Unit 7: Complex Series

1.7.1(L)

Our aim here is to emphasize the point made in the lecture that the structure of the limit concept is the same in the complex case as it was in the real case.

Recall that in the real case, our proof went somewhat as follows: Given $\varepsilon > 0$ we can find N_1 and N_2 such that

$$n > N_1 \rightarrow |L_1 - a_n| < \frac{\varepsilon}{2}$$

and (1)

$$n > N_2 \rightarrow |a_n - L_2| < \frac{\varepsilon}{2}$$

Hence, letting $N = \max\{N_1, N_2\}$ we see that

$$n > N \rightarrow |L_1 - a_n| + |a_n - L_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad (2)$$

But,

$$|L_1 - L_2| = |(L_1 - a_n) + (a_n - L_2)| \leq |L_1 - a_n| + |a_n - L_2| ,$$

so by (2),

$$n > N \rightarrow |L_1 - L_2| < \varepsilon. \quad (3)$$

Since $\varepsilon > 0$ was arbitrarily chosen, equation (3) tells us that for sufficiently large n, $|L_1 - L_2|$ is less than <u>any</u> prescribed positive number. Hence, since $|L_1 - L_2|$ is a constant which is independent of n, we may conclude that, since $|L_1 - L_2|$ is less than any prescribed positive amount and since $|L_1 - L_2| \geq 0$, that $|L_1 - L_2| = 0$, or $L_1 = L_2$.

The point is that every statement in our above proof remains valid when the a_n's, L_1, and L_2 are complex numbers.

Even the flavor of the geometric proof is preserved provided

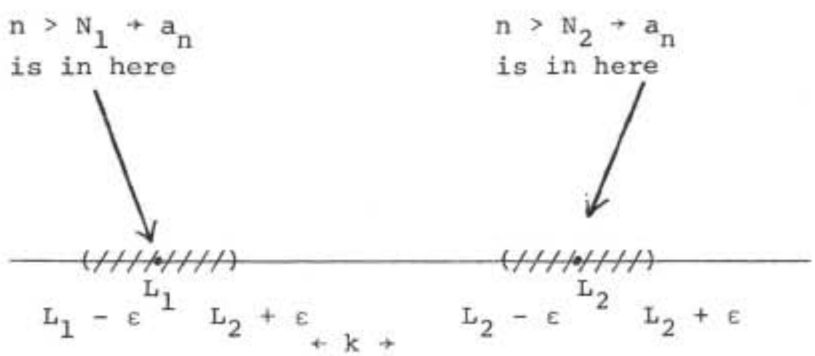
1.7.1(L) continued

only that we remember that intervals are replaced by discs.

For example, in the real case the geometric proof went as follows: Assume $L_1 \neq L_2$ and let $|L_1 - L_2| = k > 0$. Pick $\varepsilon > 0$ arbitrarily subject only to the condition that

$$\varepsilon < \frac{k}{2} .$$

We then have,



1. $\varepsilon < \frac{k}{2}$ $\quad (L_1 - \varepsilon, L_1 + \varepsilon) \cap (L_2 - \varepsilon, L_2 + \varepsilon) = \emptyset$

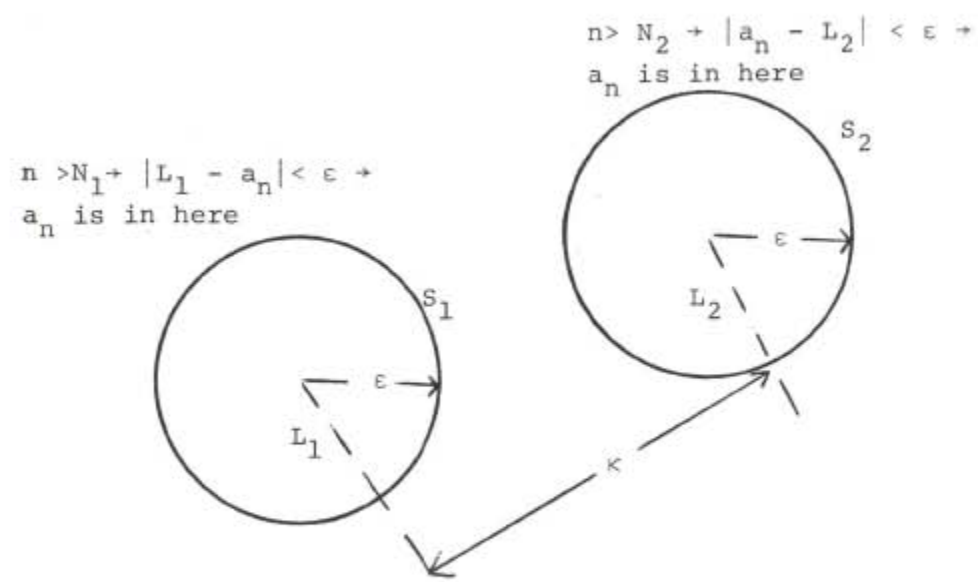
2. Hence $n > \max\{N_1, N_2\} \to a_n$ is in two <u>mutually-disjoint</u> intervals at the same time, which is a contradiction.

In the complex case, our domain is the z-plane rather than the x-axis, but we still begin as before, namely, assume $L_1 \neq L_2$ and let $|L_1 - L_2| = k > 0$. Pick $\varepsilon > 0$ arbitrarily subject only to the condition that

$$\varepsilon < \frac{k}{2}.$$

we then have

1.7.1(L) continued



1. $\varepsilon < \frac{k}{2} \rightarrow S_1 \cap S_2 = \emptyset$.

2. $n > \max\{N_1, N_2\} \rightarrow a_n \in S_1 \cap S_2 = \emptyset \rightarrow$ contradiction.

With this exercise as a specific illustration we now hope that it's clear as to how the structure of limits is transmitted (extended) from the real numbers to the complex numbers verbatim except that,pictorially, intervals are replaced by discs.

1.7.2(L)

a. Here we make use of the fact that our proofs, in the real case, for absolute and/or uniform convergence involved only $|f(x)|$. The point is that even though f(z) need not be (a non-negative) real number, $|f(z)|$ is! Consequently, since f(x) and f(z) share the same crucial properties concerning absolute values, we may again mimick the procedure used in the real case. Namely, suppose

$$f(z) = \sum_{n=0}^{\infty} (-1)^n (n + 1) z^n.$$

1.7.2(L) continued

To test for absolute converge we apply the ratio test to the <u>positive series</u>,

$$\sum_{n=0}^{\infty} |(-1)^n (n + 1) z^n|$$

$$= \sum_{n=0}^{\infty} (n + 1) |z|^n \ . \tag{1}$$

Equation (1) denotes a real positive power series (i.e. $|z| \geq 0$), so we may apply the ratio test to

$$\sum_{n=0}^{\infty} a_n$$

with $a_n = (n + 1)|z|^n$. This leads to

$$\frac{a_{n+1}}{a_n} = \frac{(n + 2)|z|^{n + 1}}{(n + 1)|z|^n} \ .$$

Hence, for $z \neq 0$

$$\frac{a_{n+1}}{a_n} = \frac{n + 2}{n + 1} |z|$$

so that

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \overbrace{\lim_{n\to\infty} \left(\frac{n + 2}{n + 1}\right)}^{1} \overbrace{\lim_{n\to\infty} |z|}^{z}$$

$$= |z| \tag{2}$$

and since to have convergence we must have that $\rho<1$, we may conclude from (2) that

$\rho < 1 \leftrightarrow |z| < 1.$

1.7.2(L) continued

Therefore, the radius of convergence for

$$\sum_{n=0}^{\infty} (-1)^n (n + 1) z^n$$

is R = 1. That is f(z) converges uniformly and absolutely for all z such that $|z| < 1$ and f(z) diverges for all z such that $|z| > 1$. As for what happens on the circle $|z| = 1$, all we know is that there is at least one point z such that f(z) diverges.

Note

The more astute reader may have recalled

$$1 - 2x + 3x^2 - 4x^3 + \ldots \equiv \frac{1}{(1 + x)^2} . \tag{3}$$

Thus, by replacing x by z in (3) we obtain

$$\frac{1}{(1 + z)^2} \equiv 1 - 2z + 3z^2 - 4z^3 + \ldots \qquad |z| < 1$$
$$\equiv \sum_{n=0}^{\infty} (-1)^n (n + 1) z^n \tag{4}$$

In this form we observe that "trouble" occurs when z = -1 (which is, as claimed above, on the circle $|z| = 1$).

We picked an example in which

$$\sum_{n=0}^{\infty} a_n z^n$$

would be expressed in closed form only to emphasize the theory more concretely. The key point is that we did not have to recognize that

$$\sum_{n=0}^{\infty} (-1)^n (n+1) z^n$$

was

$$\frac{1}{(1 + z)^2}$$

1.7.2(L) continued

to obtain the required result.

b. Here again we only wish to re-emphasize the property of absolute convergence that we may add terms in any order we wish.

In particular,

$$f\left(\frac{i}{12}\right) = 1 - 2\left(\frac{i}{12}\right) + 3\left(\frac{i}{12}\right)^2 - 4\left(\frac{i}{12}\right)^3 + \ldots \quad (5)$$

$$= 1 - \frac{i}{6} + \frac{3(-1)}{144} - \frac{4(-i)}{1728} + \ldots$$

$$\approx 1 - \frac{i}{6} - \frac{1}{48} + \frac{i}{432} \quad (6)$$

and the error in using (6) cannot exceed the magnitude of the next term in (5). That is, equation (6) yields $f(\frac{i}{12})$ with an error no greater than $\left|5\left(\frac{i}{12}\right)^4\right| = \frac{5}{20,736} \approx 0.00025$.

Regrouping the terms in (6) we see that

$$f\left(\frac{i}{12}\right) \approx \frac{47}{48} - \frac{71}{432}i. \quad (7)$$

In the present example we can check the accuracy of equation (7) because we know that in closed form, for $|z| < 1$, $f(z) = \frac{1}{(1 + z)^2}$.

Hence,

$$f\left(\frac{i}{12}\right) = \frac{1}{\left(1 + \frac{i}{12}\right)^2}$$

$$= \frac{144}{(12 + i)^2}$$

$$= \frac{144}{143 + 24i}$$

$$= \frac{144(143 - 24i)}{(143 + 24i)(143 - 24i)}$$

1.7.2(L) continued

$$= \frac{144(143)}{(143)^2 + (24)^2} - \frac{144(24)i}{(143)^2 + (24)^2}$$

$$= \frac{20592}{21025} - \frac{3456}{21025}\, i. \qquad (8)$$

Rewriting (7) and (8) in decimal form we have

$$f\left(\frac{i}{12}\right) = 0.9791^{+} - 0.1644i \qquad (7')$$

$$f\left(\frac{i}{12}\right) = 0.9794 - 0.1643i. \qquad (8')$$

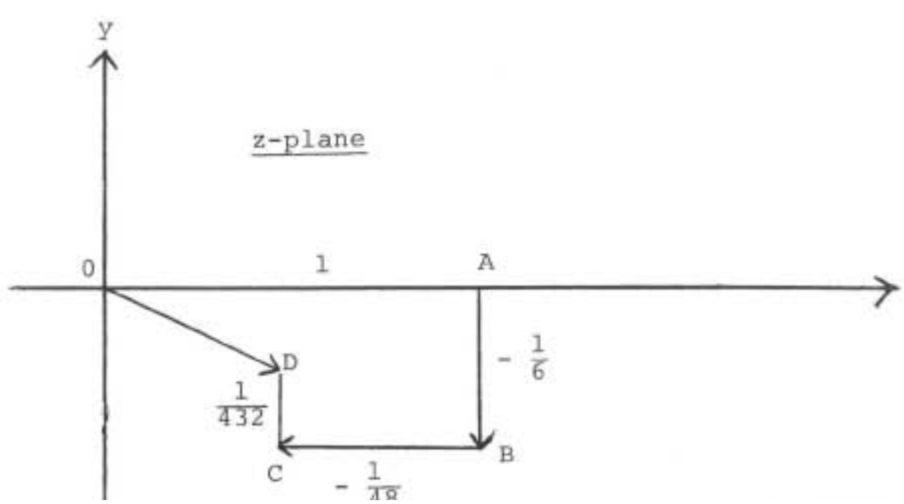
Hence to three decimal place accuracy

$$f\left(\frac{i}{12}\right) = 0.979 - 0.164i.$$

Note

If equations (8) and (8') were not available to us, we could still have concluded that if we had drawn a circle with radius $\frac{5}{20,736}$ centered at the point $\left(\frac{47}{48}, -\frac{71}{432}\right)$ in the z-plane, $f\left(\frac{i}{12}\right)$ would have to be within this circle.

