Unit 7: Variation of Parameters

2.7.1(L)

There are two difficult aspects to using the method of variation of parameters. The first is to understand why the method works (and hopefully this was made sufficiently clear in the lecture). The second is to have the computational tools necessary to grind out the specific answer. In many cases, the required functions $g_1(x)$ and $g_2(x)$ must be left in integral form.

The easiest part of the method is writing down the solution once the validity of the method is accepted.

More specifically, if $y = u_1(x)$ and $y = u_2(x)$ are two linearly independent solutions of

$$y'' + p(x)y' + q(x)y = 0^* \tag{1}$$

then

$$y_p = g_1(x)u_1(x) + g_2(x)u_2(x) \tag{2}$$

is a particular solution of

$$y'' + p(x)y' + q(x)y = f(x) \tag{3}$$

provided that

$$\left.\begin{array}{l} g_1'(x)u_1(x) + g_2'(x)u_2(x) \equiv 0 \\ \text{and} \\ g_1'(x)u_1'(x) + g_2'(x)u_2'(x) \equiv f(x) \end{array}\right\}^{**} . \tag{4}$$

*Notice in (1) that there is no restriction that we must have constant coefficients.

**We shall talk about higher order equations in Exercise 2.7.7. The general theory is the same, but the computations become more difficult when the order is greater than 2. For this reason, we prefer to emphasize the second order equation, lest the method become lost in the vast amount of computational detail.

2.7.1(L) continued

Thus, to find the general solution of

$$y'' - y = \frac{1}{1 + e^x} \tag{5}$$

we still want $y_h + y_p$. Since (5) has constant coefficients, we may obtain y_h by the method of the previous units. Namely,

$$y'' - y = 0 \rightarrow$$

$$y_h = c_1e^x + c_2e^{-x}. \tag{6}$$

From (6), we have that e^x and e^{-x} are a pair of linearly independent solutions of the reduced equation $y'' - y = 0$. Letting $e^x = u_1(x)$, $e^{-x} = u_2(x)$, and $\frac{1}{1 + e^x} = f(x)$, we may invoke (2) and (4) to conclude that

$$y_p = g_1(x)e^x + g_2(x)e^{-x} \tag{7}$$

is a particular solution of (5) provided that

$$\left.\begin{aligned} &g_1'(x)e^x + g_2'(x)e^{-x} \equiv 0 \\ &\text{and} \\ &g_1'(x)e^x - g_2'(x)e^{-x} \equiv \frac{1}{1 + e^x}{}^{*} \end{aligned}\right\} . \tag{8}$$

Solving (8) is a matter of manipulative skill (and/or a bit of luck). By adding the two equations in (8), we see that

$$2g_1'(x)e^x = \frac{1}{1 + e^x},$$

*I.e., this is equation (4) with $u_1(x) = e^x$, $u_2(x) = e^{-x}$, and $f(x) = \frac{1}{1 + e^x}$. Notice that we cannot use undetermined coefficients here since $\frac{1}{1 + e^x}$ is not of the "right form." That is, it is not a linear combination of terms of the form $x^k e^{\alpha x} \sin \beta x$ and/or $x^k e^{\alpha x} \cos \beta x$.

2.7.1(L) continued

hence,

$$g_1'(x) = \frac{e^{-x}}{2(1 + e^x)}. \tag{9}$$

Subtracting the equations in (8), we obtain

$$2g_2'(x)e^{-x} = \frac{-1}{1 + e^x},$$

hence,

$$g_2'(x) = \frac{-e^x}{2(1 + e^x)}. \tag{10}$$

From (9) and (10), it follows that

$$g_1(x) = \int \frac{e^{-x}dx}{2(1 + e^x)}$$

and

$$g_2(x) = \int \frac{-e^x dx}{2(1 + e^x)}$$

so that from (7)

$$y_p = e^x \int \frac{e^{-x}dx}{2(1 + e^x)} + e^{-x} \int \frac{-e^x dx}{2(1 + e^x)} \tag{11}$$

is a particular solution of (5).

We have deliberately written (11) in integral form to emphasize that the solution exists independently of whether the integrals may be evaluated explicitly.

In other words, as long as $\frac{e^{\pm x}}{2(1 + x)}$ is integrable (which it is because all continuous functions are integrable), the general solution of (5) is given by $y = y_h + y_p$, so that by (6) and (11)

2.7.1(L) continued

$$y = c_1e^x + c_2e^{-x} + e^x \int \frac{e^{-x}dx}{2(1 + e^x)} + e^{-x} \int \frac{-e^x dx}{2(1 + e^x)}. \qquad (12)$$

Notice that we may group the terms in (12) such that the constants of integration are included in c_1 and c_2. Namely,

$$y = \left[\int \frac{e^{-x}dx}{2(1 + e^x)} + c_1\right]e^x + \left[\int \frac{-e^x dx}{2(1 + e^x)} + c_2\right]e^{-x}.$$

"Officially" (12) is an acceptable form for the correct answer, but since the integrals are not difficult to evaluate in this case, perhaps it would be worthwhile to express y_p in (11) more explicitly.

To compute

$$\int \frac{e^{-x}dx}{2(1 + e^x)}$$

we may use tables or we may make the substitution $u = e^{-x}$, whereupon

$$du = -e^{-x}dx.$$

Moreover,

$$u = e^{-x} \rightarrow e^x = \frac{1}{u}^* \rightarrow 1 + e^x = \frac{u + 1}{u}.$$

Hence,

*Since $u = e^{-x}$, $u > 0$ for all x. In particular, we do not have to worry about u being equal to 0.

2.7.1(L) continued

$$\int \frac{e^{-x}dx}{2(1 + e^{x})} = \int \frac{-u\ du}{2(u + 1)}$$

$$= -\frac{1}{2}\int \frac{u\ du}{u + 1}$$

$$= -\frac{1}{2}\int \left(1 - \frac{1}{u + 1}\right)du$$

$$= -\frac{1}{2}u + \frac{1}{2}\ln |u + 1|$$

and since $u = e^{-x}$,

$$\int \frac{e^{-x}dx}{2(1 + e^{x})} = -\frac{1}{2}e^{-x} + \frac{1}{2}\ln (1 + e^{-x})$$

$$= -\frac{1}{2}e^{-x} + \frac{1}{2}\ln \left(\frac{e^{x} + 1}{e^{x}}\right)$$

$$= -\frac{1}{2}e^{-x} + \frac{1}{2}\ln (e^{x} + 1) - \frac{1}{2}\ln e^{x}$$

$$= -\frac{1}{2}e^{-x} + \frac{1}{2}\ln (e^{x} + 1) - \frac{1}{2}x. \qquad (13)$$

