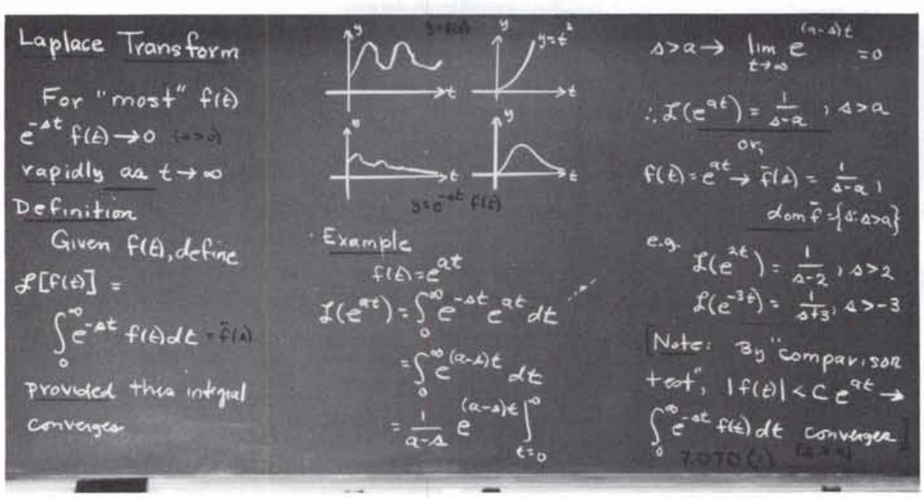
Unit 9: The Laplace Transform, Part 1

1. Overview

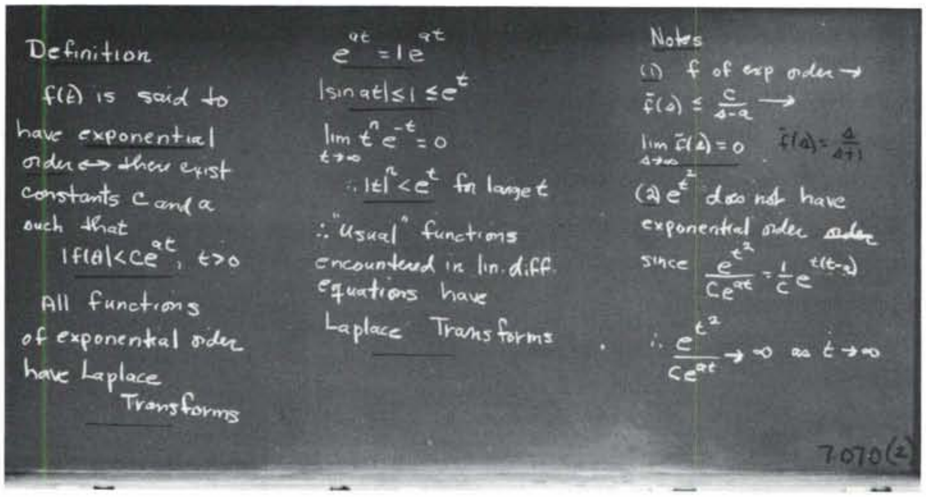
The Laplace transform has application far beyond its present role in this block of being a useful device for solving certain types of linear differential equations (usually ones in which we have constant coefficients) for prescribed initial conditions. In this unit, we introduce the concept in our lecture and we then divide the exercises into two categories. The first five exercises are designed just to help you become more familiar with the definition itself, and the last three exercises illustrate how the concept is used to solve differential equations.

Additional fine points concerning the Laplace transform are left for the next unit (a unit which is optional since it is not necessary for the student who for one reason or another prefers not to study this concept in any more depth at this time).

2. Lecture 2.070

a. 