Again we may view this result pictorially by observing that $1 - \frac{i}{6} - \frac{1}{48} + \frac{i}{432}$ represents the vector sum (not drawn to scale):



1.7.2(L) continued

1. OD = OA + AB + BC + CD = $1 - \frac{i}{6} - \frac{1}{48} + \frac{i}{432}$. The next vector in our sum would have been $\frac{5}{20,736}$ and each remaining term has a smaller magnitude than its predescessor, so the entire sum is within $\frac{5}{20,736}$ of D.

2. Notice that this parallels our treatment of the real case, except that our vectors are in the plane now, rather than along the x-axis.

c. Here we are emphasizing the property of uniform convergence whereby we are allowed to compute $f'(z)$ by differentiating $\Sigma\, a_n z^n$ term by term within the circle of convergence.

Since $f(z)$ converges uniformly to $1 - 2z + 3z^2 - 4z^2 + \ldots$ in $|z| < 1$, we have that for all z such that $|z| < 1$,

$$f'(z) = -2 + 6z - 12z^2 + z_o z^3. \tag{9}$$

From (9) we see that

$$f'\left(\frac{i}{12}\right) \approx -2 + 6\left(\frac{i}{12}\right) - 12\left(\frac{i}{12}\right)^2 \tag{10}$$

with an error no greater than $\left|20\left(\frac{i}{12}\right)^3\right| = \left|\frac{-20i}{1728}\right| \approx \frac{1}{86}$; so that from (10),

$$f'\left(\frac{i}{12}\right) = -2 + \frac{i}{2} + \frac{1}{12} + k \text{ where } |k| < \frac{1}{86} \text{ or } f'\left(\frac{i}{12}\right) = -\frac{23}{12} + \frac{i}{2}. \tag{11}$$

From (11) we may conclude that $f'\left(\frac{i}{12}\right)$ is located within the circle centered at $\left(-\frac{23}{12}, \frac{1}{2}\right)$ with radius

$$\frac{20}{1728} = \frac{5}{435} \approx \frac{1}{86}$$

so that our answer has one decimal accuracy.

Again, as a check in this exercise we know that

$$f(z) = \frac{1}{(1 + z)^2} .$$

1.7.2(L) continued

Hence

$$f'(z) = \frac{-2}{(1+z)^3},$$

so that

$$f'\left(\frac{i}{12}\right) = \frac{-2}{\left(1+\frac{i}{12}\right)^3}$$

$$= \frac{-2\ (1728)}{(12+i)^3}$$

$$= \frac{-3456}{1728 + 432i - 36 - i}$$

$$= \frac{-3456}{1692 + 431i}$$

$$= \frac{-3456\ (1692 - 431i)}{(1692)^2 + (431)^2}, \text{ etc.}$$

A Note on Analytic Functions

If the function f(z) may be represented by the power series

$$\sum_{n=o}^{\infty} \frac{f^{(n)}(0)}{n!} z^n ,$$

we say that f(z) is analytic or regular. That is, in the present context, an analytic function is one which possesses an nth derivative for all (integral) values of n. Now in our earlier treatment of differentiation, we agreed to call a function analytic if it obeyed the apparently-less-stringent condition that f'(z) exist. Thus, it would appear that we now have two difinitions of analytic, one of which is more "powerful" than the other.

The interesting point, which we shall not prove here but rather in the next Unit, is that for a complex function of a complex variable to be differentiable a very strict condition is imposed

1.7.2(L) continued

(i.e., $\lim_{\Delta z \to 0} \left[\frac{f(z + \Delta z) - f(z)}{\Delta z}\right]$

must exist <u>and be the same</u> in each possible direction). The upshot of this condition is that it is so restrictive that once a complex function of a complex variable possesses a first derivative it possesses derivatives of every order. This is a very glaring difference between differentiable functions of a real variable and differentiable functions of a complex variable. In summary, if f'(x) exists there is no guarantee that f"(x) will exist [e.g., if $f(x) = x^{3/2}$, the f'(x) exists when x = 0 but not f"(x)]. On the other hand, if f'(z) exists so also will $f^{(n)}(z)$ for every positive integer, n.

1.7.3

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \frac{z^6}{6!} + \frac{z^7}{7!} + \frac{z^8}{8!} + \cdots$$

Hence,

$$e^i \approx 1 + i + \frac{i^2}{2!} \quad \frac{i^3}{3!} \quad \frac{i^4}{4!} \quad \frac{i^5}{5!} \quad \frac{i^6}{6!} \quad \frac{i^7}{7!} \quad \frac{i^8}{8!}$$

$$= 1 + i - \frac{1}{2} - \frac{i}{6} + \frac{1}{24} + \frac{i}{120} - \frac{1}{720} - \frac{i}{5040} + \frac{1}{40,320}$$

$$= (1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} + \frac{1}{40,320}) + i(1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040}) \qquad (1)$$

$$\approx .5403 + .8414i. \qquad (2)$$

On the other hand, $e^{iz} \equiv \cos z + i \sin z$ with z = 1 implies

$$e^i = \cos 1 + (\sin 1)i \qquad (3)$$

and from the tables

cos 1(radian) $\approx$ .54
sin 1 $\approx$ $\approx$ .84

so that (3) becomes

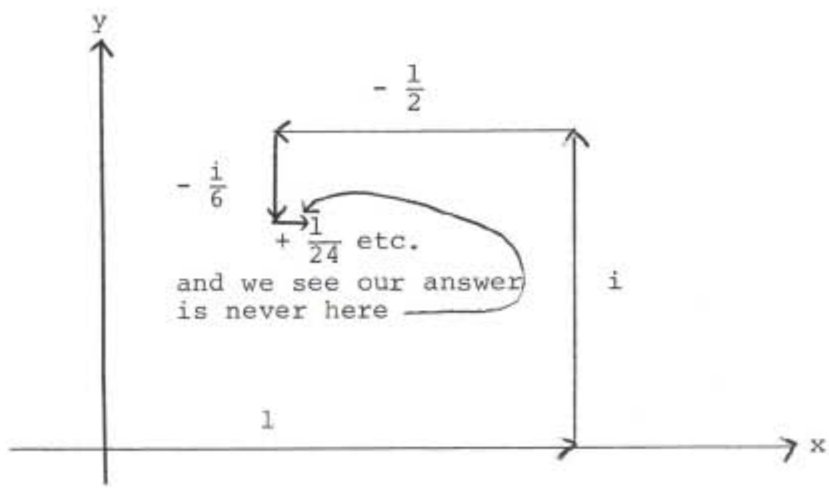
1.7.3 continued

$e^i \approx .54 + .84i$

which checks very well with our result in (2).

It is also worth noting that when we collected terms to form equation (1), the real part is $\cos 1 \approx 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!}$ and $\sin 1 \approx 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!}$, etc. so that equation (1) is the (turncated) equivalent of equation (3).

Again, pictorially,equation (1) is equivalent with



1.7.4(L)

Here we are showing other examples of how complex variables are used in real situations.

a. In an earlier unit we mentioned that if f and g were integrable functions of the <u>real</u> variable t then the complex-valued function f(t) + i g(t) was also integrable, and, in fact

$$\int_a^b [f(t) + ig(t)]dt = \int_a^b f(t)dt + i \int_a^b g(t)dt. \quad (1)$$

In the present exercise we observe that $e^{ax}\cos bx$ is the real part of $e^{(a + ib)x}$.

1.7.4(L) continued

Namely,

$$e^{(a + ib)x} = e^{ax + ibx}$$

$$= e^{ax}e^{ibx}$$

$$= e^{ax}[\cos bx + i \sin bx], \qquad (2)$$

where equation (2) was derived by using the usual exponential properties.

From (2), we conclude that

$$\int e^{(a + ib)}dx = \int e^{ax}[\cos bx + i \sin bx]dx$$

$$= \int [e^{ax}\cos bx + ie^{ax}\sin bx]dx \qquad (3)$$

and by (1) we conclude from (3) that

$$\int e^{(a + ib)x}dx = \int e^{ax}\cos bx\, dx + i\int e^{ax}\sin bx\, dx \qquad (4)$$

$$\left(= \int \text{Re}[e^{(a + ib)x}]dx + i\int \text{Im}[e^{(a + ib)x}]dx\right).$$

Now the left side of (4) yields

$$\int e^{(a + ib)x}dx = \frac{1}{a + ib}\, e^{(a + ib)x} \ (+ c)$$

$$= \frac{a - ib}{a^2 + b^2}\, e^{ax}e^{ibx} \ (+ c).$$

Thus,

$$\int e^{(a + ib)x}dx = \frac{a - ib}{a^2 + b^2}\, [e^{ax}\cos bx + ie^{ax}\sin bx] \ (+ c)$$

$$= \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}$$

$$+ ie^{ax}\, \frac{(a \sin bx - b \cos bx)}{a^2 + b^2} \ (+ c) \qquad (5)$$

1.7.4(L) continued

Replacing $\int e^{(a + ib)x}dx$ in (4) by its value in (5) and equating real and imaginary parts, we conclude that

$$\left.\begin{aligned} \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2} \quad (+ c_1) &= \int e^{ax}\cos bx\, dx \\ &\text{and} \\ \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2} \quad (+ c_2) &= \int e^{ax}\sin bx\, dx \end{aligned}\right\} \qquad (6)$$

To be sure, we could have obtained (6) [as we did in Part 1 of our course] by integration by parts; but what we wanted to show here was a rather elegant, fairly simple application of complex numbers.

b. This identity is a bit tougher to handle directly (although we did do it in Part 1 during our treatment of the definite integral).

The approach to use here is to observe that

$$\cos k\theta \equiv \text{Re}(e^{ik\theta}). \qquad (7)$$

In other words,

$$e^{ik\theta} \equiv \cos k\theta + i \sin k\theta.\text{*}$$

Thus,

$$\sum_{k=0}^{n} \cos k\theta = \sum_{k=0}^{n} \text{Re}(e^{ik\theta}) \qquad (8)$$

and because

$\sum_{k=0}^{n} \text{Re}(z_k) = \text{Re}(\sum_{k=0}^{n} z_k)$, we may rewrite (8) as

*In one form or another, this identity is used over and over to allow us to express sines and cosines as exponentials.

