CALCULUS REVISITED
PART 3
A Self-Study Course

STUDY GUIDE
Block 2
Ordinary Differential
Equations

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BLOCK 2:
ORDINARY DIFFERENTIAL EQUATIONS

Pretest

1. a. Find the first order differential equation (in which c does not appear) satisfied by each hyperbola of the family $y = \frac{1}{c - x}$ where c is an arbitrary constant and $x \neq c$.

b. Find a line which satisfies the same differential equation found in part (a).

2. Find the general solution of

$$\frac{dy}{dx} = 1 + \frac{y}{x} + \frac{y^2}{x^2} \quad (x \neq 0).$$

3. Express, in polar form, the family of curves which has the property that each member of this family intersects each line $y = mx$ at a 45° angle.

4. Find the general solution of

$$\frac{d^5y}{dx^5} - 2\frac{d^3y}{dx^3} + \frac{dy}{dx} = 0.$$

5. Find the general solution of

$$y'' - 6y' + 9y = 3e^{4x} + \sin 3x.$$

6. Find a particular solution of $y'' - y = \frac{1}{1 + e^x}$.

7. Use power series to find the particular solution, $y = f(x)$, of the equation $y'' - xy = 0$ if $f(0) = 0$ and $f'(0) = 1$.

8. Find the solution of the system:

$$\begin{cases} \frac{dx}{dt} - y = e^t \\ \frac{dy}{dt} + x = \sin t \end{cases}$$

subject to the initial conditions that $x(0) = 1$ and $y(0) = 0$.

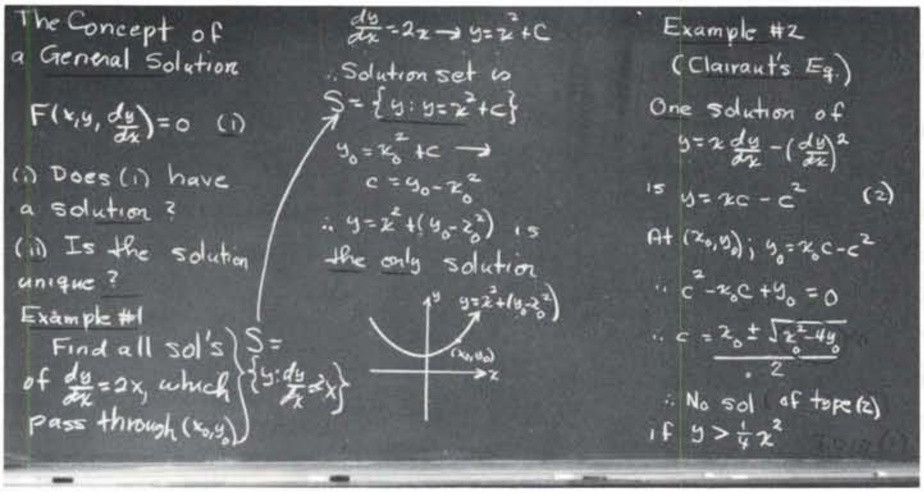
Unit 1: The Concept of a General Solution

1. Overview

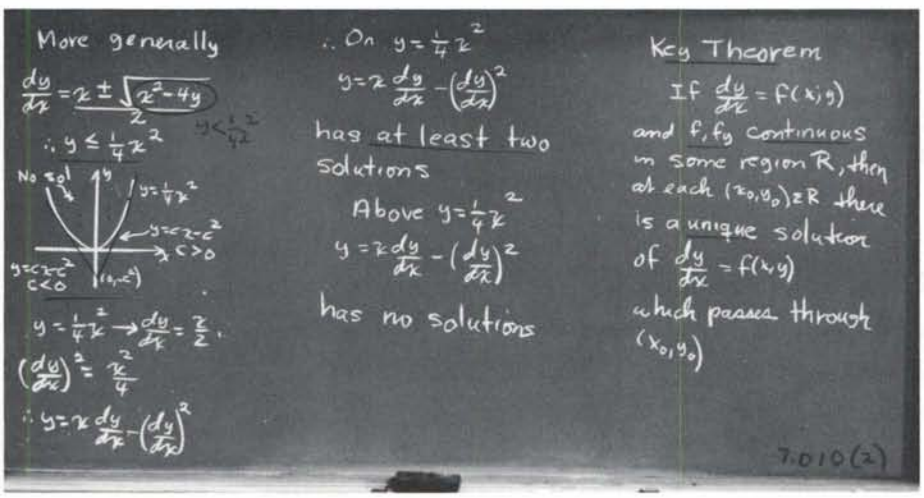
   In a way, differential equations are the inverse of differential calculus. In differential calculus we started with a given relationship between x and y and then found how the various derivatives of y with respect to x were related. In differential equations, we start with a relationship between the function and its various derivatives, from which we try to deduce what the function was. The problem is that some differential equations have no solutions while others have "too many" solutions. In this unit it is our aim to make it clear as to just what is meant by a solution of a differential equation. In the next unit we shall show how to solve certain types of differential equations; and in Unit 3 we shall try to show how differential equations occur in "nature".

