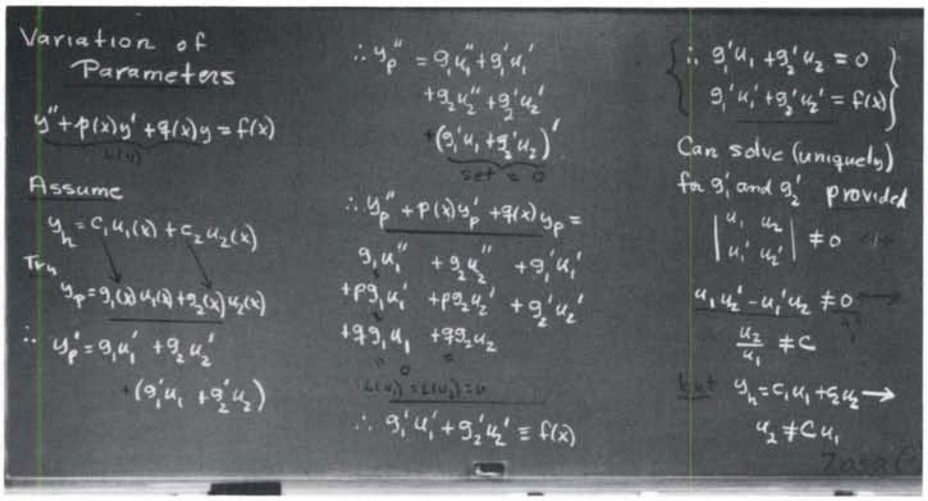
Unit 7: Variation of Parameters

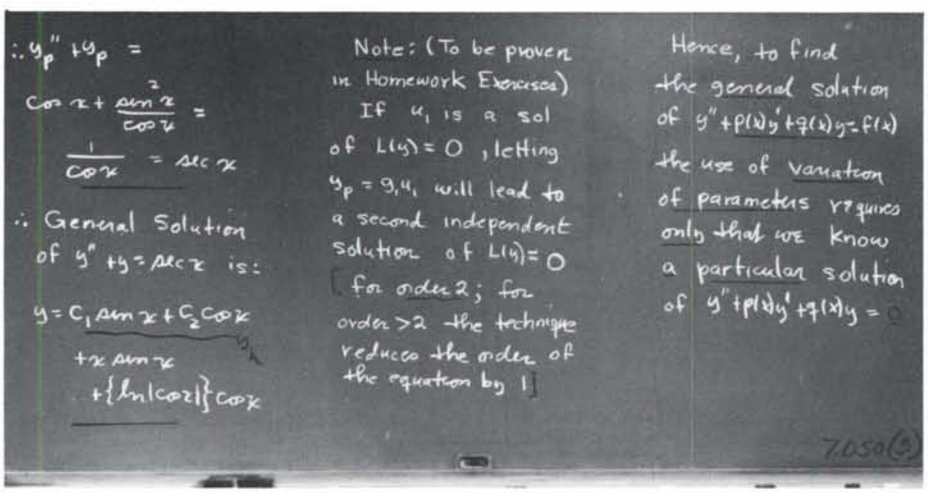
1. Overview

In this unit, we show how to find the general solution of any nth order linear differential equation once we know the general solution of the reduced equation. The method employed is known as Variation of Parameters and while it is often computationally cumbersome, it always works even if our equation does not have constant coefficients. Moreover, it does not require the right side of the equation to have any special form (other than to be integrable).