1.7.4(L) continued

$$\sum_{k=0}^{n} \cos k\theta = \mathrm{Re}\left[\sum_{k=0}^{n} e^{ik\theta} \right]. \tag{9}$$

But,

$$\sum_{k=0}^{n} e^{ik\theta} = i + e^{i\theta} + \ldots + e^{in\theta} = \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}$$

$$= \frac{1 - \cos(n+1)\theta - i \sin(n+1)\theta}{1 - \cos\theta - i \sin\theta}$$

$$= \frac{[1 - \cos(n+1)\theta - i \sin(n+1)\theta][1 - \cos\theta + i \sin\theta]}{(1 - \cos\theta - i \sin\theta)(1 - \cos\theta + i \sin\theta)}$$

$$= \frac{[1 - \cos(n+1)\theta][1 - \cos\theta] + \sin(n+1)\theta \sin\theta}{(1 - \cos\theta)^2 + \sin^2\theta}$$

$$+ i\frac{[\sin\theta(1 - \cos[n+1]\theta - (1-\cos\theta)\sin(n+1)\theta}{(1 - \cos\theta)^2 + \sin^2\theta}.$$

Hence,

$$\mathrm{Re}\left[\sum_{k=0}^{n} e^{ik\theta} \right] = \frac{[1 - \cos(n+1)\theta][1 - \cos\theta] + \sin(n+1)\theta \sin\theta}{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= \frac{1 - \cos(n+1)\theta - \cos\theta}{2 - 2\cos\theta}$$

$$+ \frac{\cos(n+1)\theta \cos\theta + \sin(n+1)\theta \sin\theta}{2 - 2\cos\theta}$$

$$= \frac{1 - \cos\theta - \cos(n+1)\theta + \cos n\theta}{2(1 - \cos\theta)}$$

1.7.4(L) continued

$$= \frac{[1 - \cos\theta] - [\cos(n+1)\theta - \cos n\theta]}{2(1 - \cos\theta)}. \qquad (10)$$

Substituting (10) into (9) yields

$$\sum_{k=0}^{n} \cos k\theta = \frac{1}{2} - [\frac{\cos(n+1)\theta - \cos n\theta)}{2(1 - \cos\theta)}],\ \cos\theta \neq 1. \qquad (11)$$

1.7.5

a. $e^{iz} = \cos z + i \sin z$ (1)

$$e^{-iz} = e^{i(-z)} = \cos(-z) + i\sin(-z)$$

$$= \cos z - i \sin z^*.$$

Subtracting (2) from (1) yields $e^{iz} - e^{-iz} = 2i \sin z$.

Hence

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}. \qquad (3)$$

If we now write $z = x + iy$, equation (3) yields

$$w = \sin(x + iy) = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$$

$$= \frac{e^{-y+ix} - e^{-ix+y}}{2i}$$

$$= \frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i \sin x)}{2i}$$

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*Note that from the power series definition

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots = 1 - \frac{(-z)^2}{2!} + \frac{(-z)^4}{4!} - \frac{(-z)^6}{6!} + \ldots$$

$$= \cos(-z), \text{ etc.}$$

1.7.5 continued

$$= \frac{-\cos x(e^y - e^{-y}) + i \sin x(e^y + e^{-y})}{2i}.$$

Hence,

$$w = \sin(x + iy) = \frac{1}{i}\left[-\cos x\left(\frac{e^y - e^{-y}}{2}\right) + i \sin x\left(\frac{e^y + e^{-y}}{2}\right)\right]$$

$$= -i\left[-\cos x\left(\frac{e^y - e^{-y}}{2}\right) + i \sin x\left(\frac{e^y + e^{-y}}{2}\right)\right]$$

$$= \sin x\left(\frac{e^y + e^{-y}}{2}\right) + i \cos x\left(\frac{e^y - e^{-y}}{2}\right) \qquad (4)$$

$$= \underbrace{\sin x \cosh y}_{u} + i \underbrace{\cos x \sinh y}_{v}. \qquad (4')$$

Therefore, w = sin z corresponds to the mapping of the xy-plane into the uv-plane defined by

$$\left.\begin{array}{l} u = \sin x \cos hy \\ v = \cos x \sin hy \end{array}\right\} \qquad (5)$$

Moreover, since $w = \sin z$ implies $\frac{dw}{dz} = \cos z$. [Here again notice our reliance on the power series' definition of the trigonometric functions. Namely, $\sin z \equiv z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots \rightarrow \frac{d(\sin z)}{dz} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots = \cos z$.]

Our mapping is everywhere analytic.

Hence, the mapping defined by (5) is conformal except in neighborhoods of those points z for which cos z = 0 (i.e., for the mapping f to be conformed, f' must be unequal to 0).

b. From equation (5) we have

$$\left.\begin{array}{l} u^2 = \sin^2 x \cos h^2 y \\ v^2 = \cos^2 x \sin h^2 y \end{array}\right\} \qquad (5')$$

1.7.5 continued

or

$$\begin{cases} \dfrac{u^2}{\cosh^2 y} = \sin^2 x \\ \dfrac{v^2}{\sinh^2 y} = \cos^2 x \end{cases} \quad . \tag{6}$$

Adding the two equations in (6) yields

$$\frac{u^2}{\cos h^2 y} + \frac{v^2}{\sin h^2 y} = \sin^2 x + \cos^2 x = 1. \tag{7}$$

From (7) we see that the line $y = y_o$ maps into the <u>ellipse</u>

$$\frac{u^2}{\cosh^2 y_o} + \frac{v^2}{\sinh^2 y_o} = 1$$

(unless $y_o = 0$ in which case $\sin h^2 y_o = 0$ and hence,

$$\frac{v^2}{\sinh^2 y_o}$$

is not defined. However when $y = 0$, $\cos hy = 1$, so equation (5) yields

$$\left.\begin{matrix} u = \sin x \\ v = 0 \end{matrix}\right\} .$$

Since $-1 \leq \sin x \leq 1$, we conclude that the image of the x-axis [i.e., $y = 0$] is the segment of the u-axis [i.e., $v = 0$] defined by $-1 \leq u \leq 1$).

We next observe that (5') can also be written in the form

$$\left.\begin{matrix} \dfrac{u^2}{\sin^2 x} = \cosh^2 y \\ \dfrac{v^2}{\cos^2 x} = \sinh^2 y \end{matrix}\right\} \tag{8}$$

1.7.5 continued

and since $\cosh^2 y - \sinh^2 y = 1$, equation (8) implies that the image of the line $x = x_o$ is the hyperbola

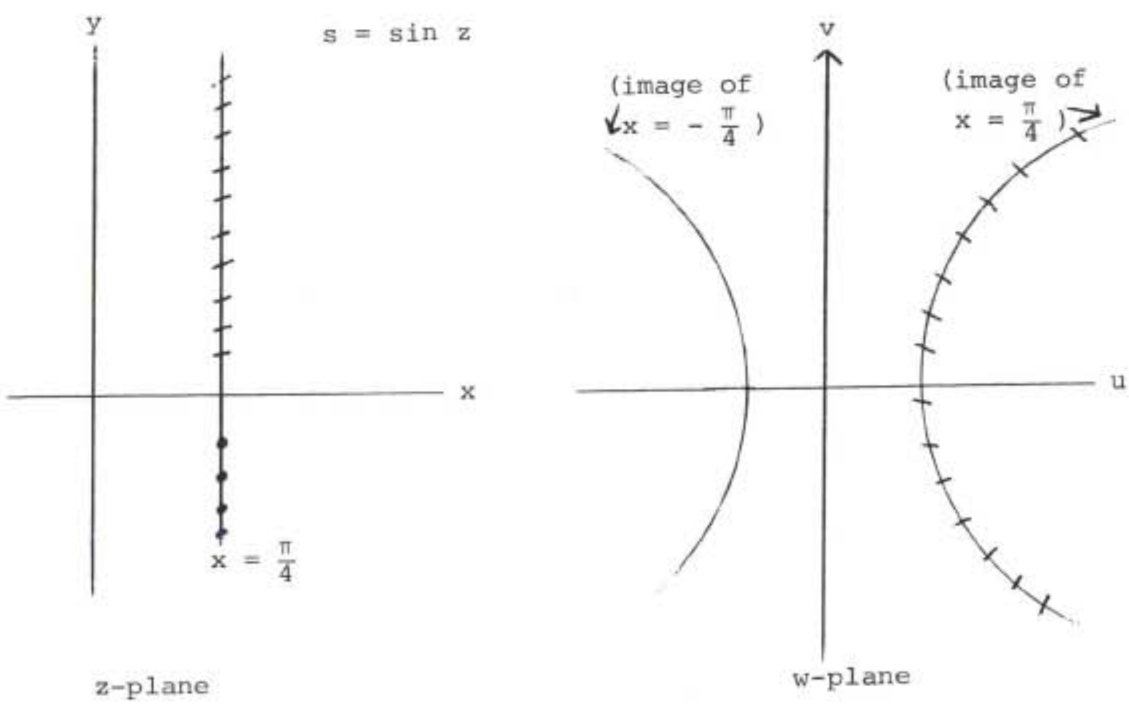
$$\frac{u^2}{\sin^2 x_o} - \frac{v^2}{\cos^2 x_o} = 1 \qquad (9)$$

unless $\sin x_o$ or $\cos x_o$ is 0. Again from equation (5) we have that if $\sin x = 0$ then

$$\begin{cases} u = 0 & (\text{since } \sin x = 0 \leftrightarrow \cos x = \pm 1). \\ v = \pm \sinh y \end{cases}$$

In other words, if $\sin x_o = 0$ the line $x = x_o$ is mapped onto the v-axis and if $\cos x_o = 0$ it is mapped onto the u-axis.

Pictorially,



1.7.6(L)

The overall aim of this exercise is to show how the use of power series representations of analytic functions allows us to solve some rather difficult problems. Just as in the real case, the fundamental building block of much of the technique rests on the fact that if two analytic functions are identical on a given region then their coefficients, term by term, must be equal.

Our approach in tackling this problem is to extend a result that we already know to be true for both real and complex power series. The idea is that we know that if

$$\sum_{n=0}^{\infty} a_n x^n$$

is identically zero and if the a_n's are real then $a_n = 0$ for each n. In the same way that we proved this result, it also follows immediately that if

$$\sum_{n=0}^{\infty} a_n z^n$$

is identically zero and each of the a_n's is complex then $a_n = 0$ for all n.

