Unit 4: Linear Differential Equations

2.4.1(L)

Notice that the notation

$$L(y) = x^2y'' - 3xy' + 3y \qquad (1)$$

implies that the "input" of the "L-machine" must be a twice-differentiable function of x [otherwise the right side of (1) is not defined]. That is, L is a function whose domain is the set of all twice-differentiable functions of x and whose range is the set of functions of x. In other words, L maps any twice-differentiable function of x into some function of x [but this mapping need not be onto, i.e., there may be functions g(x) such that $L(y) \neq g(x)$ for all twice-differentiable functions y*].

Our main aim in this exercise is to supply you with sufficient drill so that you feel at ease with the new notation. Part (a) requires that we compute L(y), or L[f(x)], for some specific twice-differentiable functions y = f(x). Part (b) asks us to verify certain "linear" properties of L.

a. As usual, in our treatment of functions, we want to view the "input" independently of any specific "label." Thus, we view (1) as

$$\left.\begin{array}{l} L([\ ]) = x^2[\ ]'' - 3x[\ ]' + 3[\ ] \\ \\ \text{or} \\ \\ L([\ ]) = x^2 \dfrac{d^2[\ ]}{dx^2} - 3x\dfrac{d[\ ]}{dx} + 3[\ ] \end{array}\right\} \qquad (2)$$

*This may, of course, be restated without reference to y, using only function notation. That is, (1) is equivalent to writing

$$L[f(x)] = x^2f''(x) - 3x\,f'(x) + 3\,f(x) \qquad (1')$$

where f is any twice-differentiable function of x. Ultimately, the problem of solving linear equations reduces to finding f(x) such that L[f(x)] = g(x) for a given function g(x). We shall say more about this shortly.

2.4.1(L) continued

where [ ] may be replaced on both sides of (2) by any twice-differentiable function of x.*

At any rate, from (2), we have

(i) $L[\sin x] = x^2[\sin x]'' - 3x[\sin x]' + 3[\sin x]$

or

$$L(\sin x) = x^2(-\sin x) - 3x(\cos x) + 3\sin x$$

$$= (3 - x^2)\sin x - 3x\cos x. \qquad (3)$$

[Notice that sin x is a twice-differentiable function of x and that the right side of (3) is a function of x. Reversing the steps involved in arriving at (3), what we have shown is that since $L(\sin x) = (3 - x^2)\sin x - 3x\cos x$, then $y = \sin x$ is <u>one</u> solution of the linear differential equation

$$x^2y'' - 3xy' + 3y = (3 - x^2)\sin x - 3x\cos x. \qquad (4)$$

The (much) more difficult problem is to start with (4) and to try to deduce (3). This is the problem with which we must ultimately become concerned, but for now we are satisfied just to drill on the notation.]

(ii) $L(e^x) = x^2(e^x)'' - 3x(e^x)' + 3(e^x)$

$$= x^2e^x - 3xe^x + 3e^x$$

$$= e^x(x^2 - 3x + 3). \qquad (5)$$

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*For example, if $y = x^{\frac{3}{2}}$, y" does not exist at x = 0; hence $L\left(x^{\frac{3}{2}}\right)$ is undefined at x = 0. Notice however, at least in this case, if we restrict x to an interval R which does not include 0, $L\left(x^{\frac{3}{2}}\right)$ is defined on R, since $x^{\frac{3}{2}}$ is twice-differentiable whenever $x \neq 0$.

2.4.1(L) continued

(iii) $L(x) = x^2(x)'' - 3x(x)' + 3(x)$

$= x^2(0) - 3x(1) + 3x,$

or

$$L(x) = 0 - 3x + 3x \equiv 0. \tag{6}$$

[In other words, $y = x$ is a solution of $x^2y'' - 3xy' + 3y = 0$.]

(iv) $L(x^2) = x^2(x^2)'' - 3x(x^2)' + 3(x^2)$

$= x^2(2) - 3x(2x) + 3x^2$

$$= -x^2. \tag{7}$$

(v) $L(x^3) = x^2(x^3)'' - 3x(x^3)' + 3(x^3)$

$= x^2(6x) - 3x(3x^2) + 3x^3$

$$\equiv 0. \tag{8}$$

[In other words, $y = x^3$ is also a solution of

$x^2y'' - 3xy' + 3y = 0$.]

b. (vi) We could solve (vi) just as we did in part (a). Namely,

$$L(e^x + \sin x) = x^2(e^x + \sin x)'' - 3x(e^x + \sin x)' + + 3(e^x + \sin x) \tag{9}$$

$$= x^2(e^x - \sin x) - 3x(e^x + \cos x) + + 3(e^x + \sin x) \tag{10}$$

(Notice the key step in going from (9) to (10). Namely, we used the fact that the derivative of a sum is the sum of the derivatives. More symbolically, $[f(x) + g(x)]' = f'(x) + g'(x)$;

2.4.1(L) continued

$[f(x) + g(x)]'' = [f'(x) + g'(x)]' = f''(x) + g''(x)$, etc. In this particular exercise, $f(x) = e^x$ and $g(x) = \sin x$.)

Hence

$$L(e^x + \sin x) = x^2e^x - x^2\sin x - 3xe^x - 3x\cos x + 3e^x + 3\sin x$$

$$= e^x(x^2 - 3x + 3) + (3 - x^2)\sin x - 3x\cos x. \quad (11)$$

The crucial point is to observe, however, that the right side of (11) is the sum of the right sides of (3) and (5). That is, we have verified by direct computation in this exercise that

$$L(e^x + \sin x) = L(e^x) + L(\sin x). \quad (12)$$

Equation (12) is a special case of the result derived in our lecture that if $L(y) = y'' + p(x)y' + q(x)y$* then for any twice-differentiable functions $u(x)$ and $v(x)$,

$$L(u + v) = L(u + v).$$

In this exercise, we have rederived the result of the lecture only more concretely for a specific choice of L, u, and v.

