \quad {}^{t+\Delta t}\underline{F}^{(0)} = {}^{t}\underline{F}, \quad i = 1$

Calculate $^{t+\Delta t}\underline{K}^{(i-1)}$

$^{t+\Delta t}\underline{K}^{(i-1)} \Delta\bar{\underline{U}}^{(i)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)}$

Is $^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)} \doteq \underline{0}$ with $\Delta\bar{\underline{U}}^{(i)}$ small?

No

Perform line search to determine β

$^{t+\Delta t}\underline{U}^{(i)} = {}^{t+\Delta t}\underline{U}^{(i-1)} + \beta \Delta\bar{\underline{U}}^{(i)}$

$i = i + 1$

Yes

Transparency 10-43

Convergence criteria:

- These measure how well the obtained solution satisfies equilibrium.
- We use
  1) Energy
  2) Force (or moment)
  3) Displacement

Transparency 10-44

Transparency 10-45

On energy:

$$\frac{\Delta\underline{\bar{U}}^{(i)T}\left({}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)}\right)}{\Delta\underline{\bar{U}}^{(1)T}\left({}^{t+\Delta t}\underline{R} - {}^{t}\underline{F}\right)} \leq \text{ETOL}$$

(Note : applied prior to line searching)

Transparency 10-46

On forces:

$$\frac{\|{}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)}\|_2}{\underbrace{\text{RNORM}}_{\text{reference force}}} \leq \text{RTOL}$$

(for moments, use RMNORM)

Typically, $\text{RTOL} = 0.01$

$$\text{RNORM} = \max \underbrace{\|{}^{t}\underline{R}\|_2}_{}$$

considering only translational degrees of freedom

Note: $\|\underline{a}\|_2 = \sqrt{\sum_k (a_k)^2}$

Transparency 10-47

On displacements:

$$\frac{\|\Delta \underline{\bar{U}}^{(i)}\|_2}{\text{DNORM}} \leq \text{DTOL}$$

reference displacement
(for rotations, use DMNORM)

Topic 11

Solution of the Nonlinear Finite Element Equations in Static Analysis—Part II

Contents:

- **Automatic load step incrementation for collapse and post-buckling analysis**
- **Constant arc-length and constant increment of work constraints**
- **Geometrical interpretations**
- **An algorithm for automatic load incrementation**
- **Linearized buckling analysis, solution of eigenproblem**
- **Value of linearized buckling analysis**
- **Example analysis: Collapse of an arch—linearized buckling analysis and automatic load step incrementation, effect of initial geometric imperfections**

Textbook: Sections 6.1, 6.5.2

Reference: The automatic load stepping scheme is presented in

Bathe, K. J., and E. Dvorkin, "On the Automatic Solution of Nonlinear Finite Element Equations," *Computers & Structures, 17,* 871–879, 1983.

- WE DISCUSSED IN THE PREVIOUS LECTURE SOLUTION SCHEMES TO SOLVE

$$^{t+\Delta t}\underline{R} = {}^{t+\Delta t}\underline{F}$$

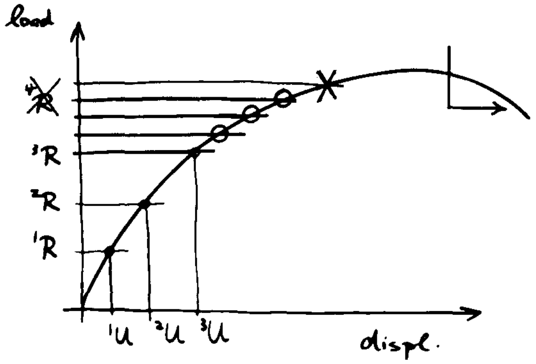
WITH $^{t+\Delta t}\underline{R}$ PRESCRIBED FOR EACH LOAD LEVEL

<u>EXAMPLE</u>:

$$^{t}\underline{K}\,\Delta\underline{U}^{(i)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)}$$

$$^{t+\Delta t}\underline{U}^{(i)} = {}^{t+\Delta t}\underline{U}^{(i-1)} + \Delta\underline{U}^{(i)}$$

SCHEMATICALLY:



- DIFFICULTIES ARE ENCOUNTERED TO CALCULATE COLLAPSE LOADS

Markerboard 11-1

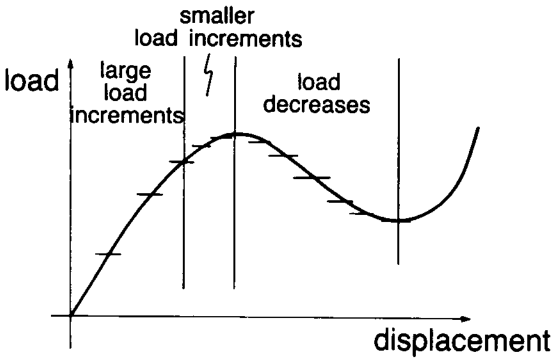
Transparency 11-1

AUTOMATIC LOAD STEP INCREMENTATION

- To obtain more rapid convergence in each load step
- To have the program select load increments automatically
- To solve for post-buckling response

Transparency 11-2

An effective solution procedure would proceed with varying load step sizes:

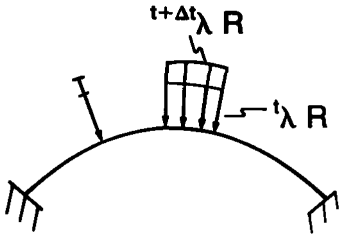


- Load increment for each step is to be adjusted in magnitude for rapid convergence.

We compute $^{t+\Delta t}\underline{R}$ using

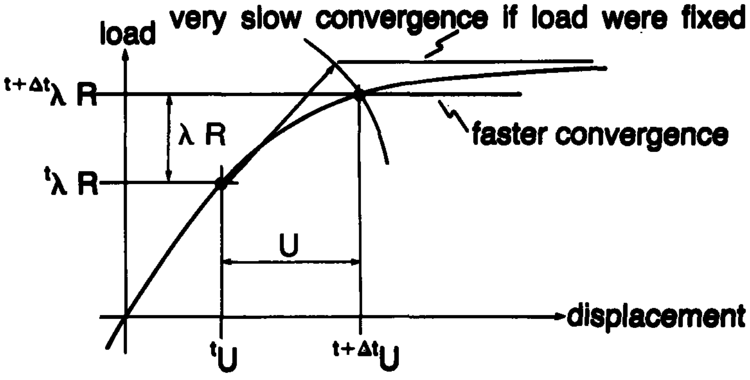
$$^{t+\Delta t}\underline{R} = {}^{t+\Delta t}\lambda \underbrace{\underline{R}}_{\text{a constant vector}}$$

Hence we assume: Deformation-independent loading.
All loads are identically scaled.



Transparency 11-3

The basic approach:



$$^{t+\Delta t}\lambda^{(i)} = {}^{t}\lambda + \underbrace{\lambda^{(i)}}_{\sum \Delta\lambda^{(k)}}$$

$$^{t+\Delta t}U^{(i)} = {}^{t}U + \underbrace{U^{(i)}}_{\sum \Delta U^{(k)}}$$

Transparency 11-4

Transparency 11-5

The governing equations are now:

$$ {}^{\tau}\underline{K}\,\Delta\underline{U}^{(i)} = \underbrace{({}^{t+\Delta t}\lambda^{(i-1)} + \Delta\lambda^{(i)})}_{{}^{t+\Delta t}\lambda^{(i)}}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)} $$

with a constraint equation

$$ f(\Delta\lambda^{(i)}, \Delta\underline{U}^{(i)}) = 0 $$

The unknowns are $\Delta\underline{U}^{(i)}$, $\Delta\lambda^{(i)}$.

$\tau = t$ in the modified Newton-Raphson iteration.

Transparency 11-6

We may rewrite the equilibrium equations to obtain

$$ {}^{\tau}\underline{K}\,\Delta\bar{\underline{U}}^{(i)} = {}^{t+\Delta t}\lambda^{(i-1)}\,\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)} $$

$$ \underbrace{{}^{\tau}\underline{K}\,\Delta\bar{\bar{\underline{U}}} = \underline{R}} $$

only solve this once for each load step.

Hence, we can add these to obtain

$$ \Delta\underline{U}^{(i)} = \Delta\bar{\underline{U}}^{(i)} + \Delta\lambda^{(i)}\,\Delta\bar{\bar{\underline{U}}} $$

Transparency 11-7

Constraint equations:

$\boxed{1}$ Spherical constant arc-length criterion

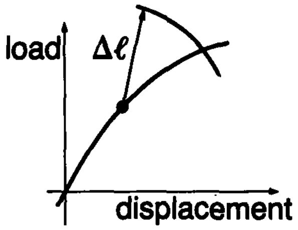
$$(\lambda^{(i)})^2 + (\underline{U}^{(i)})^T\,(\underline{U}^{(i)})/\beta = (\Delta\ell)^2$$

where

$\lambda^{(i)} = {}^{t+\Delta t}\lambda^{(i)} - {}^{t}\lambda$

$\underline{U}^{(i)} = {}^{t+\Delta t}\underline{U}^{(i)} - {}^{t}\underline{U}$

$\beta =$ A normalizing factor applied to displacement components (to make all terms dimensionless)



Transparency 11-8

This equation may be solved for $\Delta\lambda^{(i)}$ as follows:

Using $\lambda^{(i)} = \lambda^{(i-1)} + \Delta\lambda^{(i)}$

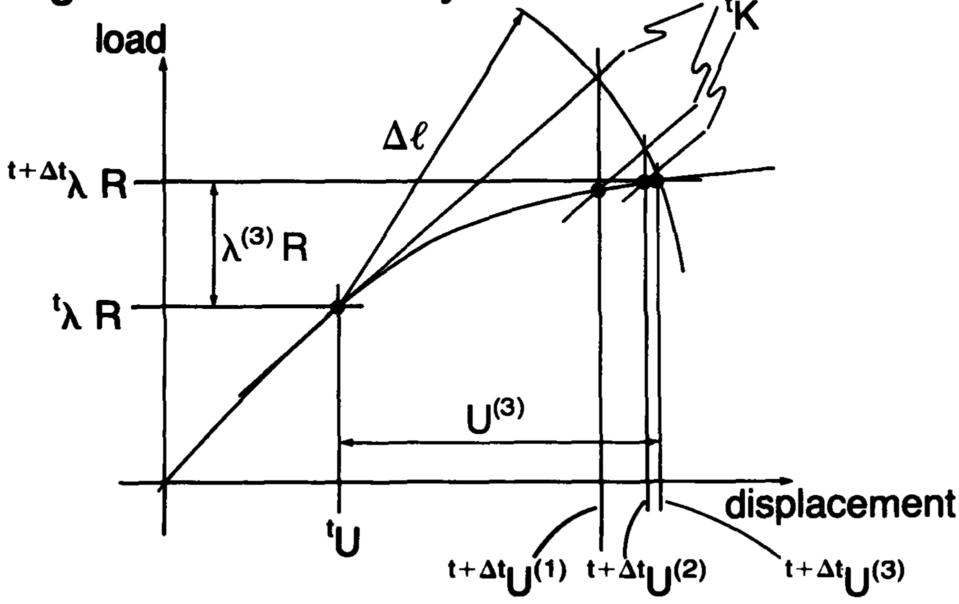
and $\underline{U}^{(i)} = \underline{U}^{(i-1)} + \Delta\underline{U}^{(i)}$

$= \underline{U}^{(i-1)} + \Delta\bar{\underline{U}}^{(i)} + \Delta\lambda^{(i)}\,\Delta\bar{\bar{\underline{U}}}$

we obtain a quadratic equation in $\Delta\lambda^{(i)}$ ($\Delta\bar{\underline{U}}^{(i)}$ and $\Delta\bar{\bar{\underline{U}}}$ are known vectors).