What we are trying to do in part (a) of this exercise (as a building block to the rest of the exercise) is to show that if

$$\sum_{n=0}^{\infty} a_n x^n$$

is identically zero but now the a_n's are complex then it is still true that $a_n = 0$ for each n. In other words, the aim of part (a) is to show that if x is a real variable and

$$\sum_{n=0}^{\infty} a_n x^n$$

is identically equal to zero then each a_n is equal to zero regardless of whether we look at the domain of the a_n's as being the

1.7.6(L) continued

real numbers or the complex numbers. Notice that we do not have to worry about the "companion" case of what happens when z is complex and the a_n's are real. Namely, the real numbers are a sub-structure of the complex numbers so that the case in which the a_n's happen to be real is included in the general category;

$$\sum_{n=0}^{\infty} a_n z^n$$

where the a_n's are complex. In this context it is wise to distinguish between "complex" and "non-real". The complex numbers include the reals as a subset. That is, by way of review, the real numbers are those complex numbers a + bi in which b = 0. On the other hand, a + bi being non-real means that b is <u>not equal to 0</u>. In other words, if we specifically require that a number not be real then we must call it non-real. It is not enough to say that it is complex. Moreover, in this case, we should also avoid saying that the number is imaginary since the term "imaginary" (or "purely imaginary") is often reserved for complex numbers of the form 0 + bi.

a. Given that

$$\sum_{n=0}^{\infty} c_n x^n = 0$$

where x is real and the c_n's are complex, we write c_n in the form $a_n + ib_n$ where a_n and b_n are real. We then have:

$$\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (a_n + ib_n)x^n. \qquad (1)$$

Notice now that it makes sense to ask whether the series in (1) converges absolutely, etc. That is, since all real numbers are complex, the terms $a_n x^n$ are complex numbers, so that

$$\sum_{n=0}^{\infty} a_n x^n$$

1.7.6(L) continued

represents a complex series; and we can talk about the convergence of complex series. At any rate, assuming that our series in (1) converges absolutely in some interval, $|x| < R$ (the interval possibly being the set of all real numbers) we may perform our usual change of order in the terms to conclude that

$$\sum_{n=0}^{\infty} (a_n + ib_n)x^n = \sum_{n=0}^{\infty} a_n x^n + i \sum_{n=0}^{\infty} b_n x^n. \tag{2}$$

Hence the fact that

$$\sum_{n=0}^{\infty} c_n x^n = 0$$

implies (by combining the information in equations (1) and (2)) that

$$\underbrace{\sum_{n=0}^{\infty} a_n x^n}_{\substack{\text{real} \\ \text{(since } a_n\text{'s} \\ \text{are real)}}} + i \underbrace{\sum_{n=0}^{\infty} b_n x^n}_{\substack{\text{real} \\ \text{(since } b_n\text{'s} \\ \text{are real)}}} = 0 = 0 + 0i. \tag{3}$$

Comparing the real and imaginary parts in equation (3), we conclude that

$$\sum_{n=0}^{\infty} a_n x^n \equiv 0 \text{ and } \sum_{n=0}^{\infty} b_n x^n \equiv 0. \tag{4}$$

Since the a_n's and b_n's are real we may apply our theorems about real series to conclude that $a_n = b_n = 0$ for each n.

Finally, since $c_n = a_n + ib_n$, we may conclude from the fact that $a_n = b_n = 0$, that $c_n = 0$; and we have proven the required result.

b. The implication of the result in this exercise is more impressive than the proof of the result. What we are going to show in this exercise is that if an analytic function is identically zero on

1.7.6(L) continued

even one line (in the present example, the line is taken as the real axis but this is not important) then it must be identically zero in its entire domain of definition. The proof of this result is word-for-word the same as the corresponding result in the real case with the exception that we must now invoke the result of part (a) to conclude that

$$\sum_{n=0}^{\infty} a_n x^n = 0$$

implies that $a_n = 0$. Namely, in the real case studied in Part 1 of our course, the a_n's were real, but now the a_n's are complex.

At any rate, the proof is straight-forward; namely assuming that

$$\sum_{n=0}^{\infty} a_n z^n \equiv 0 \text{ for all } \underline{\text{real}}\ z \qquad (5)$$

we may invoke the result of part (a) to conclude at once that $a_n = 0$ for each n. [If you wish to proceed a bit more slowly, think of $z = x + iy$, so that if z is real we may write $z = x$. In this way equation (5) becomes

$$\sum_{n=0}^{\infty} a_n x^n = 0,$$

in which case our expression looks exactly the same as the one we discussed in part (a).]

The final step consists of the observation that since $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and each $a_n = 0$,* then

$$f(z) = \sum_{n=0}^{\infty} 0z^n \equiv 0.$$

*Here it is important to realize that a_n is a constant. Hence, the fact that $a_n = 0$ when z is real means that a_n is zero for all values of z since a_n is independent of n.

1.7.6(L) continued

c. Here we wish to emphasize the importance of the assumption that $f(z)$ is analytic. We do this by exhibiting a non-analytic (yet well-defined and continuous) function $f(z)$ such that $f(z) = 0$ whenever z is real but $f(z) \not\equiv 0$. In particular, z being real means that its imaginary part, $\mathrm{Im}(z)$, is zero.

Hence,

$$f(z) = \mathrm{Im}(z) \tag{6}$$

implies that $f(x) = \mathrm{Im}(x) = 0$ whenever x is real. Yet it is clear that $f(z) \not\equiv 0$ since every non-real number z is, by definition, of the form $\mathrm{Im}(z) \neq 0$.

As an aside, notice that we may write $\mathrm{Im}(z)$ as $\frac{z - \bar{z}}{2i}$ [i.e., $\bar{z} = x + iy$ and $z = x - iy$ implies $z - \bar{z} = 2iy$]. Putting this result in (6) yields

$$f(z) = \frac{z - \bar{z}}{2i}$$

$$= \frac{z}{2i} - \frac{\bar{z}}{2i}$$

and we have already proven that if $a \neq 0$, $a\bar{z}$ is non-analytic. Thus, this result does not contradict the result of part (b) since part (b) requires the hypothesis that f be analytic.

d. Parts (a), (b), and (c) of this exercise have been "preliminaries", and part (d) now shows us how we may draw important conclusions from our newly-acquired knowledge.

We proceed as follows. Since $f(z)$ is analytic, we know that

$$f(z) \equiv \sum_{n=0}^{\infty} a_n z^n. \tag{7}$$

[In fact, we know much more. Namely, $a_n = \frac{f^{(n)}(0)}{n!}$]

1.7.6(L) continued

Since (by definition of analytic) f(z) converges uniformly and absolutely to

$$\sum_{n=0}^{\infty} a_n z^n$$

the usual rules for "conjugation" apply and we may conclude from (3) that

$$f(\bar{z}) = \sum_{n=0}^{\infty} a_n \bar{z}^n .$$

Hence,

$$\overline{f(\bar{z})} = \sum_{n=0}^{\infty} \overline{(a_n \bar{z})^n}$$

$$= \sum_{n=0}^{\infty} \bar{a}_n \overline{(\bar{z})^n} \;*$$

$$= \sum_{n=0}^{\infty} \bar{a}_n (\bar{\bar{z}})^n$$

$$= \sum_{n=0}^{\infty} \bar{a}_n z^n. \qquad (8)$$

*This is the key step. Namely we already know that $\overline{a_1 + \ldots + a_n} = \bar{a}_1 + \ldots + \bar{a}_n$. [I.e.,

$$\overline{\sum_{k=1}^{n} a_k} = \sum_{k=1}^{n} \bar{a}_k]$$

but this is for a finite number of terms. Just as in the real case where we wanted to extend results from finite to infinite sums [e.g.,

$$\int_a^b \sum_{n=0}^{\infty} a_n x^n dx = \sum_{n=0}^{\infty} \int_a^b a_n x^n dx]$$

we need the fact that we have uniform convergence.

1.7.6(L) continued

When z is real equations (7) and (8) become

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{7'}$$

$$\overline{f(\bar{x})} = \sum_{n=0}^{\infty} \bar{a}_n x^n. \tag{8'}$$

Now, since x is real $x = \bar{x}$; and since f(x) is real (this is where we use the given hypothesis that f(z) is real whenever z is real) we know that $f(x) = \overline{f(x)}$. Hence, $\overline{f(\bar{x})} = \overline{f(x)} = f(x)$, so that (8') becomes

$$f(x) = \sum_{n=0}^{\infty} \bar{a}_n x^n. \tag{8"}$$

If we equate the expressions given for f(x) in equations (7') and (8") we conclude that

$$\sum_{n=0}^{\infty} a_n x^n - \bar{a}_n x^n \equiv 0. \tag{9}$$

Since our series are absolutely convergent we may add them term-by-term, etc. so that equation (9) becomes

$$\sum_{n=0}^{\infty} (a_n - \bar{a}_n) x^n \equiv 0. \tag{10}$$

At this point we confess that parts (a), (b), and (c) were invented by hind-sight so that we could solve equation (10) without having to degress.

By part (a) we can conclude from (10) that

$$a_n - \bar{a}_n = 0 \quad \text{for each } n \tag{11}$$

<u>but</u> $a_n - \bar{a}_n = 2i\,\mathrm{Im}(a_n)$.

1.7.6(L) continued

Hence, $a_n - \bar{a}_n = 0$ implies $\mathrm{Im}(a_n) = 0$; so from the equation we conclude that a_n is real. Thus, we now have the valuable additional information that if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and f maps the real numbers into the real numbers then each a_n is real (the converse is trivial. Namely, it is obvious that if each a_n is real then

$$\sum_{n=0}^{\infty} a_n z^n$$

is real whenever z is real).

At any rate, we now have that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (12)$$

where a_n is real and since a_n is real, $a_n = \bar{a}_n$.

Thus equation (8) now becomes

$$f(\bar{z}) = \sum_{n=0}^{\infty} a_n \bar{z}^n \qquad (13)$$

and since

$$\sum_{n=0}^{\infty} a_n z^n = f(z)$$

equation (13) allows to conclude that

$$f(z) = \overline{f(\bar{z})} \quad \text{for all } z \qquad (14)$$

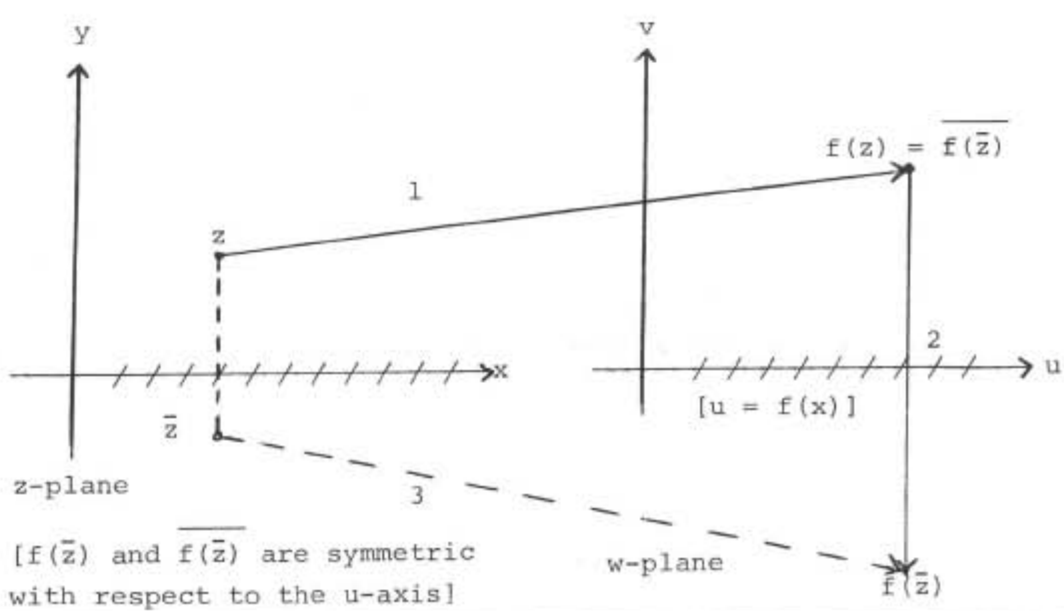
1.7.6(L) continued

whenever f is analytic and f(x) is real for all real x. If f is not analytic or if f(x) is not real for all real x then it need not be true, of course, that (14) holds. In the next exercise, we shall give an example in which (14) is false.