(vii) Again we could write

$$L(6\sin x) = x^2(6\sin x)'' - 3x(6\sin x)' + 3(6\sin x)$$

$$= x^2(-6\sin x) - 3x(6\cos x) + 18\sin x$$

$$= 6[(3 - x^2)\sin x - 3x\cos x],$$

or, again by (3),

*In the lecture, we wrote the standard (2nd order) linear equation in the form $y'' + p(x)y' + q(x)y = f(x)$. It could just as easily have been $r(x)y'' + p(x)y' + q(x)y = f(x)$, but as long as $r(x) \neq 0$, we can divide by $r(x)$ to obtain $y'' + p_1(x)y' + q_1(x)y = f_1(x)$, where $p_1(x) = \frac{p(x)}{r(x)}$, $q_1(x) = \frac{q(x)}{r(x)}$, and $f_1(x) = \frac{f(x)}{r(x)}$.

2.4.1(L) continued

$$L(6 \sin x) = 6L(\sin x). \tag{13}$$

Equation (13) verifies the result of our lecture that

$$L(cu) = cL(u)$$

at least for the $c = 6$, $u = \sin x$, and $L(u) = x^2u'' - 3xu' + 3x$. Moreover, the general, abstract proof is exactly the same as our procedure in deriving (13).

(viii) The proof given in the lecture tells us that

$$L(7e^x + 6 \sin x) = L(7e^x) + L(6 \sin x)$$

$$= 7L(e^x) + 6L(\sin x),$$

so again by (3) and (5),

$$L(7e^x + 6\sin x) = 7e^x(x^2 - 3x + 3) + 6[(3 - x^2)\sin x -$$

$$- 3x \cos x]. \tag{14}$$

That (14) agrees with the result of using (2) to compute $L(7e^x + 6 \sin x)$ is left to the interested(?) student.

(ix) The property that for a linear L,

$$L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2) \tag{15}$$

hinged on the facts that $(u_1 + u_2)' = u_1' + u_2'$ and $(cu)' = cu'$ provided c is constant. If c is not constant, $(cu)'$ is not cu' but, rather by the product rule, $(cu)' = cu' + \underline{c'u}$.

Hence, the luxury of using (15) requires that c_1 and c_2 be constants. The aim of this part of the exercise is to show that when c_1 or c_2 is not constant, then (15) need not apply. In particular, if $y = xe^x$, then $y' = xe^x + e^x$ and $y'' = xe^x + 2e^x$, so we obtain from (2) that

2.4.1(L) continued

$$L(xe^x) = x^2(xe^x)'' - 3x(xe^x)' + 3(xe^x)$$

$$= x^2(xe^x + 2e^x) - 3x(xe^x + e^x) + 3(xe^x)$$

$$= (x^3 - x^2)e^x. \tag{16}$$

On the other hand, from (5),

$$xL(e^x) = xe^x(x^2 - 3x + 3)$$

$$= e^x(x^3 - 3x^2 + 3x). \tag{17}$$

Comparing (16) and (17) shows that

$$L(xe^x) \neq xL(e^x).$$

That is, if c is not constant (in the present example, c = x) then $L(ce^x)$ need not equal $cL(e^x)$. [The "sophisticated" way of memorizing this is to say that the product rule for differentiation is not linear.]

2.4.2(L)

Here our aim is to stress the fact that the nice properties that $L(u_1 + u_2) = L(u_1) + L(u_2)$ and $L(cu) = cL(u)$ depend on $L(y)$ denoting an expression of the form

$$p_n(x)y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \ldots + p_1(x)y' + p_o(x)y,$$

and that once the coefficients of y, y', ..., and $y^{(n)}$ are not all functions of x alone, these properties need not hold.

Here we have

$$L(y) = y'' - 3yy' + 3y. \tag{1}$$

[Note that the coefficient of y' is not a function of only x. In fact, it is a function of y alone, but this is not what's really important. What's really important is that the coefficient of y'

2.4.2(L) continued

is not a function of x alone; consequently, L(y) as defined in (1) is not linear.]

Letting $y = x^3$ in (1) we obtain

$$L(x^3) = (x^3)'' - 3(x^3)(x^3)' + 3(x^3)$$

$$= 6x - 3(x^3)(3x^2) + 3x^3$$

$$= 6x + 3x^3 - 9x^5 = 3x(2 + x^2 - 3x^4). \qquad (2)$$

Letting $y = 2x^3$ in (1) yields

$$L(2x^3) = (2x^3)'' - 3(2x^3)(2x^3)' + 3(2x^3)$$

$$= 12x - 6x^3(6x^2) + 6x^3$$

$$= 6x(2 + x^2 - 6x^4). \qquad (3)$$

From (2)

$$2L(x^3) = 6x(2 + x^2 - 3x^4), \qquad (4)$$

and comparing (3) and (4) shows that

$$L(2x^3) \neq 2L(x^3)^*. \qquad (5)$$

Moreover, since $2x^3 = x^3 + x^3$ and $2L(x^3) = L(x^3) + L(x^3)$, we may rewrite (5) as

$$L(x^3 + x^3) \neq L(x^3) + L(x^3). \qquad (6)$$

Equations (5) and (6) show us that with L defined in this exercise it need not be true that

*Do not be misled by the fact that (3) and (4) look "a lot alike." To simplify the various computations, we chose u rather conveniently to be x^3. Had u (or L) been more involved, the difference between L(cu) and cL(u) could have been much more drastic. What is important is that L(cu) need not equal cL(u).

2.4.2(L) continued

$L(cu) = cL(u)$

and

$L(u_1 + u_2) = L(u_1) + L(u_2)$.

For this reason, we reserve the notation L to mean

$$L(y) = p_n(x)\frac{d^n y}{dx^n} + \ldots + p_1(x)\frac{dy}{dx} + p_o(x)y$$

and in this way, whenever we see

$L(c_1u_1 + c_2u_2)$

where c_1 and c_2 are constants and u_1 and u_2 are n-times differential functions of x (in the lecture we viewed the special case n = 2 for computational brevity, but the results hold for any order n) we may replace it by

$c_1L(u_1) + c_2L(u_2)$.