Transparency 11-9

Geometrical interpretation for single degree of freedom system:

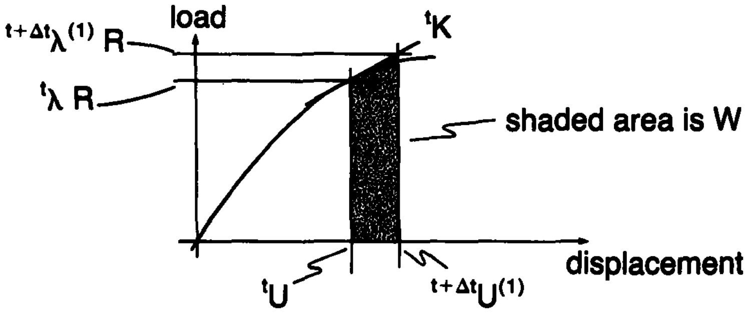


Transparency 11-10

② "Constant" increment of external work criterion

First iteration: $\left({}^{t}\lambda + \frac{1}{2}\Delta\lambda^{(1)}\right)\underline{R}^{T}\,\Delta\underline{U}^{(1)} = W$

where W is the (preselected) increment in external work:



Transparency 11-11

Successive iterations (i = 2, 3, ...)

$$\left({}^{t+\Delta t}\lambda^{(i-1)} + \frac{1}{2}\Delta\lambda^{(i)}\right) \underline{R}^T \, \Delta\underline{U}^{(i)} = 0$$

This has solutions:

- $\underline{R}^T \Delta\underline{U}^{(i)} = 0 \qquad \left(\Delta\lambda^{(i)} = -\dfrac{\underline{R}^T \, \Delta\underline{\bar{U}}^{(i)}}{\underline{R}^T \, \Delta\underline{\bar{\bar{U}}}}\right)$
- $\underbrace{{}^{t+\Delta t}\lambda^{(i)} = -{}^{t+\Delta t}\lambda^{(i-1)}}_{}$

load reverses direction
(This solution is disregarded)

Transparency 11-12

Our algorithm:

- Specify $\underline{R}$ and the displacement at one degree of freedom corresponding to ${}^{\Delta t}\lambda$. Solve for ${}^{\Delta t}\underline{U}$.
- Set $\Delta\ell$.
- Use $\boxed{1}$ for the next load steps.
- Calculate W for each load step. When W does not change appreciably, or difficulties are encountered with $\boxed{1}$, use $\boxed{2}$ for the next load step.

Transparency 11-13

- Note that $\Delta \ell$ is adjusted for the next load step based on the number of iterations used in the last load step.

- Also, ${}^{\tau}\underline{K}$ is recalculated when convergence is slow. Full Newton-Raphson iterations are automatically employed when deemed more effective.

Transparency 11-14

Linearized buckling analysis:

The physical phenomena of buckling or collapse are represented by the mathematical criterion

$$\det({}^{\tau}\underline{K}) = 0$$

where τ denotes the load level associated with buckling or collapse.

Transparency 11-15

The criterion det $({}^{T}\underline{K}) = 0$ implies that the equation

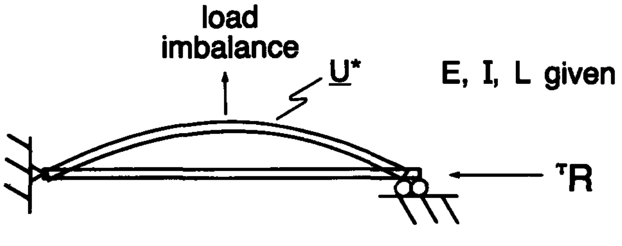
$$ {}^{T}\underline{K}\,\underline{U}^{*} = \underline{0} $$

has a non-trivial solution for $\underline{U}^{*}$ (and α U* is a solution with α being any constant). Hence we can select a small load ε for which very large displacements are obtained.

This means that the structure is unstable.

Transparency 11-16

Physically, the smallest load imbalance will trigger the buckling (collapse) displacements:



$$ {}^{T}R = \frac{\pi^{2}\,EI}{L^{2}} $$

Pinned-pinned beam

Transparency 11-17

We want to predict the load level and mode shape associated with buckling or collapse. Hence we perform a linearized buckling analysis.

We assume

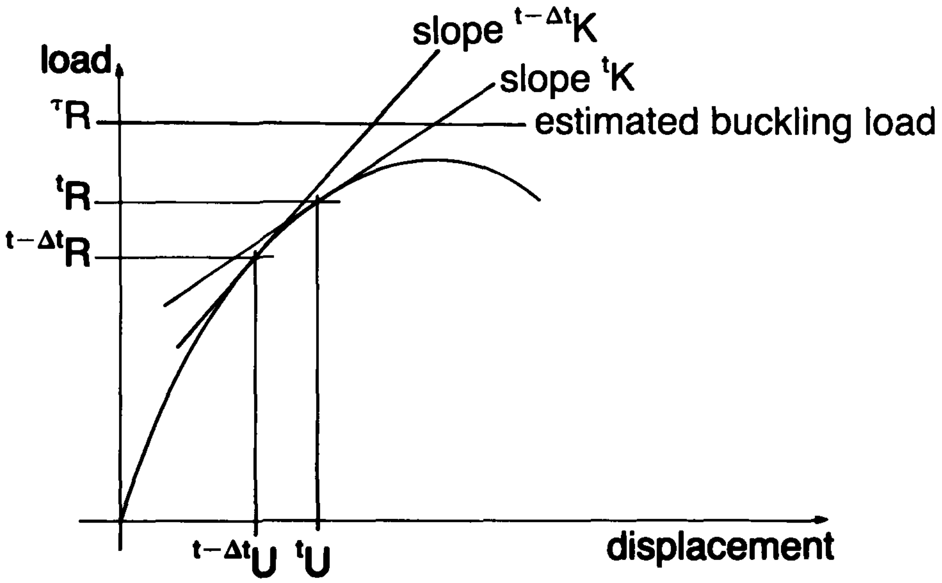
$$ {}^{\tau}\underline{K} = {}^{t-\Delta t}\underline{K} + \lambda \left({}^{t}\underline{K} - {}^{t-\Delta t}\underline{K} \right) $$

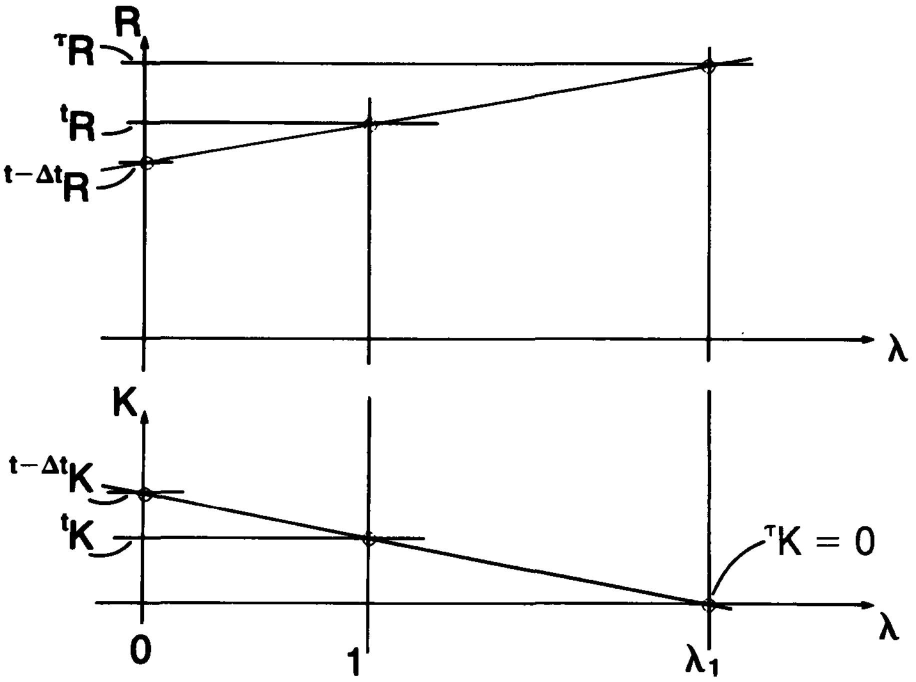
$$ {}^{\tau}\underline{R} = {}^{t-\Delta t}\underline{R} + \lambda \left({}^{t}\underline{R} - {}^{t-\Delta t}\underline{R} \right) $$

λ is a scaling factor which we determine below. We assume here that the value λ we require is greater than 1.

Transparency 11-18

Pictorially, for one degree of freedom:





Transparency 11-19

The problem of solving for λ such that $\det({}^{\tau}\underline{K}) = 0$ is equivalent to the eigenproblem

$$^{t-\Delta t}\underline{K}\,\underline{\phi} = \lambda\,({}^{t-\Delta t}\underline{K} - {}^{t}\underline{K})\,\underline{\phi}$$

where $\underline{\phi}$ is the associated eigenvector (buckling mode shape).

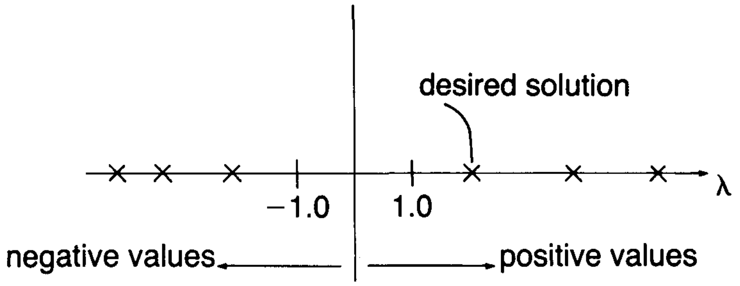
In general, $^{t-\Delta t}\underline{K} - {}^{t}\underline{K}$ is indefinite, hence the eigenproblem will have both positive and negative solutions. We want only the smallest positive λ value (and perhaps the next few larger values).

