DEVELOPMENT OF THE COMPLEX NUMBERS

In this chapter, we shall try to motivate how the structure of the complex number system developed. We shall investigate this development both from an algebraic and a geometric point of view. We shall not be concerned with the practical applications of the complex numbers here in the sense that these will be stressed as they occur throughout this block.

A

Complex Numbers From an Algebraic Viewpoint

When only the natural numbers were considered to be real, it turned out that there could be polynomial equations with real (i.e., natural) coefficients that did not possess real solutions. For example, using the language of sets, if our universe of discourse is the set of natural numbers, the solution set of the equation

$$x + 4 = 3 \tag{1}$$

is the empty set. More symbolically,

$$\{x: x + 4 = 3\} = \emptyset. \tag{2}$$

Now suppose we decide that we would like equation (1) to have solutions. Why we would make such a decision is quite subjective, but, for example, we might be interested in knowing what the temperature is if when it warms up 4°F the new temperature will be 1°F.

Be this as it may, once we elect to extend the number system so that (1) has a solution, we wind up inventing the integers.

If we now look at the integers, we see that the integers include the natural numbers, and of even greater importance, the rules of arithmetic for the integers agrees with the previous rules of arithmetic for the natural numbers in all those cases where the integer also happens to be a natural number. Indeed, this is the meaning of an extended number system. It must not only be a super-set of the original number system, but it must preserve the structure of the original system.

At any rate, if we agree to consider the integers as being the new real number system (i.e., we extend our universe of discourse to be the set of integers rather than the set of natural numbers), we now find that equation (1) does have a solution (that's why we invented the integers), and in fact the solution set is given by

$$\{x: x + 4 = 3\} = \{-1\}. \tag{3}$$

Notice that equation (3) is not a contradiction of equation (2) since the equations are in reference to different universes of discourse.

If we now look at the equation

$$2x - 3 = 0 \tag{4}$$

we see that this equation has real (integer) coefficients but it does not have a real solution since the double of an integer cannot be odd while 3 is odd.

By the way, to emphasize how we preserve structure, notice that in trying to solve equation (4), we use the rules of arithmetic which apply to the integers, and we conclude that $2x - 3 = 0$ is equivalent to $2x = 3$ (since we just add -3 to both sides of the equation), and that this is equivalent to $x = 3 \div 2$ since we just divide equals by equals. Thus, if we decide to invent a new number which is denoted by $3 \div 2$, we must make sure that these new numbers obey the old rules since the expression $3 \div 2$ was arrived at by using the old rules.

To make a long story short, the solution of equation (4) yields the invention of the rational numbers. Summarized in terms of sets again,

$$\{x: 2x - 3 = 0\} = \emptyset$$

if our universe of discourse is the integers, while

$$\{x: 2x - 3 = 0\} = \{\tfrac{3}{2}\}$$

if the universe of discourse is the rational numbers.

Proceeding in this way in the last chapter we finally came to a new batch of numbers which we called <u>the</u> real number system. By definition, these were the set of numbers defined by

$$\{x: x^2 \geqslant 0\}. \tag{5}$$

Notice that the test for membership given in (5) is basically no more real than any of the previous number systems, except, perhaps, in the sense that (5) includes all those numbers which we previously called real; namely, the natural numbers, the integers, the rational numbers, and the irrational numbers.

Suppose now we look at the equation

$$x^2 + 1 = 0. \tag{6}$$

Equation (6) has real coefficients (where now for the first time, real means "real" as defined in (5); in other words, it is in a sense a quirk of vocabulary that the term "real" ultimately came to rest upon the set (5) rather than upon any other set of numbers), but according to the criterion established by (5), equation (6) cannot have a real solution since from equation (6), it follows that

$$x^2 = -1 \tag{7}$$

and there is no real number which can satisfy (7).

In summary, if our universe of discourse is the set denoted by equation (5) [we hate to use the term "real" numbers because it seems prejudicial] then

$$\{x: x^2 + 1 = 0\} = \emptyset. \tag{8}$$

BUT IT IS CRUCIAL THAT YOU NOTICE THAT A SOLUTION OF $x^2 + 1 = 0$ IS NO MORE "UNREAL" THAN WAS A SOLUTION OF $x + 4 = 3$ WHEN THE ONLY "REAL" NUMBERS WERE THE NATURAL NUMBERS.

Now based on our "real-life" experience, it is probably more difficult to rationalize why we would like $x^2 + 1 = 0$ to have solutions than it was to rationalize why we would want, say, $2x - 3 = 0$ to have solutions. Nevertheless, if we should decide that $x^2 + 1 = 0$ should have solutions (and we hope that the lecture was sufficient motivation for you to feel that such a decision is justified, but even if not, the fact remains that from a purely philosophical view we have this right), then we must have more numbers. Moreover, since the existing numbers are being called "real," any new number is, to say the least, non-"real."

Let us invent the symbol (numeral) i to denote <u>one</u> solution of the equation $x^2 + 1 = 0$. (As an interesting aside to the number-versus-numeral theme, one often uses the numeral j rather than i in electrical engineering since i is usually used in that field to denote current. As a further aside, notice that in either case, if one is also using planar vectors in Cartesian coordinates, the "numerals" $\vec{i}$ and $\vec{j}$ are also present).

If we now impose the usual restriction that our extended number system must obey the same structure as the system it extends, we find that since $i^2 = -1$ (by definition of being a solution of $x^2 + 1 = 0$) then $(-i)^2 = -1$ also.