Similarly, letting $u = 1 + e^{x}$, $du = e^{x}dx$. Hence,

$$\int \frac{-e^{x}dx}{2(1 + e^{x})} = \int \frac{-du}{2u}$$

$$= -\frac{1}{2}\ln |u|$$

$$= -\frac{1}{2}\ln (1 + e^{x}). \qquad (14)$$

Putting (13) and (14) into (11) yields

2.7.1(L) continued

$$y_p = e^x\left[-\frac{1}{2}e^{-x} + \frac{1}{2}\ln(e^x + 1) - \frac{1}{2}x\right] + e^{-x}\left[-\frac{1}{2}\ln(1 + e^x)\right]$$

$$= -\frac{1}{2} + \frac{1}{2}e^x\ln(e^x + 1) - \frac{1}{2}xe^x - \frac{1}{2}e^{-x}\ln(1 + e^x)$$

$$= -\frac{1}{2}(1 + xe^x) + \left(\frac{1}{2}e^x - \frac{1}{2}e^{-x}\right)\ln(1 + e^x)$$

$$= -\frac{1}{2}(1 + xe^x) + [\sinh x]\ln(1 + e^x). \qquad (15)$$

2.7.2

Since

$$y'' - 2y' + y = e^x \ln x \quad (x > 0) \qquad (1)$$

we have that

$$y_h = c_1e^x + c_2xe^x. \qquad (2)$$

Letting $u_1 = e^x$ and $u_2 = xe^x$, it follows that

$$u_1'(x) = e^x$$

and

$$u_2'(x) = xe^x + e^x.$$

Hence,

$$y_p = g_1(x)e^x + g_2(x)xe^x \qquad (3)$$

is a particular solution of (1), where

2.7.2 continued

$$\left.\begin{array}{l} g_1'(x)e^x + g_2'(x)xe^x = 0 \\ \text{and} \\ g_1'(x)e^x + g_2'(x)[xe^x + e^x] = e^x \ln x \end{array}\right\} . \qquad (4)$$

Subtracting the top equation in (4) from the bottom yields

$$e^x g_2'(x) = e^x \ln x$$

or

$$g_2'(x) = \ln x. \qquad (5)$$

Replacing $g_2'(x)$ by ln x in the top equation of (4) yields

$$g_1'(x)e^x + xe^x \ln x = 0$$

or

$$g_1'(x) = -x \ln x. \qquad (6)$$

From (5) and (6), using tables or integrating by parts, we obtain

$$g_2(x) = x \ln x - x \qquad (7)$$

and

$$g_1(x) = \frac{1}{4} x^2 - \frac{1}{2} x^2 \ln x. \qquad (8)$$

Replacing $g_1(x)$ and $g_2(x)$ in (3) by their values in (7) and (8) yields

2.7.2 continued

$$y_p = e^x\left[\frac{1}{4}x^2 - \frac{1}{2}x^2 \ln x\right] + xe^x[x \ln x - x] = \frac{3}{4}x^2e^x + \frac{1}{2}x^2e^x \ln x$$

$$= x^2e^x\left(\frac{1}{2}\ln x - \frac{3}{4}\right).^* \qquad (9)$$

Combining (2) and (9) yields that

$$y = c_1e^x + c_2xe^x + x^2e^x\left(\frac{1}{2}\ln x - \frac{3}{4}\right)$$

is the general solution of (1).

2.7.3(L)

Our main aim here is to show how one can find the general solution of $y'' + p(x)y' + q(x)y = 0$** once one non-zero particular solution of the equation is known. We shall handle the general case as a note at the conclusion of this exercise, but for now we would like to illustrate the method more concretely.

We are given the equation

$$y'' + \frac{1}{x^2}y' - \frac{1}{x^3}y = 0,\ x \neq 0^{***} \qquad (1)$$

and we assume that by "hook or crook" we stumbled across the fact that $y = x$ happens to be a solution of (1). [In the next unit, we

*Had the domain of (1) been $x < 0$, (9) would have been replaced by $x^2e^x\left(\frac{1}{2}\ln|x| - \frac{3}{4}\right)$. The important point is that $x = 0$ is excluded from the domain of (1) since then $e^x \ln x$ is undefined.

**Recall from our earlier lectures that the existence of the general solution in this case is guaranteed as soon as $p(x)$ and $q(x)$ are continuous.

***Since $p(x) = \frac{1}{x^2}$ and $q(x) = -\frac{1}{x^3}$ here, the condition that $x \neq 0$ guarantees that the general solution exists, since only at $x = 0$ are $\frac{1}{x^2}$ and $-\frac{1}{x^3}$ discontinuous.

2.7.3(L) continued

shall show a method for finding this solution, but in terms of our immediate objective in this exercise, how we know that $y = x$ is a solution of (1) is irrelevant.]

Since (1) is homogeneous (i.e., the right side equals 0), we know that $y = cx$ is a 1-parameter family of solutions of (1). Thus, all we need is one additional solution which does not belong to this family in order to obtain the general solution of (1).

In other words, if $u(x)$ is any function which is not a constant multiple of x [i.e., $\{x,u(x)\}$ is linearly independent] then

$$y = c_1x + c_2u(x)$$

is the general solution of (1).

The major problem is that of finding a technique which yields $u(x)$, and it turns out that the method of variation of parameters is again the solution.

As before, we replace c by $g(x)$ in $y = cx$ to obtain

$$y = xg(x) \tag{2}$$

and we now try to see what $g(x)$ must look like if (2) is to be a solution of (1) [observing that if such a $g(x)$ can be found and if $g(x)$ is not constant then $xg(x)$ is not a constant multiple of x, so that $\{x,xg(x)\}$ is linearly independent].

At any rate, from (2), we obtain

$$y' = xg'(x) + g(x) \tag{3}$$

and

$$y'' = [xg''(x) + g'(x)] + g'(x)$$

$$= xg''(x) + 2g'(x). \tag{4}$$

Using (2), (3), and (4) in (1) yields

2.7.3(L) continued

$$[xg''(x) + 2g'(x)] + \frac{1}{x^2}[xg'(x) + g(x)] - \frac{1}{x^3}[xg(x)] = 0$$

or

$$xg''(x) + 2g'(x) + \frac{1}{x}\,g'(x) + \frac{g(x)}{x^2} - \frac{g(x)}{x^2} = 0$$

or

$$xg''(x) + \left(2 + \frac{1}{x}\right)g'(x) = 0.^{*} \qquad (5)$$

Letting $p = g'(x)$ in (5), we see that

$$x\,\frac{dp}{dx} + \left(2 + \frac{1}{x}\right)p = 0, \qquad (6)$$

and since $x \neq 0$, (6) may be written in the equivalent standard form

$$\frac{dp}{dx} + \left(\frac{2}{x} + \frac{1}{x^2}\right)p = 0. \qquad (6')$$

We may solve (6') either by observing that the variables are separable or by observing that the equation is linear in p. If we elect to separate variables, we obtain

$$\frac{dp}{p} = -\left(\frac{2}{x} + \frac{1}{x^2}\right)dx$$

or

$$\ln|p| = -2\ln|x| + \frac{1}{x} + k_1$$

(k_1 an arbitrary constant), or

*Notice that g(x) conveniently disappeared from our equation, so that (5) is now easily handled by the substitution $p = g'(x)$. In our note following this exercise, we shall show that our method guarantees that g(x) will <u>always</u> be missing from our final equation, and this is why this technique always works.

