Unit 6: The Method of Undetermined Coefficients

2.6.1(L)

We want to emphasize here that the method of undetermined coefficients requires not only that

$$L(y) = f(x) \tag{1}$$

but also that f(x) must be a linear combination of terms of the form

$$x^k e^{\alpha x} \cos \beta x \text{ or } x^k e^{\alpha x} \sin \beta x \tag{2}$$

where k is a whole number; and α and β are real.*

You may notice that the examples used in the lecture were special cases of (2). For example, if $L(y) = e^x$, then e^x has the form $x^k e^{\alpha x} \cos \beta x$ where $k = 0$, $\alpha = 1$, and $\beta = 0$.

The question that we want to analyze in this exercise is why the right side of (1) must be restricted to having the form described by (2). The point is that the method of undetermined coefficients requires that the right side of (1) be a function which has only a finite number of linearly independent derivatives. As we shall discuss in more detail in the next (optional) exercise, a function has this property if and only if it has the form described by (2), or else is a linear combination of terms of type (2).**

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*If we allow the use of non-real numbers, the two types of terms described in (2) can be written in the single form:

$x^k e^{rx}$

where k is still a whole number, but r is now any complex number.

**What we mean here is the superposition principle discussed in the lecture. Namely, if f(x) is a sum of terms of the type (2); say $f(x) = g_1(x) + \ldots + g_m(x)$, where $g_1(x), \ldots, g_m(x)$ each have the form described in (2); then solving (1) requires that we solve each of the equations: $L(y) = g_1(x)$, $L(y) = g_2(x)$, ..., and $L(y) = g_m(x)$ separately. Then, if $L(u_1) = g_1(x)$, $L(u_2) = g_2(x), \ldots$, and $L(u_m) = g_m(x)$; by linearity $L(u_1 + \ldots + u_m) = g_1(x) + \ldots + g_m(x) = f(x)$.

2.6.1(L) continued

Since the reasoning and the concept itself is a bit subtle, we shall indicate what we mean in terms of the various parts of this exercise.

a. 1. With

$$f(x) = e^{3x} \tag{1}$$

we obtain

$$f'(x) = 3e^{3x}$$

$$f''(x) = 9e^{3x}$$

and proceeding inductively,

$$f^{(n)}(x) = 3^n e^{3x}. \tag{2}$$

From (2) we conclude that $\{f, f', f'', \ldots\}$ is linearly dependent; and even more, that each member of the set is a constant multiple of e^{3x}. In this respect, then, $\{f, f', f'', \ldots\}$ is a 1-dimensional vector space spanned by e^{3x}.

2. With

$$f(x) = xe^{3x} \tag{3}$$

we obtain

$$f'(x) = e^{3x} + 3xe^{3x}. \tag{4}$$

Looking at (4) we see that $f'(x)$ is now a linear combination of e^{3x} and xe^{3x}.

We could now, by "brute force" extablish that each derivative of xe^{3x} is a linear combination of e^{3x} and xe^{3x} simply by computing the various derivatives and seeing what happens inductively. Namely, we may rewrite (4) by replacing xe^{3x} by $f(x)$, so that we now have

2.6.1(L) continued

$f'(x) = e^{3x} + 3f(x).$ (4')

Using (4') to find f"(x), we obtain

$f''(x) = 3e^{3x} + 3f'(x),$

or, replacing f'(x) by its value in (4),

$f''(x) = 3e^{3x} + 3(e^{3x} + 3xe^{3x})$

$= 6e^{3x} + 9xe^{3x}.$ (5)

Clearly, we conclude from (5) that f"(x) is a linear combination of e^{3x} and xe^{3x}; and even more, we have probably begun to sense that each time we differentiate 3^{3x} or xe^{3x} we will continue to obtain linear combinations of e^{3x} and xe^{3x}.

Thus, in this case it appears that the set consisting of xe^{3x} and its various derivatives is a 2-dimensional vector space, with the basic "vectors" being e^{3x} and xe^{3x}. That is, every derivative of xe^{3x} can be written in one and only one way in the form $Axe^{3x} + Be^{3x}$, where A and B are constants.

Optional Note:
A "cuter", more mathematical, inductive approach would be to let $g(x) = e^{3x}$. Then, from (1) we already know that $g^{(k)}(x) = 3^k e^{3x} = 3^k g(x)$.

We may now show that each derivative, $f^{(n)}(x)$, of (3) is a linear combination of the preceeding derivative, $f^{(n-1)}(x)$ and g(x). Namely, we rewrite (4') in the form

$f'(x) = g(x) + 3f(x)$ (6)

whereupon

$f''(x) = g'(x) + 3f'(x)$
$f'''(x) = g''(x) + 3f''(x),$

2.6.1(L) continued

and, in general

$$f^{(n)}(x) = g^{(n-1)}(x) + 3f^{(n-1)}(x). \qquad (7)$$

Since $g^{(k)}(x) = 3^k e^{3x}$ and since $e^{3x} = g(x)$, we may let $k = n - 1$ to conclude that $g^{(n-1)}(x) = 3^{n-1}e^{3x} = 3^{n-1}g(x)$.

Consequently, (7) takes the form

$$f^{(n)}(x) = 3^{n-1}g(x) + 3f^{(n-1)}(x). \qquad (8)$$

Without worrying about the specific details at this moment, notice how (8) tells us that each $f^{(n)}(x)$ is a linear combination of e^{3x} and xe^{3x}. Namely, since we already have shown that g(x) and f'(x) are linearly combinations of e^{3x} and xe^{3x}, we may use (8) with n = 2 to conclude that f"(x) was also a linear combination of e^{3x} and xe^{3x}. Once we knew this we could again invoke (8), but now with n = e, to conclude that $f^{(3)}(x)$ was a linear combination of e^{3x} and xe^{3x}.

In more computational detail, using (8) with n = 1 yields

$$f'(x) = g(x) + 3f(x)$$

and since $f(x) = xe^{3x}$ and $g(x) = e^{3x}$, this means that

$$f'(x) = e^{3x} + 3xe^{3x}, \qquad (9)$$

which agrees with (4) [as it should].