As a final note on the importance of linearity, notice how closely L(y) in this exercise resembles L(y) of the previous exercise (the only difference is that we replaced x by y in the coefficient of y'). Yet this rather "small" change was enough to "nullify" the linearity of L(y).

2.4.3(L)

Our aim here is to emphasize what is meant by the general solution of a 2nd order (linear) differential equation.

We want to find all solutions of

$$x^2y'' - 3xy' + 3y = 0 \quad (x \neq 0). \tag{1}$$

2.4.3(L) continued

From parts (iii) and (v) of Exercise 2.4.1, we saw that $y = x$ and $y = x^3$ were two solutions of (1).* That is, letting

$$L(y) = x^2y'' - 3xy' + 3y$$

we know that

$$L(x) = 0 \quad \text{and} \quad L(x^3) = 0. \tag{2}$$

Therefore, since $L(c_1x) = c_1L(x)$ and $L(c_2x^3) = c_2L(x^3)$, we see from (2) that if c_1 and c_2 are arbitrary constants

$$L(c_1x) = 0 \quad \text{and} \quad L(c_2x^3) = 0. \tag{3}$$

Then since $L(u + v) = L(u) + L(v)$, we obtain from (3) that

$$L(c_1x + c_2x^3) = 0.^{**} \tag{4}$$

To give (4) more concreteness, we shall actually test whether $y = c_1x + c_2x^3$ satisfies (1) [which is the "translation" of (4)]. From

$$y = c_1x + c_2x^3 \tag{5}$$

we obtain

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*In a manner of speaking, we have "cheated" by starting with an equation, some of whose solutions we found (by being given them) in a previous exercise. This is fine since all we want to do here is discuss what is meant by the general solution of an equation without worrying about the techniques for finding particular solutions. As mentioned earlier, we shall see later how one could have deduced that $y = x$ and $y = x^3$ were solutions of (1) directly from (1).

**It is <u>crucial</u> to stress that deriving (4) from (3) required linearity, not simply "equals added to equals are equal." That is, even if L had not been linear, by equals added to equals, we could conclude that $L(c_1x) + L(c_2x^3) = 0$. It is linearity which allows us to replace $L(c_1x) + L(c_2x^3)$ by $L(c_1x + c_2x^3)$ to obtain (4).

2.4.3(L) continued

$$y' = c_1 + 3c_2x^2 \tag{6}$$

and

$$y'' = 6c_2x. \tag{7}$$

From (5), (6), and (7), we have

$$\begin{aligned} x^2y'' - 3xy' + 3y &= x^2(6c_2x) - 3x(c_1 + 3c_2x^2) + 3(c_1x + c_2x^3) \\ &= 6c_2x^3 - 3c_1x - 9c_2x^3 + 3c_1x + 3c_2x^3 \\ &\equiv 0 \end{aligned} \tag{8}$$

so that from (8) we see that (5) satisfies (1).

Now that we know that every member of the 2-parameter family (5), i.e., $y = c_1x + c_2x^3$, is a solution of (1), we now want to see whether for a given point (x_0,y_0) and a given number y_0' we can choose c_1 and c_2 in one and only one way so that a member of (5) passes through (x_0,y_0) with slope y_0'.

Well from (5) and (6), letting $x = x_0$, $y = y_0$, and $y' = y_0'$, we have

$$\left.\begin{aligned} y_0 &= c_1x_0 + c_2x_0^3 \\ y_0' &= c_1 + 3c_2x_0^2 \end{aligned}\right\}. \tag{9}$$

Notice that in (9) we have two <u>linear</u> <u>algebraic</u> in two unknowns where c_1 and c_2 are the "<u>unknowns</u>," since x_0, y_0, and y_0' are given constants. Thus, from our previous theory of linear algebraic equations, we see that equation (9) has one and only one solution for c_1 and c_2 provided the determinant of coefficients

$$\begin{vmatrix} x_0 & x_0^3 \\ 1 & 3x_0^2 \end{vmatrix} \neq 0,$$

2.4.3(L) continued

or

$$3x_o^3 - x_o^3 \neq 0. \qquad (10)$$

Now (10) will be satisfied unless $x_o = 0$.*

What we have now shown is that there is one and only one member of (5) which passes through a given point (x_o,y_o) with slope y_o' and which satisfies (1).

The next question is, can there is other solutions of (1) which aren't members of (5)? Here we invoke the "Crucial Theorem" of the lecture by noting that as long as $x \neq 0$,** we may rewrite (1) as

$$y'' = \frac{3y'}{x} - \frac{3y}{x^2}. \qquad (11)$$

Letting $F(x,y,z) = \frac{3z}{x} - \frac{3y}{x^2}$ [so that (11) has the form $y'' = F(x,y,y')$], we see that $F_y = -\frac{3}{x^2}$ and $F_z = \frac{3}{x}$. Hence, as long as $x \neq 0$, F, F_y, and F_z exist and are continuous. Therefore, the "crucial theorem" tells us that $x^2y'' - 3xy' + 3y = 0$ has one and only one solution through a given point $(x_o,y_o)\varepsilon R$ with slope y_o' as long as $x \neq 0$ is excluded from R.

If $x = 0$ then $x^2y'' - 3xy' + 3y = 0$ reduces to $y = 0$. Hence, if $x = 0$, the equation $x^2y'' - 3xy' + 3y = 0$ is satisfied at the single point (0,0). For obvious reasons, we demand that a solution exist in a neighborhood of a point. More strongly, if a single point (or a union of isolated points) satisfy a differential equation, we do not call this set of points a solution of the equation. Namely, the very concept of a derivative at a given point requires that we know what is happening "just before" and "just after" the point. In other words, the concept of derivative requires that we know what is happening in a non-zero neighborhood of a point (no matter how small this neighborhood is).