2.7.3(L) continued

$$\ln|p| = \ln|x|^{-2} + \frac{1}{x} + k_1$$

or

$$|p| = e^{\ln|x|^{-2} + \frac{1}{x} + k_1}$$

$$= e^{\ln|x|^{-2}} e^{\frac{1}{x}} e^{k_1}$$

$$= e^{k_1} |x|^{-2} e^{\frac{1}{x}}. \qquad (7)$$

Hence, since e^{k_1} is an arbitrary positive constant and $|x|^{-2} = \frac{1}{|x|^2} = \frac{1}{x^2}$, (7) becomes

$$|p| = k_2 \frac{e^{\frac{1}{x}}}{x^2} \text{ where } k_2 = e^{k_1} > 0.$$

Hence

$$p = k_3 \frac{e^{\frac{1}{x}}}{x^2} \text{ where } k_3 = \pm k_2.\text{*} \qquad (8)$$

Recalling that $p = g'(x)$, we have from (8) that

$$g'(x) = k_3 \frac{e^{\frac{1}{x}}}{x^2}. \qquad (9)$$

*We could have obtained (8) by viewing (6') as linear in which case $e^{\int\left(\frac{2}{x} + \frac{1}{x^2}\right)dx} = e^{2\ln|x| - \frac{1}{x}} = x^2 e^{-\frac{1}{x}}$ is an integrating factor in which case (6') becomes

$\frac{d}{dx}\left(px^2 e^{-\frac{1}{x}}\right) = 0$, or $px^2 e^{-\frac{1}{x}} = k$; whence

$$p = \frac{ke^{\frac{1}{x}}}{x^2}.$$

2.7.3(L) continued

Now from (9)

$$g(x) = \int k_3 \frac{e^{\frac{1}{x}}}{x^2}\,dx. \tag{10}$$

In the present exercise, we have chosen coefficients which allow us to compute (10) explicitly (but as we've said before, this is not crucial). Namely, letting $v = \frac{1}{x}$, we see that $dv = -\frac{dx}{x^2}$, hence

$$\int k_3 \frac{e^{\frac{1}{x}}dx}{x^2} = \int -k_3\, e^u du$$

$$= -k_3\, e^u + k_5$$

or since $-k_3$ is still arbitrary,

$$g(x) = k_4\, e^{\frac{1}{x}} + k_5. \tag{11}$$

Returning to (2), (11) tells us that

$$y = x\left(k_4\, e^{\frac{1}{x}} + k_5\right) \tag{12}$$

is also a solution of (1).

In fact, (12) contains two arbitrary constants k_4 and k_5 so that it appears that

$$y = k_4\, xe^{\frac{1}{x}} + k_5 x \tag{12'}$$

is the general solution of (1). In this regard, notice that (11) was more "complete" than was necessary. Namely, all we needed was one specific solution of (1) which was not a constant multiple of x, or, equivalently, any one function g(x) which was not a

2.7.3(L) continued

constant. In particular, had we chosen $k_4 = 1$ and $k_5 = 0$, we would have obtained $g(x) = e^{\frac{1}{x}}$; whence a particular solution of (1) would be

$$y = xg(x) = xe^{\frac{1}{x}} \tag{13}$$

which is not a constant multiple of x. Notice that (13) checks with (12') in the sense that since x and $xe^{\frac{1}{x}}$ are linearly independent solutions of (1),

$$y = c_1x + c_2xe^{\frac{1}{x}} \tag{14}$$

is the general solution of (1) [(14) is (12') with $c_1 = k_5$ and $c_2 = k_4$].

As a check that (13) is a solution of (1),* we have

$$y = xe^{\frac{1}{x}} \rightarrow$$

$$y' = e^{\frac{1}{x}} + x\left[-\frac{1}{x^2}\, e^{\frac{1}{x}}\right]$$

$$= e^{\frac{1}{x}} - \frac{1}{x}\, e^{\frac{1}{x}} \rightarrow$$

$$y'' = -\frac{1}{x^2}\, e^{\frac{1}{x}} - \left[\frac{1}{x}\left(-\frac{1}{x^2}\, e^{\frac{1}{x}}\right) + \left(-\frac{1}{x^2}\right)e^{\frac{1}{x}}\right]$$

$$= \frac{1}{x^3}\, e^{\frac{1}{x}}.$$

*Again, technically speaking, all we have shown is that if there exists a solution of the form xg(x) where g is not constant, then any such solution is essentially given by (13).

2.7.3(L) continued

Hence

$$y'' + \frac{1}{x^2}y' - \frac{1}{x^3}y = \frac{1}{x^3}e^{\frac{1}{x}} + \frac{1}{x^2}\left[e^{\frac{1}{x}} - \frac{1}{x}e^{\frac{1}{x}}\right] - \frac{1}{x^3}\left[xe^{\frac{1}{x}}\right]$$

$$= \frac{1}{x^3}e^{\frac{1}{x}} + \frac{1}{x^2}e^{\frac{1}{x}} - \frac{1}{x^3}e^{\frac{1}{x}} - \frac{1}{x^2}e^{\frac{1}{x}}$$

$$= 0.$$

<u>Note</u>

The technique used in this exercise may be generalized as follows.

Suppose we are given the <u>homogeneous</u> linear differential equation (where the coefficients need not be constants)

$$y'' + p(x)y' + q(x)y = 0 \qquad (1)$$

and we "happen to know" that $y = u_1(x)$ is a non-zero solution of (1). We then write

$$u_2(x) = g(x)u_1(x) \qquad (2)$$

and try to determine $g(x)$ so that $u_2(x)$ is also a solution of (1). From (2), we obtain

$$u_2'(x) = g(x)u_1'(x) + g'(x)u_1(x) \qquad (3)$$

and

$$u_2''(x) = [g(x)u_1''(x) + g'(x)u_1'(x)] + [g'(x)u_1'(x) + g''(x)u_1(x)] \qquad (4)$$

$$= g(x)u_1''(x) + 2g'(x)u_1'(x) + g''(x)u_1(x).$$

Replacing y by u_2 in (1) and using (2), (3), and (4), we obtain

2.7.3(L) continued

$$\left.\begin{aligned}&[g(x)u_1''(x) + 2g'(x)u_1'(x) + g''(x)u_1(x)]\\&+ p(x)[g(x)u_1'(x) + g'(x)u_1(x)]\\&+ q(x)[g(x)u_1(x)]\end{aligned}\right\} = 0. \qquad (5)$$

While the left side of (5) may seem a bit cumbersome, let us observe that $u_1(x)$ was not just "any old function" but rather was a solution of (1). This means that

$$u_1''(x) + p(x)u_1'(x) + q(x)u_1(x) \equiv 0. \qquad (6)$$

Herein lies the key as to why every term involving $g(x)$ vanishes from the left side of (5). Namely, the portion of the left side of (5) which involves $g(x)$ is

$$g(x)u_1''(x) + p(x)g(x)u_1'(x) + q(x)g(x)u_1(x)$$

or

$$g(x)[u_1''(x) + p(x)u_1'(x) + q(x)u_1(x)],$$

and, from (6), this is zero!