If we next use (8) with n = 2, we obtain

$$f''(x) = 3g(x) + 3f'(x)$$

or, from (9),

$$f''(x) = 3e^{3x} + 3(e^{3x} + 3xe^{3x}).$$

Hence,

2.6.1(L) continued

$$f''(x) = 6e^{3x} + 9xe^{3x} \text{ [which agrees with (5)].} \quad (10)$$

Again, returning to (8), but now with n = 3, we have

$$f'''(x) = 9g(x) + 3f''(x)$$

so that by (10),

$$f'''(x) = 9e^{3x} + 3(6e^{3x} + 9xe^{3x})$$

$$= 27e^{3x} + 27xe^{3x}. \quad (11)$$

With the arch of (11) we use (8) with n = 4 to obtain

$$f^{(4)}(x) = 27e^{3x} + 3f'''(x)$$

$$= 27e^{3x} + 3(27e^{3x} + 27xe^{3x})$$

$$= 108e^{3x} + 81xe^{3x}. \quad (12)$$

Notice how nicely this inductive method minimizes the amount of actual computation. At the same time, notice how (9), (10), (11), and (12) tell us specifically how f', f'', f''', and $f^{(4)}$ are expressed as linear combinations of e^{3x} and xe^{3x}.

With some additional experience and/or luck, we might even discover in time the more general result that

$$f^{(n)}(x) = n\,3^{n-1}e^{3x} + 3^n\,xe^{3x}. \quad (13)$$

Of course, even if we never discovered (13), once given it as a conjecture, we can verify it by mathematical induction.

The really important point, however, is that by use of (8) we can virtually immediately compute $f^{(n)}(x)$ as a linear combination of e^{3x} and xe^{3x} as soon as we know $f^{(n-1)}(x)$ as a linear combination of e^{3x} and xe^{3x}. Thus, e^{3x}, xe^{3x} is a linearly independent set with the property that the derivatives of xe^{3x} are all linear combinations of the members of this set.

2.6.1(L) continued

b. Given the equation

$$L(y) = f(x) \tag{1}$$

with

$$L(y) = y'' - 8y' + 7y \tag{2}$$

and

$$f(x) = e^{3x}$$

we see that the right conditions for the method of undetermined coefficients exist. Namely, our equation is linear; it has constant coefficients; and its right side is of the required type.

Hence, as outlined in our lecture, we look for a particular solution of (1) in this case of the form

$$y_p = Ae^{3x}. \tag{4}$$

From (4) we obtain

$$y'_p = 3Ae^{3x} \tag{5}$$

and

$$y''_p = 9Ae^{3x}. \tag{6}$$

Therefore, if we replace y in (1) by y_p as defined by (4); and use the results of (5) and (6), we obtain

$$(9Ae^{3x}) - 8(3Ae^{3x}) + 7(Ae^{3x}) = e^{3x},$$

or

$$-8Ae^{3x} = e^{3x} \; [= 1e^{3x}]. \tag{7}$$

2.6.1(L) continued

Equating the coefficients of e^{3x} in (7) we obtain

$A = -\frac{1}{8}$,

so by (4)

$$y_p = -\frac{1}{8}e^{3x} \qquad (8)$$

is a particular solution of (1).

Check:

$$y_p = -\frac{1}{8}e^{3x} \rightarrow y'_p = -\frac{3}{8}e^{3x} \rightarrow y''_p = -\frac{9}{8}e^{3x}.$$

Hence,

$$y''_p - 8y'_p + 7y_p = -\frac{9}{8}e^{3x} - 8(-\frac{3}{8}e^{3x}) - \frac{7}{8}e^{3x} = e^{3x}.$$

The rest of this problem involves "old" stuff. Namely, our reduced equation is

$$y'' - 8y' + 7y = 0 \qquad (9)$$

and in the previous unit we learned that the general solution of (9) is given by

$$y_h = c_1e^{7x} + c_2e^x. \qquad (10)$$

The general solution of (1) is given by

$y = y_h + y_p$,

so from (8) and (10) we have that

$$y = c_1e^{7x} + c_2e^x - \frac{1}{8}e^{3x} \qquad (11)$$

is our general solution.

Note #1
All that is "new" in this unit is that we have a special technique for finding y_p that works for certain types of

2.6.1(L) continued

equations. Other than that, the rest of the theory in this problem has been discussed more generally in previous units.

Note #2

While (11) expresses y as a sum of 3 terms and (10) as a sum of 2 terms, (11) still has only the same two arbitrary constants as does (10). That is, if we want to find the solution of (1) which passes through (x_o, y_o) with slope y'_o, we see from (11) that c_1 and c_2 are determined by the system of equations.

$$\left.\begin{aligned} y_o &= c_1 e^{7x_o} + c_2 e^{x_o} - \frac{1}{8} e^{3x_o} \\ y'_o &= 7c_1 e^{7x_o} + c_2 e^{x_o} - \frac{3}{8} e^{3x_o} \end{aligned}\right\} \qquad (12)$$

Since x_o, y_o, and y'_o are given numbers, the only unknowns in (12) are c_1 and c_2. Thus, the uniqueness of the solution in (12) depends only on the determinant of coefficients of c_1 and c_2; i.e.,

$$\begin{vmatrix} e^{7x_o} & e^{x_o} \\ 7e^{7x_o} & e^{x_o} \end{vmatrix}$$

which is the same determinant that characterizes (10). This is why we only needed one solution y_p once we knew the general solution, y_n of $y'' - 8y' + 7y = 0$.

c. Our main aim in this part of the exercise is to emphasize our earlier remarks about the fact that f(x) and its various derivatives can be expressed in terms of linear combinations of an appropriate finite subset of derivatives. To set up our claim, let us try to lead you into a trap. Namely, suppose we tried to solve this exercise in the same way as we did the previous part. That is, given that

$$y'' - 8y' + 7y + xe^{3x} \qquad (1)$$

we try for a particular solution of (1) in the form

$$y_p = Axe^{3x}. \qquad (2)$$

2.6.1(L) continued

From (2), we conclude [see part (a)]

$$y'_p = A(e^{3x} + 3xe^{3x}) \qquad (3)$$

and

$$y''_p = A(6e^{3x} + 9xe^{3x}). \qquad (4)$$

Using (2), (3), and (4) and letting $y = y_p$, equation (1) yields

$$A(6e^{3x} + 9xe^{3x}) - 8A(e^{3x} + 3xe^{3x}) + 7Axe^{3x} = xe^{3x},$$

or,

$$-8Axe^{3x} - 2Ae^{ex} = xe^{3x} = 1xe^{3x} + 0e^{3x}. \qquad (5)$$

Since xe^{3x} and e^{3x} are linearly independent, we know, again from the previous unit, that equation (5) can be satisfied if and only if "like" coefficients are equal. Equating the coefficients of xe^{3x}, we see that

$$-8A = 1 \qquad (6)$$

and equating the coefficients of e^{3x}, we see that

$$-2A = 0. \qquad (7)$$

Comparing (6) and (7) we see that we have arrived at a contradiction. Namely A must be constant, yet would have to be $-\frac{1}{8}$ to satisfy (6), but 0 to satisfy (7).