Transparency 11-20

Transparency 11-21

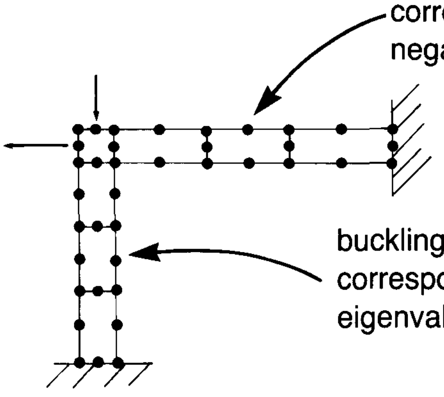
Solution of problem

$$ {}^{t-\Delta t}\underline{K}\,\underline{\phi} = \lambda\,({}^{t-\Delta t}\underline{K} - {}^{t}\underline{K})\,\underline{\phi} $$


desired solution

Transparency 11-22

Example of model with both positive and negative eigenvalues:


buckling of this member corresponds to a negative eigenvalue

buckling of this member corresponds to a positive eigenvalue

Transparency 11-23

We rewrite the eigenvalue problem as follows:

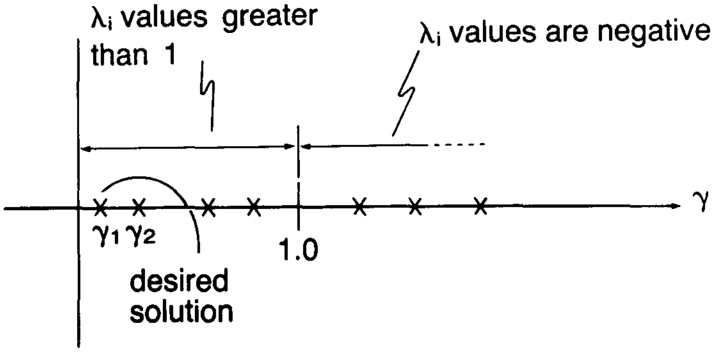
$${}^{t}\underline{K}\,\underline{\phi} = \underbrace{\left(\frac{\lambda - 1}{\lambda}\right)}_{\gamma}\,{}^{t-\Delta t}\underline{K}\,\underline{\phi}$$

Now we note that the critical buckling mode of interest is the one for which γ is small and positive.

Transparency 11-24

Solution of problem

$${}^{t}\underline{K}\,\underline{\phi} = \gamma\,{}^{t-\Delta t}\underline{K}\,\underline{\phi}; \quad \gamma = \frac{\lambda - 1}{\lambda}$$



Transparency 11-25

Value of linearized buckling analysis:

- Not expensive
- Gives insight into possible modes of failure.
- For applicability, important that pre-buckling displacements are small.
- Yields collapse modes that are effectively used to impose imperfections.
  - — To study sensitivity of a structure to imperfections

Transparency 11-26

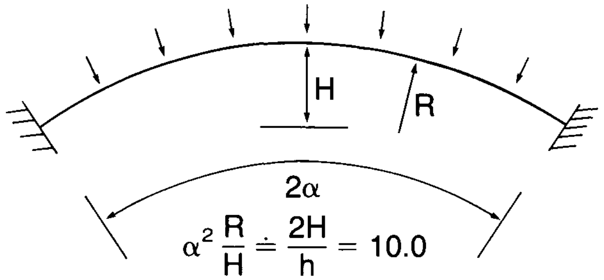
But

— procedure must be employed with great care because the results may be quite misleading.

— procedure only predicts physically realistic buckling or collapse loads when structure buckles "in the Euler column type".

Example: Arch

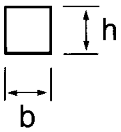
uniform pressure load ${}^tp$



$$\alpha^2 \frac{R}{H} \doteq \frac{2H}{h} = 10.0$$

$R = 64.85$
$\alpha = 22.5°$
$E = 2.1 \times 10^6$
$\nu = 0.3$
$h = b = 1.0$

Cross-section:



Transparency 11-27

Finite element model:

- Ten 2-node isoparametric beam elements
- Complete arch is modeled.

Purpose of analysis:

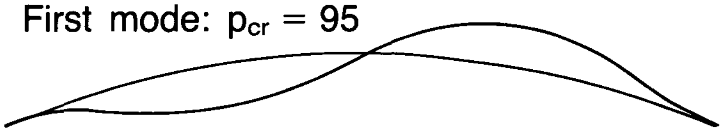
- To determine the collapse mechanism and collapse load level.
- To compute the post-collapse response.

Transparency 11-28

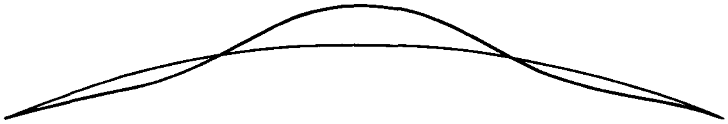
Transparency 11-29

Step 1: Determine collapse mechanisms and collapse loads using a linearized buckling analysis ($^{\Delta t}p = 10$).

First mode: $p_{cr} = 95$

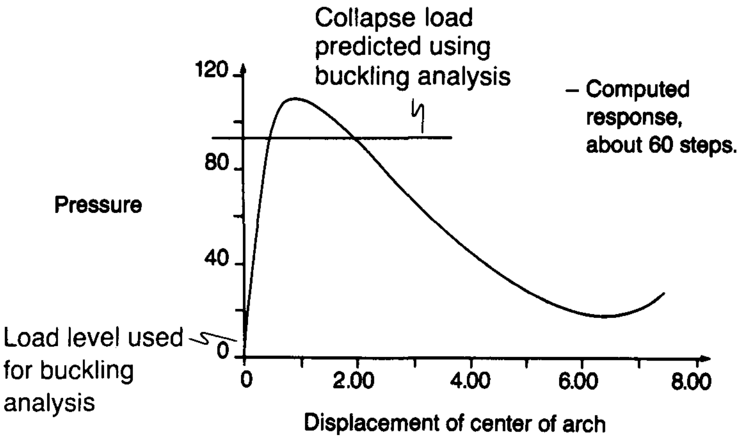


Second mode: $p_{cr} = 150$



Transparency 11-30

Step 2: Compute the response of the arch using automatic step incrementation.



We have computed the response of a perfect (symmetric) arch. Because the first collapse mode is antisymmetric, that mode is not excited by the pressure loading during the response calculations.

However, a real structure will contain imperfections, and hence will not be symmetric. Therefore, the antisymmetric collapse mode may be excited, resulting in a lower collapse load.

Transparency 11-31

Hence, we adjust the initial coordinates of the arch to introduce a geometric imperfection. This is done by adding a multiple of the first buckling mode to the geometry of the undeformed arch.

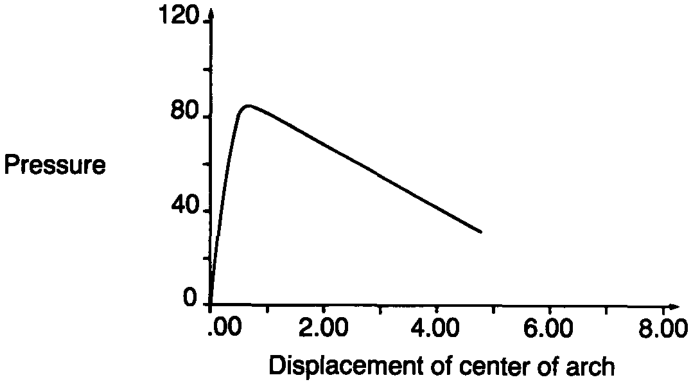
The collapse mode is scaled so that the magnitude of the imperfection is less than 0.01.

The resulting "imperfect" arch is no longer symmetric.

Transparency 11-32

Transparency 11-33

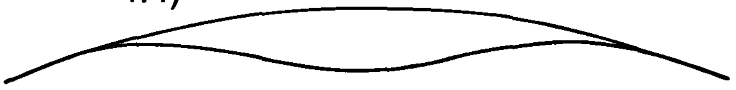
Step 3: Compute the response of the "imperfect" arch using automatic step incrementation.



Transparency 11-34

Comparison of post-collapse displacements:

"Perfect" arch: (disp. at center of arch = −4.4)



"Imperfect" arch: (disp. at center of arch = −4.8)

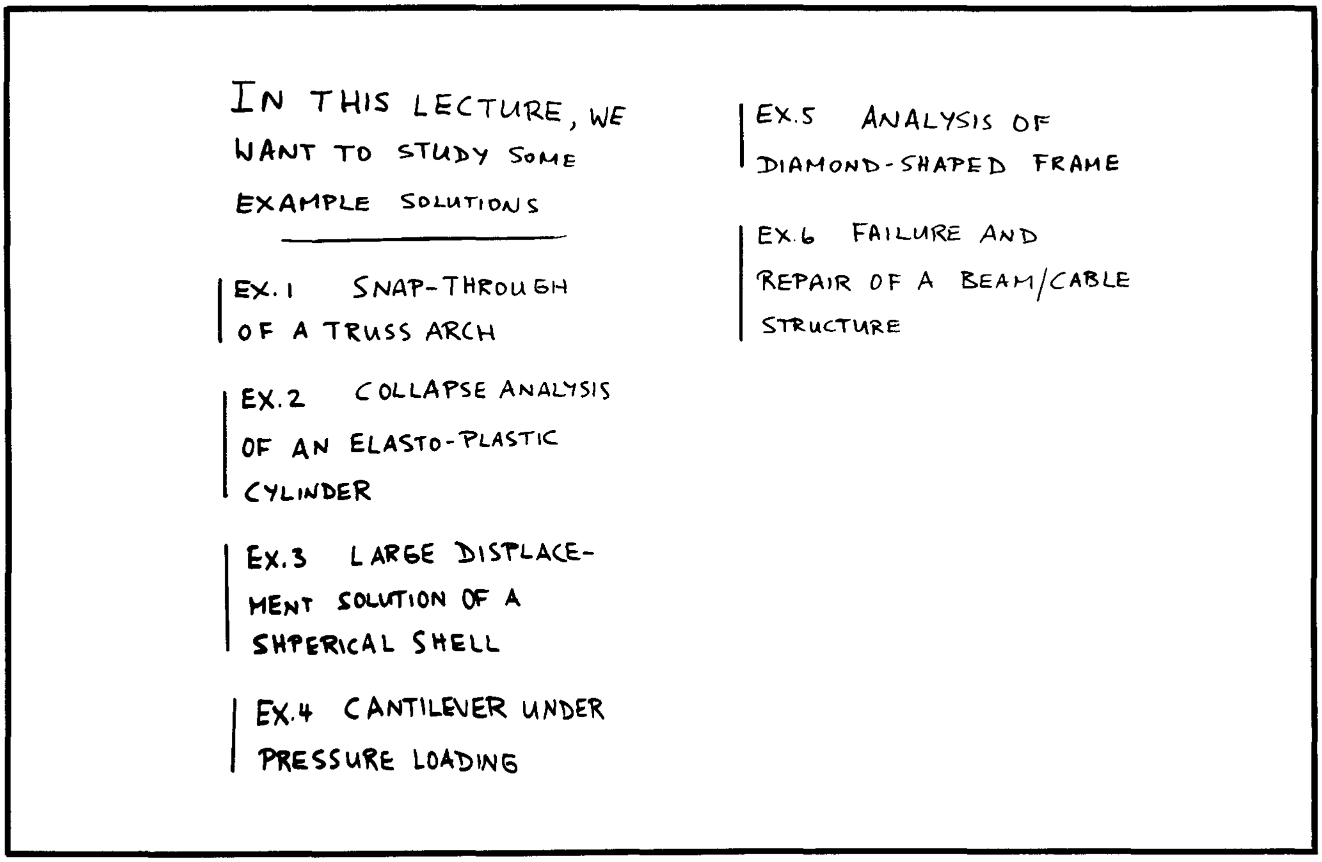
Topic 12

Demonstrative Example Solutions in Static Analysis

Contents:

- Analysis of various problems to demonstrate, study, and evaluate solution methods in statics
- Example analysis: Snap-through of an arch
- Example analysis: Collapse analysis of an elastic-plastic cylinder
- Example analysis: Large displacement response of a shell
- Example analysis: Large displacements of a cantilever subjected to deformation-independent and deformation-dependent loading
- Example analysis: Large displacement response of a diamond-shaped frame
- Computer-plotted animation: Diamond-shaped frame
- Example analysis: Failure and repair of a beam/cable structure

Textbook: Sections 6.1, 6.5.2, 8.6, 8.6.1, 8.6.2, 8.6.3



IN THIS LECTURE, WE WANT TO STUDY SOME EXAMPLE SOLUTIONS

EX.1 SNAP-THROUGH OF A TRUSS ARCH

EX.2 COLLAPSE ANALYSIS OF AN ELASTO-PLASTIC CYLINDER

EX.3 LARGE DISPLACEMENT SOLUTION OF A SHPERICAL SHELL

EX.4 CANTILEVER UNDER PRESSURE LOADING

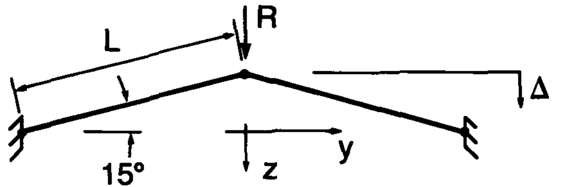
EX.5 ANALYSIS OF DIAMOND-SHAPED FRAME

EX.6 FAILURE AND REPAIR OF A BEAM/CABLE STRUCTURE

Markerboard 12-1

Transparency 12-1

Example: Snap-through of a truss arch



$L = 10.0$

$k = \frac{EA}{L} = 2.1 \times 10^5$

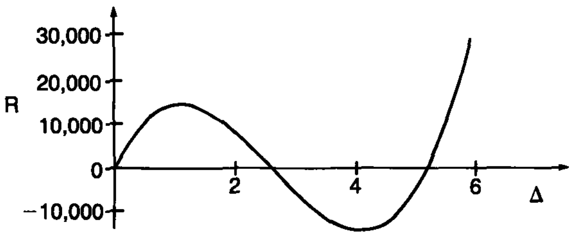
- Perform post-buckling analysis using automatic load step incrementation.
- Perform linearized buckling analysis.

Transparency 12-2

Postbuckling analysis:

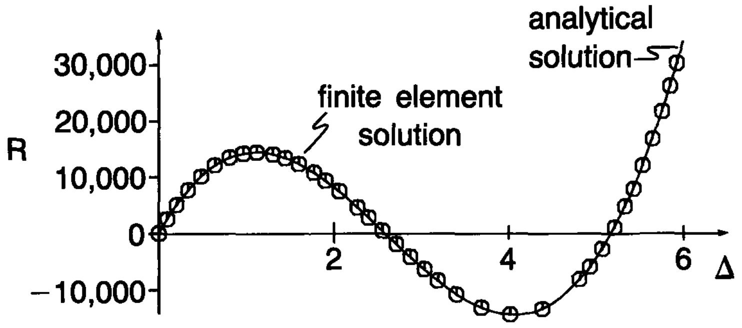
The analytical solution is

$$R = 2kL\left[\frac{1}{\sqrt{1 - 2\left(\frac{\Delta}{L}\right)\sin 15^\circ + \left(\frac{\Delta}{L}\right)^2}} - 1\right]\left(\sin 15^\circ - \frac{\Delta}{L}\right)$$



The automatic load step incrementation procedure previously described may be employed.

Using ${}^{1}\Delta = {}^{1}U = -0.1$, we obtain



Transparency 12-3

Solution details for load step 7:

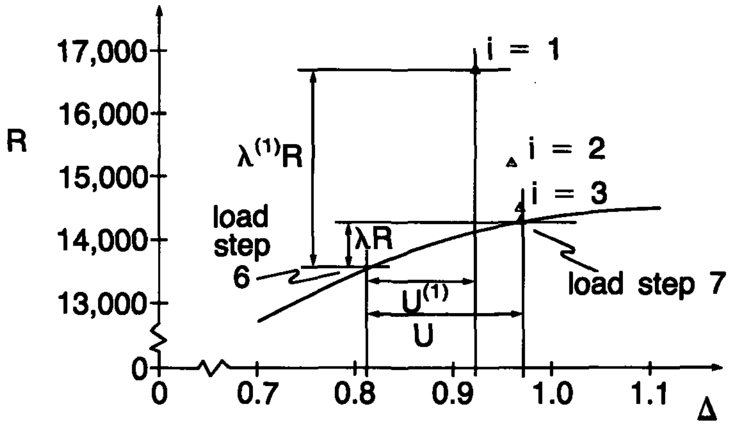
- The spherical constant arc-length algorithm is employed.
- The initial stiffness matrix is employed for all iterations, ${}^{t}U = .8111$, ${}^{t}R = 13{,}580$.

i	${}^{t+\Delta t}U^{(i)}$	${}^{t+\Delta t}\lambda^{(i)}R$	$U^{(i)}$	$\lambda^{(i)}R$
1	.9220	16,690	.1109	3,120
2	.9602	15,220	.1491	1,640
3	.9686	14,510	.1575	936
4	.9699	14,340	.1588	763
5	.9701	14,310	.1590	734
6	.9701	14,310	.1590	731

Transparency 12-4

Transparency
12-5

Pictorially, for load step 7,



Transparency
12-6

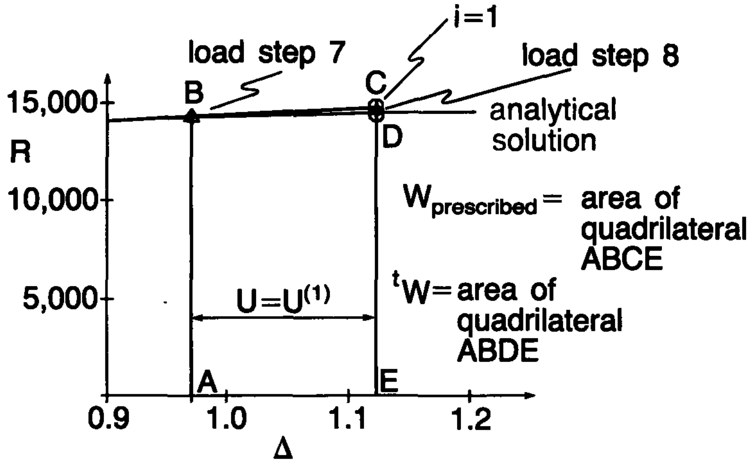
Solution details for load step 8:

- The constant increment of external work algorithm is employed.
- Modified Newton iterations are used, ${}^{t}U = .9701$, ${}^{t}R = 14{,}310$.

i	${}^{t+\Delta t}U^{(i)}$	${}^{t+\Delta t}\lambda^{(i)}R$	$U^{(i)}$	$\lambda^{(i)}R$
1	1.1227	14,740	.1526	440
2	1.1227	14,500	.1526	200

Transparency 12-7

Pictorially, for load step 8,

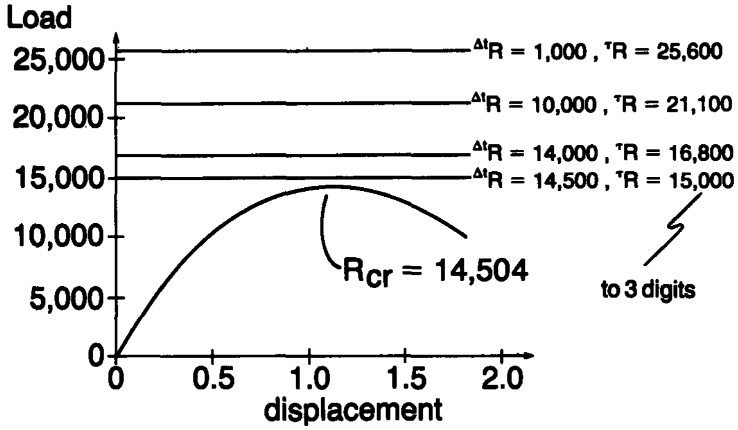


$W_{prescribed}$ = area of quadrilateral ABCE

${}^{t}W$ = area of quadrilateral ABDE

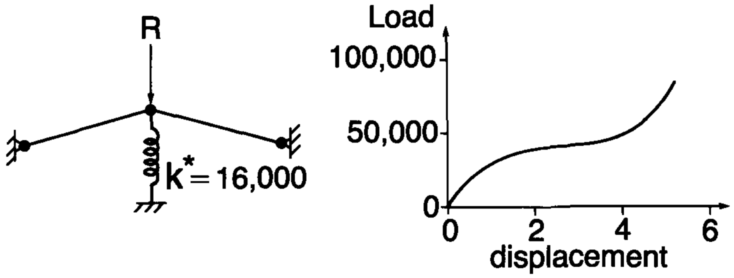
Transparency 12-8

We now employ a linearized buckling analysis to estimate the collapse load for the truss arch.



Transparency 12-9

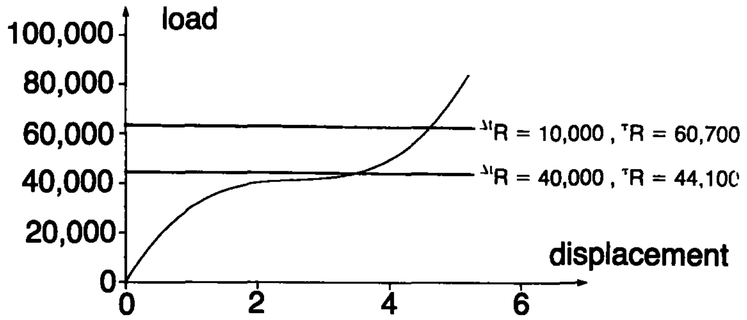
There are cases for which linearized buckling analysis gives buckling loads for stable structures. Consider the truss arch reinforced with a spring as shown:



This structure is always stable.

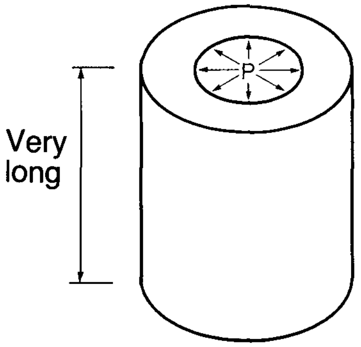
Transparency 12-10

We perform a linearized buckling analysis. When the load level is close to the inflection point, the computed collapse load is also close to the inflection point.