*This was partially the hindsight that had us define (1) in a (connected) region R which excluded $x = 0$. The other part of our hindsight will be shown in a moment.

**This is the other fact that motivated us to write $x \neq 0$ in (1).

2.4.3(L) continued

At any rate, we may summarize this exercise by saying that in any region R which excludes $x = 0$, there is one and only one curve which passes through $(x_0,y_0) \varepsilon R$ with slope y_0' and this curve is a member of (5).

If $x_0 = 0$, then $x^2y'' - 3xy' + 3y = 0$ cannot be satisfied at any point (x_0,y_0) unless y_0 also equals 0 (since this is the only way that $3y = 0$), and if $y_0 = 0$, the "solution" consists of the single point (0,0) which is not called a solution of the equation.

As a final point, notice that we have shown that the given equation has no <u>singular</u> solutions. In other words, one and only one curve can pass through (x_0,y_0) with slope y_0' and satisfy the given equation. Since the family defined by (5) supplies us with one such curve, there can be no others. This is an important property of linear differential equations. Namely, they possess no singular solutions.

2.4.4

Since

$$y = c_1x + c_2x^3 \qquad (1)$$

implies

$$y' = c_1 + 3c_2x^2 \qquad (2)$$

we see that letting $x = 1$, $y = 4$, and $y' = 2$ in (1) and (2)* yields

$$\left.\begin{array}{l} 4 = c_1 + c_2 \\ 2 = c_1 + 3c_2 \end{array}\right\} \qquad (3)$$

or

*Notice here that we are repeating our theoretical discussion of equations (9) and (10) of the previous exercise, but only in terms of a concrete illustration.

2.4.4 continued

$$\left.\begin{aligned} 12 &= 3c_1 + 3c_2 \\ 2 &= c_1 + 3c_2 \end{aligned}\right\}$$

Hence,

$$c_1 = 5$$

and

$$c_2 = -1.$$

Therefore,

$$y = 5x - x^3$$

is the only curve which passes through (1,4) with slope 2 and satisfies $x^2y'' - 3xy' + 3y = 0$.

2.4.5(L)

Here we investigate the difference between arbitrary constants and "independent" constants in a fairly trivial way. (More sophisticated remarks are reserved for the Supplementary Notes.)

Certainly since $x^2y'' - 3xy' + 3y = 0$ is linear and $y = x$ is a solution, so also are $y = c_1x$ and $y = c_2x$ where c_1 and c_2 are arbitrary constants.

Hence, the family

$$y = c_1x + c_2x \qquad (1)$$

is a solution of

$$x^2y'' - 3xy' + 3y = 0 \qquad (2)$$

and (1) contains two arbitrary constants.

2.4.5(L) continued

The trouble with (1) is that the two arbitrary constants are "dependent" in the sense that they behave like one constant. That is, we may rewrite (1) as

$$y = cx \quad \text{where} \quad c = c_1 + c_2. \tag{3}$$

The "bad thing" about (3) is that since we have only one "free" constant at our disposal we "use it up" by either specifying the value of y when $x = x_o$ <u>or</u> the value of y' when $x = x_o$. In general, we will arrive at a contradiction if we wish to specify both y and y' at $x = x_o$.

For example, in this particularly simple contrived exercise, we see from (3) that

$$y' = c \tag{4}$$

so that given the value of y' at $x = x_o$ (in fact, in this simple example, equation (4) shows us that y' can't even vary with x) is the value that c must have.

But once $c = y_o'$, equation (3) becomes

$$y = y_o'x \tag{5}$$

and we see from (5) that y is no longer arbitrary once $x = x_o$ is specified.

For example, suppose we want to find a member of (1) which passes through (1,4) with slope equal to 2.

From (1) we have

$$y' = c_1 + c_2, \tag{6}$$

so letting x = 1, y = 4, and y' = 2 in (1) and (6), we obtain

$$\left.\begin{aligned} 4 &= c_1 + c_2 \\ 2 &= c_1 + c_2 \end{aligned}\right\} \tag{7}$$

2.4.5(L) continued

and the equations in (7) are incompatible.

What happened here was that the family (1) was a "pseudo - 2 parameter" family in the sense that it was identical to the 1-parameter family given by (3).

With $y = cx$, $y' = c$ so that $y' = 2 \leftrightarrow c = 2$, so that the only member of (3) which is <u>eligible</u> to be a solution of (2) is

$$y = 2x$$

but this fails on the ground that when $x = 1$, $y = 2$ rather than 4.

Since the only eligible member of (3) has been eliminated, there is no member of (3) which is a solution to the given problem.

Again further details are left for the supplementary notes, but for now notice that the number of initial conditions that can be specified at $x = x_0$ (i.e., the values of y', y'', etc. at $x = x_0$) depend <u>not</u> on the number of <u>arbitrary</u> constants but rather on the number of <u>independent</u> arbitrary constants.

2.4.6

Using the technique of the lecture, we let

$$y = e^{rx} \qquad (1)$$

so that $y' = re^{rx}$ (2)

and

$$y'' = r^2e^{rx}. \qquad (3)$$

Substituting (1), (2), and (3) into

$$y'' + 7y' - 8y = 0 \qquad (4)$$

we obtain

2.4.6 continued

$$r^2e^{rx} + 7re^{rx} - 8e^{rx} = 0$$

or

$$e^{rx}(r^2 + 7r - 8) = 0. \tag{5}$$

Since e^{rx} is never 0, equation (5) implies that

$$r^2 + 7r - 8 = 0$$

or

$$(r + 8)(r - 1) = 0. \tag{6}$$

From (6), we see that the only solutions of (4) which have the form of (1) are

$$y = e^{-8x} \quad \text{and} \quad y = e^x. \tag{7}$$

Note that (7) has given us the eligible members of (1) which may satisfy (4). It still remains to be seen that $y = e^{-8x}$ and $y = e^x$ are solutions of (4), but the verification is trivial. Namely,