This contradiction is not the end of the world, but it does tell us that equation (1) has no solution of the form: $y = Axe^{3x}$.

Where, then, did we go wrong? How come we used the same method in part (b) and didn't get into any trouble but that the method got into trouble here?

2.6.1(L) continued

The answer lies in the fact that when we differentiate xe^{3x}, the derivative includes the new, linearly independent term e^{3x}. This did not happen in part (b). That is, in part (b) every derivative of e^{3x} was a constant multiple of e^{3x} so that no new family of linearly independent terms was introduced by taking the various derivatives.

Herein lies the importance of the work discussed in part (a) of this exercise. In part (a) we saw every derivative of xe^{3x} was a linear combination of the linearly independent functions, xe^{3x} and e^{3x}. Thus, to protect ourselves against new, unforeseen terms from "creeping in", our trial solution should not have been $y_p = Axe^{3x}$, but rather

$$y_p = Axe^{3x} + Be^{3x}. \tag{8}$$

Using a bit of hindsight, we see that with (8) replacing (2), the fact that we are going to have to compare like coefficients of two sets of terms no longer forces us into a contradiction since (8) gives us two undetermined coefficients at our disposal.

More computationally, from (8) we have

$$\begin{aligned} y'_p &= A(e^{3x} + 3xe^{3x}) + 3Be^{3x} \\ &= 3Axe^{3x} + (A + 3B)e^{3x} \end{aligned} \tag{9}$$

and

$$\begin{aligned} y''_p &= A(6e^{3x} + 9xe^{3x}) + 9Be^{3x} \\ &= 9Axe^{3x} + (6A + 9B)e^{3x}. \end{aligned}$$

If we now replace y by y_p in (1), only now using (8), (9), and (10) instead of (2), (3), and (4), we obtain

$$\begin{aligned} 9Axe^{3x} + (6A + 9B)e^{3x} - 8[3Axe^{3x} + (A + 3B)e^{3x}] \\ + 7(Axe^{3x} + Be^{3x}) = xe^{3x} \end{aligned}$$

2.6.1(L) continued

or,

$$-8Axe^{3x} + (-2A - 8B)e^{3x} = 1xe^{3x} + 0e^{3x}. \tag{11}$$

Equating coefficients of "like" terms in (11) yields

$$\left.\begin{aligned} &-8A = 1 \text{ [just as in (6)]} \\ &\underline{\text{and}} \\ &-2A - 8B = 0 \end{aligned}\right\} \tag{12}$$

With $A = -\frac{1}{8}$, the bottom equation in (12) becomes

$$B = -\frac{1}{4}A = \frac{1}{32}.$$

Using these values of A and B in (8), we obtain

$$y_p = -\frac{1}{8}xe^{3x} + \frac{1}{32}e^{3x}. \tag{13}$$

As a check that (13) is a solution of (1), we have

$$\begin{aligned} y'_p &= -\frac{1}{8}e^{3x} - \frac{3}{8}xe^{3x} + \frac{3}{32}e^{3x} \\ &= -\frac{1}{32}e^{3x} - \frac{3}{8}xe^{3x}; \end{aligned}$$

hence,

$$\begin{aligned} y''_p &= -\frac{3}{32}e^{3x} - \frac{3}{8}e^{3x} - \frac{9}{8}xe^{3x} \\ &= -\frac{15}{32}e^{3x} - \frac{9}{8}xe^{3x}. \end{aligned}$$

Therefore,

$$\left.\begin{aligned} y''_p - 8y'_p + 7y_p = &-\frac{15}{32}e^{3x} - \frac{9}{8}xe^{3x} \\ &+\frac{1}{4}e^{3x} + 3xe^{3x} \\ &+\frac{7}{32}e^{3x} - \frac{7}{8}xe^{3x} \end{aligned}\right\} 0 + xe^{3x}.$$

2.6.1(L) continued

Since the reduced equation is still

$$y'' - 8y' + 7y = 0$$

we see from (13) that the required solution is

$$y = c_1e^{7x} + c_2e^{x} - \frac{1}{8}xe^{3x} + \frac{1}{32}e^{3x}.$$

This helps to explain why we do not have to go beyond the family of derivatives of the right side in looking for y_p by the method of undetermined coefficients. A good rule of thumb, however, is that when in doubt as to whether a particular term should be included in the trial solution of y_p, always include the term. If the term was unnecessary its coefficient will simply turn out to be zero, in which case we have wasted some time but no damage is done in the sense that we still get the answer. On the other hand, if we leave out a term which should have been included, then we will arrive at a contradiction [as in (6) and (7)] in which case we must start over.

Note: (optional)
Let us observe that we do not have to worry about whether additional term should have been included in (8). For example, had we tried

$$y_p = Axe^{3x} + Be^{3x} + c\,h(x) \qquad (14)$$

then, if h(x) is a linear combination of e^{3x} and xe^{3x}, then ch(x) is redundant in (14) because h(x) could have been written as $k_1xe^{3x} + k_2e^{3x}$ in which case (14) would be back in the form (8). If h(x) is not a linear combination of e^{3x} and xe^{3x}, then $\{h(x), e^{3x}, xe^{3x}\}$ is linearly independent. Since h(x) does not appear on the right side of (1) but ch(x) appears on the left side, we must have that c = 0, since the coefficient of h(x) on the left side must equal the coefficient on the right side (which is 0).

2.6.1(L) continued

d. Here it appears that we are back to the type of problem in part (b) rather than the type of part (c). That is, looking at the right side of

$$y'' - 8y' + 7y = e^{7x} \tag{1}$$

it seems that our trial solution should be

$$y_p = Ae^{7x} \tag{2}$$

and that no other terms are necessary since all the derivatives of e^{7x} are (constant) multiples of e^{7x}.

Nevertheless, using (2) we see that

$$y'_p = 7Ae^{7x} \tag{3}$$

and

$$y''_p = 49Ae^{7x}. \tag{4}$$

Replacing y by y_p in (1) now yields

$$49Ae^{7x} - 8(7Ae^{7x}) + 7(Ae^{7x}) = e^{7x}$$

or

$$0 = e^{7x} . \tag{5}$$

Obviously (5) is a contradiction, since e^{7x} cannot be zero for any value of x - let alone identically zero. Where we went wrong in this problem (unless you saw through our approach and did it correctly on your own) was that we did not recognize that our trial solution was itself part of the solution of the reduced equation.