Transparency 12-11

Example: Elastic-plastic cylinder under internal pressure

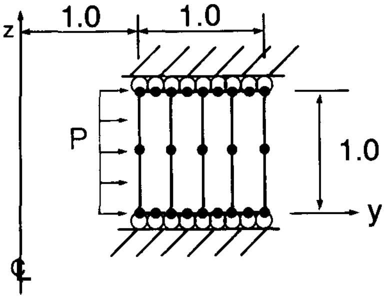


$E = 8667$
$\nu = 0.3$
$E_T = 0$
$\sigma_y = 17.32$
Inner radius $= 1$
Outer radius $= 2$

— Goal: Determine the limit load.

Transparency 12-12

Finite element mesh: Four 8-node axisymmetric elements

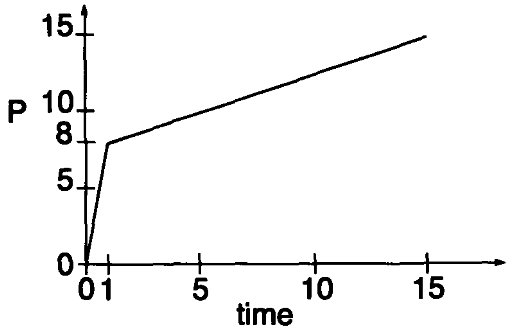


We note that, due to the boundary conditions and loading used, all stresses are constant in the z direction. Hence, 6-node elements could also have been used.

Transparency 12-13

Since the displacements are small, we use the M.N.O. formulation.

- We employ the following load function:



(the elastic limit load is P=7.42)

Transparency 12-14

Now we compare the effectiveness of various solution procedures:

- — Full Newton method with line searches
- — Full Newton method without line searches
- — BFGS method
- — Modified Newton method with line searches
- — Modified Newton method without line searches
- — Initial stress method

Transparency 12-15

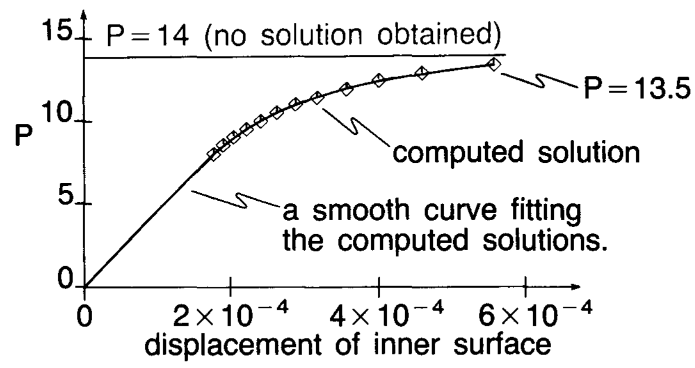
The following convergence tolerances are employed:

$$\frac{\Delta \underline{U}^{(i)T}\,[{}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)}]}{\Delta \underline{U}^{(1)T}\,[{}^{t+\Delta t}\underline{R} - {}^{t}\underline{F}]} \le \underbrace{0.001}_{\text{ETOL}}$$

$$\frac{\|{}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)}\|_2}{\underbrace{1.0}_{\text{RNORM}}} \le \underbrace{0.01}_{\text{RTOL}}$$

Transparency 12-16

When any of these procedures are used, the following force-deflection curve is obtained. For P = 14, no converged solution is found.



Transparency 12-17

We now compare the solution times for these procedures. For the comparison, we end the analysis when the solution for $P = 13.5$ is obtained.

Method	**Normalized time**
Full Newton method with line searches	1.2
Full Newton method	1.0
BFGS method	0.9
Modified Newton method with line searches	1.1
Modified Newton method	1.1
Initial stress method	2.2

Transparency 12-18

Now we employ automatic load step incrementation.

- No longer need to specify a load function
- Softening in force-deflection curve is automatically taken into account.

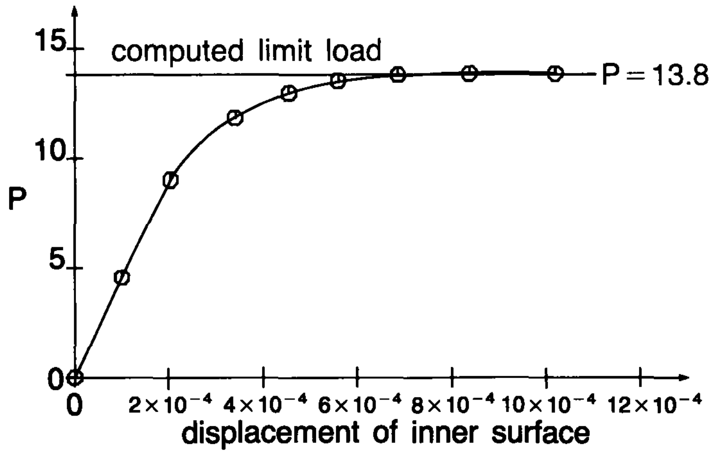
Here we use

$\mathrm{ETOL} = 10^{-5}$

$\mathrm{RTOL} = 0.01$

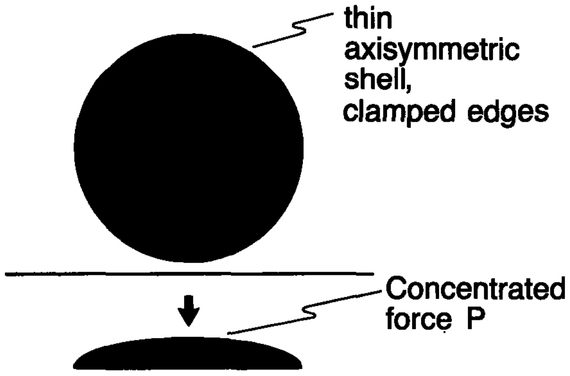
$\mathrm{RNORM} = 1.0$

Result: Here we selected the displacement of the inner surface for the first load step to be 10^{-4}.



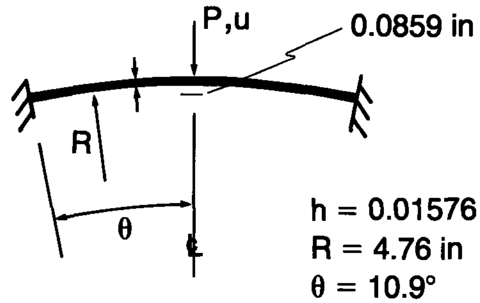
Transparency 12-19

Example: Spherical Shell



Transparency 12-20

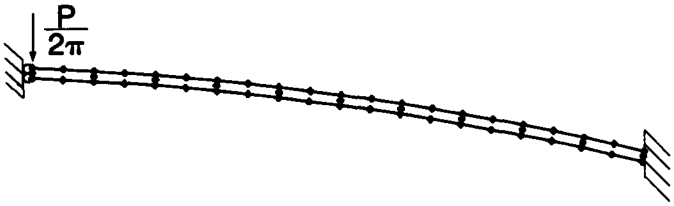
Transparency 12-21



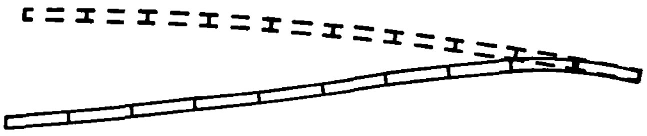
h = 0.01576 in
R = 4.76 in
θ = 10.9°
E = 10^7 psi
ν = 0.3

Transparency 12-22

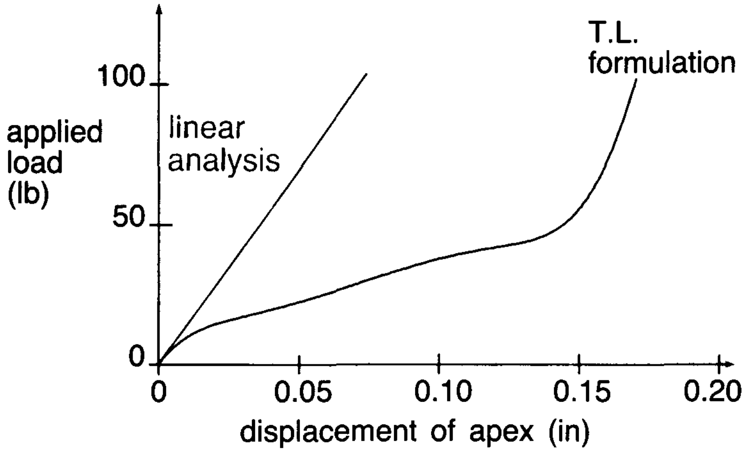
Finite element mesh: Ten 2-D axisymmetric elements



Deformed configuration for P = 100 lb:



Force-deflection curve obtained using 10 element mesh:



**Transparency
12-23**

Comparison of solution procedures:

1) Apply full load (100 lb) in 10 equal steps:

Solution procedure	Normalized solution time
Full Newton with line searches	1.4
Full Newton without line searches	1.0
BFGS method	did not converge
Modified Newton with line searches	did not converge
Modified Newton without line searches	did not converge

**Transparency
12-24**

Transparency 12-25

2) Apply full load in 50 equal steps:

Solution procedure	**Normalized solution time**
Full Newton with line search	1.3
Full Newton without line search	1.0
BFGS method	1.6
Modified Newton with line search	1.9
Modified Newton without line search	did not converge

Transparency 12-26

Convergence criterion employed:

$$\frac{\Delta\underline{U}^{(i)T}\,[{}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)}]}{\Delta\underline{U}^{(1)T}[{}^{t+\Delta t}\underline{R} - {}^{t}\underline{F}]} \leq \underbrace{0.001}_{\text{ETOL}}$$

Maximum number of iterations permitted = 99

Transparency 12-27

We may also employ automatic load step incrementation:

Here we use

$$\text{ETOL} = 10^{-5}$$

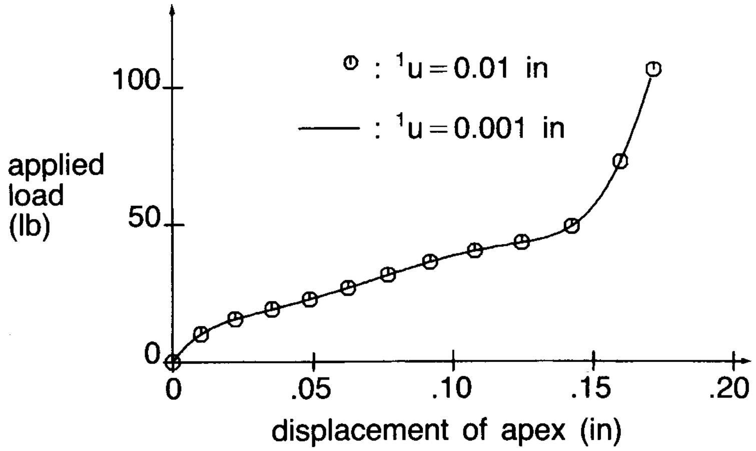
and

$$\frac{\| {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)} \|_2}{\underbrace{1.0}_{\text{RNORM}}} \leq \underbrace{0.01}_{\text{RTOL}}$$

as convergence tolerances.