In concluding this exercise it seems appropriate to interpret our result geometrically. Indeed, in terms of complex functions of a complex variable being an excellent vehicle for studying real mappings of the xy-plane into the uv-plane, geometrical interpretations are always valuable to us.

Let us observe that z and $\bar{z}$ are located symmetrically with respect to the real axis (the x-axis). Similarly, f(z) and $\overline{f(z)}$ are symmetric with respect to the u-axis. In this context, simply by replacing z by $\bar{z}$, we see that $f(\bar{z})$ and $f(\bar{z})$ are symmetric with respect to the u-axis. Hence, since $f(z) = f(\bar{z})$ we may conclude that f maps points which are symmetric with respect to the x-axis into points which are symmetric with respect to the u-axis. In the language of complex variables, if f is analytic and f is real on the real numbers then f maps conjugate pairs of complex numbers into conjugate pairs of complex numbers.

Pictorially,



1.7.6(L) continued

In summary, we have used the power series representation of analytic functions to prove the following rather powerful result. If f is analytic and f(x) is real whenever x is real, then $f(z) = \overline{f(\bar{z})}$ for every complex number z. Pictorially this means that if z and $\bar{z}$ are complex conjugates, then their images under f will also be complex conjugates.

As a final note on this exercise, let us observe that there is a very special catagory of function f for which f(z) is real whenever z is real. Namely, suppose we extend a real function to the complex numbers. For example, this is precisely what we have done in this unit when we defined cos z, sin z, e^z, etc. The fact that f(z) was an extension of f(x) makes it a truism that f(z) is real whenever z is real. Thus, whenever f(z) is an extension of f(x) we can be sure, based on the result of this exercise, that $f(z) = \overline{f(\bar{z})}$ for each complex number z.

1.7.7

a. $f(z) = iz \rightarrow$ (1)

$f(\bar{z}) = i\bar{z} \rightarrow$

$$\overline{f(\bar{z})} = \overline{i\bar{z}}$$

$$= \bar{i}\bar{\bar{z}}$$

$$= \bar{i}z$$

$$= -iz. \qquad (2)$$

Comparing (1) and (2) we see that $f(z) \neq \overline{f(\bar{z})}$. In fact in this example $f(z) = \overline{f(\bar{z})} \leftrightarrow z = 0$, what is true in this example is that

$$f(z) \equiv -\overline{f(\bar{z})}. \qquad (3)$$

This result does not contradict our result in the previous exercise since we see from (1) that f(x) = ix = purely imaginary whenever x is real; while the result derived in **Exercise** 1.7.6(d)

1.7.7 continued

required that $f(x)$ be real whenever x is real. Geometrically speaking, in the previous exercise the image of the x-axis was required to be the u-axis while in this exercise the image of the x-axis is the v-axis.

b. Here we are trying to generalize the conditions under which equation (3) is true.

Since x is real $x = \bar{x}$, but since $f(x)$ is purely imaginary $f(x) = -\overline{f(x)}$ [i.e., if $z = ib$, $\bar{z} = -ib$]. Thus,

$$\overline{f(\bar{x})} = -f(\bar{x}) = f(x). \tag{4}$$

Therefore,

$$f(x) + \overline{f(\bar{x})} = 0 \text{ whenever } x \text{ is real.} \tag{5}$$

But $f(z) + \overline{f(\bar{z})}$ is analytic [i.e.,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \rightarrow \overline{f(\bar{z})} = \sum_{n=0}^{\infty} \bar{a}_n z^n \rightarrow f(z) + \overline{f(\bar{z})} = \sum_{n=0}^{\infty} (a_n + \bar{a}_n) z^n]$$

Since $f(z) + \overline{f(\bar{z})}$ is analytic and it is zero whenever z is real, we may use the result of Exercise 1.7.6(b) to conclude that $f(z) + \overline{f(\bar{z})} \equiv 0$

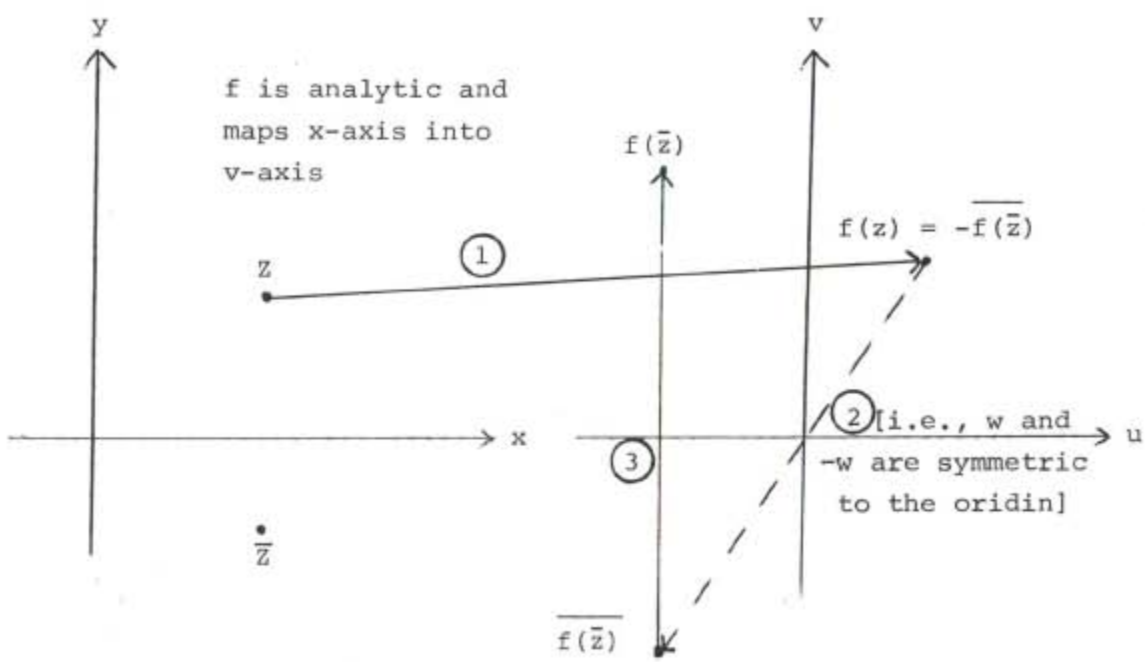
or

$$f(z) = -\overline{f(\bar{z})}. \tag{6}$$

Equation (6) required, in addition to f being analytic, only that $f(x)$ was purely imaginary whenever x was real.

1.7.7 continued

c.



Therefore, if z and $\bar{z}$ are symmetric with respect to the x-axis, f(z) and $f(\bar{z})$ are symmetric with respect to the v-axis.

Note:
We have deliberately tried to avoid any "cute solution of this exercise because it seems that the straight-forward approach gives us much needed drill and experience with some new ideas. There is, however, a rather simple, elegant technique (in terms of mappings) that may make this exercise a bit simpler. Namely, if f maps the real axis onto the imaginary axis, we may think of f as the product (composition) of two mappings; namely, a mapping that carries the real axis into the real axis (for example, if the given function is denoted by f then if we let g be defined by $g(z) = -if(z)$ then g is simply a $-90°$ rotation of the plane thus sending the imaginary axis into the real axis) followed by a $90°$ rotation. Summarized more succinctly, we write $w = f(z)$ in the form

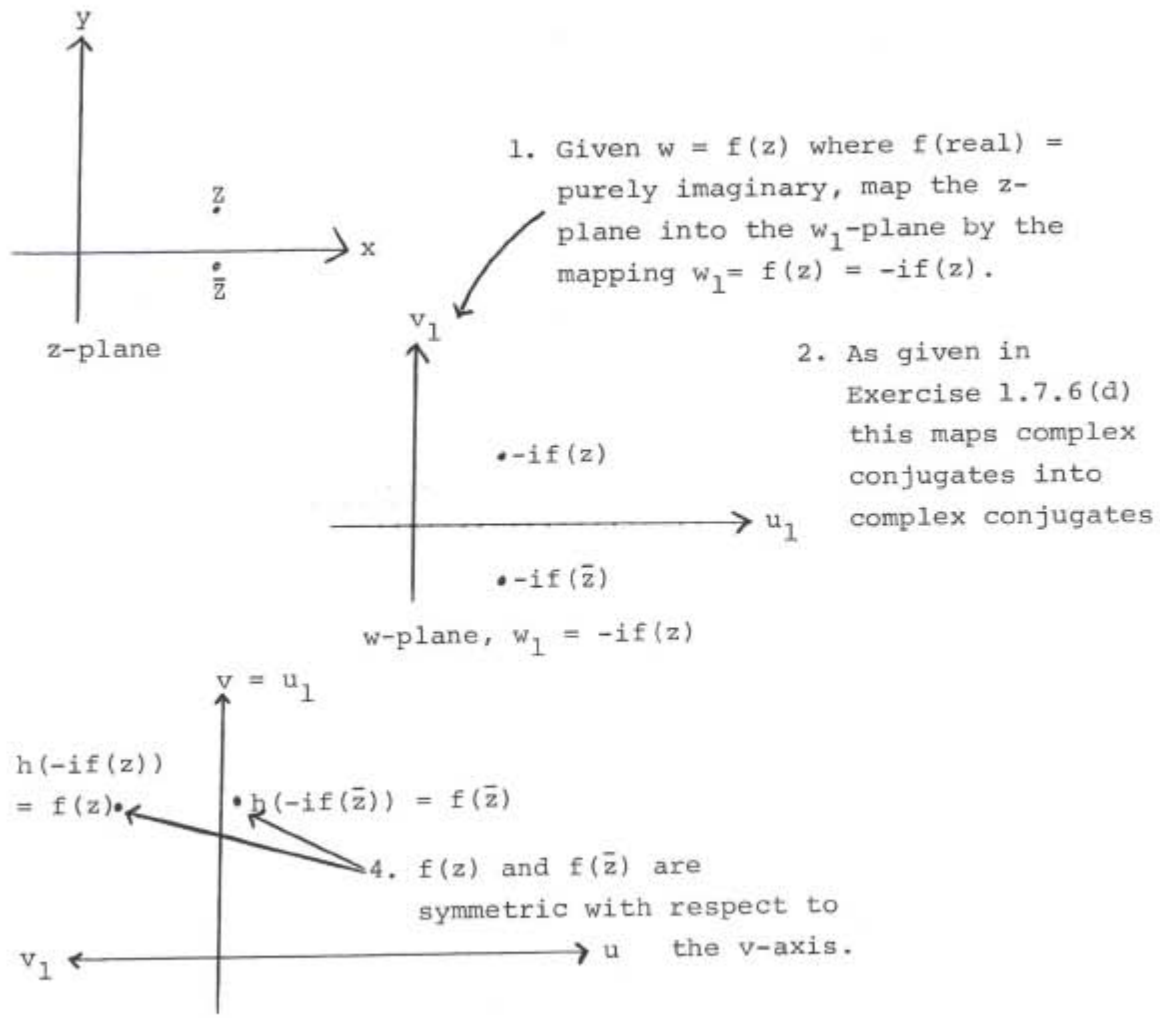
$$w = i[-if(z)]. \qquad (7)$$

The point is that since f maps the reals into the imaginaries,

1.7.7 continued

$-i(f)$ maps the reals into the reals. Thus, the study of $-if(z)$ is a special case of our investigation made in the previous exercise; and we see that (7) is the result obtained in part (d) of the previous exercise - but rotated through +90°.