Thus, (5) may be simplified to read

$$2g'(x)u_1'(x) + g''(x)u_1(x) + p(x)g'(x)u_1(x) = 0. \qquad (7)$$

If we now let $v = g'(x)$ [in the exercise, we let $p = g'(x)$ but this would be confusing in the present context because p or $p(x)$, is being used to denote the coefficient of y' in (1)], (7) becomes

$$2u_1'(x)v + u_1(x)\frac{dv}{dx} + p(x)u_1(x)v = 0$$

or

$$u_1(x)\frac{dv}{dx} + [2u_1'(x) + p(x)u_1(x)]v = 0 \qquad (8)$$

or, if we assume our equation is defined on an interval for which

2.7.3(L) continued

$u_1(x) \neq 0$, we may rewrite (8) as

$$\frac{dv}{dx} + \left[\frac{2u_1'(x)}{u_1(x)} + p(x)\right]v = 0. \tag{9}$$

Notice that, just as in the exercise, (9) may be viewed either as variables separable or as linear in v [keeping in mind that p(x), $u_1(x)$, and $u_1'(x)$ are <u>known</u> functions of x].

Treating (9) as linear, we have that an integrating factor is

$$e^{\int\left[2\frac{u_1'(x)}{u_1(x)} + p(x)\right]dx} = e^{2\ln|u_1(x)| + \int p(x)dx}$$

$$= u_1^2(x)e^{\int p(x)dx},$$

so that (9) is equivalent to

$$\frac{d\left[vu_1^2(x)e^{\int p(x)dx}\right]}{dx} = 0$$

or

$$vu_1^2(x)e^{\int p(x)dx} = k.$$

Hence

$$v = \frac{ke^{-\int p(x)dx}}{u_1^2(x)} = g'(x);$$

so that

$$g(x) = \int \frac{ke^{-\int p(x)dx}}{u_1^2(x)}\,dx. \tag{10}$$

Simplifying (10), of course, depends on p(x) and $u_1(x)$, but (10) supplies us with the solution, from (2),

2.7.3(L) continued

$u_2 = xg(x)$, with $g(x)$ as in (10). (11)

2.7.4

Given

$L(y) = y'' - 2y' + y,$

$L(e^{rx}) = e^{rx}(r^2 - 2r + 1) = e^{rx}(r - 1)^2.$

Hence, $r = 1$ is a root of $L(e^{rx}) = 0$, and we may conclude that

$$y = e^x \quad (1)$$

is a solution of

$$y'' - 2y' + y = 0. \quad (2)$$

Using variation of parameters, we see from (1) that there should be a different solution of (2) in the form

$$y = g(x)e^x. \quad (3)$$

From (3),

$$y' = g(x)e^x + g'(x)e^x \quad (4)$$

and

$$y'' = g(x)e^x + 2g'(x)e^x + g''(x)e^x. \quad (5)$$

Using (3), (4), and (5) in (2), we obtain

$$[g(x)e^x + 2g'(x)e^x + g''(x)e^x] - 2[g(x)e^x + g'(x)e^x] + [g(x)e^x] = 0,$$

or

$$e^x[g(x) + 2g'(x) + g''(x) - 2g(x) - 2g'(x) + g(x)] = 0,$$

2.7.4 continued

or

$$g''(x) = 0. \qquad (6)$$

From (6),

$$g(x) = c_1x + c_2$$

and; hence, choosing $c_1 = 1$ and $c_2 = 0$, we see that one simple choice for $g(x)$ is

$$g(x) = x,$$

whereupon (3) becomes

$$y = xe^x,$$

which as we already knew by other methods was a solution of (2) which was not a constant multiple of e^x.

2.7.5

a. Once we write

$$y'' + (x^2 - 4)y' - 4x^2y = 0 \qquad (1)$$

in the equivalent form

$$(y'' - 4y') + x^2(y' - 4y) = 0, \qquad (2)$$

it is not hard to notice that

$$y'' - 4y' = (y' - 4y)'.$$

Thus, letting

$$u = y' - 4y \qquad (3)$$

2.7.5 continued

in (2) yields

$$u' + x^2u = 0,$$

for which u = 0 is trivially solution.

Since $u = y' - 4y$, $u = 0$ means

$$y' - 4y = 0. \qquad (4)$$

From (4), which is a linear, first-order, homogeneous differential equation with constant coefficients, it is easy to conclude that $y = c_1e^{4x}$. In particular, letting $c_1 = 1$, we see that

$$y = e^{4x} \qquad (5)$$

is a particular solution of (4); hence, also of (1).

[<u>Check</u>: $y = e^{4x} \rightarrow y' = 4e^{4x} \rightarrow y'' = 16e^{4x}$.

Hence,

$$y'' + (x^2 - 4)y' - 4x^2y = 16e^{4x} + (x^2 - 4)4e^{4x} - 4x^2e^{4x} = 0.]$$

b. By variation of parameters, we conclude from (5) that there exists a second linearly independent solution of (1) with the form

$$y = g(x)e^{4x}. \qquad (6)$$

From (6)

$$y' = 4g(x)e^{4x} + g'(x)e^{4x} \qquad (7)$$

and

$$y'' = [16g(x)e^{4x} + 4g'(x)e^{4x}] + [4g'(x)e^{4x} + g''(x)e^{4x}]$$

$$= 16g(x)e^{4x} + 8g'(x)e^{4x} + g''(x)e^{4x}. \qquad (8)$$

2.7.5 continued

Using (6), (7), and (8) in (1), we obtain

$$\left.\begin{aligned}&[16g(x)e^{4x} + 8g'(x)e^{4x} + g''(x)e^{4x}]\\ &+ (x^2 - 4)[4g(x)e^{4x} + g'(x)e^{4x}]\\ &+ (-4x^2)[g(x)e^{4x}].\end{aligned}\right\} = 0. \qquad (9)$$