In other words, notice that once we pick our y_p, the approach is to compute $L(y_p)$ and then look at the right side, f(x), and compare coefficients. The point is that once y_p is a

2.6.1(L) continued

solution of the reduced equation, $L(y_p)$ will automatically be zero and this will lead to a contradiction unless $f(x)$ is also identically zero (if $f(x)$ is identically zero then we would not have been using undetermined coefficients in the first place, but rather the method of the previous unit).

This is why we emphasized in the lecture that one should solve the reduced equation before trying to find y_p by the method of undetermined coefficients.

The trick in this case is that whenever the logical choice we would have made for y_p turns out to be a solution of the reduced equation, we replace y_p by xy_p. If xy_p is still a solution of the reduced equation we then try x^2y_p, etc., and we ultimately wind up with a value of k for which x^ky_p does not satisfy the reduced equation. In this case, x^ky_p will work as our trial solution.

With respect to the present exercise, we see that with $y_p = Ae^{7x}$, y_p is a particular solution of the reduced equation since the general solution of the reduced equation is

$$y_h = c_1e^{7x} + c_2e^x,$$

from which we may obtain y_p simply by letting $c_1 = A$ and $c_2 = 0$.

Therefore, rather than (2) we try

$$y_p = Axe^{7x} \tag{6}$$

as our trial solution, This will work since xe^{7x} is not a linear combination of e^x and e^{7x}; and consequently Axe^{7x} cannot be a solution of the reduced equation (since all such solutions are linear combinations of e^x and e^{7x}).

Before carrying out the solution of this problem, let us make sure that we see the basic difference between this part of the exercise and the previous part. In the previous part, we

2.6.1(L) continued

tried for a solution of the form

$$y_p = Axe^{3x} + Be^{3x}$$

not because e^{3x} was a solution of the reduced equation but because the right side of the equation had the term xe^{3x}. In this part of the exercise, we used (6) not because the right side of (1) had a term of the form xe^{7x}, but because the right side was e^{7x} which was a solution of the reduced equation. In this context, notice that (6) is written just as it is. We do not write (6) as

$$y_p = Axe^{7x} + Be^{7x} \tag{6'}$$

since (6') would just cause us extra work (and a contradiction unless B = 0). Namely, in computing

$$L(Axe^{7x} + Be^{7x})$$

We obtain by linearity

$$AL(xe^{7x}) + BL(e^{7x})$$

and this, in turn, is simply

$$AL(xe^{7x})$$

since $L(e^{7x}) = 0$.

Thus, had we used (6'), when we went to equate coefficients, B would have been missing!

At any rate, returning to (6) we obtain

$$y'_p = Ae^{7x} + 7Axe^{7x} \tag{7}$$

and

$$y''_p = 7Ae^{7x} + 7Ae^{7x} + 49Axe^{7x} = 14\ Ae^{7x} + 49Axe^{7x}. \tag{8}$$

2.6.1(L) continued

Thus, using (6) in (1) yields

$$(14Ae^{7x} + 49Axe^{7x}) - 8(Ae^{7x} + 7Axe^{7x}) + 7(Axe^{7x}) = e^{7x}$$

or

$$6Ae^{7x} = e^{7x} \tag{9}$$

and from (9) we conclude that

$$6A = 1, \text{ or } A = \frac{1}{6}$$

so that, from (6)

$$y_p = \frac{1}{6} xe^{7x}. \tag{10}$$

From (10) we conclude that the general solution of (1) is

$$y = c_1e^{7x} + c_2e^x + \frac{1}{6} xe^{7x}.$$

Summary:

Admittedly this was an unusually long exercise (even by our standards). The point is that this one exercise hits at virtually every problem that can arise when one uses the method of undetermined coefficients.

We may summarize our results as follows. Assume for the remainder of this discussion that

$$L(y) = f(x)$$

is a linear differential equation with constant coefficients.

1. We investigate to see whether $f(x)$ is a solution of the reduced equation; that is, we see whether $L(f(x)) = 0$.

2.6.1(L) continued

2. In the event that f(x) is not a solution of the reduced equation, we choose our trial solution as follows:

If $f(x) = e^{rx}$, then $y_p = Ae^{rx}$

If f(x) = sin mx, then y_p = A sin mx + B cos mx* (and the same y_p is used if f(x) = cos mx).

3. If $f(x) = x^k$, then $y_p = A_k x^k + \ldots + A_1 x + A_o$ [i.e., f and its derivatives are all linear combinations of 1, x..., and x^k].

4. If $f(x) = f_1(x) + \ldots + f_m(x)$, where $f_1, \ldots, f_m$ are of the types (1), (2), and/or (3), we solve $L(y) = f_1(x), \ldots, L(y) = f_m(x)$ separately and then use linearity to find the solution of L(y) = f(x).

5. If $f(x) = x^k e^{\alpha x} \cos \beta x$ (the most general case), we take the families $\{1, x, \ldots, x^k\}$, $\{e^{\alpha x}\}$, $\{\cos \beta x, \sin \beta x\}$ and form all possible terms which consist of a member of each set. Our trial solution is then the general linear combination of the members of the resulting set.

By way of illustration, if

$f(x) = x^3 \cos 2x$

we form one family $\{1, x, x^2\ x^3\}$ [by (3)], and the other family {cos 2x, sin 2x} and obtain the set of eight terms

$\{\cos 2x, x \cos 2x, x^2 \cos 2x, x^3 \cos 2x, \sin 2x, x \sin 2x, x^2 \sin 2x, x^3 \sin 2x\}$

All derivatives of $x^3 \cos 2x$ are linear combinations of these eight members.

*Notice that this is consistent with our claim that y_p must include all linearly independent derivatives of f(x). In the case that f(x) = sin mx or cos mx, f and all its derivatives are linear combinations of sin mx and cos mx.

2.6.1(L) continued

6. If f(x) is a solution of the reduced equation then the corresponding y_p given in (2) is multiplied by the smallest power of x, say x^k, for which $x^k y_p$ is not a solution of the reduced equation. Our new trial solution is then $x^k y_p$.

For example, suppose we wanted to find a particular solution of

$$L(y) = y''' - 3y'' + 3y' - y = e^x.$$

In this case,

$$L(e^{rx}) = e^{rx}(r^3 - 3r^2 + 3r - 1)$$

from which we see that r = 1 is a 3-fold root of $L(e^{rx})$.

Hence, it is "foolish" to try linear combinations of e^x, xe^x, and x^2e^x since each of these functions is a solution of the reduced equation.