2. Do Exercises 2.1.1, 2.1.2 and 2.1.3. The lecture deals with the concept of a general solution to a differential equation. While the lecture is self-contained, there is a certain amount of experience or "sophistication" that is required on the part of the student if the notion of general solution is to be as meaningful as possible. It is for this reason that you are asked to do these three problems before viewing the lecture. You may also wish to review these three exercises after viewing the lecture before tackling the remaining exercises in this Unit.

3. Lecture 2.010



a.



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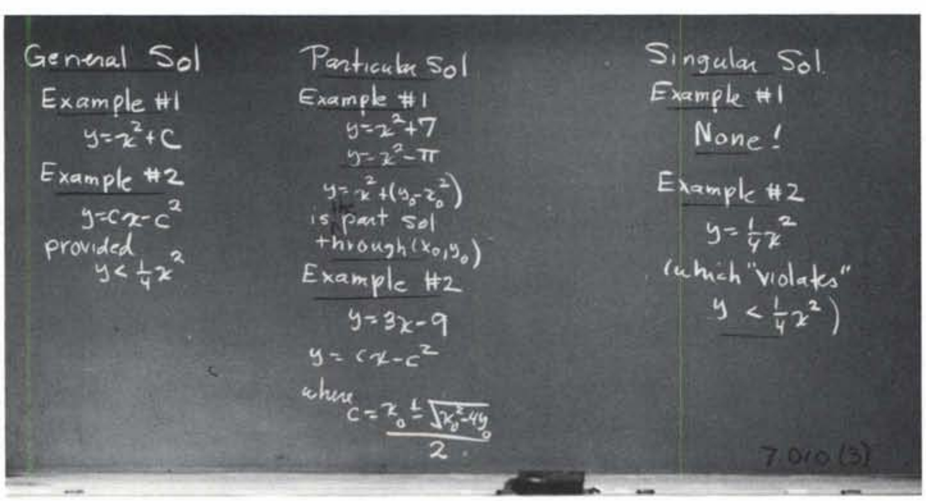
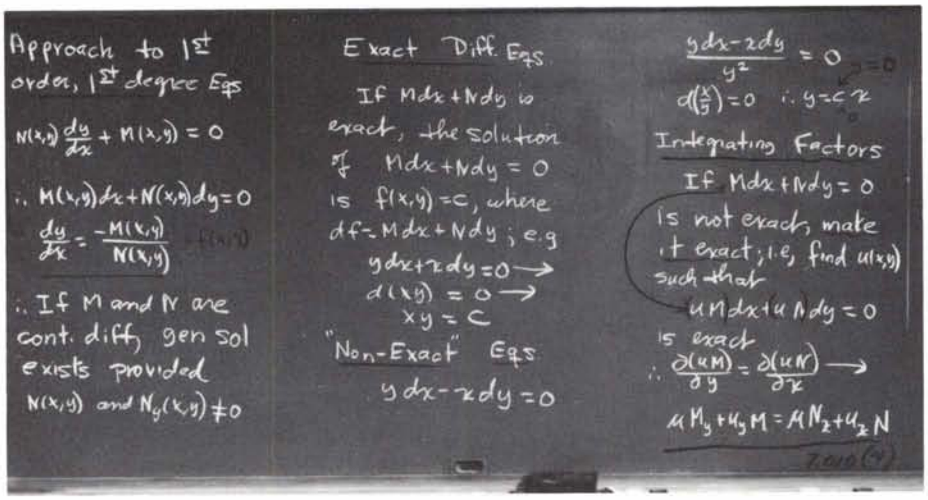


c.