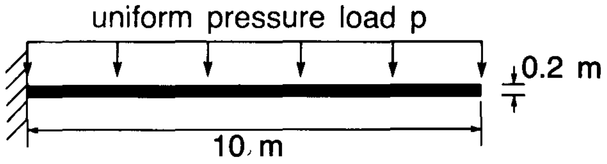
Transparency 12-28

Results: Using different choices of initial prescribed displacements, we obtain



Transparency 12-29

Example: Cantilever under pressure loading

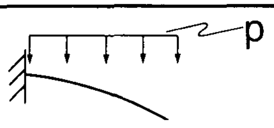
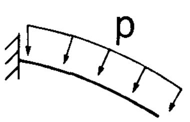


E = 207000 MPa
$\nu = 0.3$
Plane strain, width = 1.0 m

- Determine the deformed shape of the cantilever for p = 1 MPa.

Transparency 12-30

— Since the cantilever undergoes large displacements, the pressure loading (primarily the direction of loading) depends on the configuration of the cantilever:

	
deformation-independent	deformation-dependent

Transparency 12-31

The purpose of this example is to contrast the assumption of deformation-independent loading with the assumption of deformation-dependent loading.

Transparency 12-32

Finite element model: Twenty-five two-dimensional 8-node elements
(1 layer, evenly spaced)

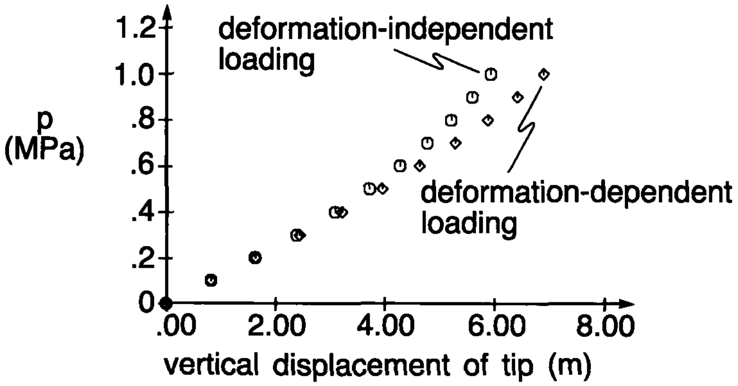
Solution details:

- Full Newton method without line searches is used.
- Convergence tolerances are
  - $\text{ETOL} = 10^{-3}$
  - $\text{RTOL} = 10^{-2}$, $\text{RNORM} = 1.0$ MN

Transparency 12-33

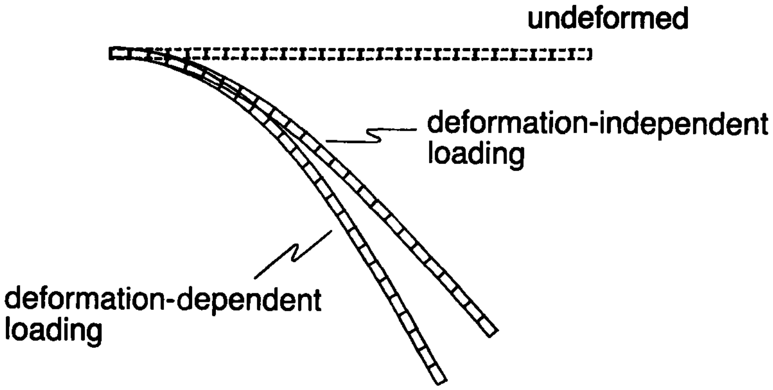
Results: Force-deflection curve

- For small deflections, there are negligible differences between the two assumptions.

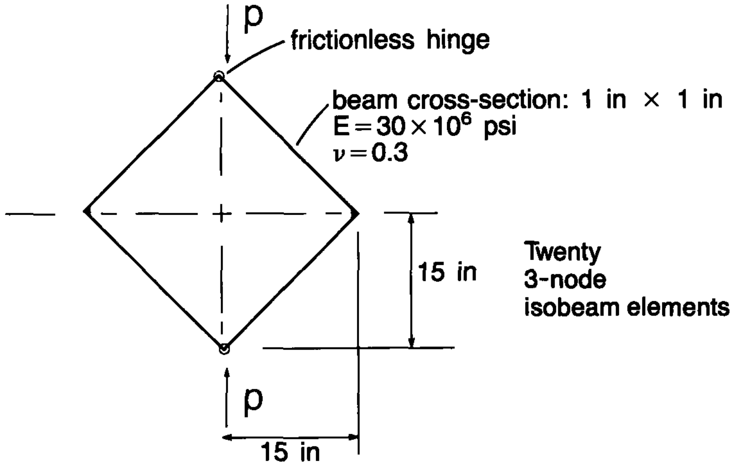


Transparency 12-34

Pictorially, for $p = 1.0$ MPa,



Example: Diamond-shaped frame

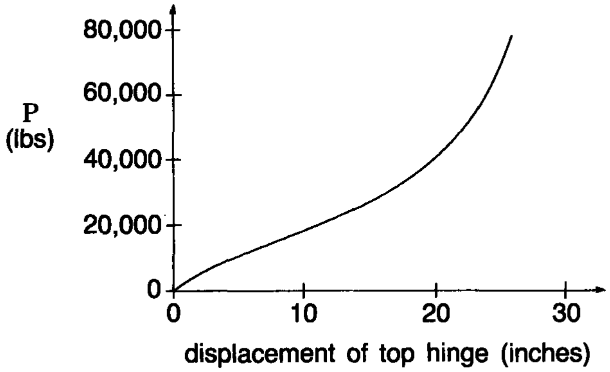


Twenty
3-node
isobeam elements

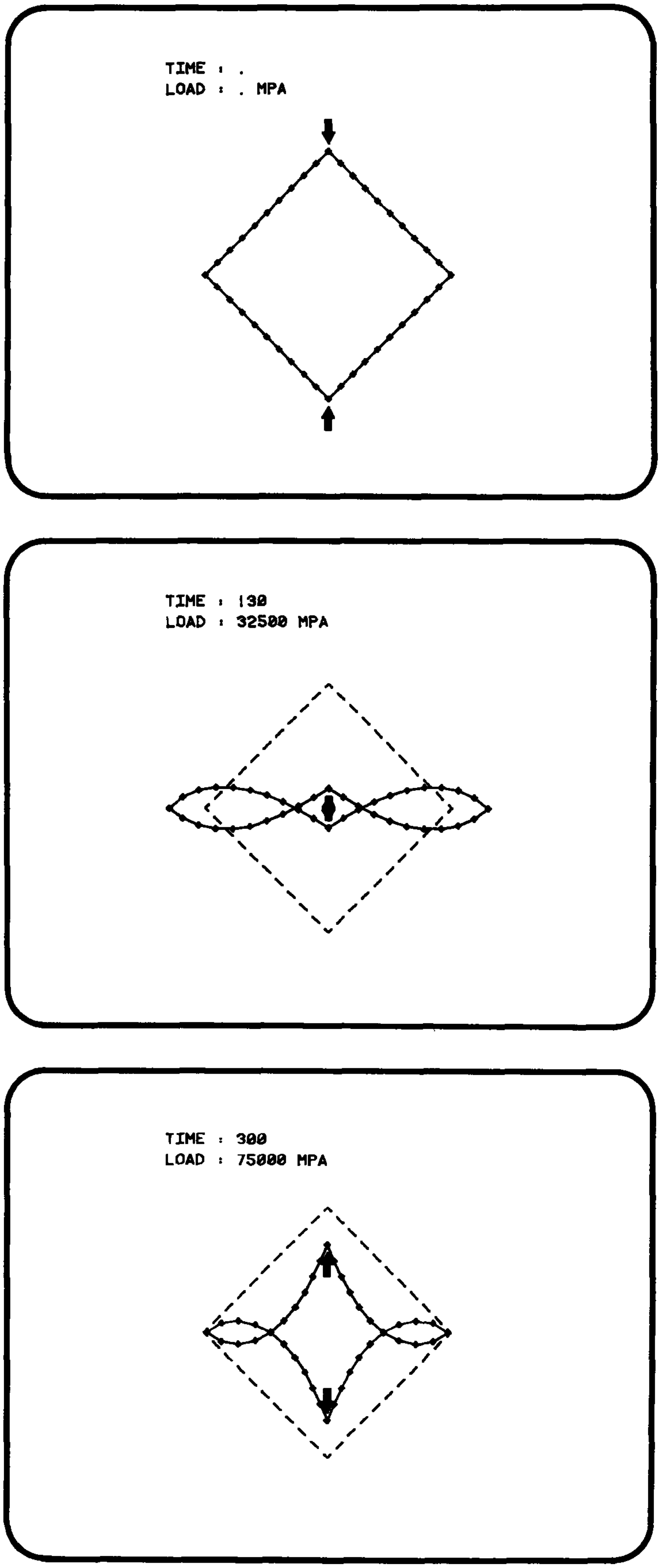
Transparency 12-35

Force-deflection curve, obtained using the T.L formulation:

- A constant load increment of 250 lbs is used.



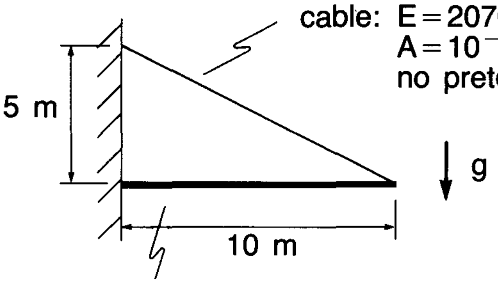
Transparency 12-36



Computer Animation
Diamond shaped frame

Transparency 12-37

Example: Failure and repair of a beam/cable structure



cable: $E = 207000$ MPa
$A = 10^{-4}$ m
no pretension

beam: $E = 207000$ MPa
$\nu = 0.3$
$\sigma_y = 200$ MPa
$E_T = 20700$ MPa

$\rho = 7800$ kg/m^3
cross-section:
0.1 m × 0.1 m

Transparency 12-38

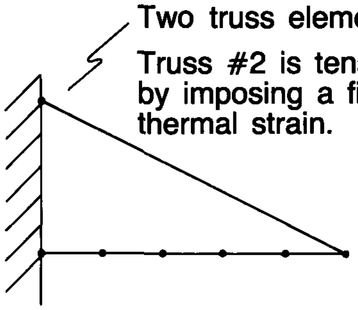
In this analysis, we simulate the failure and repair of the cable.

Steps in analysis:

Load step	Event
1	Beam sags under its weight, but is supported by cable.
1 to 2	Cable snaps, plastic flow occurs at built-in end of beam.
2 to 4	A new cable is installed, and is tensioned until the tip of the beam returns to its location in load step 1.

Transparency 12-39

Finite element model:



Two truss elements:

Truss #2 is tensioned by imposing a fictitious thermal strain.