Thus, our trial solution in this illustration would be

$$y_p = Ax^3e^x.$$

Further drill is left to the remaining exercises.

2.6.2 (optional)

In the last exercise we showed how the method of undetermined coefficients worked in the case L(y) = f(x) where L had constant coefficients and f(x) had either of the two forms: $x^k e^{\alpha x}\cos\beta x$ or $x^k e^{\alpha x}\sin\beta x$.

We showed why it was necessary that f(x) and its various derivatives all be linear combinations of a finite subset of this set. In this exercise we want to show that f(x) must be restricted to the type discussed in the previous exercise if this is to happen.

2.6.2 continued

To prove our assertion, let us assume that that f(x) is any analytic function which together with all its derivatives forms a linearly dependent set. By definition of linearly dependent, this means that there exists a derivative of f, say the nth derivative, $f^{(n)}(x)$, which is a linear combination of the previous derivatives.

In other words, where exist constants $c_1, \ldots,$ and c_{n-1} such that

$$f^{(n)}(x) = c_{n-1}f^{(n-1)}(x) + \ldots + c_1 f'(x) + c_o f(x),$$

and if we transpose all terms onto the left side of this equations, we obtain

$$f^{(n)}(x) + a_{n-1}f^{(n-1)}(x) + \ldots + a_1 f'(x) + a_o f(x) = 0 \qquad (1)$$

where we have replaced $-c_k$ by a_k simply for the sake of convenient notation.

Looking at (1) we see that y = f(x) is a solution of the homogeneous linear differential equation with constant coefficients:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_o = 0. \qquad (2)$$

In the last Unit we showed that every solution of (2) was a linear combination of terms of the form $x^k e^{\alpha x} \cos \beta x$ and/or $x^k e^{\alpha x} \sin \beta x$. In other words, if we insist that the set {f, f",..., } be linearly dependent, then f(x) must have the form described in the previous exercise.

For example, if we refer to the equation

$$y'' + y = \sec x, \qquad (3)$$

mentioned at the end of the lecture, and try for a solution in the form

$$y_p = A \sec x \qquad (4)$$

2.6.2 continued

we obtain

$$y'_p = A \sec x \tan x \qquad (5)$$

and

$$y''_p = A \sec x(\sec^2 x) + A(\sec x \tan x) \tan x$$

$$= A \sec x(\sec^2 x + \tan^2 x). \qquad (6)$$

Letting $y = y_p$ in (3) then yields

$$A \sec x(\sec^2 x + \tan^2 x) + A \sec x = \sec x + 0 \sec x(\sec^2 x + \tan^2 x). \qquad (7)$$

Thus, by comparing coefficients in (7) we obtain A = 1 and A = 0* which is a contradiction since A is constant.

If we now try to adjust (4) by trying

$y_p = A \sec x + B(\sec x)'$
$(= A \sec x + B \sec x \tan x)$

we wind up in the same trouble because each time we differentiate, a new (linearly independent) derivative enters the picture. Thus, we keep imposing too many conditions of our given undetermined coefficients. This will always happen because we never get to the stage that any finite sum of the form

$$A_o \sec x + A_1(\text{xec } x)' + A_2(\sec x)'' + \ldots + A_n(\sec x)^{(n)}$$

characterizes all of the remaining derivatives of sec x.

This is not to say that we cannot find the general solution of y" + y = sec x but rather that this general solution

*We may only compare like coefficients when our functions are linearly independent. The fact that sec x is not a linear combination of terms of the family $x^k e^{\alpha x} \cos \beta x$ and $x^k e^{\alpha x} \sin \beta x$ means that {sec x,(sec x)', (xec x)"} is linearly independent. Thus, in particular {sec x, (sec x)"} is linearly independent. Thus, what (7) says is

$$A(\sec x)'' + A \sec x = 0 (\sec x)'' + 1 \sec x.$$

2.6.2 continued

cannot be found by the method of undetermined coefficients. In this example, we still have constant coefficients but the right side is of the wrong form.

2.6.3

Since $L(y) = y'' - 6y' + 9y$ in each part of this exercise, we have that the general solution of the reeuced equation will always be

$$y_h = (c_o + c_1x)e^{3x}. \qquad (1)$$

Thus, in each part of this exercise we must find one particular solution, y_p, of $L(y) = f(x)$ whereupon the general solution will be

$$y = y_p + y_k$$
$$= y_p + (c_o + c_1x)e^{3x}. \qquad (2)$$

a. Given

$$y'' - 6y' + 9y = e^{4x} \qquad (3)$$

our trial solution is $y_p = Ae^{4x}$. Hence

$y_p' = 4Ae^{4x}$ and $y_p'' = 16Ae^{4x}$.

We then obtain from (3)

$(16Ae^{4x}) - 6(4Ae^{4x}) + 9(Ae^{4x}) = e^{4x}$ or $19Ae^{4x} = 1e^{4x}$

so that $A = \frac{1}{19}$; whence $y_p = \frac{1}{19}e^{4x}$.

From (2) our general solution is given by

$$y = (c_o + c_1x)e^{3x} + \frac{1}{19}e^{4x}. \qquad (4)$$

2.6.3 continued

b. With

$$y'' - 6y' + 9y = \sin 3x, \tag{5}$$

our trial solution becomes

$$y_p = A \sin 3x + B \cos 3x. \tag{6}$$

Hence,

$$y'_p = 3A \cos 3x - 3B \sin 3x$$

and

$$y''_p = -9A \sin 3x - 9B \cos 3x.$$

Thus (5) now leads to

$$(-9A \sin 3x - 9B \cos 3x) - 6(3A \cos 3x - 3B \sin 3x) + 9(A \sin 3x + B \cos 3x) = \sin 3x$$

or

$$18B \sin 3x - 18A \cos 3x = 1 \sin 3x + 0 \cos 3x,$$

from which we conclude that $B = \frac{1}{18}$ and $A = 0$. Thus, from (6),

$$y_p = \frac{1}{18} \cos 3x.$$

Referring again to (2), our general solution is

$$y = (c_o + c_1 x)e^{3x} + \frac{1}{18} \cos 3x.$$

c. With

$$y'' - 6y' + 9y = xe^x \tag{7}$$

our trial solution must be of the form

2.6.3 continued

$$y_p = Axe^x + Be^x \qquad (8)$$

since all derivatives of xe^x are linear combinations of xe^x and e^x.