Lecture 2.010 continued

d. 

4. Read: Thomas, Sections 20.1 and 20.2. The main role of these two sections is to help you establish the basic vocabulary that is used in the study of differential equations.

5. Exercises:

2.1.1(L)

Find the first order differential equation which is satisfied by each member of the family, $y = x^2 + c$, where c is an arbitrary constant.

2.1.2(L)

Find the first order differential equation which is satisfied by each hyperbola of the family $y = \frac{1}{c - x}$ $(x \neq c)$ where c is an arbitrary constant. Show that the differential equation can be satisfied by a curve which does not belong to the given family of hyperbolas.

2.1.3(L)

Find the first order differential equation satisfied by the family of circles $(x - c)^2 + y^2 = 1$, and show that this differential equation is also satisfied by the lines $y = \pm 1$.

2.1.4(L)

Consider the differential equation $\frac{dy}{dx} = x^2y$.

a. Find a 1-parameter family of curves which satisfies this differential equation.

b. Can the given equation have solutions which do not belong to the family found in (a)? Explain.

c. Find all solutions of the given equation which pass through a given point (x_0,y_0) in the plane.

2.1.5(L)

The equation $\frac{dy}{dx} = 3y^{2/3}$ is defined at all points (x_0,y_0) in some region R. Describe R if it is known that there is one and only one solution of the equation that passes through a given point $(x_0,y_0)\varepsilon R$, and describe the solution.

2.1.6

Consider the differential equation $x\frac{dy}{dx} - 3y = 0$.

a. Describe the most general (connected) region R for which the given equation has a general solution.

b. With R as in (a), find the solution of the given equation which passes through $(x_0,y_0)\varepsilon R$.

c. In particular, find the solution which passes through (1,1).

2.1.7(L)

Given a 1-parameter family of curves, γ defined by $y = f(x,c)$, E is called an envelope of the family γ if and only if for each point (x_0,y_0) on E, E is tangent to at least one member of γ at

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2.1.7(L) continued

(x_0,y_0). If γ has an envelope it is found by solving the system of equations

$$\begin{cases} y = f(x,c) \\ f_c(x,c) = 0 \end{cases}$$

and eliminating c.

a. Find the envelope of the family $y = cx - c^2$.

b. Find the envelope of the family $(x - c)^2 + y^2 = 1$.

The final two exercises are optional. While they are not particularly difficult, they deal with the concept of envelopes; and this concept is not vital to the material which follows. Nevertheless, if you have the time it is worthwhile to obtain the additional experience in working with envelopes. Moreover, Exercise 2.1.9 outlines the procedure for solving Clairaut's equation in general.

2.1.8 (optional)

a. Find the differential equation which is satisfied by the 1-parameter family of circles $(x - c)^2 + y^2 = 4c + 4$.

b. Describe the region R in which the differential equation of part (a) has a solution.

c. Find the members of the family in (a) which pass through (3,4). Explain why two different members of this family pass through (3,4) and satisfy the same differential equation.

d. Find the envelope of the family of curves in (a) and show that the envelope satisfies the same differential equation as does the given family of circles.

2.1.9 (optional)

a. Find a 1-parameter family of lines which satisfy the Clairaut Equation $y = x\frac{dy}{dx} - \frac{1}{4}\left(\frac{dy}{dx}\right)^4$.

b. Find the envelope of the family of straight lines defined in (a).

c. Use the result of (b) to find another solutions of the Clairaut Equation of part (a).

d. Graph the envelope of part (b) and explain how it is related to the family of lines in part (a).

e. Solve the Clairaut Equation of part (a) directly by differentiating the equation with respect to x and letting u denote $\frac{dy}{dx}$.