Pictorially,



3. Now map the w_1-plane onto the w-plane by the mapping $w = h(w_1) = iw_1$ [i.e., $w = i(u_1 + iv_1) = -v_1 + iu_1$ so $\left.\begin{array}{l} u = -v_1 \\ v = u_1 \end{array}\right\} = +\ 90°$ rotation]

1.7.8(L)

Since $e^{i\theta} = \cos\theta + i\sin\theta$, we now have the rather interesting result that the complex number whose polar form was $(r,\theta) = r\cos\theta + ir\sin\theta$ may now be written as $r(\cos\theta + i\sin\theta) = re^{i\theta}$. In other words, the complex number z may be written as $re^{i\theta}$ where $r = |z|$ and $\theta = \arg(z)$. In still other words, $z = |z|\, e^{i \arg z}$ which is quite consistent with the idea that when we multiply complex numbers we multiply their magnitudes and add their arguments. Indeed by the "inherited" structure of the complex numbers

$$z_1 = r_1e^{i\theta_1},\ z_2 = r_2e^{i\theta_2} \rightarrow z_1z_2 = (r_1e^{i\theta_1})(r_2e^{i\theta_2})$$

$$= r_1r_2e^{i\theta_1 + i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}]$$

Using polar notation, then

$$z = re^{i\theta}. \tag{1}$$

Hence,

$$z^n = (re^{i\theta})^n$$

$$= r^n(e^{i\theta})^n$$

$$= r^ne^{in\theta}.$$

Therefore

$$f(z) = \sum_{n=0}^{\infty} a_nz^n \tag{2}$$

implies that

$$f(z) = \sum_{n=0}^{\infty} a_nr^ne^{in\theta}. \tag{2'}$$

1.7.8(L)

Similarly,

$z = re^{i\theta} \rightarrow z = r\cos\theta + ir\sin\theta.$

Hence,

$$\begin{aligned}\bar{z} &= r\cos\theta - ir\sin\theta \\ &= r(\cos\theta - i\sin\theta) \\ &= r(\cos[-\theta] + i\sin[-\theta]) \\ &= re^{i(-\theta)} \\ &= re^{-i\theta}.\end{aligned}$$

Geometrically, if $z = (r,\theta)$ then $\bar{z} = (r,-\theta) = re^{i(-\theta)}$. Therefore,

$\bar{z}^n = (re^{-i\theta})^n = r^n e^{-in\theta},$

so that, from (2)

$$f(\bar{z}) = \sum_{n=0}^{\infty} a_n \bar{z}^n \tag{3}$$

$$= \sum_{n=0}^{\infty} a_n r^n e^{-in\theta}. \tag{3'}$$

The fact that $f(z)$ is analytic guarantees that the power series representation of $f(z)$, as given in (2) and (2'), exists. Moreover, the fact that we are told that $f(z) \equiv f(\bar{z})$ guarantees that $f(\bar{z})$ must also be analytic [i.e., $f(z)$ and $f(\bar{z})$ are synonyms], so that the power series representation of $f(\bar{z})$, as given by (3) and (3') also exists.

Since $f(z) \equiv f(\bar{z})$ we may equate (2') and (3') to obtain

$$\sum_{n=0}^{\infty} a_n r^n e^{in\theta} \equiv \sum_{n=0}^{\infty} a_n r^n e^{-in\theta} \tag{4}$$

or

$$\sum_{n=0}^{\infty} a_n r^n e^{in\theta} - \sum_{n=0}^{\infty} a_n r^n e^{-in\theta} \equiv 0. \tag{4'}$$

1.7.8(L)

Since the series in (4') are assumed to be uniformly convergent (since f(z) is analytic), we may combine the series term by term to conclude that

$$\sum_{n=1}^{\infty} a_n r^n (e^{in\theta} - e^{-in\theta}) \equiv 0.^* \quad (5)$$

But,

$$e^{in\theta} - e^{-in\theta} = (\cos n\theta + i \sin n\theta) - [\cos n(-\theta) + i \sin n(-\theta)]$$

$$= (\cos n\theta + i \sin n\theta) - (\cos n\theta - i \sin n\theta)$$

$$= 2i \sin n\theta.$$

Hence, equation (5) becomes

$$\sum_{n=0}^{\infty} a_n r^n (2i \sin n\theta) \equiv 0$$

or

$$2i \sum_{n=0}^{\infty} [a_n \sin n\theta] r^n \equiv 0 \quad (6)$$

and since $2i \neq 0$, we conclude from (6) that

$$\sum_{n=0}^{\infty} [a_n \sin n\theta] r^n \equiv 0. \quad (7)$$

Now r is real (since $r = |z|$), even though $a_n \sin n\theta$ need not be (i.e., $\sin n\theta$ is real but all we know about a_n is that it's complex). Thus, the left side of (7) is a power series in which the coefficients are complex but the variable, r, is real.

Accordingly, we may apply the result of Exercise 1.7.6(a) to equation (7) and conclude that

*Notice that our sum begins with r = 1 not n = 0. When n = 0, $e^{in\theta} = e^{-in\theta} = 1$ so the $e^{in\theta} - e^{-in\theta} = 0$.

1.7.8(L) continued

$a_n \cos n\theta = 0$ for each n. (8)

The "trouble" with (8) is that we cannot conclude that each $a_n = 0$. Namely, it is possible that we have chosen $z = re^{i\theta}$ in such a way that $\cos n\theta = 0$, in which case $a_n \cos n\theta = 0$ even if $a_n \neq 0$.

We notice, however, that $\cos n\theta = 0$ if and only if $n\theta = \frac{\pi}{2} + 2k$ where k is an arbitrary integer.

The point is that any number of the form $\frac{\pi}{2} + 2k\pi$ must be irrational [because itself in irrational;in other words: $\frac{\text{irrational}}{\text{integer}}$ + integer (irrational) = irrational]

So, for example, if we chose z such that $\theta = 1$ (radian) we may conclude that $n\theta$ is never equal to $\frac{\pi}{2} + 2k\pi$ since $n\theta$(=n) is rational while $\frac{\pi}{2} + 2k\pi$ cannot be rational.

Thus, for example, along the ray $\theta = 1$ radian, equation (8) tells us that

$a_n \cos n\theta = a_n \cos n = 0$ for n = 1,2,3 (9)

and since $\cos n \neq 0$ for n = 1,2,3,..., we conclude from (8) that

$a_n = 0$ for n = 1, 2, 3,... (10)

Equation (10) puts no restriction on a_o*; so combining the result of (10) with the fact that a_o may be arbitrarily chosen, we obtain

*In case you missed this point in our earlier footnote,

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n \bar{z}^n,\ a_o + a_1 z + a_2 z^2 + \ldots \equiv a_o + a_1 \bar{z} + a_2 \bar{z}^2 + \ldots$$

and the a_o cancels when we combine terms.

1.7.8(L) continued

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$= a_o + \sum_{n=1}^{\infty} a_n z^n$$

$$= a_o + \sum_{n=1}^{\infty} 0 z^n$$

$$= a_o = \text{constant.} \qquad (11)$$

As another reminder, notice that the validity of our result requires that f be analytic. For example, if f is defined by $f(z) = z + \bar{z}$, then f is not analytic (since $\bar{z}$ is not); <u>yet</u>

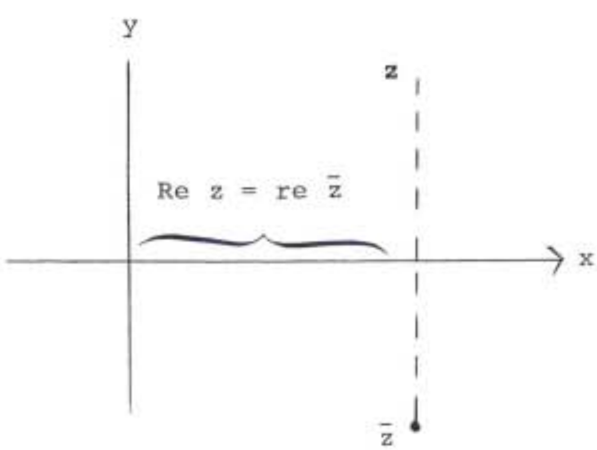
$$f(\bar{z}) = \bar{z} + \bar{\bar{(z)}}$$

$$= \bar{z} + z$$

$$= z + \bar{z}$$

$$= f(z).$$

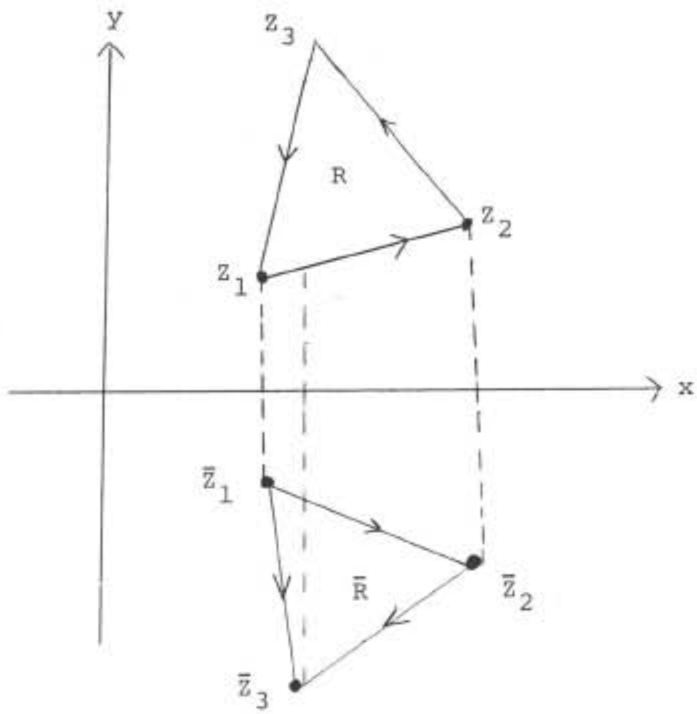
Geometrically, all this says is that the mapping $f(z) = 2\,\mathrm{Re}(z)$ is not conformed (or even analytic) but since $\mathrm{Re}(z) = \mathrm{Re}(\bar{z})$ for <u>every</u> complex number z, $f(z) \equiv f(\bar{z})$, i.e.,



1.7.9 (optional)

a. In the previous exercise we derived a rather interesting result based on analyzing a suitable power series. Let us now show how an understanding of conformal mappings can show us very quickly that if $f(z)$ is conformal then $f(z)$ cannot be identical to $f(\bar{z})$.

For example pick three points z_1, z_2, and z_3 in the z-plane, not on the same line.