Lecture 2.070 continued



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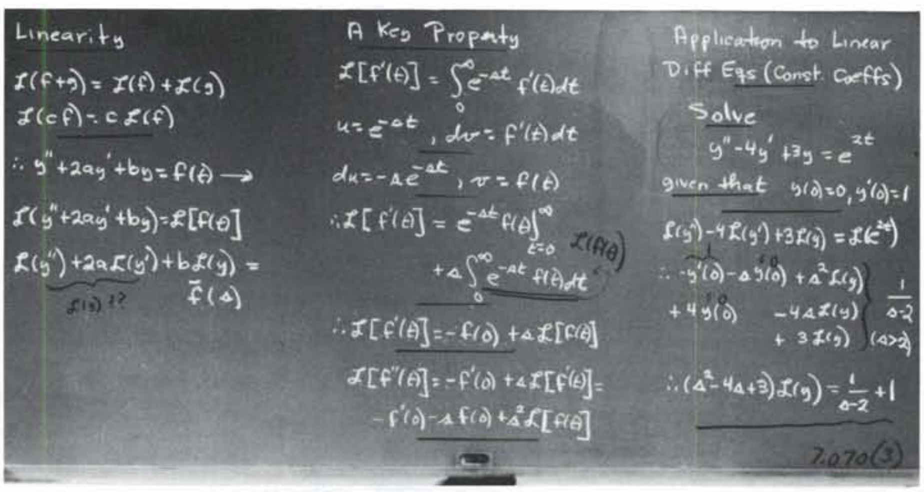
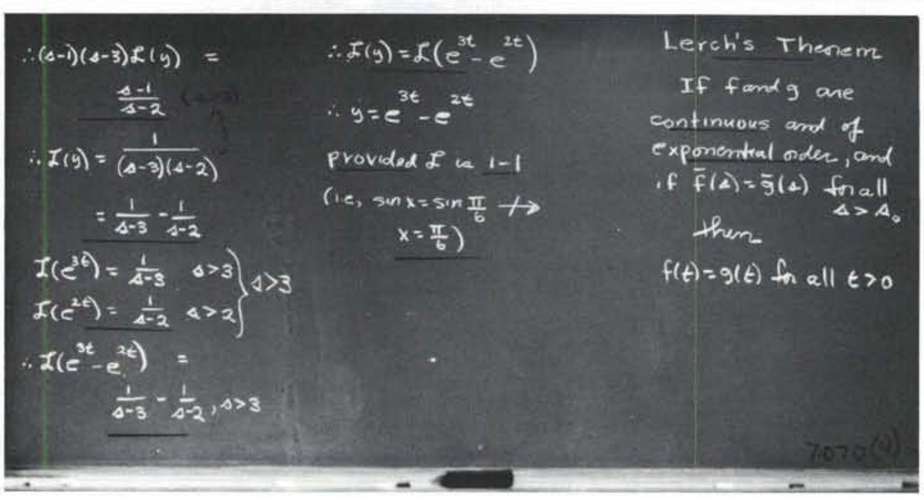


c.



d.

3. Exercises:

2.9.1(L)

a. Use the linear properties of the Laplace transform to compute $\mathcal{L}(\cosh bt)$, knowing that $\mathcal{L}(e^{at}) = \frac{1}{s - a}$ $(s > a)$.

b. Prove that if $\mathcal{L}\big(f(t)\big) = \bar{f}(s)$ then $\mathcal{L}[e^{at}f(t)] = \bar{f}(s - a)$.

c. Use the results of (a) and (b) to compute

$$\mathcal{L}^{-1}\left[\frac{s - a}{(s - a)^2 - b^2}\right].$$

That is, determine g(t) if

$$\mathcal{L}\big(g(t)\big) = \frac{s - a}{(s - a)^2 - b^2}.$$

2.9.2

a. Use the identity that $\sinh 3t = \frac{1}{2}(e^{3t} - e^{-3t})$ to determine $\mathcal{L}(\sinh 3t)$.

b. Use (a) together with (b) of the previous exercise to determine $g(t)$ if $\mathcal{L}\big(g(t)\big) = \frac{3}{(s - 4)^2 - 9}$.

2.9.3(L)

a. Compute $\mathcal{L}(\cos bx)$ by using the fact that

$$\cos bx = \frac{1}{2}(e^{ibx} + e^{-ibx}).$$

b. Determine f(x) if f is continuous and

$$\mathcal{L}\big(f(x)\big) = \frac{s}{s^2 + 9} + \frac{3}{s^2 - 9}.$$

2.9.4(L)

By writing $s^2 - 4s + 20$ in the form $(s - a)^2 + b^2$, use the tables at the end of the solution of the previous exercise to find f(t) if f is continuous and

$$\mathcal{L}\big(f(t)\big) = \frac{2s + 3}{s^2 - 4s + 20}.$$

2.9.5

Determine f(t) if it is known that f is continuous and that $\mathcal{L}\big(f(t)\big)$ is

(a) $\frac{1}{s(s + 1)}$

(b) $\frac{1}{s^2 + 4s + 29}$

(c) $\frac{1}{s(s + 2)^2}$

(d) $\frac{3s + 1}{(s + 1)(s + 2)(s + 3)}$

2.9.6(L)

Use the Laplace transform method to find the particular solution of $y'' + 2y' + y = e^t$ which satisfies the initial conditions $y(0) = y'(0) = 0$.

2.9.7

Use Laplace transforms to find the solution of

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 2$$

which satisifes the initial conditions $y(0) = 0$ and $y'(0) = 1$.

2.9.8

Use Laplace transforms to find the solution of

$$y''' - y' = e^{2t}$$

which satisfies $y(0) = 0$, $y'(0) = 0$, and $y''(0) = 0$.

Unit 10: The Laplace Transform, Part 2

1. Overview

As far as defining the Laplace transform and seeing how it is used to solve linear differential equations with constant coefficients, our task is satisfactorily completed in the previous unit. Yet Laplace transform and other related forms occur very often both in theory and in application. For this reason we have elected to supply additional exercises involving the Laplace transform. The exercises were chosen not only to give you drill in computing transforms, but also because they bring up important new areas of discussion.

2. Exercises:

2.10.1

a. Compute $\mathcal{L}[u_a(t)]$ where

$$u_a(t) = \begin{cases} 0 & t \leq a \\ 1 & t < a \end{cases}$$

b. If $\mathcal{L}(f(t)) = \bar{f}(s)$, show that $[u_a(t)f(t-a)] = e^{-as}\,\bar{f}(s)$.

c. Determine $f(t)$ if f is continuous and $(f(t)) = e^{-3s}/s^2 + 4s + 5$.