(i) $y = e^{-8x} \rightarrow y' = -8e^{-8x} \rightarrow y'' = 64e^{-8x}$

Hence

$$y'' + 7y' - 8y = 64e^{-8x} - 56e^{-8x} - 8e^{-8x} \equiv 0$$

(ii) $y = e^x \rightarrow y' = y'' = e^x$

Hence

$$y'' + 7y' - 8y = e^x + 7e^x - 8e^x \equiv 0.$$

Referring to the L(y)-notation, we may let $L(y) = y'' + 7y' - 8y$ in which case (4) becomes

$$L(y) = 0. \tag{8}$$

2.4.6 continued

Equation (7) says that $L(e^{-8x}) = 0$ and $L(e^x) = 0$. Hence, by linearity

$$L(c_1e^{-8x} + c_2e^x) = 0 \tag{9}$$

[i.e.,

$$\begin{aligned} L(c_1e^{-8x} + c_2e^x) &= L(c_1e^{-8x}) + L(c_2e^x) \\ &= c_1L(e^{-8x}) + c_2L(e^x) \\ &= c_1 0 + c_2 0 \\ &= 0.] \end{aligned}$$

From (9), we see that the family

$$y = c_1e^{-8x} + c_2e^x \tag{10}$$

is a solution of (4).

Moreover, from (10)

$$y' = -8c_1e^{-8x} + c_2e^x. \tag{11}$$

Hence, if we require that (10) and (11) be satisfied with $y = y_o$ and $y' = y_o'$* when $x = x_o$ we obtain the system

*Perhaps this should have been mentioned earlier but y_o' is simply a constant which denotes the value of y' when $x = x_o$. It should not be construed to mean the derivative of y_o since the derivative of a constant is always zero. In other words, we use y_o', y_o'' etc. suggestively to denote the values of y', y'', etc., at the initial condition $x = x_o$.

2.4.6 continued

$$\left.\begin{aligned} y_o &= e^{-8x_o}c_1 + e^{x_o}c_2 \\ y_o' &= -8e^{-8x_o}c_1 + e^{x_o}c_2 \end{aligned}\right\} \quad (12)$$

for which the determinant of coefficients is

$$\begin{vmatrix} e^{-8x_o} & e^{x_o} \\ -8e^{-8x_o} & e^{x_o} \end{vmatrix} = e^{-7x_o} - (-8e^{-7x_o}) = 9e^{-7x_o}$$

which cannot be zero for any value of x_o.

Hence, we see from (12) that there is one and only one way of determining c_1 and c_2 so that $y = c_1e^{-8x} + c_2e^x$ satisfies (4) and passes through (x_o,y_o) with slope y_o'.

To show that no other curve can pass through (x_o,y_o) with slope y_o', we invoke the "crucial theorem" of the lecture by rewriting (4) as

$$y'' = 8y - 7y' = F(x,y,y')$$

where $F(x,y,z) = 8y - 7z$. Hence, $F_y = 8$, $F_z = -7$ and we see that the equation has a unique solution [since F, F_y, and F_z exist and are continuous always].

2.4.7

Given

$$y'' - 8y' + 15y = 0, \quad (1)$$

we let $y = e^{rx}$ to obtain

$$e^{rx}(r^2 - 8r + 15) = 0. \quad (2)$$

2.4.7 continued

Since $e^{rx} \neq 0$, it follows from (2) that r = 3 or r = 5. Hence, $y = e^{3x}$ and $y = e^{5x}$ are both solutions of (1). Since (1) is linear, we may conclude that

$$y = c_1e^{3x} + c_2e^{5x} \tag{3}$$

is a family of solutions of (1).*

From (3)

$$y' = 3c_1e^{3x} + 5c_2e^{3x}. \tag{4}$$

Letting x = 0, y = 1, and y' = -3 in (4), we obtain

$$\left.\begin{aligned} 1 &= c_1 + c_2 \\ -3 &= 3c_1 + 5c_2 \end{aligned}\right\}$$

whence

$$\left.\begin{aligned} 3 &= 3c_1 + 3c_2 \\ -3 &= 3c_1 + 5c_2 \end{aligned}\right\}$$

or

*Again, be careful to note that we've used linearity. For example, consider the nonlinear equation $\frac{dy}{dx} - y^2 = 0$ for which the variables are separable and we obtain the general solution $y = \frac{1}{c - x}$ $(y_o \neq 0)$ and $y = 0$ (if $y_o = 0$). Thus, both $y = \frac{1}{1 - x}$ and $y = \frac{1}{-1 - x}$ are solutions of $\frac{dy}{dx} - y^2 = 0$ $(y \neq 0)$. The sum of these two solutions is $y = \frac{1}{1 - x} + \frac{1}{-1 - x} = \frac{1}{1 - x} - \frac{1}{1 + x} = \frac{2x}{1 - x^2}$, but $y = \frac{2x}{1 - x^2}$ does not satisfy $\frac{dy}{dx} - y^2 = 0$. Namely, $\frac{dy}{dx} = \frac{2(1 + x^2)}{(1 - x^2)^2}$ while $y^2 = \frac{4x^2}{(1 - x^2)^2}$.

2.4.7 continued

$-2c_2 = 6$

or

$c_2 = -3.$

Hence,

$c_1 = 4.$

Therefore,

$$y = 4e^{3x} - 3e^{5x} \tag{5}$$

passes through (0,1) with slope -3 and satisfies (1).

By rewriting (1) as

$$y'' = 8y' - 15y = F(x,y,y')$$

where $F(x,y,z) = 8z - 15y$ may we conclude from the crucial theorem that (5) is the only curve with the required properties.

2.4.8(L)

a. This exercise is a learning exercise only in the sense that we are dealing with a third-order linear equation. In the lecture, we mentioned that all the techniques worked for any n but that for the sake of simplicity, we limited ourselves to n = 2.