Load step	Active truss
1	#1
2	none
3	#2
4	#2

Five 2-node Hermitian beam elements

5 Newton-Cotes integration points in r direction
3 Newton-Cotes integration points in s direction

Transparency 12-40

Solution details: The U.L. formulation is employed for the truss elements and the beam elements.

Convergence tolerances:

$\text{ETOL} = 10^{-3}$

$\text{RTOL} = 10^{-2}$

$\text{RNORM} = 7.6 \times 10^{-3}$ MN

$\text{RMNORM} = 3.8 \times 10^{-2}$ MN-m

Transparency 12-41

Comparison of solution algorithms:

Method	**Results**
Full Newton with line searches	All load steps successful, normalized CPU time = 1.0.
Full Newton	Stiffness matrix not positive definite in load step 2.
BFGS	All load steps successful, normalized CPU time = 2.5.
Modified Newton with or without line searches	No convergence in load step 2.

Transparency 12-42

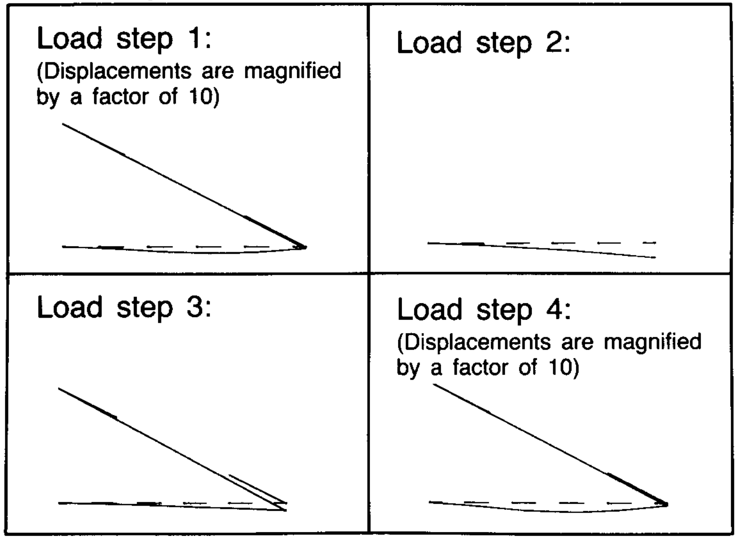
Results:

Load step	**Disp. of tip**	**Stress in cable**	**Moment at built-in end**
1	−.008 m	64 MPa	9.7 KN-m
2	−.63 m	—	38 KN-m
3	−.31 m	37 MPa	22 KN-m
4	−.008 m	72 MPa	6.2 KN-m

Note: The elastic limit moment at the built-in end of the beam is 33 KN-m.

Transparency 12-43

Pictorially,



Load step 1:

(Displacements are magnified by a factor of 10)

Load step 2:

Load step 3:

Load step 4:

(Displacements are magnified by a factor of 10)

Topic 13

Solution of Nonlinear Dynamic Response—Part I

Contents:

- **Basic procedure of direct integration**
- **The explicit central difference method, basic equations, details of computations performed, stability considerations, time step selection, relation of critical time step size to wave speed, modeling of problems**
- **Practical observations regarding use of the central difference method**
- **The implicit trapezoidal rule, basic equations, details of computations performed, time step selection, convergence of iterations, modeling of problems**
- **Practical observations regarding use of trapezoidal rule**
- **Combination of explicit and implicit integrations**

Textbook: Sections 9.1, 9.2.1, 9.2.4, 9.2.5, 9.4.1, 9.4.2, 9.4.3, 9.4.4, 9.5.1, 9.5.2

Examples: 9.1, 9.4, 9.5, 9.12

SOLUTION OF DYNAMIC EQUILIBRIUM EQUATIONS

- Direct integration methods
  Explicit
  Implicit
- Mode superposition
- Substructuring

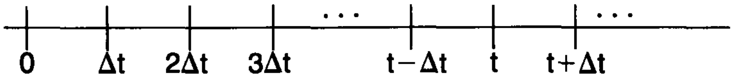
Transparency 13-1

The governing equation is

$$\underbrace{\underline{F}_I(t)}_{\substack{\text{Inertia}\\ \text{forces}}} + \underbrace{\underline{F}_D(t)}_{\substack{\text{Damping}\\ \text{forces}}} + \underbrace{\underline{F}_E(t)}_{\substack{\text{“Elastic”}\\ \text{forces}}} = \underbrace{\underline{R}(t)}_{\substack{\text{Externally}\\ \text{applied loads}}}$$

↯
nodal point
forces equivalent to
element stresses

This equation is to be satisfied at the discrete times



Transparency 13-2

Transparency 13-3

Issues to discuss:

- What are the basic procedures for obtaining the solutions at the discrete times?
- Which procedure should be used for a given problem?

Transparency 13-4

Explicit time integration:

Central difference method

$$\underline{M}\,{}^t\ddot{\underline{U}} + \underline{C}\,{}^t\dot{\underline{U}} + {}^t\underline{F} = {}^t\underline{R}$$

$${}^t\dot{\underline{U}} = \frac{1}{2\Delta t}\left({}^{t+\Delta t}\underline{U} - {}^{t-\Delta t}\underline{U}\right)$$

$${}^t\ddot{\underline{U}} = \frac{1}{(\Delta t)^2}\left({}^{t+\Delta t}\underline{U} - 2\,{}^t\underline{U} + {}^{t-\Delta t}\underline{U}\right)$$

- Used mainly for wave propagation problems
- An explicit method because the equilibrium equation is used at time t to obtain the solution for time t+Δt.

Transparency 13-5

Using these equations,

$$\left(\frac{1}{\Delta t^2}\underline{M} + \frac{1}{2\Delta t}\underline{C}\right){}^{t+\Delta t}\underline{U} = {}^t\underline{\hat{R}}$$

where

$${}^t\underline{\hat{R}} = {}^t\underline{R} - {}^t\underline{F} + \frac{2}{(\Delta t)^2}\underline{M}\,{}^t\underline{U} - \left(\frac{1}{\Delta t^2}\underline{M} - \frac{1}{2\Delta t}\underline{C}\right){}^{t-\Delta t}\underline{U}$$

- The method is used when $\underline{M}$ and $\underline{C}$ are diagonal:

$${}^{t+\Delta t}U_i = \left(\frac{1}{\frac{1}{\Delta t^2}m_{ii} + \frac{1}{2\Delta t}c_{ii}}\right){}^t\hat{R}_i$$

and, most frequently, $c_{ii} = 0$.

Transparency 13-6

Note:

- We need $m_{ii} > 0$! (assuming $c_{ii} = 0$)
- $${}^t\underline{F} = \sum_m {}^t\underline{F}^{(m)}$$
  where m denotes an element.
- To start the solution, we use
  $${}^{-\Delta t}\underline{U} = {}^0\underline{U} - \Delta t\,{}^0\underline{\dot{U}} + \frac{\Delta t^2}{2}\,{}^0\underline{\ddot{U}}$$

Transparency 13-7

The central difference method is only conditionally stable. The condition is

$$\Delta t \leq \Delta t_{cr} = \frac{T_n}{\pi} \quad \text{smallest period in finite element assemblage}$$

In nonlinear analysis, T_n changes during the time history

— becomes smaller when the system stiffens (for example, due to large displacement effects),

— becomes larger when the system softens (for example, due to material nonlinearities).

Transparency 13-8

We can estimate T_n:

$$\underbrace{(\omega_n)}_{\text{frequency}}{}^2 \leq \max\{(\omega_n^{(m)})^2\} \text{ over all elements } m$$

Hence the largest frequency of all individual elements, $(\omega_n^{(m)})_{max}$, is used:

$$T_n \geq \frac{2\pi}{(\omega_n^{(m)})_{max}}$$

In nonlinear analysis $(\omega_n^{(m)})_{max}$ will in general change with the response.

The time integration step, Δt, used can be

$$\Delta t = \frac{2}{(\omega_n^{(m)})_{max}} \leq \Delta t_{cr}$$

We may call $\frac{2}{\omega_n^{(m)}}$ the critical time step of element m.

Hence $\frac{2}{(\omega_n^{(m)})_{max}}$ is the smallest of these "element critical time steps."

Transparency 13-9

Proof that $(\omega_n)^2 \leq (\omega_n^{(m)})^2_{max}$:

Using the Rayleigh quotient (see textbook), we write

$$(\omega_n)^2 = \frac{\underline{\phi}_n^T \sum_m \underline{K}^{(m)} \underline{\phi}_n}{\underline{\phi}_n^T \sum_m \underline{M}^{(m)} \underline{\phi}_n} \quad \left(\begin{array}{l}\text{the summation is}\\ \text{taken over all}\\ \text{finite elements}\end{array}\right)$$

Let $\mathcal{U}^{(m)} = \underline{\phi}_n^T \underline{K}^{(m)} \underline{\phi}_n$, $\mathcal{I}^{(m)} = \underline{\phi}_n^T \underline{M}^{(m)} \underline{\phi}_n$,

then

$$(\omega_n)^2 = \frac{\sum_m \mathcal{U}^{(m)}}{\sum_m \mathcal{I}^{(m)}}$$

Transparency 13-10

Transparency 13-11

Consider the Rayleigh quotient for a single element:

$$\rho^{(m)} = \frac{\underline{\phi}_n^T \underline{K}^{(m)} \underline{\phi}_n}{\underline{\phi}_n^T \underline{M}^{(m)} \underline{\phi}_n} = \frac{\mathcal{U}^{(m)}}{\mathcal{J}^{(m)}}$$

Using that $\rho^{(m)} \leq (\omega_n^{(m)})^2$ where $\omega_n^{(m)}$ is the largest frequency (rad/sec) of element m, we obtain

$$\mathcal{U}^{(m)} \leq (\omega_n^{(m)})^2 \, \mathcal{J}^{(m)}$$

Transparency 13-12

Therefore $(\omega_n)^2$ is also bounded:

$$(\omega_n)^2 \leq \frac{\sum_m (\omega_n^{(m)})^2 \, \mathcal{J}^{(m)}}{\sum_m \mathcal{J}^{(m)}}$$

$$\leq \frac{(\omega_n^{(m)})^2_{max} \cancel{\sum_m \mathcal{J}^{(m)}}}{\cancel{\sum_m \mathcal{J}^{(m)}}}$$

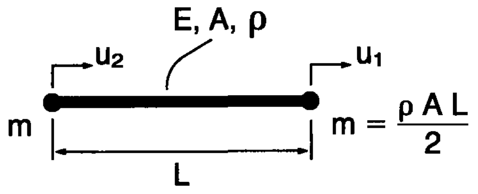
resulting in

$$(\omega_n)^2 \leq (\omega_n^{(m)})^2_{max}$$

Transparency 13-13

The largest frequencies of simple elements can be calculated analytically (or upper bounds can be estimated).

