

GILBERT

OK. So speaking today about separable equations. These are, in principle, the easiest to solve. They include nonlinear equations but they have a special feature that makes them easy, makes them approachable. And that special feature is that the right hand side of the equation separates into some function of t divided by or multiplied by some function of y .

STRANG:

The t and the y have separated on the right hand side. And for example, dy/dt equal y plus t would not be separable. They'd be very simple but not separable. Separable means that we can keep those two separately and do an integral of f and an integral of g and we're in business.

OK. Examples. Suppose that f of y is 1. Then we have this simplest differential equation of all, dy/dt is some function of t . That's what calculus is for. y is the integral of g .

Suppose there was no t . Just a 1 over f of y , with g of t equal one. Then I bring the f of y up. I integrate the $[1/f] dy$.

And moving the dt there, I'm just integrating dt . So the right hand side would just be t . And the left hand side is an integral we have to do. That's the minimum amount of work to solve a differential equation.

But the point is, with y and t separate, we just have integration to do. And here is the case when there is both a g of t and an f of y . Then let me just emphasize what's happening here.

The f of y I am moving up with a dy . The dt I'm moving up with a g of t dt . So g of t dt equals f of y dy and I integrate both sides.

The left side is an integral of y with respect to y . The right hand side is an integral with respect to t or a dummy variable s . The integral going from 0 to t . This integral going from y of 0 to y of t .

Those are the two integrations to be done. And you will get examples of separable equations. And what you have to do is two integrals.

And then there's this one little catch at the end. This is some function of y when I integrate. But I usually like to have the solution to a differential equation just y equal something.

And you'll see in the examples. I have to solve it for y because this isn't going to give me just y . It's going to give me some expression involving y .

So let me do examples. Let me do examples. You see why it's correct. OK. So here are examples.

So let me take, what about the equation dy/dt equals t over y . Clearly separable. The function's f . g of t is just t . f of y is just y . I combine those to y dy equalling t dt .

You see I've picked a pretty straightforward example. Now I'm integrating both sides from y of 0 to y of t on the left. And from 0 to t on the right.

And of course that is $1/2$ t squared. And the left hand side is $1/2$ y squared between these two limits. So I'm getting the integral of that is $1/2$ y squared.

So up top I have $1/2$ y of t squared minus, at the bottom end, $1/2$ y of 0 squared equalling the right hand side $1/2$ t squared.

So you see, we got a function of y equal to a function of t . And the equation is solved, really.

That differential equation is solved. But I haven't found it in the form y of t equal something. But I can do that. I just move this to the other side. So that will go to the other side with a plus.

And then I'll cancel the $1/2$. And then I'll take the square root. So the solution y of t is the square root of y of 0 squared plus t squared.

That's the solution to the differential equation. Maybe I make a small comment on this equation. Because it's essential to begin to look for dangerous points. Singular points where things are not quite right.

Here the dangerous point is clearly y equal zero. If I start at y of 0 equals zero then I'm not sure what. What's the solution to that equation if I start at y of 0 equals 0 ?

I'm starting with a 0 over 0 . What a way to begin your life, starting with a 0 over 0 . This, well, actually the solution would still be correct. If y of 0 is 0 , I would get the square root of t squared. I would get t .

So y of 0 equals 0 allows the solution y equals t . And that is a solution. That if y is equal to t

then dy/dt is 1. And on the right hand side t over y is t over t is 1. So the equation is solved. But my point was, there's got to be something going a little strange when y of 0 is 0. And what happens strangely is there are other solutions.

I like, I think, y equal negative t . And more, probably. But if y is equal to negative t , then its derivative is minus 1. And on the right hand side, I have t over negative t minus 1 again. So the equation is solved. That's a perfectly good solution. That's a second solution. It's an equation with more than one solution. And we'll have to think, when can we guarantee there is just one solution, which is of course what we want.

OK. I'd better do another example going beyond this. And maybe the logistic equation is a good one. So that's separable. And it's going to be a little harder. So let me do that one.

dy/dt is y minus y squared, let's say. The logistic equation. Linear term minus a quadratic term.

That's separable because the g of t part is 1. And what's the f of y ? Remember f of y -- I want to put that on the y side. But it's going to show up in the denominator. So I have dy over y minus y squared equaling dt . And I have to integrate both sides to get the solution y .

Now, integrating the right hand side is of course a picnic. I get t . But integrating the left hand side, I have to either know how, or look up, or figure out the integral of 1 over y minus y squared.

So let me just make a little comment about integrating, because examples often have this problem. Integrating when there is a polynomial a quadratic in the denominator. There are different ways to do it. And the time that we'll really see this type of problem is when we discuss Laplace transforms.

So I'm going to save the details of the method until then. But let me give the name of the method. The name is partial fractions, which is a method of integration. Partial fractions.

And I'll just say here what it means. It means that I want to write this 1 over y minus y squared in a nicer way. What over y minus y squared can be split up into two fractions? Those are the partial fractions.

One fraction is-- so I'm going to factor that y minus y squared factors into y and 1 minus y . The partial fractions will be some number over the y and some other number over the 1 minus y .

This is just algebra now. Partial fractions is just algebra. It's not calculus. So I factored the y minus y squared into these two terms. You see that if I come to a common denominator, if I put these two fractions together, then the denominator is going to be that. And the numerator, if I choose a and b correctly, will be 1.

So, integrating this, I can separately integrate a over y dy and b dy over 1 minus y . And those are easy. So partial fractions, after you go to the effort of finding the fractions, then you have separate integrations that you can do. That integral is just a times the log of y . And this is maybe b times-- maybe it's minus b times the log of 1 minus y .

So we've integrated. Just remember though. That with this particular equation, the logistic equation, we didn't have to use partial fractions. We could have done-- we've just seen how, thinking of it as a separable equation.

But that logistic equation had the very neat approach. Much quicker, much nicer. We just introduced z equal 1 over y . We looked at the unknown 1 over y , called it z , found the equation for z , and it was linear.

And we can write down its solution. So when we can do that it wins. But if we don't see how to do that, partial fractions is the systematic way. One fraction, another fraction. Integrate those fractions. Put the answer together. And then, and then, at the end, this is some integral depending on y equal to t .

And to finish the problem perfectly, I would have to solve for y as a function of t . And that was what came out so beautifully by letting 1 over y be z . We got an easy formula for z and then we had the formula for y . This we would integrate easily enough. But then we have to solve to find that formula for y .

OK. That's a more serious example. This example was a very simple one. You can do other examples of separable equations. y and t integrated separately. Good. Thank you.