Unit 2: Special Types of First Order Equations

1. Overview

   In many respects, at least from an engineer's point of view, differential equations is viewed as a "cookbook" course. The main reason for this is that once the general theory concerning the existence of solutions of differential equations is established, we must then turn our attention to the "nitty-gritty" of finding these solutions.

   Our aim in this unit is to help you learn a few different techniques. Since each exercise illustrates a different technique, we have decided to view each exercise in this unit as a learning exercise.

   As indicated at the end of Lecture 2.010, our approach is to test first for exactness, then to look for integrating factors and then to look for various other techniques. Our approach does not follow quite the same order as the reading material in the text. Consequently, we suggest the following approach.

2. Read (fairly quickly) the following sections of the Thomas text: 20.3, 20.4, 20.5, 20.6, and 20.7.

3. Then, re-read these sections in more detail after the appropriate learning exercise. More specifically, where appropriate, each exercise will tell the section of the text from which the exercise is drawn. After solving the exercise, read the appropriate section of the text, practicing on several of the exercises given there. There is no substitute for experience in learning how to solve differential equations.

4. Exercises:

2.2.1 (L) [Section 20.6]

Find the general solution of

$$(2xy + x^3)dx + (x^2 + y^2 + 1)dy = 0.$$

2.2.2 (L) [Section 20.3]

Find the general solution of

$$(1 + y^2)dx + (1 + x^2)dy = 0.$$

2.2.3 (L) [Section 20.4]

Find the general solution of

$$\frac{dy}{dx} = 1 + \frac{y}{x} + \frac{y^2}{x^2}$$

in the region R for which $x > 0$.

2.2.4 (optional)

This exercise is optional only because it is not exactly of a type solved in the reading material. However, it is a slight refinement of the homogeneous equation discussed in Section 20.4 and is worth trying if only to show you how even in "cookbook" situations some ingenuity is helpful.

a. Assuming that $a_1b_2 - a_2b_1 \neq 0$, find an appropriate substitution of the form

$$\left.\begin{aligned} x &= x_1 + h \\ y &= y_1 + k \end{aligned}\right\} \text{ h and k suitably are chosen constants}$$

which reduces

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

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2.2.4 continued

to an equation of the form

$$\frac{dy_1}{dx_1} = f\left(\frac{y_1}{x_1}\right).$$

b. Given that

$$\frac{dy}{dx} = \frac{10 - 2x + 2y}{2x - y - 9}$$

in any connected region R which excludes any portion of the line $y = 3x - 9$, find the general solution of the equation.

c. Solve the equation

$$(2x + 3y + 4)dx - (4x + 6y + 1)dy = 0.$$

d. Find the particular solution of the equation in part (c) which passes through (-2,1).

2.2.5(L) [Section 20.5]

a. Show that we can find an integrating factor u(x) [i.e., an integrating factor that is independent of y] if $Mdx + Ndy = 0$ is not exact but $M_y - N_x/N$ is a function of only x (not y). Find u(x) explicitly in this case.

b. Apply the method of part (a) to solve

$$(y - xe^x)dx - xdy = 0.$$

c. Use the method of part (a) to solve

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where P and Q are continuous functions of x.

d. Solve the equation $\frac{dy}{dx} - \frac{y}{x} = x^5$ $(x > 0)$.

2.2.6(L) [Section 20.5]

a. Show that there is one and only one solution of $dy/dx + p(x)y = g(x)$ through any given point (x_0, y_0) in the plane provided only that $p(x)$ and $g(x)$ are continuous functions of x.

b. Show that the 1-parameter family, $y = f(x) + c\,g(x)$ [where f and g are given differentiable functions of x, and c is an arbitrary constant] is always a solution of a first order linear differential equation.

c. Find the general solution of

$$\frac{dx}{dy} + \frac{x}{y} = y^6 \quad (y > 0).$$

d. The equation $dy/dx + y/x = x^3y^4$ is called a Bernoulli equation. Show that multiplying both sides of this equation by y^{-4} converts the equation into one which is linear. Then, solve the original equation.

