Unit 7: Complex Series

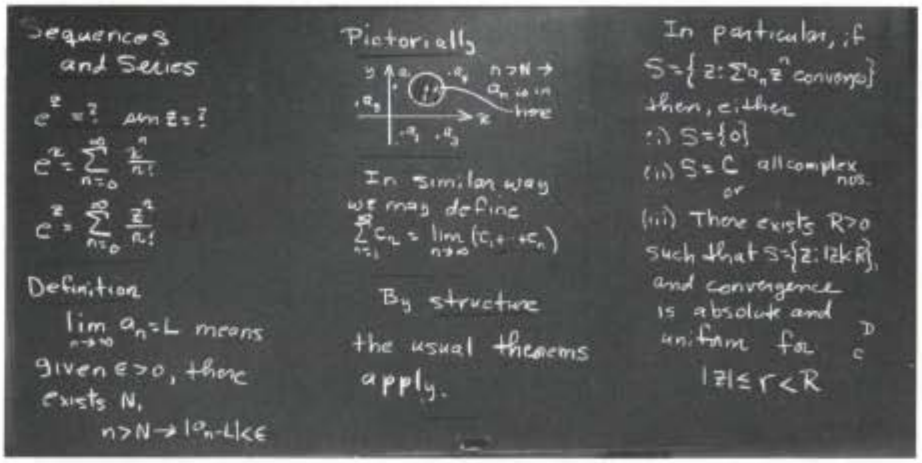
1. Overview

The definition of

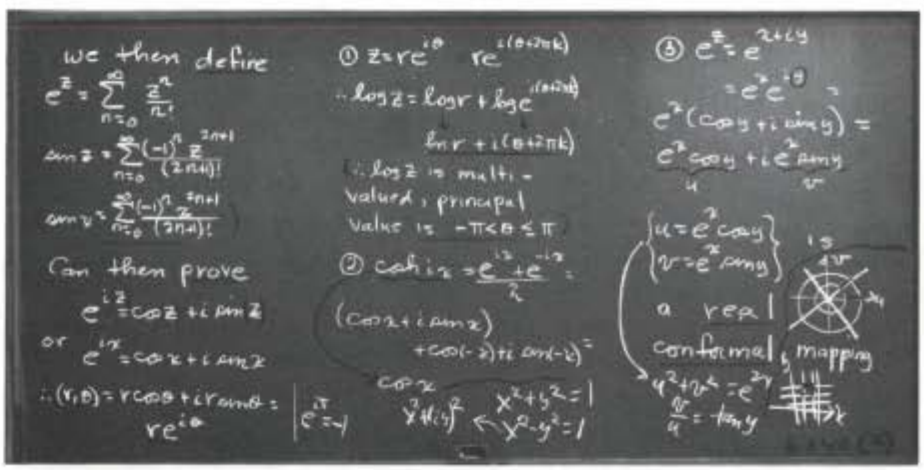
$$\lim_{n \to \infty} a_n = L$$

is word for word the same if a_n and L are complex as when a_n and L were real. Moreover, since the properties of the absolute value function which were necessary for proving limit theorems in the real case also apply in the complex case, the general theory of series carries over, essentially verbatim, from the real case to the complex case. In this Unit we investigate some of the consequences of this carry-over.

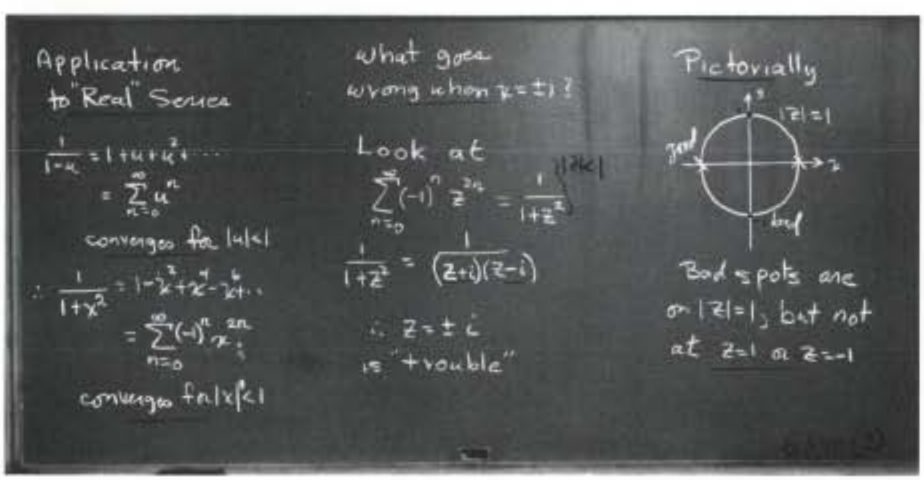
2. Lecture 1.040



a.



b.



c.