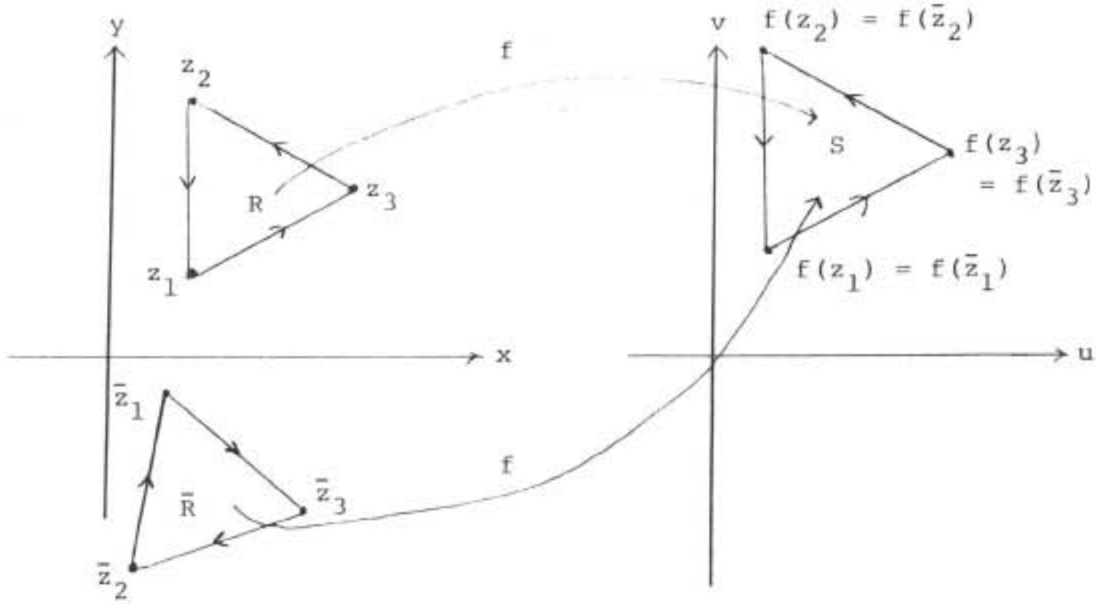


Notice that if R is traversed in the counter-clockwise sense, the image of this path traverses $\bar{R}$ in the clockwise sense.

Thus R and $\bar{R}$ (where $\bar{R} = \{\bar{z}:z \ R\}$) have opposite orientation. Consequently, if $f(R) = f(\bar{R}) = S$, then either R or $\bar{R}$ cannot have the same orientation as S since R and $\bar{R}$ have opposite orientations.

1.7.9 continued

Again, pictorially, suppose



[We have assumed a sufficiently small neighborhood so the R, $\bar{R}$, and S may be assumed to have the same shape.]

In our diagram the contradiction lies in the fact that $\bar{R}$ and $f(\bar{R}) = S$ have opposite orientations, contrary to the fact that a conformal mapping must preserve sense.

b. Despite our efforts to have you become more at ease with complex functions of a complex variable, many of us still prefer to work in terms of the real and imaginary parts. That is, we look at f(z) in the form

$$w = f(z) = f(x + iy) = f(x,y) = u(x,y) + i\ v(x,y) \tag{1}$$

and then feel more at home because u and v are real-valued functions of the two real variables.

From equation (1) we conclude that

$$f(\bar{z}) = f(x - iy) = f(x, -y) = u(x, -y) + i\ v(x, -y). \tag{2}$$

1.7.9 continued

The fact that f is analytic tells us, from (1), that

$$\left.\begin{aligned} &\frac{\partial u(x,y)}{\partial x} \equiv \frac{\partial v(x,y)}{\partial y} \\ &\text{and} \\ &-\frac{\partial u(x,y)}{\partial y} \equiv \frac{\partial v(x,y)}{\partial x} \end{aligned}\right\} \quad (3)$$

That is, the usual Cauchy-Riemann conditions imply that $u_x = v_y$ and $u_y = -v_x$.

Now the fact that $f(z) \equiv f(\bar{z})$ implies that f(z) also satisfies the Cauchy-Riemann conditions (sin f(z) does).

From equation (2) this means that

$$\left.\begin{aligned} &u_x = v_{-y} \\ &\text{and} \\ &u_{-y} = -v_x \end{aligned}\right\} \quad (4)$$

Notice the subtlety in (4). The Cauchy-Riemann conditions involve derivatives with respect to the real and imaginary parts of the variable and in (2) these are x and -y (not y).

We may use the chain rule to avoid this difficulty. Namely we let, for example

$$\left.\begin{aligned} x &= s \\ y &= -t \end{aligned}\right\} \quad (5)$$

so that

$u(x,-y) = u(s,t)$ and $v(x,-y) = v(s,t)$. Then equation (2) becomes $f(z) = u(s,t) + i\,v(s,t)$ from the Cauchy-Riemann conditions allow us to conclude that

1.7.9 continued

$$\left.\begin{aligned} u_s &= v_t \\ &\text{and} \\ u_t &= -v_s \end{aligned}\right\} \quad (6)$$

Clearly $u_s \equiv u_x$ and $-v_s \equiv -v_x$ since $s = x$. On the other hand, by the chain rule

$$\left.\begin{aligned} v_t &= v_x x_t + v_y y_t \\ u_t &= u_x x_t + u_y y_t \end{aligned}\right\} \quad (7)$$

and from (5) we know that $x_t = 0$ and $y_t = -1$.

Thus (7) is equivalent to

$$\left.\begin{aligned} v_t &= -v_y \\ u_t &= -u_y \end{aligned}\right\} \quad (7')$$

Putting the results of (7') into (6), we obtain

$$\left.\begin{aligned} u_x &= -v_y \\ -u_y &= -v_x \end{aligned}\right\}$$

or

$$\left.\begin{aligned} u_y &= v_x \\ u_x &= -v_y \end{aligned}\right\} \quad (8)$$

[Most likely we could have written down (8) by inspection after looking at (4) but we felt the review of the chain rule was worthwhile]

If we now combine the results of (3) and (8) we have

$$u_x = v_y = -u_x \quad (9)$$

and

$$u_y = -v_x = -u_y. \quad (10)$$

1.7.9 continued

Equation (9) yields $2u_x = 0$ so that $u_x = 0$. Hence,

$$u = g(y). \tag{11}$$

On the other hand, equation (10) implies that

$$u_y = 0. \tag{12}$$

But from (11)

$$u_y = g'(y) \tag{13}$$

and a comparison of (12) and (13) shows that $g'(y) = 0$ or

$$g(y) = \text{constant}. \tag{14}$$

Putting this information into (11) allows us to conclude that

$$u(x,y) = \text{constant}. \tag{15}$$

In a completely analogous manner we may show that

$$v(x,y) = \text{constant}. \tag{16}$$

Namely,

$$\left.\begin{array}{l} v_x = -u_y = -v_x \\ v_y = u_x = -v_y \end{array}\right\} \rightarrow v_x = v_y = 0 \rightarrow v = \text{constant}.$$

Combining the results of (15) and (16) we conclude that

$$\begin{aligned} f(z) = f(x + iy) = f(x,y) &= u(x,y) + i\,v(x,y) \\ &= c_1 + ic_2 \\ &= \text{(complex) constant} \end{aligned}$$

and we have now obtained a proof which is "pseudo-complex" in the sense that we never referred to the complex numbers other than in the sense of identifying the xy-plane with the z-plane and the uv-plane with the w-plane.

1.7.10(L)

a. $|1 + i| = \sqrt{1 + 1} = \frac{\pi}{2}$

$\arg(1 + i) = \arctan \frac{1}{1} = 45° = \frac{\pi}{4}$ radians

$1 + i = \sqrt{2}\ e^{i\frac{\pi}{4}}$.

Hence,

$$\log(1 + i) = \log[\sqrt{2}\ e^{\frac{i\pi}{4}}] \quad (1)$$

$$= \log \sqrt{2} + \log e^{\frac{i\pi}{4}} \quad (2)$$

$$= \ln\sqrt{2} + i\ \frac{\pi}{4} \quad (3)$$

so that

$$\log(1 + i) = a + ib \text{ where } \begin{cases} a = \ln \sqrt{2} = \frac{1}{2} \ln 2 \\ b = \frac{\pi}{4} \end{cases}$$

Notes:

1. When we write arctan x we imply that $-\frac{\pi}{2} < x \leq \frac{\pi}{2}$, but there are infinitely many aryles (numbers) which have the same tangent. Thus, without the restriction to principal values $\arg(1 + i) = \frac{\pi}{4} + 2\pi n$ where n is any integer. The principal value occurs when we choose n to be 0. In any event, if we pick n to be any integer, not necessarily 0, then $\arg(1 + i) = \frac{\pi}{4} + 2\pi n$ so that $(1 + i) = \frac{\pi}{2}\ e^{i(\pi/4 + 2\pi n)}$, and

$$\log\ (1 + i) = \ln\sqrt{2} + i(\frac{\pi}{4} + 2\pi n). \quad (4)$$

In other words, if $z = \ln\sqrt{2} + i(\frac{\pi}{4} + 2\pi n)$, then $e^z = 1 + i$. That is, log z is an infinitely-valued function, one value for each integer n in (4).

2. In going from equation (1) to equation (2) we used the fact that log (ab) = log a + log b. This properly was true when we dealt with real numbers; and since the properties of the log are preserved in going to the complex numbers (i.e., these properties

1.7.10(L) continued

were derivable form the power series representation) we may still use these results. The more theory-oriented student may wish to derive these results a bit more formally.

3. In going from equation (2) to equation (3) we used the fact that log z was an extension of ln x. That is, when z is real log z = ln z.

4. We also use the fact that since log z is the inverse of the exponential (at least when we restrict ourselves to principal values), $e^{-\log z} = z$ for each complex number z.

b. Given i^i we write it as $e^{\log i^i}$. Then since $\log i^i = i \log i$ (i.e., just as in the real case $\log b^c = c \log b$) we have that

$$i^i = e^{i \log i}. \tag{5}$$

Since $|i| = 1$ and the (principal value) argument of i is $\frac{\pi}{2}$

we have that

$$\log i = \log (1e^{i\frac{\pi}{2}})$$

$$= \log 1 + \log e^{i\frac{\pi}{2}}$$

$$= 0 + i\frac{\pi}{2}$$

$$= \frac{i\pi}{2}.$$

Putting this result into (6) yields

$$i^i = e^{i(i\frac{\pi}{2})}$$

$$= e^{\frac{i^2\pi}{2}}$$

$$= e^{-\frac{\pi}{2}}$$

(a rather "weird" result).

1.7.10(L) continued

Without restriction to principal values $i = e^{i(\frac{\pi}{2} + 2\pi n)}$ so that $\log i = i(\frac{\pi}{2} + 2\pi n)$ and $i \log i = -(\frac{\pi}{2} + 2\pi n)$ so that $i^i = e^{-(\pi/2 + 2\pi n)}$ $n = 0, \pm 1, \pm 2, \ldots$

c. Let $z = re^{i\theta}$ where $0 \le 0 < 2\pi$.