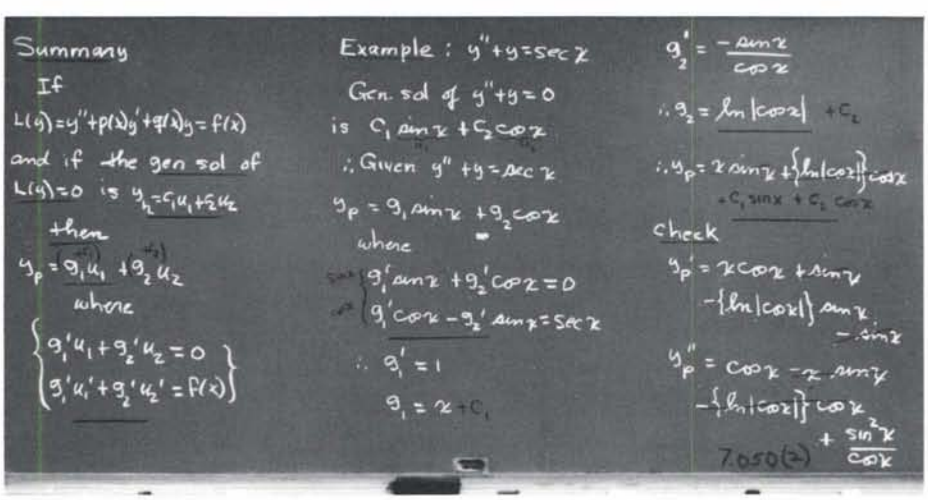
2. Lecture 2.050



a.



b.



c.

3. (Optional) Read Thomas, Sections 20.10 and 20.11.

(The reading material offers nothing different from the discussion in the lecture, but what is interesting is that every exercise lends itself to the method of undetermined coefficients discussed by us in the previous unit. It might be of interest to you to try a few of these exercises, doing them both ways, but the real value of the method is in the case where either the coefficients are not constant or the right side is more "complicated.")

4. Exercises:

2.7.1(L)

Find a particular solution of $y'' - y = \frac{1}{1 + e^x}$.

2.7.2

Find the general solution of

$y'' - 2y' + y = e^x \ln x \ (x > 0)$.

2.7.3(L)

Suppose we discovered that $y = x$ is a solution of

$y'' + \frac{1}{x^2} y' - \frac{1}{x^3} y = 0 \ (x \neq 0)$.

Find another solution of this equation which has the form $y = xg(x)$ where $g(x)$ is not constant.

2.7.4

Use the technique of Exercise 2.7.3 to find a solution of $y'' - 2y' + y = 0$ which is not a constant multiple of e^x. (We solved this type of problem in Unit 5. This exercise supplies us with another technique for obtaining the same result, and also supplies us with additional drill in the use of variation of parameters.)

2.7.5

a. By writing

$$y'' + (x^2 - 4)y' - 4x^2y = 0$$

in the form

$$(y'' - 4y') + x^2(y' - 4y) = 0$$

find one (non-zero) solution of the equation.

b. Use the answer in (a) and the method of Exercise 2.7.3 to find the general solution of

$$y'' + (x^2 - 4)y' - 4x^2y = 0.$$

2.7.6(L)

Observing that $y = x$ is a solution of

$$y'' - \frac{x}{1 - x^2}\,y' + \frac{y}{1 + x^2} = 0 \quad (|x| < 1)$$

find the general solution of

$$y'' - \frac{x}{1 - x^2}\,y' + \frac{y}{1 - x^2} = 1 \quad (|x| < 1).$$

2.7.7 (Optional)

[This problem is not difficult but rather lengthy and perhaps a bit abstract (at least part (a)). The aim is to show how the method of variation of parameters works in higher order equations. If you have time try the entire exercise or at least read the solution. If you are pressed for time but want to get an idea of what is happening, accept the result of part (a) and try to solve (b) using this result.]

(continued on next page)

2.7.7 continued

a. Suppose $y_h = c_1u_1(x) + c_2u_2(x) + c_3u_3(x)$ is the general solution of $L(y) = y''' + p(x)y'' + q(x)y' + r(x)y = 0$. Show that $y_p = g_1(x)u_1(x) + g_2(x)u_2(x) + g_3(x)u_3(x)$ is a particular solution of $L(y) = f(x)$ provided that

$$g_1'u_1 + g_2'u_2 + g_3'u_3 = 0$$

$$g_1'u_1' + g_2'u_2' + g_3'u_3' = 0$$

$$g_1'u_1'' + g_2'u_2'' + g_3'u_3'' = f$$

b. Find a particular solution of $x^3y''' + xy' - y = x \ln x$ $(x > 0)$ knowing that the general solution of the reduced equation is $c_1x + c_2x \ln x + c_3x(\ln x)^2$.