Example:



$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \underline{\phi} = \omega^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \underline{\phi}$$

$$(\omega_1)^2 = 0\,,\ (\omega_2)^2 = (\omega_n)^2 = 4\,\frac{E}{\rho}\,\frac{1}{L^2} = 4\,\frac{c^2}{L^2} \quad c \leftarrow \text{the wave speed}$$

Transparency 13-14

We note that hence the critical time step for this element is

$$\left(\frac{2}{\omega_n}\right) = \left(\frac{2}{\left(\frac{2c}{L}\right)}\right)$$

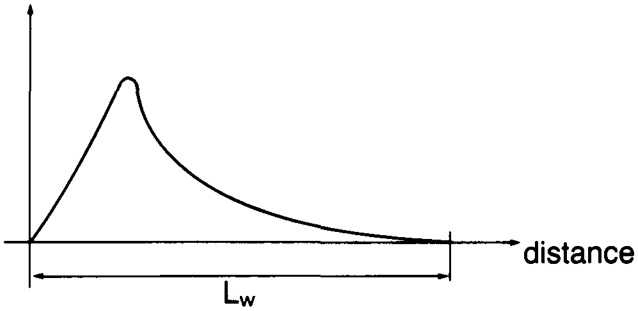
$$= \frac{L}{c}\,;\ L = \text{length of element!}$$

Note that $\frac{L}{c}$ is the time required for a wave front to travel through the element.

Transparency
13-15

Modeling:

Let the applied wavelength be L_w



Transparency
13-16

Then $t_w = \dfrac{L_w}{\underbrace{c}}$ wave speed

Choose $\Delta t = \dfrac{t_w}{\underbrace{n}}$ number of time steps used to represent the wave

$$\underbrace{L_e}_{\text{related to element length}} = c\,\Delta t$$

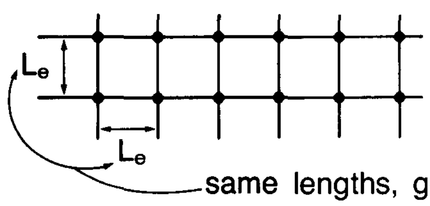
Transparency 13-17

Notes:

- In 1-D, $c = \sqrt{\frac{E \text{ — Young's modulus}}{\rho \text{ — density}}}$
- In nonlinear analysis, Δt must satisfy the stability limit throughout the analysis. Since c changes, use the largest value anticipated.
- It may also be effective to change the time step during the analysis.

Transparency 13-18

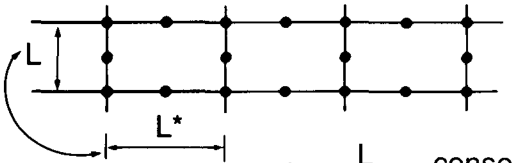
- Low-order elements:



same lengths, good

Usually preferable

- Higher-order elements:

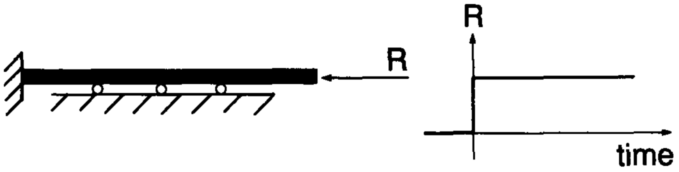


different lengths, not good $L_e = \frac{L}{8}$ — conservative

Transparency 13-19

Some observations:

1) Linear elastic 1-D analysis



For this special case the exact solution is obtained for any number of elements provided $L_e = c\,\Delta t$.

Wave travels one element per time step.

Transparency 13-20

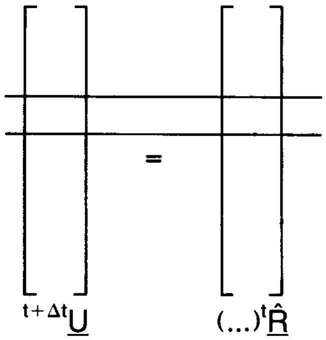
2) Uniform meshing is important, so that with the time step selected, no unduly small time step in any region of the total mesh is used.

   Different time steps for different parts of the mesh could be used, but then special coupling considerations must be enforced.

3) A system with a very large bandwidth may also be solved efficiently using the central difference method, although the problem may not be a wave propagation problem.

Transparency 13-21

4) Explicit time integration lends itself to parallel processing.



Can consider a certain number of equations in parallel (by element groups)

Transparency 13-22

Implicit time integration:

Basic equation (assume modified Newton-Raphson iteration):

$$\underline{M}\,^{t+\Delta t}\underline{\ddot{U}}^{(k)} + \underline{C}\,^{t+\Delta t}\underline{\dot{U}}^{(k)} + {}^{t}\underline{K}\,\Delta\underline{U}^{(k)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(k-1)}$$

$$^{t+\Delta t}\underline{U}^{(k)} = {}^{t+\Delta t}\underline{U}^{(k-1)} + \Delta\underline{U}^{(k)}$$

We use the equilibrium equation at time $t+\Delta t$ to obtain the solution for time $t+\Delta t$.

Transparency 13-23

Trapezoidal rule:

$${}^{t+\Delta t}\underline{U} = {}^{t}\underline{U} + \frac{\Delta t}{2}\left({}^{t}\dot{\underline{U}} + {}^{t+\Delta t}\dot{\underline{U}}\right)$$

$${}^{t+\Delta t}\dot{\underline{U}} = {}^{t}\dot{\underline{U}} + \frac{\Delta t}{2}\left({}^{t}\ddot{\underline{U}} + {}^{t+\Delta t}\ddot{\underline{U}}\right)$$

Hence

$${}^{t+\Delta t}\dot{\underline{U}} = \frac{2}{\Delta t}\left({}^{t+\Delta t}\underline{U} - {}^{t}\underline{U}\right) - {}^{t}\dot{\underline{U}}$$

$${}^{t+\Delta t}\ddot{\underline{U}} = \frac{4}{(\Delta t)^2}\left({}^{t+\Delta t}\underline{U} - {}^{t}\underline{U}\right) - \frac{4}{\Delta t}\,{}^{t}\dot{\underline{U}} - {}^{t}\ddot{\underline{U}}$$

Transparency 13-24

In our incremental analysis, we write

$${}^{t+\Delta t}\dot{\underline{U}}^{(k)} = \frac{2}{\Delta t}\left({}^{t+\Delta t}\underline{U}^{(k-1)} + \Delta\underline{U}^{(k)} - {}^{t}\underline{U}\right) - {}^{t}\dot{\underline{U}}$$

$${}^{t+\Delta t}\ddot{\underline{U}}^{(k)} = \frac{4}{(\Delta t)^2}\left({}^{t+\Delta t}\underline{U}^{(k-1)} + \Delta\underline{U}^{(k)} - {}^{t}\underline{U}\right)$$
$$- \frac{4}{\Delta t}\,{}^{t}\dot{\underline{U}} - {}^{t}\ddot{\underline{U}}$$

and the governing equilibrium equation is

$$\underbrace{\left({}^{t}\underline{K} + \frac{4}{\Delta t^2}\underline{M} + \frac{2}{\Delta t}\underline{C}\right)}_{{}^{t}\hat{\underline{K}}} \Delta\underline{U}^{(k)}$$

$$= {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(k-1)}$$

$$-\underline{M}\left[\frac{4}{\Delta t^2}\left({}^{t+\Delta t}\underline{U}^{(k-1)} - {}^{t}\underline{U}\right) - \frac{4}{\Delta t}\,{}^{t}\dot{\underline{U}} - {}^{t}\ddot{\underline{U}}\right]$$

$$-\underline{C}\left[\frac{2}{\Delta t}\left({}^{t+\Delta t}\underline{U}^{(k-1)} - {}^{t}\underline{U}\right) - {}^{t}\dot{\underline{U}}\right]$$

Transparency 13-25

Some observations:

1) As Δt gets smaller, entries in ${}^{t}\hat{\underline{K}}$ increase.
2) The convergence characteristics of the equilibrium iterations are better than in static analysis.
3) The trapezoidal rule is unconditionally stable in linear analysis. For nonlinear analysis,
   - — select Δt for accuracy
   - — select Δt for convergence of iteration

Transparency 13-26

Transparency 13-27

Convergence criteria:

Energy:

$$\frac{\Delta \underline{U}^{(i)T}\left({}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)} - \underline{M}\,{}^{t+\Delta t}\ddot{\underline{U}}^{(i-1)} - \underline{C}\,{}^{t+\Delta t}\dot{\underline{U}}^{(i-1)}\right)}{\Delta \underline{U}^{(1)T}\left({}^{t+\Delta t}\underline{R} - {}^{t}\underline{F} - \underline{M}\,{}^{t+\Delta t}\ddot{\underline{U}}^{(0)} - \underline{C}\,{}^{t+\Delta t}\dot{\underline{U}}^{(0)}\right)}$$

$$\leq \text{ETOL}$$

Transparency 13-28

Forces:

$$\frac{\left\|{}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)} - \underline{M}\,{}^{t+\Delta t}\ddot{\underline{U}}^{(i-1)} - \underline{C}\,{}^{t+\Delta t}\dot{\underline{U}}^{(i-1)}\right\|_2}{\text{RNORM}}$$

$$\leq \text{RTOL}$$

(considering only translational degrees of freedom, for rotational degrees of freedom use RMNORM).

Note: $\|\underline{a}\|_2 = \sqrt{\sum_k (a_k)^2}$

Transparency 13-29

Displacements:

$$\frac{\|\Delta \underline{U}^{(i)}\|_2}{\text{DNORM}} \leq \text{DTOL}$$

(considering only translational degrees of freedom, for rotational degrees of freedom, use DMNORM).

Transparency 13-30

Modeling:

- Identify frequencies contained in the loading.
- Choose a finite element mesh that can accurately represent the static response and all important frequencies.
- Perform direct integration with

$$\Delta t \doteq \frac{1}{20} T_{co}$$

(T_{co} is the smallest period (secs) to be integrated).

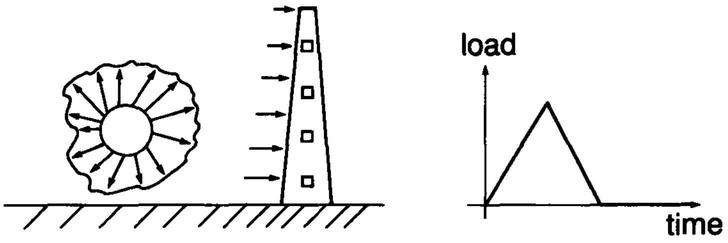
Transparency 13-31

- Method used for structural vibration problems.
- Typically it is effective to use higher-order elements.
- It can also be effective to use a consistent mass matrix.

Because a structural dynamics problem is thought of as a "static problem including inertia forces".

Transparency 13-32

Typical problem:



Analysis of tower under blast load

- We assume that only the structural vibration is required.
- Perhaps about 100 steps are sufficient to integrate the response.

Combination of methods: explicit and implicit integration

- Use central difference method first, then switch to trapezoidal rule, for problems which show initially wave propagation, then structural vibration.

- Use central difference method for certain parts of the structure, and implicit method for other parts; for problems with "stiff" and "flexible" regions.

Transparency 13-33

Resource: Finite Element Procedures for Solids and Structures
Klaus-Jürgen Bathe