Simplifying (9) yields

$$8g'(x)e^{4x} + g''(x)e^{4x} + x^2g'(x)e^{4x} - 4g'(x)e^{4x} = 0$$

or

$$e^{4x}[4g'(x) + g''(x) + x^2g'(x)] = 0$$

or

$$g''(x) + [x^2 + 4]g'(x) = 0. \qquad (10)$$

Equation (10) is linear in $g'(x)$, so that an integrating factor is

$$e^{\int(x^2 + 4)dx} = e^{\frac{1}{3}x^3 + 4x}.$$

Hence, (10) is equivalent to

$$\frac{d\left[e^{\frac{1}{3}x^3 + 4x}g'(x)\right]}{dx} = 0,$$

so that

$$g'(x) = ke^{-\frac{1}{3}x^3 - 4x}. \qquad (11)$$

From (11), one choice of $g(x)$ is

2.7.5 continued

$$g(x) = \int e^{-\frac{1}{3}x^3 - 4x} dx.*$$

Hence from (6)

$$y = e^{4x} \int e^{-\frac{1}{3}x^3 - 4x} dx$$

is also a solution of (1).

Accordingly

$$y = c_1 e^{4x} + c_2 e^{4x} \int e^{-\frac{1}{3}x^3 - 4x} dx$$

is the general solution of (1).

2.7.6(L)

Our main aim here is to put the two previously-discussed uses of variation of parameters together and show by means of a specific example what it means when we say that we can find the general solution of

$$y'' + p(x)y' + q(x)y = f(x)$$

once we know one particular solution of the reduced equation. Other than for this, the present exercise comes under the heading of additional drill.

We are given

$$y'' - \frac{x}{1 - x^2} y' + \frac{y}{1 - x^2} = 1 \qquad (1)$$

*Here we have an example in which $g(x)$ (up to an additive constant) is well-defined because our integrand is continuous, but we cannot express $g(x)$ in a more explicit closed form. We could expand the integrand as a power series and integrate term-by-term etc., but our main point is that $g(x)$ may not be convenient from an explicit point of view.

2.7.6(L) continued

and told that one solution of the reduced equation is $y = x$ [and this may be easily checked once given; namely, $y = x \to y' = 1 \to y'' = 0 \to y'' - \frac{x}{1 - x^2} y' + \frac{y}{1 - x^2} = 0 - \frac{x}{1 - x^2} + \frac{x}{1 - x^2} = 0$].

Since $y = x$ is one solution of

$$y'' - \frac{x}{1 - x^2} y' + \frac{y}{1 - x^2} = 0 \qquad (2)$$

the method of variation of parameters tells us that another linearly independent solution of (2) exists in the form

$$y = xg(x). \qquad (3)$$

From (3),

$$y' = g(x) + xg'(x) \qquad (4)$$

and

$$\begin{aligned} y'' &= g'(x) + [xg''(x) + g'(x)] \\ &= xg''(x) + 2g'(x). \qquad (5) \end{aligned}$$

Putting (3), (4), and (5) into (2) yields

$$xg''(x) + 2g'(x) - \frac{x}{1 - x^2} [g(x) + xg'(x)] + \frac{1}{1 - x^2} [xg(x)] = 0,$$

or

$$xg''(x) + \left[2 - \frac{x^2}{1 - x^2}\right] g'(x) = 0,$$

and since $x \neq 0$,

$$g''(x) + \left[\frac{2}{x} - \frac{x}{1 - x^2}\right] g'(x) = 0. \qquad (6)$$

2.7.6(L) continued

Equation (6) is linear in $g'(x)$, so that an integrating factor of (6) is $e^{\int\left(\frac{2}{x} - \frac{x}{1 - x^2}\right)dx}$.

Since

$$\int\left(\frac{2}{x} - \frac{x}{1 - x^2}\right)dx = 2\ \ln|x| + \frac{1}{2}\ln(1 - x^2)$$

$$= \ln x^2 + \ln\sqrt{1 - x^2}\ ^*$$

$$= \ln x^2\sqrt{1 - x^2},$$

$$e^{\int\left(\frac{2}{x} - \frac{x}{1 - x^2}\right)dx} = e^{\ln x^2\sqrt{1 - x^2}} = x^2\sqrt{1 - x^2}.$$

Hence, (6) is equivalent to

$$\frac{d\left[x^2\sqrt{1 - x^2}\ g'(x)\right]}{dx} = 0,$$

or

$$g'(x) = \frac{k}{x^2\sqrt{1 - x^2}} \tag{7}$$

(where k is an arbitrary constant).

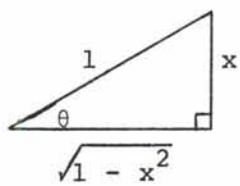
Hence

$$g(x) = \int\frac{k\ dx}{x^2\sqrt{1 - x^2}}\ (0 < x < 1). \tag{8}$$

*We are in no trouble here since $|x| < 1$ so that $\sqrt{1 - x^2}$ is real and non-zero. Had x not been so restricted, we would have had to remember to observe that $\int -\frac{x}{1 - x^2}\,dx = \frac{1}{2}\ln|1 - x^2|$ and so the only trouble occurs when $x = \pm 1$.

2.7.6(L) continued

The integral in (8) lends itself nicely to trigonometric substitution. Namely,



$$\left.\begin{aligned}&\sin\theta = x\\&\cos\theta\,d\theta = dx\\&\sqrt{1-x^2} = \cos\theta\end{aligned}\right.$$

so that

$$\int \frac{k\,dx}{x^2\sqrt{1-x^2}} = \int \frac{k\cos\theta\,d\theta}{\sin^2\theta\cos\theta}$$

$$= k\int \csc^2\theta\,d\theta$$

$$= -k\cot\theta\ (+k_1). \qquad (9)$$

From our diagram (or else analytically)

$$\cot\theta = \frac{\sqrt{1-x^2}}{x},$$

so that from (8) and (9),

$$g(x) = \frac{-k\sqrt{1-x^2}}{x} + k_1. \qquad (10)$$

Since we need only <u>one</u> <u>non-constant</u> choice of g(x) in (10), we may let $k = -1$ and $k_1 = 0$, and we then obtain

$$g(x) = \frac{\sqrt{1-x^2}}{x}. \qquad (11)$$

2.7.6(L) continued

Using g(x), as given by (11), in (3), we obtain

$$y = x\left(\frac{\sqrt{1 - x^2}}{x}\right) = \sqrt{1 - x^2}$$

is our second linearly independent solution of (2).

Thus, the general solution of (2) is

$$y = c_1x + c_2\sqrt{1 - x^2}. \tag{12}$$

Observe that in deriving (12), we were doing the same thing as we did in Exercises 2.7.3, 2.7.4, and 2.7.5; namely, finding the general solution of L(y) = 0 knowing one non-zero solution.