In this exercise, since our coefficients are constants, we again try for solutions of

$$y''' - y' = 0 \tag{1}$$

in the form

$$y = e^{rx}. \tag{2}$$

2.4.8(L) continued

From (2),

$y' = re^{rx}$

$y'' = r^2e^{rx}$

$y''' = r^3e^{rx}$

so that with $y = e^{rx}$, (1) becomes

$r^3e^{rx} - re^{rx} = 0$

or

$$(r^3 - r)e^{rx} = 0. \qquad (3)$$

Since e^{rx} is never zero, we conclude from (3) that

$r^3 - r = 0$

and, since

$r^3 - r = r(r^2 - 1) = r(r + 1)(r - 1)$

we conclude that

$y = e^{0x}[= 1]$, $y = e^x$, and $y = e^{-x}$

are solutions of (1).

Hence, again by linearity,

$y = c_1e^{0x} + c_2e^x + c_3e^{-x}$

or

$$y = c_1 + c_2e^x + c_3e^{-x} \qquad (4)$$

is a family of solutions of (1).

2.4.8(L) continued

Moreover (4) implies that

$$y' = c_2e^x - c_3e^{-x} \tag{5}$$

and

$$y'' = c_2e^x + c_3e^{-x}. \tag{6}$$

Thus, if we want a member of (4) for which $y = y_0$, $y' = y_0'$, and $y'' = y_0''$ when $x = x_0$, we see from (4), (5), and (6) that we must be able to solve the system of equations.

$$\left.\begin{aligned} y_0 &= c_1 + c_2e^{x_0} + c_3e^{-x_0} \\ y_0' &= c_2e^{x_0} - c_3e^{-x_0} \\ y_0'' &= c_2e^{x_0} + c_3e^{-x_0} \end{aligned}\right\} \tag{7}$$

In (7) the unknowns are c_1, c_2, and c_3, and the determinant of coefficients is

$$\begin{vmatrix} 1 & e^{x_0} & e^{-x_0} \\ 0 & e^{x_0} & -e^{-x_0} \\ 0 & e^{x_0} & e^{-x_0} \end{vmatrix} \tag{8}$$

and expanding (8) along the first column (to take advantage of the 0 entries), we see that (8) is

$$\begin{vmatrix} e^{x_0} & -e^{-x_0} \\ e^{x_0} & e^{-x_0} \end{vmatrix} = e^{x_0}e^{-x_0} - (-e^{-x_0}e^{x_0})$$

$$= 1 + 1$$

$$= 2,$$

2.4.8(L) continued

for each x.

Hence, there is one and only one member of (4) which passes through (x_o, y_o) such that when $x = x_o$, $y' = y_o'$ and $y'' = y_o''$.

Finally, by writing (1) in the form

$$y''' = y',$$

we see that one and only one solution of (1) satisfies $y = y_o$, $y' = y_o'$, and $y'' = y_o''$ when $x = x_o$.

[The general uniqueness theorem says that if

$$y^{(n)} = F(x, y, y', \ldots, y^{(n-1)}), \tag{9}$$

and if F, F_{y_1}, F_{y_2}, and F_{y_n} all exist and are continuous in a region of n-space R [where $F(x, y_1, y_2, \ldots, y_{n-1})$ is obtained from $F(x, y, y', \ldots, y^{(n-1)})$ by letting $y_k = y^{(k)}$) then there is one and only one solution of (9) that passes through each point (x_o, y_o) in the plane with prescribed values for y_o, y_o', y_o'', ..., and $y_o^{(n-1)}$ at $x = x_o$.]

b. Letting $x_o = 0$, $y_o = 1$, $y_o' = 3$, and $y_o'' = 5$, we see from (7) that

$$\left.\begin{aligned} 1 &= c_1 + c_2 + c_3 \\ 3 &= c_2 - c_3 \\ 5 &= c_2 + c_3 \end{aligned}\right\} \tag{10}$$

Adding the last two equations in (10) yields $c_2 = 4$, from which we conclude that $c_3 = 1$. Letting $c_2 = 4$ and $c_3 = 1$ in the first equation of (10), we obtain

$$1 = c_1 + 4 + 1$$

so that

$$c_1 = -4.$$

2.4.8(L) continued

Hence, the required solution, from (4), is

$$y = -4 + 4e^{x} + e^{-x}. \quad (11)$$

Note

What seems to be happening is that as the order of our linear equation increases, we can make it "behave better" (i.e., by prescribing more derivatives) in the neighborhood of a given point.

2.4.9(L)

In the last several exercises, we have restricted our attention to linear equations of the form $L(y) = 0$.

We now wish to conclude this unit with the case $L(y) = f(x)$ where $f(x)$ is not identically zero.

To simplify matters, we have elected to solve a problem in which the constituent parts have already been handled.

From equation (5) of Exercise 2.4.1, we have that $y = e^{x}$ is a particular solution of

$$L(y) = e^{x}(x^{2} - 3x + 3)$$

where

$$L(y) = x^{2}y'' - 3xy' + 3y.$$

In the notation of our lecture

$$y_p = e^{x}. \quad (1)$$

In Exercise 2.4.3, we saw that

$$y = c_1x + c_2x^{3} \quad (2)$$

was the general solution of $L(y) = 0$.

2.4.9(L) continued

Hence, again in the notation of our lecture, we conclude from (2) that

$$y_h = c_1x + c_2x^3. \qquad (3)$$

Since $y_h + y_p$ is then the general solution of $L(y) = f(x)$, we have from (1) and (3) that

$$y = c_1x + c_2x^3 + e^x \qquad (4)$$

is the general solution of

$$x^2y'' - 3xy' + 3y = e^x(x^2 - 3x + 3). \qquad (5)$$

Notice that the crucial existence theorem of the lecture is essentially of the same form as in Exercise 2.4.3. That is, the "crucial theorem" when applied to

$$y'' + p(x)y' + q(x)y = f(x) \qquad (6)$$

requires only that p, q, and f be defined and continuous in some region R. Thus, if we replace f(x) by any other continuous function of x, the region R in which the general solution exists doesn't change.