Namely,

$$(-i)^2 = (-i)(-i) \tag{9}$$

$$= [(-1)i][(-1)i] \tag{9.1}$$

$$= [(-1)(-1)]i^2 \tag{9.2}$$

$$= [(-1)(-1)](-1) \tag{9.3}$$

$$= 1(-1) \tag{9.4}$$

$$= -1. \tag{9.5}$$

Notice that while every step may have seemed "obvious" in going from (9) to (9.5), we were assuming certain structural properties. For example, while it is true that $-a = -1(a)$ for any real number a, we were also assuming that this rule applied to our extended number system when we wrote $-i = (-1)i$ since i is not a real number.

Thus, in the extended number system the equation $x^2 + 1 = 0$ has two roots (actually we have only proved that it has at least two roots, i and -i, but we do not want to become any more rigorous at this point). If we elect to call the extended number system in this case the complex numbers, we are saying that

$$\{x: x^2 + 1 = 0\} = \{i, -i\} \tag{10}$$

if our universe of discourse is the complex numbers.

Before we decide to pursue the computational aspects of the complex numbers any further, let us observe that we could next call the complex numbers the new "real" numbers, and it appears that we could continue this sequence of extensions of the number system ad infinitum (ad nauseum?). An amazing result, which shall be discussed as a prelude to the solution of Exercise 1.3.7 in the next unit, is that complex numbers end the chain! In other words, any polynomial equation which has complex numbers as coefficients has all its roots as complex numbers. In other words, there is no need to extend the complex numbers if all we want to be able to do is express the solution set of each polynomial equation with complex coefficients!

B

Complex Numbers From a Geometric Viewpoint

Very early in Part 1 of our course, we established the theme that a picture is worth a thousand words. We would now like to revive this idea in terms of the development of the complex number system. We began this chapter with an algebraic treatment of the complex numbers in order to emphasize another central theme of our course - namely, that of mathematical structure, but we feel that a geometric development might make things a bit easier to visualize.

To begin with, notice that when we used the number line, we were in effect visualizing numbers either as points on the line or else as lengths (and notice in this respect the notion of vectors; that is if we assume that a line segment originates at 0 it terminates at the point which names its length).

To motivate the complex numbers from the number line motif, all we need ask is whether a point should be denied the "privilege" of naming a number simply because it was not located on the number line? In other words, since a point in the plane which is not on the number line is as "legitimate" a point as one which is on the number line, shouldn't these points also be allowed to name numbers?

Stated in terms of lengths, shouldn't a length drawn from the origin to any point in the plane be as valid a way of denoting a number as a line drawn from the origin to a point on the x-axis?

Once we agree to answer the question in the affirmative, we have agreed to extend the number system, at least from a geometrical point of view. This means that we must also extend the geometric versions of the rules of arithmetic from the x-axis to the plane.

For example, two numbers were said to be equal if as lengths starting at the origin, they terminated at the same point on the number line. We would now define two of our new numbers to be equal simply by deleting reference to the number line and replacing it by reference to the plane. That is, if we agree to view lengths in the plane as denoting the complex numbers, we define two complex numbers to be equal if when they originate at the origin they terminate at the same point in the plane.

Notice that this gives us a way of describing what we have called the real and the imaginary parts of a complex number in terms of the plane. Namely, suppose we identify the x-axis with what we will call the real axis, and the y-axis with the (purely) imaginary axis. Then the point (a,b) in the xy-plane denotes the complex number a + bi. We may also, in this context, view the complex number as being the vector from the origin to the point (a,b). When we use this interpretation the xy-plane becomes known as the Argand Diagram. In other words, the Argand Diagram is the xy-plane viewed as depicting the complex numbers.

How shall we add two complex numbers? Well, if we had never invented the concept of vector addition previously, we would have been tempted to do so now because this is precisely how numbers are added as lengths. In other words, viewed as vectors, we add two complex numbers as we would add vectors; and this, too, captures the flavor of what it means to add the real parts and the imaginary parts to form the sum of complex numbers.

If we now define multiplication of complex numbers to be obtained geometrically by multiplying the magnitudes and adding the arguments (as described in Lecture 1.010) we see that this not only agrees with the definition given in terms of the last section, but that also it extends the idea of multiplication in the real case. Namely, when we multiply real numbers, we multiply their magnitudes and add their arguments. The key point for real numbers is that the argument is either 0° (i.e., when the number is positive) or 180° (i.e., when the number is negative). Thus, the product of two positive numbers is still positive since 0° + 0° = 0°; positive times negative is negative because 180° + 0° = 180°; and negative times negative is positive since 180° + 180° = 360°, which is equivalent (in position in the plane) to 0°.

We also notice that the concept of absolute value can also be extended from the x-axis to the plane by defining the absolute value of a complex number to be its distance from the origin (when viewed as a point) or as its length (when viewed as a vector). Again, we leave the computational details to the exercises in this unit, but we would like to close this brief discussion with the same point we brought up in the previous section.

It is obvious that from this geometrical point of view there is a natural extension of the complex number system (the Argand Diagram). Namely, in the same way that a point in the plane is as worthy of being named as is a point on the x-axis, it is clear that any point in 3-space is as worthy as being named as is a point in the plane. Why, then, can't we in turn extend the complex numbers, at least geometrically, to include three dimensional space? Again, as we shall explain in the solution of Exercise 1.3.7, there is no need, at least in terms of the usual mathematical applications, to do this.