Now we use variation of parameters, starting with (12), as we did in Exercises 2.7.1 and 2.7.2; namely, to find a particular solution of L(y) = f(x) once we know the general solution of L(y) = 0. Namely, letting $u_1(x) = x$ and $u_2(x) = \sqrt{1 - x^2}$ we have that $u_1'(x) = 1$ and $u_2'(x) = \frac{-x}{\sqrt{1 - x^2}}$. Hence, there exists a particular solution of (1) in the form

$$y_p = xh_1(x) + \sqrt{1 - x^2}\, h_2(x)^* \tag{13}$$

where

$$\left.\begin{aligned} &xh_1'(x) + \sqrt{1 - x^2}\, h_2'(x) = 0 \\ &\text{and} \\ &h_1'(x) - \frac{x}{\sqrt{1 - x^2}}\, h_2'(x) = 1 \end{aligned}\right\} . \tag{14}$$

*In our lecture and previous exercises, we used g_1 and g_2 rather than h_1 and h_2. Since the names of our functions is not important, we elected to use h rather than g simply so as not to become confused with g as used in equation (3).

2.7.6(L) continued

Multiplying the bottom equation of (14) by x and then subtracting the bottom equation from the top, we obtain

$$\left(\sqrt{1 - x^2} + \frac{x^2}{\sqrt{1 - x^2}}\right) h_2'(x) = -x, \text{ or } \frac{h_2'(x)}{\sqrt{1 - x^2}} = -x$$

or

$$h_2'(x) = -x\sqrt{1 - x^2} \tag{15}$$

Hence, from (15),

$$h_2(x) = -\int x\sqrt{1 - x^2}\,dx$$

$$= \frac{1}{3}\left(1 - x^2\right)^{\frac{3}{2}}. \tag{16}$$

From the top equation in (14)

$$h_1'(x) = \frac{-\sqrt{1 - x^2}\,h_2'(x)}{x},$$

so from (15),

$$h_1'(x) = \frac{-\sqrt{1 - x^2}}{x}\left[-x\sqrt{1 - x^2}\right]$$

or

$$h_1'(x) = 1 - x^2.$$

Hence,

$$h_1(x) = x - \frac{1}{3}x^3. \tag{17}$$

Using (16) and (17) in (13), we obtain

2.7.6(L) continued

$$y_p = x\left(x - \frac{1}{3}x^3\right) + \sqrt{1 - x^2}\left[\frac{1}{3}\left(1 - x^2\right)^{\frac{3}{2}}\right]$$

$$= x^2 - \frac{1}{3}x^4 + \frac{1}{3}\left(1 - x^2\right)^2$$

$$= x^2 - \frac{1}{3}x^4 + \frac{1}{3} - \frac{2}{3}x^2 + \frac{1}{3}x^4$$

$$= \frac{1}{3}(1 + x^2). \qquad (18)$$

<u>Check of (18)</u>

$$y = \frac{1}{3}(1 + x^2) \rightarrow y' = \frac{2}{3}x \rightarrow y'' = \frac{2}{3}$$

Therefore,

$$y'' - \frac{x}{1 - x^2}y' + \frac{y}{1 - x^2} = \frac{2}{3} - \left(\frac{x}{1 - x^2}\right)\frac{2x}{3} + \frac{(1 + x^2)}{3(1 - x^2)}$$

$$= \frac{2(1 - x^2) - 2x^2 + (1 + x^2)}{3(1 - x^2)}$$

$$= \frac{3 - 3x^2}{3(1 - x^2)}$$

$$= 1.$$

Now we combine (18) with (12) [remembering that (12) is y_h] to obtain the fact that

$$y = c_1x + c_2\sqrt{1 - x^2} + \frac{1}{3}(1 + x^2)$$

is the general solution of (1).

2.7.7 (Optional)

a. Up to now we have been stressing variation of parameters in the case of second order linear equations. Even in this relatively simple case, the arithmetic often becomes cumbersome. Obviously, then, one can expect some real "messes" to occur when we apply variation of parameters to higher order linear equations.

Nevertheless, we feel it is worthwhile to practice on at least one higher order equation in order to make sure that you understand the general theory.

The overview is as follows.

Suppose

$$L(y) = f(x) \tag{1}$$

is a linear nth-order differential equation, and that

$$y_h = c_1u_1(x) + \ldots + c_nu_n(x) \tag{2}$$

is the <u>general</u> solution of the <u>reduced</u> equation, $L(y) = 0$.

Then a particular solution of (1) <u>exists</u> which has the form

$$y_p = g_1(x)u_1(x) + \ldots + g_n(x)u_n(x)^* \tag{3}$$

<u>where</u>

$$\left.\begin{aligned} &g_1'(x)u_1(x) + \ldots + g_n'(x)u_n(x) = 0 \\ &g_1'(x)u_1'(x) + \ldots + g_n'(x)u_n(x) = 0 \\ &\quad\vdots \qquad \vdots \qquad\qquad \vdots \\ &g_1'(x)u_1^{(n-2)}(x) + \ldots + g_n'(x)u_n^{(n-2)}(x) = 0 \\ &g_1'(x)u_1^{(n-1)}(x) + \ldots + g_n'(x)u_n^{(n-1)}(x) = f(x) \end{aligned}\right\} \tag{4}$$

*We obtain (3) from (2) just as we did in the case $n = 2$. That is, we replace the arbitrary constants $c_1, \ldots, c_n$ by the arbitrary functions $g_1(x), \ldots, g_n(x)$.

2.7.7 continued

System (4) is essentially n linear equations is the n unknowns $g_1'(x)$, ..., $g_n'(x)$. [That is, the u's are known functions of x; hence, the coefficients of $g_1'(x)$, ..., $g_n'(x)$ in (4) are known constants for a given value of x.]

Notice that in (4) that each of the left sides of the first (n-1) equations is equated to 0. The last equation has the left side equated to f(x).

If we view the u's as coefficients, we see that the determinant of coefficients is

$$\begin{vmatrix} u_1(x), \ldots, u_n(x) \\ u_1'(x), \ldots, u_n'(x) \\ \cdot \\ \cdot \\ \cdot \\ u_1^{(n-1)}(x), \ldots, u_n^{(n-1)}(x) \end{vmatrix}$$

which by definition is the <u>Wronskian</u> of $\{u_1, \ldots, u_n\}$, $W(u_1, \ldots, u_n)$. Since $\{u_1, \ldots, u_n\}$ is a set of n linearly independent solutions of L(y) = 0, $W(u_1, \ldots, u_n)$ is never zero (see Supplementary Notes, Cahpter 10). Hence, (4) may always be solved to yield unique expressions for $g_1'(x)$, ..., and $g_n'(x)$, whereupon we may find $g_1(x)$, ..., and $g_n(x)$ by integration (so they are unique up to an additive constant). Then with any particular choices of $g_1(x)$, ..., $g_n(x)$, we find y_p from (3).