Hence, we need only show that we can determine c_1 and c_2 so that at any given point (x_o,y_o), we can find a member of (4) which passes through (x_o,y_o) with a prescribed slope y_o' and that (4) really does satisfy (5) [as the theory claims].

Well, from (4),

$$y' = c_1 + 3c_2x^2 + e^x \qquad (7)$$

and

$$y'' = 6c_2x + e^x. \qquad (8)$$

Hence from (4), (7), and (8), we have

2.4.9(L) continued

$$x^2y'' - 3xy' + 3y = x^2(6c_2x + e^x) - 3x(c_1 + 3c_2x^2 + e^x) + 3(c_1x + c_2x^3 + e^x)$$

$$= 6c_2x^3 + x^2e^x - 3c_1x - 9c_2x^3 - 3xe^x + 3c_1x + 3c_2x^3 + 3e^x$$

$$= x^2e^x - 3xe^x + 3e^x$$

$$= e^x(x^2 - 3x + 3)$$

which agrees with (5).

Again, it would have been quicker, once we have the confidence, to use the properties of linearity directly. That is,

$$L(c_1x + c_2x^3 + e^x) = L(c_1x) + L(c_2x^3) + L(e^x)^*$$

$$= c_1L(x) + c_2L(x^3) + L(e^x) \qquad (9)$$

and since in this problem $L(x) = L(x^3) = 0$ and $L(e^x) = e^x(x^2 - 3x + 3)$, we conclude at once from (6) that

$$L(c_1x + c_2x^3 + e^x) = 0 + L(e^x) = e^x(x^2 - 3x + 3).$$

Finally, to show that c_1 and c_2 may be chosen uniquely to find a member of (4) which passes through (x_o,y_o) with slope y_o', we see from (4) and (7) that we must have a unique solution of the system

*It is easy to prove inductively that if L is linear, $L(u_1 + \dots + u_n) = L(u_1) + \dots + L(u_n)$. For example, $L(u_1 + u_2 + u_3) = L([u_1 + u_2] + u_3) = L(u_1 + u_2) + L(u_3) = L(u_1) + L(u_2) + L(u_3)$. We have given the definition with the assumption that $n = 2$ knowing that the results can be easily extended by induction.

2.4.9(L) continued

$$\left.\begin{aligned} y_o &= c_1x_o + c_2x_o^{\,3} + e^{x_o} \\ y_o' &= c_1 + 3c_2x_o^{\,2} + e^{x_o} \end{aligned}\right\} \qquad (10)$$

The key point in (10) is that since x_o, y_o, and y_o' are given constants, c_1 and c_2 are still the only unknowns. Hence, the determinant of coefficients is still

$$\begin{vmatrix} x_o & x_o^{\,3} \\ 1 & 3x_o^{\,2} \end{vmatrix}$$

just as in Exercise 2.4.3. In other words, the determinant of coefficients depends only on $c_1u_1 + \dots + c_nu_n = y_n$ and is unchanged when y_p is added to y_h

2.4.10 (Optional)

a. Since $L(y) = 0$ is particularly convenient when the coefficients in L are constant, we often try to reduce linear equations with non-constant coefficients to ones with constant coefficients.

One very convenient type is the so called <u>equidimensional equation</u> which has the form

$$c_nx^ny^{(n)} + c_{n-1}x^{n-1}y^{(n-1)} + \dots + c_1xy' + c_oy = 0 \qquad (1)$$

i.e., the coefficient of $y^{(k)}$ is a constant times x^k; the equation is called equidimensional because the power of x equals the order of the derivative in each term.

In the special case that $n = 2$ and $c_n \neq 0$, equation (1) takes the form

$$x^2y'' + axy' + by = 0 \quad (x > 0). \qquad (2)$$

The substitution

$$z = \ln x \quad (x > 0) \qquad (3)$$

2.4.10 continued

yields

$$\frac{dz}{dx} = \frac{1}{x} \qquad (4)$$

so that we get a hunch, at least that the substitution (3) may reduce (2) to constant coefficients (at least the 2axy' term has x eliminated when we replace $\frac{dy}{dx}$ by $\frac{dz}{dx}$).

Thus, the trick is to rewrite (2) in such a way that y becomes the independent variable and z the dependent variable. For example,

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$$

or by (4),

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz}. \qquad (5)$$

From (5)

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx}$$

[i.e., $\frac{d\left(\frac{dy}{dz}\right)}{dz} = \frac{d^2y}{dz^2}$ so by the chain rule $\frac{d}{dx}\left(\frac{dy}{dz}\right) = \frac{d^2y}{dz^2}\frac{dz}{dx}$] or, again by (4),

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\left(\frac{d^2y}{dz^2}\right)\frac{1}{x}$$

$$= \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right). \qquad (6)$$

Putting the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ as given by (5) and (6) into (2), we obtain

2.4.10 continued

$$x^2\left[\frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)\right] + ax\left[\frac{1}{x}\frac{dy}{dz}\right] + by = 0$$

or

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + a\frac{dy}{dz} + by = 0$$

or

$$\frac{d^2y}{dz^2} + (a - 1)\frac{dy}{dz} + by = 0 \qquad (7)$$

where (7) is linear in y and has constant coefficients. We can then solve (7) to find y as a function of z and then use (3) to replace z by ln x.

b. Using a = -3 and b = 3, we see from (7) that z = ln x transforms

$$x^2y'' - 3xy' + 3y = 0$$

into

$$\frac{d^2y}{dz^2} - 4\frac{dy}{dz} + 3y = 0. \qquad (8)$$

Letting $y = e^{rz}$, we see that (8) yields

$$e^{rz}(r^2 - 4r + 3) = 0$$

so that r = 1 or r = 3.