3. Read Thomas; Sections 19.6, 19.7, and 19.8. (You may prefer to skim the text the first time through just enough to get an idea of how to tackle the exercises; and then read the material in more detail after finishing the exercises.)

4. Exercises:

1.7.1(L)

Suppose that $\lim_{n\to\infty} a_n = L_1$ and also that $\lim_{n\to\infty} a_n = L_2$. Prove that $L_1 = L_2$.

1.7.2(L)

Let $f(z)$ be defined by $f(z) = 1 - 2z + 3z^2 - 4z^3 + \ldots$ [i.e., $f(z) = \sum_{n=0}^{\infty} (-1)^n (n + 1) z^n$].

a. Find the radius of convergence of f.

b. Compute $f(\frac{i}{12})$ to within the thickness of a disc with radius .001.

c. Compute $f'(\frac{i}{12})$ to one decimal place accuracy.

1.7.3

Use the first nine terms of the power series expansion for e^z to obtain an estimate for e^i. Then check the accuracy of your result by writing e^i in the form $\cos\theta + i \sin\theta$ and using tables of values for sine and cosine.

1.7.4(L)

Use appropriate identities for e^z to evaluate each of the following:

a. $\int e^{ax} \cos bx \, dx$

b. $\sum_{k=0}^{n} \cos k\theta$, where $\cos\theta \neq 1$

1.7.5(L)

a. Determine the real and imaginary parts of w if $w = \sin z$. Then use this information to describe a mapping of the xy-plane into the uv-plane which is conformal except at those points (x,y) for which $\cos(x + iy) = 0$.

b. Discuss the image of the line $x = \frac{\pi}{4}$ under mapping $w = \sin z$.

1.7.6(L)

a. Suppose that

$$\sum_{n=0}^{\infty} c_n x^n$$

is identically zero, where x denotes a real variable and c_n is complex for all n. Show that c_n must be zero for each value of n.

b. Suppose that the analytic function f(z) is represented by the complex power series

$$\sum_{n=0}^{\infty} a_n z^n$$

and that f(z) is 0 for each real value of z. Show that this is enough to imply that f(z) is identically 0 (i.e., for all values of z, real or non-real).

c. By examining the function $f(z) = I_m(z)$ show that the validity of the result proved in part (b) hinges on the fact that f is analytic.

d. Suppose f is an analytic function with the property that f(z) is real whenever z is real. Prove that for each complex number z, $\overline{f(z)} = f(\overline{z})$.

1.7.7

a. Let $f(z) = iz$. Show that $f(z)$ is not identical to $\overline{f(\bar{z})}$. Why doesn't this contradict the result of Exercise 1.7.6(d)?

b. If f is analytic and $f(z)$ is purely imaginary whenever z is real, prove that for each complex number z, $f(z) = -\overline{f(\bar{z})}$.

c. Interpret the result of (b) geometrically.

1.7.8(L)

Use power series in polar coordinates to prove that if $f(z)$ is analytic and $f(z) = f(\bar{z})$ for every complex number z, then $f(z)$ must be a constant.

1.7.9 (optional)

This exercise shows two other ways of obtaining the result given in Exercise 1.7.8. One of the ways is by a geometric interpretation and the other is by means of the theory of real functions of two real variables. Thus, this exercise affords a good review of some of our previous results, but is not necessary in terms of telling us things we didn't already know.

a. Show pictorially that any mapping of the z-plane into the w-plane of the form $w = f(z)$ where $f(z) = f(\bar{z})$ cannot be conformal.

b. Use the Cauchy-Riemann conditions to conclude that if $f(z)$ and $f(\bar{z})$ are both analytic then $f(z)$ must be constant.

1.7.10(L)

a. Compute the principle value of each of the following: $\log(1 + i)$, i^i.

b. Discuss the graph $w = \log z$.

c. Compute $\sin^{-1}2$.