The derivation of (4) is very analogous to the procedure explained in the lecture for n = 2. Essentially, with n arbitrary functions, we may impose n - 1 conditions at our disposal. Rather than give the proof for an aribtrary value of n, we shall pick n = 3 and then mimic the procedure of the lecture. Our feeling is that once you see explicitly what happens when n = 3, it will be easy to understand (4) for any value of n.

Suppose, then, we are given that

$$y_h = c_1u_1(x) + c_2u_2(x) + c_3u_3(x) \qquad (5)$$

2.7.7 continued

is the general solution of

$$y''' + p(x)y'' + q(x)y' + r(x)y = 0, \quad (6)$$

and we want to find a particular solution of

$$y''' + p(x)y'' + q(x)y' + r(x)y = f(x) \quad (7)$$

Replacing the constants in (5) by arbitrary functions, we obtain the function

$$y = g_1(x)u_1(x) + g_2(x)u_2(x) + g_3(x)u_3(x). \quad (8)$$

At this point y, as described in (8), is extremely vague. We might say that y has three degrees of freedom in the sense that we are completely free to choose $g_1(x)$, $g_2(x)$, and $g_3(x)$ as we choose [except, of course, that they must each be at least thrice-differentiable, otherwise we will not be able to use (8) when we seek a solution of (7)].

From (8), we have

$$y' = g_1(x)u_1'(x) + g_1'(x)u_1(x) + g_2(x)u_2'(x) + g_2'(x)u_2(x) + $$
$$+ g_3(x)u_3'(x) + g_3'(x)u_3(x). \quad (9)$$

[Notice, in reference to our last parenthetical remark, that equation (9) requires $g_1(x)$, $g_2(x)$, and $g_3(x)$ to be differentiable.]

"Surveying" (9) [and perhaps even using a bit of hindsight by being reminded by (4) that our system of equations will not involve $g_1(x)$, $g_2(x)$, $g_3(x)$, but rather $g_1'(x)$, $g_2'(x)$, $g_3'(x)$], we now elect to impose our first condition on the g's. Namely, we assume that $g_1(x)$, $g_2(x)$, and $g_3(x)$ are chosen such that

$$g_1'(x)u_1(x) + g_2'(x)u_2(x) + g_3'(x)u_2(x) = 0. \quad (10)$$

2.7.7 continued

[This is not hard to do. For example, we may choose g_1 and g_2 at random, and then pick g_3 to be any function defined by

$$g_3'(x) = \frac{-[g_1'(x)u_1(x) + g_2'(x)u_2(x)]}{u_3(x)} \text{ etc.]}$$

Once condition (10) is imposed, equation (9) reduces to

$$y' = g_1(x)u_1'(x) + g_2(x)u_2'(x) + g_3(x)u_3'(x). \tag{9'}$$

From (9'), we obtain

$$y" = g_1(x)u_1"(x) + g_1'(x)u_1'(x) + g_2(x)u_2"(x) + g_2'(x)u_2'(x) +$$

$$g_3(x)u_3"(x) + g_3'(x)u_3'(x),$$

or, upon regrouping terms,

$$y" = [g_1(x)u_1"(x) + g_2(x)u_2"(x) + g_3(x)u_3"(x)] +$$

$$+ [g_1'(x)u_1'(x) + g_2'(x)u_2'(x) + g_3'(x)u_3'(x)]. \tag{11}$$

If we now elect to impose our next restriction on $g_1(x)$, $g_2(x)$, and $g_3(x)$, equation (11) suggest that it be

$$g_1'(x)u_1'(x) + g_2'(x)u_2'(x) + g_3'(x)u_3'(x) = 0. \tag{12}$$

[Equation (12) is suggested by (4). Had we not known this, however, our choice would have been the same, but the reasoning might have been different. Namely, somewhere along the line, we expect to have to use the fact that $u_1(x), u_2(x)$, and $u_3(x)$ are (linearly independent) solutions of the reduced equation. In other words, for k = 1, 2, or 3

$$u_k'''(x) + p(x)u_k"(x) + q(x)u_k'(x) + r(x)u_k(x) = 0 \tag{13}$$

2.7.7 continued

so we want to keep the higher order derivatives of u_1, u_2, and u_3 in (11) in the hope that we will be able to simplyfy things by use of (13).]

At any rate, assuming that condition (12) has been imposed, (11) becomes

$$y'' = g_1(x)u_1''(x) + g_2(x)u_2''(x) + g_3(x)u_3''(x) \tag{11'}$$

and from (14)

$$y''' = [g_1(x)u_1'''(x) + g_2(x)u_2'''(x) + g_3(x)u_3'''(x)] + \\ + [g_1'(x)u_1''(x) + g_2'(x)u_2''(x) + g_3'(x)u_3''(x)]. \tag{14}$$

We now replace y''', y'', y', and y in (7) by their values given in (8), (9'), (11'), and (14). This yields

$$\left.\begin{array}{l} g_1(x)u_1'''(x) + g_2(x)u_2'''(x) + g_3(x)u_3'''(x) + g_1'(x)u_1''(x) + \\ \qquad + g_2'(x)u_2''(x) + g_3'(x)u_3''(x) \\ + \; p(x)g_1(x)u_1''(x) + p(x)g_2(x)u_2''(x) + p(x)g_3(x)u_3''(x) \\ + \; q(x)g_1(x)u_1'(x) + q(x)g_2(x)u_2'(x) + q(x)g_3(x)u_3'(x) \\ + \; r(x)g_1(x)u_1(x) + r(x)g_2(x)u_2(x) + r(x)g_3(x)u_3(x) \\ \hline \underbrace{g_1(x)[u_1'''+pu_1''+qu_1'+ru_1]}_{=0, \text{ by } (13)} \; \underbrace{g_2(x)[u_2'''+pu_2''+qu_2'+ru_2]}_{=0, \text{ by } (13)} \; \underbrace{g_3(x)[u_3'''+pu_3''+qu_3'+ru_3]}_{=0, \text{ by } (13)} \end{array}\right\} = f(x)$$

Hence,

$$g_1'(x)u_1''(x) + g_2'(x)u_2''(x) + g_3'(x)u_3''(x) = f(x). \tag{15}$$

2.7.7 continued

[Notice that (15) is not arbitrarily prescribed. Rather we have shown that our first two restrictions, (10) and (12), force us to accept (15) if there is to be any hope for (8) to be a solution of (7).]