From (8) we obtain

$$y'_p = Ae^x + Axe^x + Be^x$$

$$= Axe^x + (A + B)e^x$$

and

$$y''_p = Ae^x + Axe^x + (A + B)e^x$$

$$= Axe^x + (2A + B)e^x.$$

Using this information in (7) yields

$$Axe^x + (2A + B)e^x - 6[Axe^x + (A + B)e^x] + 9(Axe^x + Be^x) = xe^x$$

or

$$4Axe^x + (-4A + 4B)e^x = 1\ xe^x + 0e^x.$$

Hence,

$$\left.\begin{aligned} 4A &= 1 \\ -4A + 4B &= 0 \end{aligned}\right\} \quad \text{or } A = B = \frac{1}{4}$$

Thus, (8) becomes

$$y_p = \frac{1}{4}\,xe^x + \frac{1}{4}\,e^x = \frac{1}{4}\,e^x(x + 1),$$

so that by (2) our general solution is

$$y = (c_1 + c_1x)e^{3x} + \frac{1}{4}\,e^x(x + 1)$$

d. Since $y'' - 6y' + 9y = 0$ is satisfied by both $y = e^{3x}$ and $y = xe^{3x}$, our trial solution for

2.6.3 continued

$$y'' - 6y' + 9y = e^{3x} \tag{9}$$

should be

$$y_p = Ax^2e^{3x}. \tag{10}$$

From (1)

$$y'_p = 3Ax^2e^{3x} + 2Axe^{3x}$$

and

$$y''_p = (9Ax^2e^{3x}) + (6Axe^{3x}) + (2Ae^{3x} + 6Axe^{3x})$$

$$= 9Ax^2e^{3x} + 12Axe^{3x} + 2Ae^{3x}.$$

Putting these results into (9) yields

$$\left.\begin{array}{l} 9Ax^2e^{3x} + 12Axe^{3x} + 2Ae^{3x} \\ -18Ax^2e^{3x} - \underset{0}{\underline{12Axe^{3x}}} \\ + \underset{0}{\underline{9Ax^2e^{3x}}} \end{array}\right\} = e^{3x}.$$

Hence,

$2Ae^{3x} = 1e^{3x}$, or $A = \frac{1}{2}$.

With $A = \frac{1}{2}$, (10) yields

$y_p = \frac{1}{2} x^2e^{3x}$;

so from (2) we obtain as our general solution

$$y = (c_o + c_1x)e^{3x} + \frac{1}{2} x^2e^{3x}.$$

2.6.4

From the previous exercise we have that

$$L(e^{4x}) = e^{4x}. \quad (1)$$

Hence,

$$3L(e^{4x}) = 3e^{4x}. \quad (2)$$

By linearity,

$$3L(e^{4x}) = L(3e^{4x});$$

hence, (2) becomes

$$L(3e^{4x}) = 3e^{4x}. \quad (3)$$

We also saw in the previous exercise that

$$L(\tfrac{1}{18} \cos 3x) = \sin 3x. \quad (4)$$

Combining (3) and (4) yields

$$L(3e^{4x}) + L(\tfrac{1}{18} \cos 3x) = 3e^{4x} + \sin 3x. \quad (5)$$

Again by linearity,

$$L(3e^{4x}) + L(\tfrac{1}{18} \cos 3x) = L(3e^{4x} + \tfrac{1}{18} \cos 3x),$$

so that (5) implies

$$L(3e^{4x} + \tfrac{1}{18} \cos 3x) = 3e^{4x} + \sin 3x.$$

In other words,

$$y = 3e^{4x} + \tfrac{1}{18} \cos 3x \quad (6)$$

is a particular solution of $L(y) = 3e^{4x} + \sin 3x$.

2.6.4 continued

Since $L(y) = y'' - 6y' + 9y$,

$$y_h = (c_o + c_1x)e^{3x}. \tag{7}$$

Hence, from (6) and (7) we have that the general solutions of

$$y'' - 6y' + 9y = 3e^{4x} + \sin 3x$$

is

$$y = (c_o + c_1x)e^{3x} + 3e^{4x} + \frac{1}{18} \cos 3x.$$

2.6.5

Since x^2e^x is not a particular solution of $y'' + 3y' + 2y = 0$, we may try as a solution of

$$y'' + 3y' + 2y = x^2e^x \tag{1}$$

the trial solution

$$y_p = Ax^2e^x + Bxe^x + Ce^x \tag{2}$$

[i.e., all derivatives of x^2e^x are linear combinations of x^2e^x, xe^x, and e^x].

From (2) we have

$$\begin{aligned} y'_p &= (Ax^2e^x + 2Axe^x) + (Be^x + Bxe^x) + Ce^x \\ &= Ax^2e^x + (2A + B)xe^x + (B + C)e^x \end{aligned}$$

and

$$\begin{aligned} y''_p &= 2Axe^x + Ax^2e^x + (2A + B)xe^x + (2A + B)e^x + (B + C)e^x \\ &= Ax^2e^x + (4A + B)xe^x + (2A + 2B + C)e^x. \end{aligned}$$

With this information, (1) becomes

2.6.5 continued

$$\left.\begin{array}{l} Ax^2e^x + (4A + B)xe^x + (2A + 2B + C)e^x \\ 3Ax^2e^x + (6A + 3B)xe^x + (3B + 3C)e^x \\ 2Ax^2e^x + 2Bxe^x + 2Ce^x \end{array}\right\} = x^2e^x$$

or

$$6Ax^2e^x + (10A + 6B)xe^x + (2A + 5B + 6C)e^x = 1x^2e^x + 0xe^x + 0\,e^x.$$

Hence,

$$\left.\begin{array}{l} 6A = 1 \\ 10A + 6B = 0 \\ 2A + 5B + 6C = 0 \end{array}\right\} \to \left\{\begin{array}{l} A = \frac{1}{6} \\ B = -\frac{5}{3}A \\ C = -\frac{1}{6}(2A + 5B) \end{array}\right. \to \left\{\begin{array}{l} A = \frac{1}{6} \\ B = -\frac{5}{18} \\ C = \frac{19}{108} \end{array}\right.$$

Therefore, one solution of (1) is

$$y = \frac{1}{6}x^2e^x - \frac{5}{18}xe^x + \frac{19}{108}e^x;$$

and since the general solution of $y'' + 3y' + 2y = 0$ is $y_h = c_1e^{-2x} + c_2e^{-x}$, the general solution of (1) is
$y = c_1e^{-2x} + c_2e^{-x} + \frac{1}{6}x^2e^x - \frac{5}{18}xe^x + \frac{19}{108}e^x$

2.6.6

Letting $L(y) = d^3y/dx^3 - dy/dx$, we see that $L(e^{rx}) = e^{rx}(r^3 - r)$; so that $L(e^{rx}) = 0 \leftrightarrow r = 0, 1, -1$.