2.2.7 (optional)

The following exercise involves a great deal of computational manipulation and a review of several ideas already discussed in this Block. Aside from this, the exercise, just as Exercise 2.2.4, tries to show how even after we know a few techniques we must often still resort to well-calculated guesses. Skipping this exercise does you no serious harm in the material which follows, except it is probably a good case to gain experience in handling "messy" situations.

Find all solutions of the equation

$$\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx} + 4y = 4x - 1.$$

2.2.8(L) [Section 20.7]

a. Find a curve which passes through (1, 3/2) with slope equal to 1 and which satisfies the equation

$$y''e^{y'} = e^x.$$

b. Find a solution of the equation

$$y\,y'' = (y')^2$$

which has the property that when $x = 0$, $y = 2$ and $y' = 4$. [I.e. find a curve which passes through (0,2) with slope 4, satisfying the given equation.]

Unit 3: (optional) Some Geometric Applications of First Order Equations

1. Overview

As you have probably long ago concluded, this course, in itself, is not designed from a practical application point of view. Nevertheless, there are certain topics that occupy a very important role in applications, and differential equations is one of the most important of these topics. Indeed, it has been said that the real world was written in the language of differential equations.

At any rate, for the student who does not care about applications this unit may be omitted without any loss of continuity. Even worse, for the student who does care about applications, this unit may also be omitted. The reason is that the types of applications of differential equations and the sophistication needed in deriving the equation are often of a very specialized nature. What is practical to one student may be impractical (or practical but beyond the realm) of another student.

For this reason we have adopted the attitude that the role of a mathematics course, at least at this level, is to teach the student the necessary mathematics; and it is the role of his field of interest to supply the equations to which the mathematical knowledge will be put to use.

As a compromise, we have used the rationalization that every first order equation has a geometric interpretation (since in the expression, say, dy/dx, we have no way of deciding a priori what physical quantities are named by x and y). Consequently, whatever applications we make in this unit (except for two optional exercises at the end of the exercises) are restricted to geometry.

For the student who is interested in applying differential equations to his other work, we strongly recommend that he do this unit. Otherwise, we must admit that this unit does nothing more than reinforce the computations done in the previous units. For this reason, the student who would like a

Overview continued

bit more experience before tackling higher order differential equations might prefer to try the exercises in this unit.

2. Exercises:

2.3.1(L)

a. Find the family of orthogonal trajectories to the family of curves $y = x + ce^{-x}$, where c is an arbitrary constant.

b. Find the member of $y = x + ce^{-x}$ and the member of the family of orthogonal trajectories which pass through (0,4).

c. Do the same as in part (b), only now let the members pass through (0,1).

2.3.2

Find the orthogonal trajectories of the family of parabolas, $y^2 = cx$, where c is an arbitrary constant.

2.3.3(L)

Find the family of curves which intersects every line of the form $y = mx$ at a 45° angle.

2.3.4

Find the family of curves with the following property. At each point P on the curve the angle made by the line tangent to the curve at P and the positive x-axis is twice the angle made up of the positive x-axis and the line OP where O is the origin of the coordinate plane.

2.3.5

A curve has the property that it passes through (0, 3/16) and its slope at each point P(x,y) is given by

$\frac{dy}{dx} = 3x - 4y$. Find the equation of this curve.

2.3.6(L)

Find all curves with the following property. The segment of the line tangent to the curve at P, between P and the x-axis, is bisected by the y-axis.

2.3.7

a. Give the equation for the family of curves with the property that at each point P(x,y) on any such curve the slope of the curve at P is equal to the y-intercept of the line tangent to the curve at P.

b. What members of this family pass through (2,9)?

The next two problems are optional and stress physical situations which may be translated geometrically.

2.3.8 (optional)

A plane mirror has the property that if any light ray from a point source O (the origin) strikes the mirror at any point P(x,y) the light is reflected parallel to the x-axis. Find the equation of the mirror.

2.3.9 (optional)

A boat A moves along the y-axis with a constant speed of <u>a</u> mph. A missile B moves in the right half of the xy-plane with a constant speed of <u>b</u> mph in such a way that B is always pointed directly toward A. Find the equation of the path followed by B.