2.10.2

Suppose there exists a positive number p such that $f(t) = f(t+p)$ for all t [in this case, f is said to be periodic with period p].

a. Show that

$$\mathcal{L}(f(t)) = \int_0^p e^{-st} f(t)\,dt/1 - e^{-ps}.$$

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2.10.2 continued

b. Compute (f(t)) if

$$f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & 1 \leq t < 2 \end{cases}$$

and f has period 2.

2.10.3

a. By computing

$$\frac{d}{ds} \int_0^\infty e^{-st} f(t)dt$$

show that if $\bar{f}(s)$ exists, then $\mathcal{L}[tf(t)] = -d\bar{f}(s)/ds$. Extend this result inductively to obtain a formula for $\mathcal{L}[t^n f(t)]$ for any positive integer n.

b. Use the result of part (a) to determine $\bar{y}(s)$ if y(t) satisfies

$$t \frac{d^2y}{dt^2} + \frac{dy}{dt} + ty = 0.$$

2.10.4

Recalling that $\Gamma(x)$ is defined by

$$\int_0^\infty t^{x-1}e^{-t}dt \quad (x > 0)$$

develop a formula for computing $\mathcal{L}(x^n)$ where n is any real number greater than -1.

2.10.5 (To reinforce the definition of $\Gamma(x)$)

a. Compute $\Gamma(\frac{1}{2})$.

b. Use the fact that $\Gamma(n + 1) = n\Gamma(n)$ to compute $\Gamma(\frac{3}{2})$ and $\Gamma(\frac{5}{2})$.

2.10.6

a. By making appropriate use of the fact that the product of two integrals may be viewed as a double integral, show that

$$\mathcal{L}(f)\,\mathcal{L}(g) = \mathcal{L}\left[\int_0^u f(v)g(u - v)dv\right].$$

b. Use (a) to determine h(t) if

$$\bar{h}(s) = \frac{1}{s(s - 1)}.$$

2.10.7 (Checking some of the properties of convolution)

Define the convolution of f and g, written f*g by

$$f*g = \int_{v=0}^{u} f(v)g(u - v)dv.$$

Show that

a. $f*g = g*f$

b. $f*(g + h) = (f*g) + (f*h)$

2.10.8

Make appropriate use of convolution to determine h(t) if

a. $\mathcal{L}(h) = \dfrac{s}{(s - 1)(s^2 + 1)}$

b. $\mathcal{L}(h) = \dfrac{1}{(s^2 + 1)^2}$

c. $\mathcal{L}(h) = \dfrac{1}{(s - 1)(s^2 + 1)}$

d. $\mathcal{L}(h) = \dfrac{s}{(s^2 + 1)^2}$

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2.10.8 continued

e. Use the results of parts (a), (b), (c), and (d) to find the solution of the system

$$\left.\begin{aligned} \frac{dx}{dt} - y &= e^t \\ \frac{dy}{dt} + x &= \sin t \end{aligned}\right\}$$

subject to the initial conditions that $x(0) = 1$ and $y(0) = 0$. [Actually (e) can be tackled by the transform method without doing (a), (b), (c) and (d). What happens, however, is that to solve (e) we ultimately wind up having to solve these four parts anyway.]

Quiz

1. Find the general solution of each of the following differential equations:

   (a) $\frac{dy}{dx} = 2xy$

   (b) $\frac{dy}{dx} = 2xy + e^{x^2}$

   (c) $\frac{dy}{dx} = \frac{2xy}{x^2 + y^2}$ (where not both x and y equal 0).

2. (a) Find the envelope of the family of lines

   $y = cx - 2c^2,$

   where c is an arbitrary (real) constant.

   (b) Find the first order differential equation which is satisfied by both the family $y = cx - 2c^2$ and its envelope.

   (c) The curve C satisfies the equation

   $y = x\frac{dy}{dx} - 2\left(\frac{dy}{dx}\right)^2.$

   Describe the curve C if it is known that C passes through the point

   (i) (4,2)

   (ii) (4,3)

   (iii) (4,0)

3. Let $L(y) = y'' + 4y' - 21y$. Find the general solution of $L(y) = f(x)$ if:

   (a) $f(x) = e^x$

   (b) $f(x) = \sin x$

   (c) $f(x) = 3e^x + 5 \sin x$

   (d) $f(x) = e^{3x}$

4. The curve C satisfies the differential equation

$$y'' + 2y' + y = \frac{e^{-x}}{x + 1}$$

(where C doesn't intersect the line $x = -1$). Find the equation of C.

5. The curve C satisfies the differential equation $y'' - 3xy' - 3y = 0$. It passes through the point (0,1) and has its slope equal to 0 at that point. Use the series technique to find the equation of C. (Write the series explicitly through the term involving x^7.)

6. A particle moves along the x-axis according to the rule

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 8 \sin t + 4 \cos t.$$

At time $t = 0$, the particle is at $x = 1$ and has speed $\frac{dx}{dt} = 3$. Use the Laplace transform method to determine x explicitly as a function of t.