Hence,

$$y_h = c_1e^{0x} + c_2e_1^{\,x} + c_3e^{-x}$$

$$= c_1 + c_2e^x + c_3e^{-x}. \qquad (1)$$

We now want to solve

$$L(y) = e^x \qquad (2)$$

2.6.6 continued

Tc find a particular solution of (2), we would ordinarily use $y_p = Ae^x$ but since e^x is a solution of the reduced equation (1), our trial solution should be

$$y_p = Axe^x. \tag{3}$$

This leads to

$$y_p' = Ae^x + Axe^x$$

$$y_p'' = 2Ae^x + Axe^x$$

$$y_p''' = 3Ae^x + Axe^x$$

so that

$$y''' - y' = e^x$$

implies

$$\left.\begin{array}{ll} 3Ae^x & + \; Axe^x \\ \underline{-Ae^x} & - \; \underline{Axe^x} \\ 2Ae^x & \quad\;\; 0 \end{array}\right\} = ie^x,$$

or $A = \frac{1}{2}$.

Hence, from (3)

$$y_p = \frac{1}{2}\,xe^x$$

is a solution of (2). Therefore, from (1) we conclude that the solution of the differential equation is

$$y = c_1 + c_2e^x + c_3e^{-x} + \frac{1}{2}\,xe^x. \tag{4}$$

From (4)

$$y' = c_2e^x - c_3e^{-x} + \frac{1}{2}\,e^x + \frac{1}{2}\,xe^x \tag{5}$$

2.6.6 continued

and

$$y'' = c_2e^x + c_3x^{-x} + \frac{1}{2}e^x + \frac{1}{2}xe^x + \frac{1}{2}e^x$$

$$= c_2e^x + c_3e^{-x} + e^x + \frac{1}{2}xe^x. \qquad (6)$$

Now we want the member of (4) such that when $x = 0$*, $y = 1$, $y' = 3/2$, and $y'' = 4$.

Putting this information into (4), (5), and (6) yields:

$$\left.\begin{aligned} 1 &= c_1 + c_2 + c_3 \\ \frac{3}{2} &= c_2 - c_3 + \frac{1}{2} \\ 4 &= c_2 + c_3 + 1 \end{aligned}\right\}$$

or

$$\left.\begin{aligned} c_1 + c_2 + c_3 &= 1 \\ c_2 - c_3 &= 1 \\ c_2 + c_3 &= 3 \end{aligned}\right\} \qquad (7)$$

From the bottom two equations in (7) we see at once that $c_2 = 2$ and $c_3 = 1$, hence from the top equation, $c_1 = -2$.

Putting these results into (4) yields

$$y = -2 + 2e^x + e^{-x} + \frac{1}{2}xe^x .$$

---

*$x = 0$ is chosen only to make the arithmetic easier. The crucial point is that since (4) is the general solution of (2), it means that one and only one curve satisfies (2) and passes through a given point (x_0, y_0) with slope y_0' and concavity y_0''; and this curve belongs to (4). Our present exercise is simply a specific application of this result.

2.6.7(L)

Here we simply want to emphasize that while most of our discussion centers about equations of order 2, the concepts apply to higher order equations as well. In particular, the equi-dimensional (or Euler-Cauchy) equation (which we mentioned in Exercise 2.4.10 in our discussion of using a change of variables to reduce linear equations with variable coefficients to linear equations with constant coefficients) has the general form

$$x^n \frac{d^n y}{dx^n} + a_{n-1}x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1 x \frac{dy}{dx} + a_o y = f(x) \qquad (1)$$

where $a_o, \ldots, a_{n-1}$ are constants.

Just as in the case n = 2, the substitution z = ln x (or x = e^z) converts (1) into an equivalent equation with constant coefficients in which y is expressed as a function of z. In our particular case, with

$$x^3 y''' + xy' - y = 0 \qquad (x > 0)^* \qquad (2)$$

we let

$$z = \ln x \text{ or } x = e^z. \qquad (3)$$

Had we allowed x to be negative, then (3) would not be real. In the event x < 0, (3) would be replaced by z = ln (-x).

Using (3) we obtain

$$y' = \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}. \qquad (4)$$

*We still want to emphasize that our theorems concerning the existence of a general solution were all stated in terms of the leading coefficient being 1. To put (2) in this form requires that we divide by x^3, and this means that $x \neq 0$. Our condition that x > 0 simply is meant to capture the flavor that we usually solve equations in a connected region. Thus, since x > 0 and x < 0 are both connected regions, we usually handle $x \neq 0$ as a union of these two cases.

2.6.7(L) continued

Since we want to eliminate x in our new equation, we might prefer to replace $1/x$ by e^{-z} [as seen from (3)] and write $y' = e^z\, dy/dz$. This is certainly permissible, but we prefer to use (4) in the sense that when we replace y' by (4) in equation (2), the 1/x will be cancelled by the coefficient of y'. This type of cancellation occurs because of the form of the equi-dimensional equation wherein the coefficient of the m^{th} derivative is a constant multiple of x^m.

From (4) we have

$$y'' = \left(\frac{1}{x}\frac{dy}{dz}\right)'$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\left(\frac{d^2y}{dz^2}\right)\frac{dz}{dx}*$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2} = \frac{1}{x^2}\left[\frac{d^2y}{dz^2} - \frac{dy}{dz}\right]**; \qquad (5)$$

and from (5) we have that

$$y''' = \left[\frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)\right]'$$

or

---

*Recall that we are differentiating with respect to x and thus must use the chain rule. That is $d/dx(dy/dz) = d(dy/dz)/dz \cdot dz/dx = d^2y/dz^2 \cdot dz/dx$.

**Again, notice that the coefficient of y" in (1) is a constant multiple of x^2 so that when y" is replaced in (1) by its value in (5), only the constant will remain as a coefficient. By coincidence the constant multiple of x^2 in (2) happens to be 0 so that (5) is not needed there, but (5) is still necessary if we are to compute y"'.