Hence, the general solution of (8) is

$$y = c_1e^z + c_2e^{3z} \qquad (9)$$

and since z = ln x, (9) becomes

2.4.10 continued

$$y = c_1e^{\ln x} + c_2e^{3 \ln x}$$

$$= c_1e^{\ln x} + c_2e^{\ln x^3}$$

$$= c_1x + c_2x^3.$$

2.4.11 (Optional)

Given

$$y'' + p(x)y' + q(x)y = 0 \qquad (1)$$

any substitution of the form $z = f(y,y')$ is recommended.

More explicitly, in the present exercise, we are asked to see what happens to (1) under the substitution

$$z = \frac{y'}{y} \qquad (2)$$

or

$$y' = zy. \qquad (2')$$

From (2'),

$$y'' = zy' + z'y, \qquad (3)$$

and replacing y' and y'' in (1) by their values in (2') and (3) we obtain

$$zy' + z'y + p(x)zy + q(x)y = 0. \qquad (4)$$

Equation (4) is nasty in that it includes the three variables x, y, and z and to help alleviate this situation, we divide both sides of (4) by y ($y \neq 0$) to obtain

2.4.11 continued

$$z\frac{y'}{y} + z' + p(x)z + q(x) = 0 \tag{5}$$

and since $\frac{y'}{y} = z$, (5) becomes

$$z^2 + z' + p(x)z + q(x) = 0 \tag{6}$$

and equation (6) is a <u>first order</u> differential equation involving x and z.

b. In general, had we started with (6), we would have a mess (meaning a rather nasty first order nonlinear equation). The point is that the substitution $z = \frac{y'}{y}$ transforms (6) into (1). Therefore, if we can solve (1), we can also solve (6).

Thus, reversing the steps (1) through (6), we see that the non-linear first degree equation

$$z^2 + z' + p(x)z + q(x) = 0$$

becomes the linear second-order equation

$$y'' + p(x)y' + q(x)y = 0$$

under the substitution $z = \frac{y'}{y}$.

Thus, if it happens that we can solve (1), we can also solve (6).

In this particular problem, we have rigged things a little by picking

$$\frac{dz}{dx} = \frac{3z}{x} - z^2 - \frac{3}{x^2} \quad (x \neq 0), \tag{7}$$

since if we write this in the form (6), we obtain

$$z^2 + z' - \frac{3z}{x} + \frac{3}{x^2} = 0 \tag{8}$$

and (8) is (6) with $p(x) = -\frac{3}{x}$ and $q(x) = \frac{3}{x^2}$. Hence, the substitution $z = \frac{y'}{y}$ transforms (8) into

2.4.11 continued

$$y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 0,$$

or, equivalently since $x \neq 0$,

$$x^2y'' - 3xy' + 3y = 0. \tag{9}$$

But we already know the general solution of (9), namely,

$$y = c_1x + c_2x^3. \tag{10}$$

From (10),

$$y' = c_1 + 3c_2x^2 \tag{11}$$

and since $z = \frac{y'}{y}$, we have from (10) and (11) that

$$z = \frac{c_1 + 3c_2x^2}{c_1x + c_2x^3}. \tag{12}$$

If we now divide both numerator and denominator on the right side of (12) by c_1 and let $c = \frac{c_2}{c_1}$, we obtain

$$z = \frac{1 + 3cx^2}{x + cx^3}. \tag{13}$$

To check that (13) is a solution of (7), we have from (13) that

$$z' = \frac{(x + cx^3)(6cx) - (1 + 3cx^2)(1 + 3cx^2)}{\left(x + cx^3\right)^2}$$

$$= \frac{6cx^2 + 6c^2x^4 - 1 - 6cx^2 - 9c^2x^4}{x^2\left(1 + cx^2\right)^2}$$

$$= \frac{-3c^2x^4 - 1}{x^2\left(1 + cx^2\right)^2} \tag{14}$$

2.4.11 continued

Also from (13)

$$z^2 = \frac{1 + 6cx^2 + 9c^2x^4}{x^2\left(1 + cx^2\right)^2}. \tag{15}$$

Using (13), (14) and (15) in (7), we obtain

$$\frac{-3c^2x^4 - 1}{x^2 \; 1 + cx^{2} \;{}^{2}} \overset{?}{=} \frac{3(1 + 3cx^2)}{x^2(1 + cx^2)} - \frac{(1 + 6cx^2 + 9c^2x^4)}{x^2\left(1 + cx^2\right)^2} - \frac{3}{x^2}$$

$$\overset{?}{=} \frac{(3 + 9cx^2)(1 + cx^2) - 1 - 6cx^2 - 9c^2x^4 - 3(1 + 2cx^2 + c^2x^4)}{x^2\left(1 + cx^2\right)^2}$$

$$\overset{?}{=} \frac{3 + 12cx^2 + 9c^2x^4 - 1 - 6cx^2 - 9c^2x^4 - 3 - 6cx^2 - 3c^2x^4}{x^2\left(1 + cx^2\right)^2}$$

$$\overset{?}{=} \frac{-3c^2x^4 - 1}{x^2\left(1 + cx^2\right)^2},$$

which completes our check.

Lest we have lost of our accomplishment, we have exhibited here an example where a change of variables transforms a quite possibly difficult nonlinear first-order equation into a second order linear equation, and since a great deal is known about second-order linear equations, we replace our unsolved problem by a simpler unsolved problem.

In particular, we have seen here that any first-order equation of the form

$$z' + z^2 + p(x)z + q(x) = 0 \tag{16}$$

is equivalent to

$$y'' + p(x)y' + q(x) = 0$$

2.4.11 continued

under the change of variable $z = \frac{y'}{y}$.

Of course, (16) denotes just one particular type of nonlinear equation, and we might not be as fortunate in other types. This is one reason that differential equations gets its "cookbook" reputation.

From our point of view, the main aim of this exercise is to open avenues for you to see how the solution of linear differential equations can sometimes help us solve certain types of nonlinear equations.