Collecting the results of (10), (12), and (15), etc., we have shown that

$$y_p = g_1(x)u_1(x) + g_2(x)u_2(x) + g_3(x)u_3(x)$$

will be a particular solution of (7) provided that

$$\left.\begin{aligned} g_1'(x)u_1(x) + g_2'(x)u_2(x) + g_3'(x)u_3(x) &= 0 \\ g_1'(x)u_1'(x) + g_2'(x)u_2'(x) + g_3'(x)u_3'(x) &= 0 \\ g_1'(x)u_1''(x) + g_2'(x)u_2''(x) + g_3'(x)u_3''(x) &= f(x) \end{aligned}\right\} . \qquad (16)$$

The determinant of coefficients in (16) is $W(u_1,u_2,u_3)$ and this is never zero since $\{u_1,u_2,u_3\}$ is a linearly independent set of solutions of (6).

Since the determinant of coefficients never vanishes, equations (16) are consistent and uniquely determine $g_1'(x)$, $g_2'(x)$, and $g_3'(x)$, from which we can now find $g_1(x)$, $g_2(x)$, and $g_3(x)$, and thus determine y_p from (8).

Notice that equations (16) would still make sense even if $\{u_1,u_2,u_3\}$ was a linearly dependent set. In this case, however, $W(u_1,u_2,u_3) \equiv 0$ and consequently equations (16) need not be consistent. In other words, we would not have enough information in (16) to determine $g_1(x)$, $g_2(x)$, or $g_3(x)$.

b. Since $x \neq 0$, we may rewrite our equation as

$$y''' + \frac{1}{x^2}y' - \frac{1}{x^3}y = \frac{1}{x^2}\ln x \quad (x > 0). \qquad (1)$$

Knowing that

$$y_h = c_1x + c_2x\ln x + c_3x(\ln x)^2 \qquad (2)$$

2.7.7 continued

is the general solution of the reduced equation, we may let

$$u_1(x) = x \tag{3a}$$

$$u_2(x) = x \ln x \tag{3b}$$

$$u_3(x) = x(\ln x)^2 \tag{3c}$$

whereupon the result of Exercise 2.7.7 tells us that

$$y_p = g_1(x)u_1(x) + g_2(x)u_2(x) + g_3(x)u_3(x) \tag{4}$$

is a particular solution of (1) provided

$$\left.\begin{aligned} &g_1'u_1 + g_2'u_2 + g_3'u_3 = 0 \\ &g_1'u_1' + g_2'u_2' + g_3'u_3' = 0 \\ &g_1'u_1'' + g_2'u_2'' + g_3'u_3'' = f(x) = \frac{1}{x^2}\ln x \end{aligned}\right\} \tag{5}$$

[Notice from (4) and (5) that it is easy to <u>specify</u> y_p. The difficult tasks are verifying the formula (which we did in the previous exercise) and carrying out the computations (which we shall do now).]

From (3), we have

$$u_1'(x) = 1,\ u_1''(x) = 0,\ u_2'(x) = 1 + \ln x,\ u_2''(x) = \frac{1}{x},$$

$$u_3'(x) = 2\ln x + (\ln x)^2,\ u_3''(x) = \frac{2}{x} + \frac{2\ln x}{x}.$$

Using these results in (5), we obtain the system

$$\left.\begin{aligned} &xg_1'(x) + x\ln x g_2'(x) + x(\ln x)^2 g_3'(x) = 0 \\ &g_1'(x) + (1 + \ln x)g_2'(x) + [2\ln x + (\ln x)^2 g_3'(x)] = 0 \\ &\frac{1}{x} g_2'(x) + \left(\frac{2}{x} + \frac{2\ln x}{x}\right) g_3'(x) = \frac{1}{x^2}\ln x \end{aligned}\right\} \tag{6}$$

2.7.7 continued

Equations (6) lend themselves nicely to a direct solution without our having to refer to determinants. For a start, since $x \neq 0$, we multiply the first equation in (6) $\frac{1}{x}$ and the third by x to obtain

$$\left.\begin{aligned} g_1'(x) + \ln x\, g_2'(x) + (\ln x)^2\, g_3'(x) &= 0 \\ g_1'(x) + (1 + \ln x)\, g_2'(x) + [2 \ln x + (\ln x)^2]\, g_3'(x) &= 0 \\ g_2'(x) + [2 + 2 \ln x]\, g_3'(x) &= \frac{\ln x}{x} \end{aligned}\right\} \quad (7)$$

Replacing the second equation in (7) by the second minus the first, we obtain

$$\left.\begin{aligned} g_1'(x) + \ln x\, g_2'(x) + (\ln x)^2\, g_3'(x) &= 0 \\ g_2'(x) + 2 \ln x\, g_3'(x) &= 0 \\ g_2'(x) + (2 + 2 \ln x)\, g_3'(x) &= \frac{\ln x}{x} \end{aligned}\right\} \quad (8)$$

We complete the diagonalization of the system by replacing the third equation in (8) by the third minus the second to obtain

$$\left.\begin{aligned} g_1'(x) + \ln x\, g_2'(x) + (\ln x)^2\, g_3'(x) &= 0 \\ g_2'(x) + 2 \ln x\, g_3'(x) &= 0 \\ 2\, g_3'(x) &= \frac{\ln x}{x} \end{aligned}\right\} \quad (9)$$

Therefore, from the third equation in (9)

$$g_3'(x) = \frac{\ln x}{2x}. \quad (10)$$

Putting this result into the second equation in (9) yields

$$g_2'(x) + 2 \ln x \left(\frac{\ln x}{2x}\right) = 0$$

or,

2.7.7 continued

$$g_2'(x) = \frac{-(\ln x)^2}{x}. \tag{11}$$

Using (10) and (11) in the first equation of (9), we finally obtain

$$g_1'(x) + \ln x \left[\frac{-(\ln x)^2}{x}\right] + (\ln x)^2 \frac{\ln x}{2x} = 0$$

or

$$g_1'(x) = \frac{(\ln x)^3}{2x}. \tag{12}$$

We now integrate equations (10), (11), and (12) to conclude that

$$\left.\begin{aligned} g_3(x) &= \tfrac{1}{4} (\ln x)^2 \\ g_2(x) &= -\tfrac{1}{3} (\ln x)^3 \\ g_1(x) &= \tfrac{1}{8} (\ln x)^4 \end{aligned}\right\} \tag{13}$$

Using equations (13) and (3) in (4) yields

$$\begin{aligned} y_p &= \tfrac{1}{8} (\ln x)^4\, x - \tfrac{1}{3} (\ln x)^3\, x \ln x + \tfrac{1}{4} (\ln x)^2\, x(\ln x)^2 \\ &= (\tfrac{1}{8} - \tfrac{1}{3} + \tfrac{1}{4})x\, (\ln x)^4 \\ &= \tfrac{1}{24}\, x\, (\ln x)^4 \end{aligned}$$

(which agrees with our result in Exercise 2.6.7, where we obtained the same answer using undetermined coefficients.