2.6.7(L) continued

$$y''' = -\frac{2}{x^3}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) + \frac{1}{x^2}\,\frac{d\left[\frac{d^2y}{dz^2} - \frac{dy}{dz}\right]}{dz}\,\frac{dz}{dx}$$

$$= -\frac{2}{x^3}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) + \frac{1}{x^2}\left(\frac{d^3y}{dz^3} - \frac{d^2y}{dz^2}\right)\frac{1}{x}$$

$$= \frac{1}{x^3}\left(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right). \qquad (6)$$

If we now rewrite (2) in light of (4), (5), and (6), we obtain,

$$x^3\left[\frac{1}{x^3}\left(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right)\right] + x\left[\frac{1}{x}\frac{dy}{dz}\right] - y = 0,$$

or

$$\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 3\frac{dy}{dz} - y = 0. \qquad (7)$$

b. Equation (7) has constant coefficients and hence may be solved in the usual way. Namely, letting $L(y)$ denote the left side of (7) we have

$$L(e^{rz})^* = e^{rz}(r^3 - 3r^2 + 3r - 1) = e^{rz}(r-1)^3.$$

Thus, $r = 1$ is a triple-root of $L(e^{rz}) = 0$.

Accordingly, the general solution of (7) is

$$y = (c_o + c_1 z + c_2 z^2)e^z. \qquad (8)$$

Recalling now that (7) is the result of making the substitution (3) into equation (2); we may now replace z by $\ln x$ in (8), and the resulting equation is the general solution of (2). That is, the general solution of (2) is:

$$y = [c_o + c_1 \ln x + c_2(\ln x)^2]e^{\ln x};$$

*The technique involves the substitution $y = e^{ru}$ where y denotes the dependent variable and u the independent variable. In (7) it is z, not x, which is the independent variable.

2.6.7(L) continued

or, since $e^{\ln x} = x$,

$$y = [c_o + c_1 \ln x + c_2(\ln x)^2]x$$

$$= c_o x + c_1 x \ln x + c_2 x (\ln x)^2. \qquad (9)$$

c. We now want to emphasize that our present technique of undetermined coefficients can only be applied to equations with constant coefficients. Thus, had we not first done part (a) and we were confronted with the equation

$$x^3 y''' + xy' - y = x \ln x \ (x > 0), \qquad (10)$$

the method of undetermined coefficients could not be used here. Namely, once the coefficients are variable, the restriction that the right side be of the form $x^k e^{\alpha x} \cos \beta x$, etc. no longer applies.

This does not mean that equations like (10) cannot be solved, but rather unless we can find a way of reducing (10) to a linear equation with constant coefficients [and this is the role played by part (a) in the solution of part (b)], we must find a different method for solving (1). This more general technique is the subject of the next Unit, but we may end this Unit on the proper note of applying the method of undetermined coefficients.

Namely, using the substitution $z = \ln x (x = e^z)$, the right side of (1) becomes ze^z; and from part (a), the left side of (10) becomes

$$\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 3\frac{dy}{dz} - y.$$

Hence, to solve (10) it is sufficient to solve

$$\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 3\frac{dy}{dz} - y = ze^z \qquad (10')$$

and then replace z by ln x in the solution.

2.6.7(L) continued

The point is that (10') does have constant coefficients and the right side has the "proper" form. Thus, since ze^z is associated with the family $\{e^z, ze^z\}$ we would ordinarily try for a particular solution of (10') in the form

$$y_p = Aze^z + Be^z \tag{11}$$

but since e^z, ze^z, and z^2e^z are all solutions of the reduced equation [which is, you will notice, equation (7) of part (a); and whose solution is equation (8)], we must "scale" y_p, as given in (11), by a factor of z^3 before we can conclude that no part of our trial solution is a solution of the reduced equation. In other words, our trial solution should be z^3y_p, where y_p is as in (11).

We therefore apply the method of undetermined coefficients to

$$y = Az^4e^z + Bz^3e^z \tag{12}$$

to obtain

$$\begin{aligned} y' &= Az^4e^z + 4Az^3e^z + Bz^3e^z + 3Bz^2e^z \\ &= Az^4e^z + (4A + B)z^3e^z + 3Bz^2e^z \end{aligned} \tag{13}$$

$$\begin{aligned} y'' = Az^4e^z &+ 4Az^3e^z \\ &+ (4A + B)z^3e^z + 3(4A + B)z^2e^z \\ &+ 3Bz^2e^z + 6Bze^z \end{aligned}$$

or

$$y'' = Az^4e^z + (8A + B)z^3e^z + (12A + 6B)z^2e^z + 6Bze^z \tag{14}$$

and

$$\begin{aligned} y''' = Az^4e^z + 4Az^3e^z &+ (8A + B)z^3e^z + 3(8A + B)z^2e^z \\ &+ (12A + 6B)z^2e^z + 2(12A + 6B)ze^z \end{aligned}$$

2.6.7(L) continued

$+ 6Bze^z + 6Be^z$

or

$$y''' = Az^4e^z + (12A + B)z^3e^z + (36A + 9B)z^2e^z + (24A + 18B)ze^z + 6Be^z. \quad (15)$$

Using (12), (13), (14) and (15) in (10') we obtain

$$\left.\begin{matrix} Az^4 + (12A + B)z^3e^z + (36A + 9B)z^2e^z + (24A + 18B)ze^z + 6Be^z \\ -3Az^4 - 3(8A + B)z^3e^z - 3(12A + 6B)z^2e^z - \underline{18Bze^z} \\ +3Az^4 + 3(4A + B)z^3e^z + \underset{0}{\underline{9Bz^2e^z}} \qquad 24Aze^z \\ -\underset{0}{\underline{Az^4e^z}} - \underset{0}{\underline{Bz^3e^z}} \end{matrix}\right\} = ze^z$$

or:

$$24Aze^z + 6Be^z = ze^z = 1ze^z + 0e^z,$$

so be comparing coefficients of like terms

$24A = 1$ and $B = 0$*,

so that from (12) our particular solution is

$$y = \frac{1}{24} z^4e^z. \quad (16)$$

Now if we combine (16) with (8) we obtain that the general solution of (10') is

$$y = (c_o + c_1z + c_2z^2)e^z + \frac{1}{24} z^4e^z. \quad (17)$$

*This tells us that $y = Az^4e^z$ was sufficient for a trial solution. Notice, however, as we've said before, including the term Bz^3e^z did nothing to prevent us from finding a solution, except that we had to perform a little more computation. In the next unit we shall obtain this particular solution by a different method.

2.6.7(L) continued

Now replacing z by ln x in (17) yields

$$y = (c_o + c_1 \ln x + c_2 \ln^2 x)e^{\ln x} + \frac{1}{24} \ln^4 x \; e^{\ln x}$$

$$= c_o x + c_1 x \ln x + c_2 x (\ln x)^2 + \frac{1}{24} x (\ln x)^4$$

is the general solution of (10).

Resource: Calculus Revisited: Complex Variables, Differential Equations, and Linear Algebra
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