Then,

$$\log z = \log re^{i\theta}$$

$$= \log r + \log e^{i\theta}$$

$$= \ln r + i\theta. \tag{1}$$

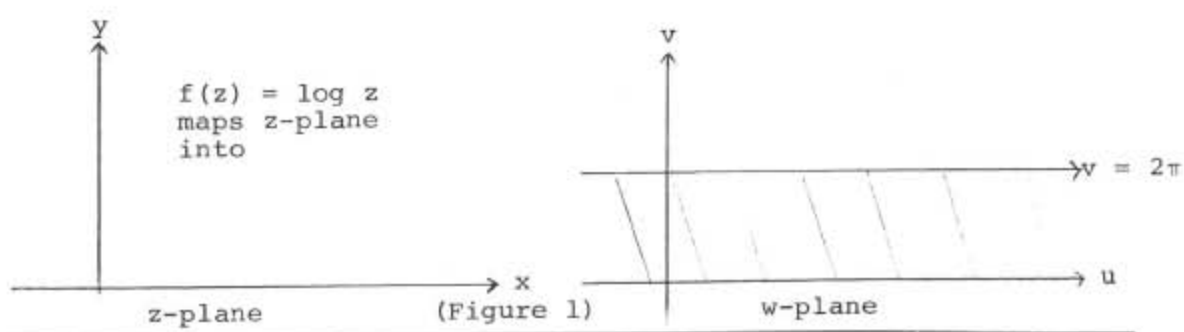
From (1) we see that

$$\begin{cases} u = \ln r = \ln\sqrt{x^2 + y^2} = \frac{1}{2}\ln(x^2 + y^2) \\ v = \theta = \tan^{-1}\frac{y}{x}\ (0 \le \theta < 2\pi) \end{cases} \tag{2}$$

Thus, as r ranges from 0 to ∞, $\ln r$ ranges from $-\infty$ to ∞[i.e., $\ln r \ge 0 \leftrightarrow r \ge 1$; as $r \to -\infty$ $\ln r \to 0$]. Hence u ranges from $-\infty$ to ∞ ; and as θ ranges from 0 to 2π so does v. Consequently, the mapping $w = \log z$ maps the entire xy-plane onto the horizontal strip defined by

$$\begin{cases} -\infty < u < \infty \\ < v < 2\pi . \end{cases}$$

That is,



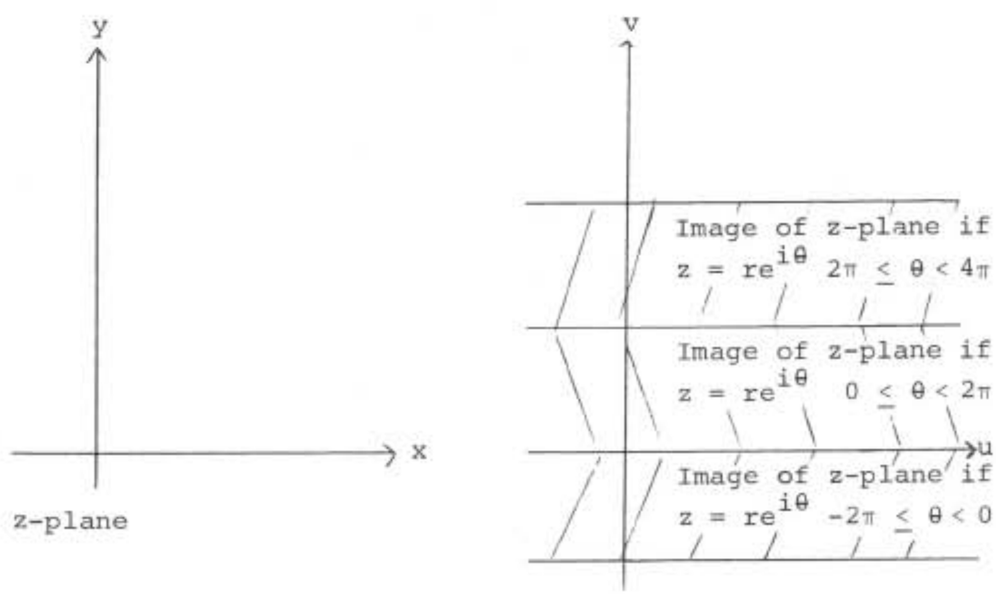
(Figure 1)

1.7.10(L) continued

We next observe that since (r,θ) and $(r,2\pi + \theta)$ name the same point, $re^{i\theta}$ and $re^{i(\theta + 2\pi)}$ are different names for the same complex number. Under the name $re^{i(\theta + 2\pi)}$, z is mapped onto $\ln r + i(\theta + 2\pi)$, rather than $\ln r + i\theta$. That is, if we look at z in the form $re^{i(\theta + 2\pi)}$ rather than in the form $re^{i\theta}$ we see that the xy-plane is now mapped onto the strip

$$\begin{cases} -\infty < u < \infty \\ 2\pi \leq v < 4\pi. \end{cases}$$

Again pictorially,



(Figure 2)

Thus log z is an infinitely-valued function in the sense that if $\log z = \ln r + i\theta$ where $0 \leq \theta < 2\pi$ then $\log z = \ln r + i(\theta + 2\pi n)$ where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

We refer to the principal value of log z as being the value for which $n = 0$. If, however, the principal value is not stressed, log z has infinitely many values. They are all on the line

1.7.10(L) continued

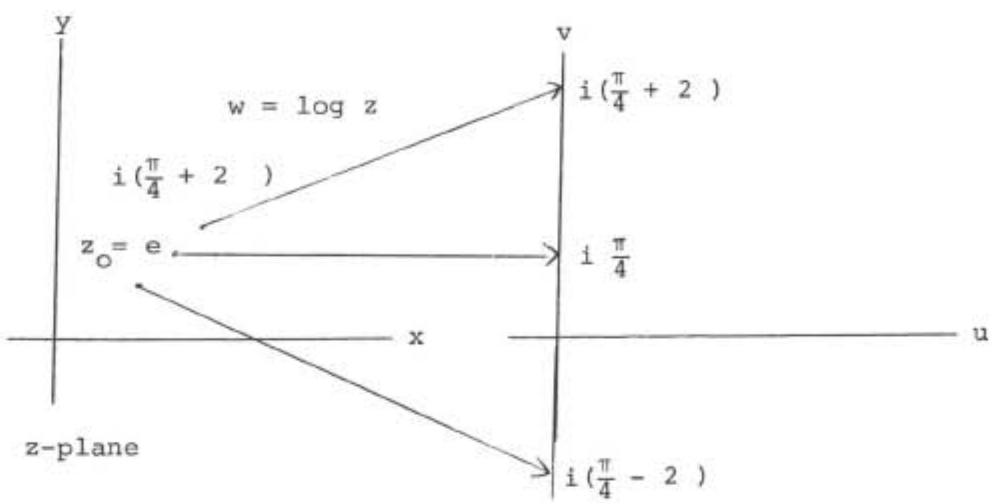
$u = \ln|z|$ and spaced at distances of 2π from the point $(\ln|z|, \arg z)$.

By way of illustration suppose $z = e^{i\frac{\pi}{4}}$ then z is also named by $e^{i(\frac{\pi}{4} + 2\pi)}$, $e^{i(\frac{\pi}{4} + 4\pi)}$, etc.

Thus,

$$\log e^{i\frac{\pi}{4}} = i\frac{\pi}{4},\ i(\frac{\pi}{4} + 2\ ),\ i(\frac{\pi}{4} - 2\pi), ..$$

Pictorially,



(Figure 3)

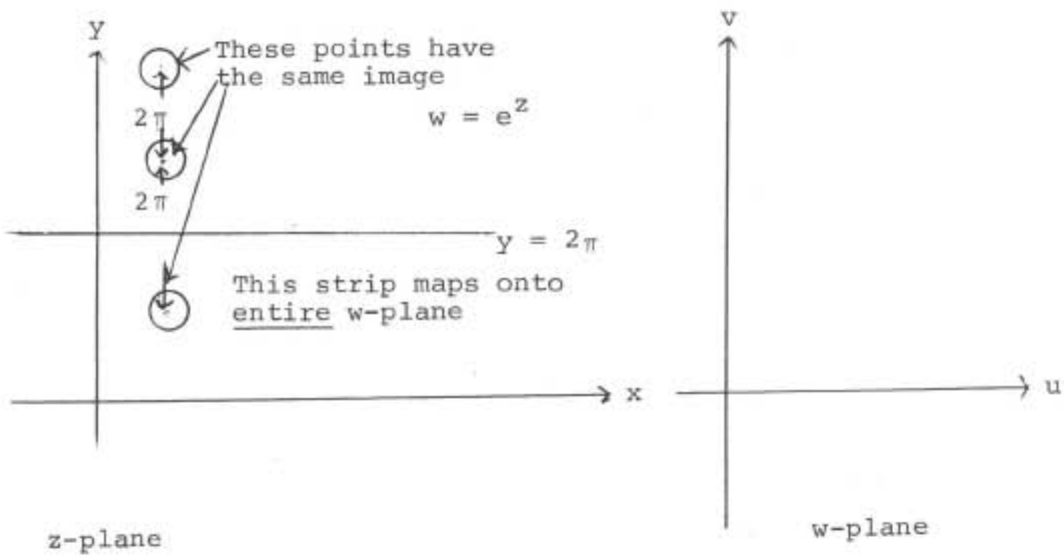
Note:

This exercise shows at least one glaring difference between logarithms of real numbers and logarithms of complex numbers. Aside from the fact that logarithms of real numbers require that the number be positive while the logarithm of a complex number requires only that the number be different from 0, the log function is single valued (in fact, 1-1) in the real case but infinitely-valued in the complex case. Thus, if we still want to think of the log as being the inverse of the exponential, we must agree that whenever we write e^z or log z, we are assuming that arg z is between 0 and 2π radians. Unless we make this convention (or an equivalent one) log z is multi-valued and

1.7.10(L) continued

consequently it is not invertible.

In this context once principal values are stressed, we have that the graph $w = e^z$ is given by



(Figure 4)

e^z and log z are inverses in the sense that Figure 4 is the same as Figure 1 with the z-plane and w-plane interchanged.

d. $z = \sin^{-1} 2$ means

$\sin z = 2$ (which is enough to tell us that z is non-real since $|\sin z| \leq 1$ if z is real). (1)

Since

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

equation (1) requires that we solve

$$\frac{e^{iz} - e^{-iz}}{2i} = 2. \qquad (2)$$

1.7.10(L) continued

From equation (2) we conclude that $e^{iz} - e^{-iz} = 4i$ or $e^{iz} - e^{\frac{1}{iz}} = 4i$ or $(e^{iz})^2 - 4i\ e^{iz} - 1 = 0$. The quadratic formula now yields

$$e^{iz} = \frac{4i \pm \sqrt{(-4i)^2 + 4}}{2}$$

$$= \frac{4i \pm \sqrt{-16 + 4}}{2}$$

or

$$e^{iz} = \frac{4i \pm \sqrt{-12}}{2}$$

$$= \frac{4i \pm 2i\sqrt{3}}{2}$$

$$= 2i + i\sqrt{3}\ .$$

Hence,

$$iz = \log(2i \pm i\sqrt{3})$$

or

$$z = -i\ \log(2i \pm i\sqrt{3}). \qquad (3)$$

Finally, since $|2i + i\sqrt{3}| = \sqrt{4 + 3} = \sqrt{7}$ and $\arg i\ (2 + \sqrt{3}) = \frac{\pi}{2}$. Therefore,

$$z = -i\ \log i\ (2 + \sqrt{3}) = -i\ (2 + \sqrt{3}) + \frac{\pi}{2}.$$

Hence from (3) we see that the two principal values z_1 and z_2 such that $\sin z_1 = \sin z_2 = 2$

$$z_1 = -i\ (2 + \sqrt{3}) + \frac{\pi}{2}$$

$$z_2 = -i\ (2 + \sqrt{3}) - \frac{\pi}{2}.$$