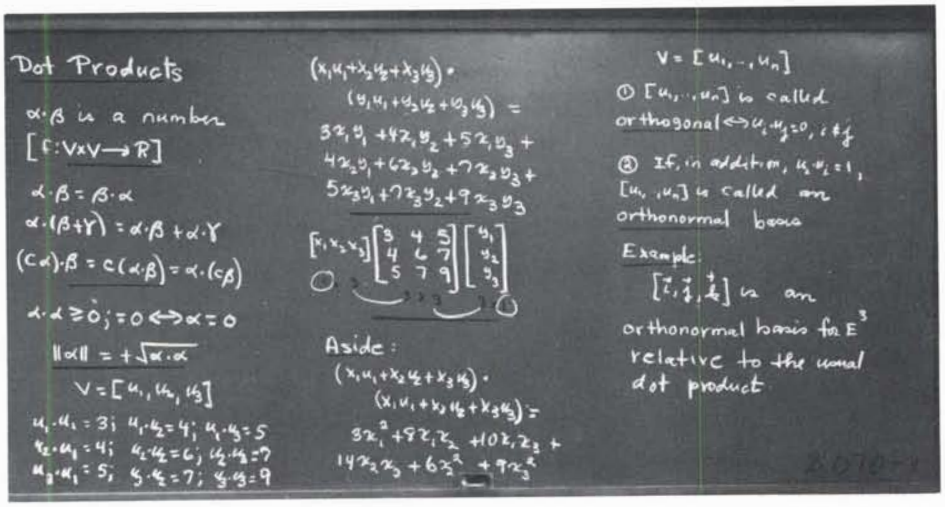
Unit 7: The Dot (Inner) Product

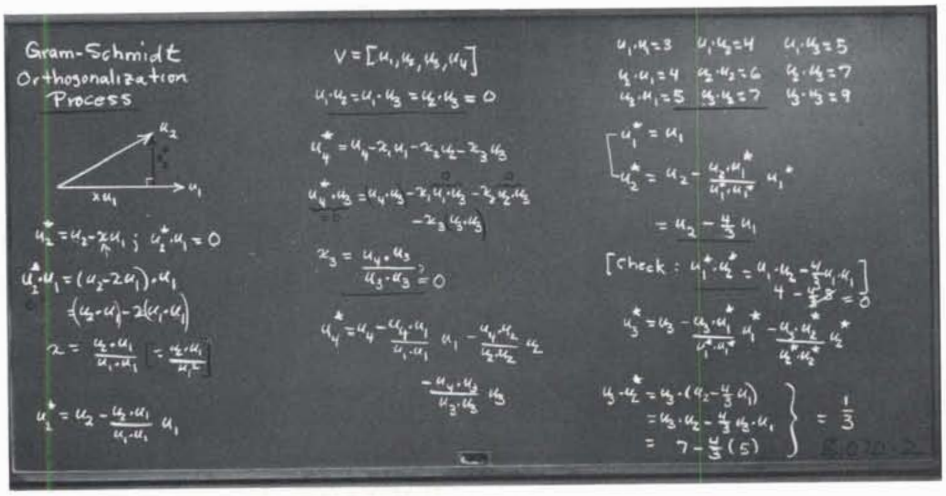
1. Overview

In this unit, we discuss how one may define the dot product on an n-dimensional vector space. The main point is that up to now there has been no need to define such a concept. That is, the concept of a vector space is well-defined without reference to a dot product. In this unit, however, we attempt to show how the added structure of the dot product gives us a better hold on the vector space.

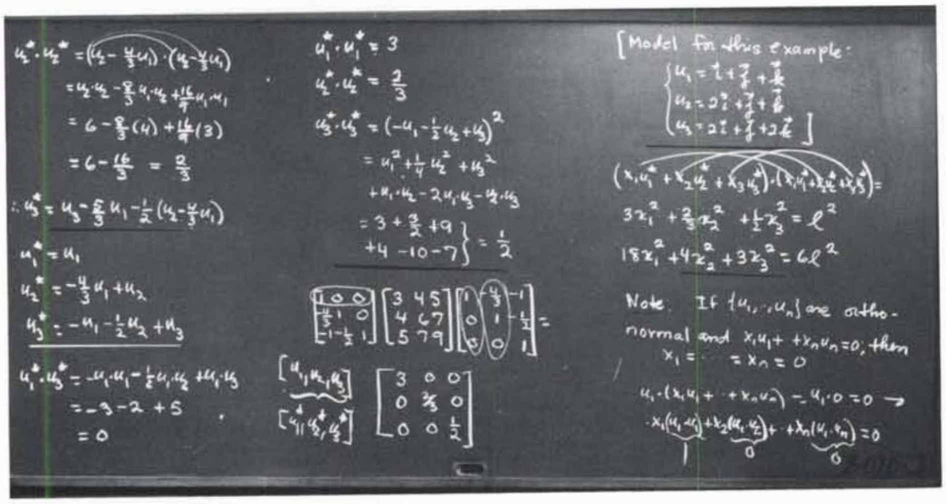
2. Lecture 3.070



a.



b.



c.

3. Do the exercises.

4. (Optional) Read Thomas, Chapter 13.

(Up to now, we have omitted this chapter for two reasons. For one thing, we felt it was too compact for a first exposure. The second reason was that the author develops Chapter 13 under the assumption that one always uses a dot product in the study of a vector space. To be sure, one often stresses the idea of a Euclidean space (i.e. a space in which there is defined a symmetric, positive definite, bilinear function) but the fact remains that the study of vector spaces is more general than this. Moreover, even after one assumes that he is dealing with a Euclidean Space, it is not made clear in the text that one must define the dot product in the usual term-by-term manner. What we have done in our treatment is to show that by the Gram-Schmidt Orthogonalization Process we may assume without loss of generality that we have chosen an orthonormal basis. Once this point is made clear, there is no harm in the special case defined in the text. At any rate, as an optional topic, it might make a good review session to now browse through Chapter 13 and use this as a concise overview of our treatment of vector spaces as developed in this block.)

5. Exercises:

3.7.1

Let $f: V \times V \to R$ be a bilinear function* and let $\alpha \cdot \beta$ denote $f(\alpha, \beta)$ where α and β belong to V. Show that

a. $\alpha \cdot \vec{0} = 0$ for all $\alpha \varepsilon V$

b. $(a_1\alpha_1 + a_2\alpha_2) \cdot (a_3\alpha_3 + a_4\alpha_4) = a_1a_3(\alpha_1 \cdot \alpha_3) + a_1a_4(\alpha_1 \cdot \alpha_4) + a_2a_3(\alpha_2 \cdot \alpha_3) + a_2a_4(\alpha_2 \cdot \alpha_4)$.

3.7.2

Let $V = [u_1, u_2]$, and let α and β denote arbitrary elements of V.

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*i.e., (i) $(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$; (ii) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$; (iii) $\alpha \cdot (c\beta) = (c\alpha) \cdot \beta = c(\alpha \cdot \beta)$.

3.7.2 continued

a. Show that $\alpha \cdot \beta$ is completely determined once we know the values of $u_1 \cdot u_1$, $u_1 \cdot u_2$, $u_2 \cdot u_1$, and $u_2 \cdot u_2$.

b. Suppose $u_1 \cdot u_1 = 1$, $u_1 \cdot u_2 = -1$, $u_2 \cdot u_1 = -5$, and $u_2 \cdot u_2 = 6$. Compute $\alpha \cdot \beta$ where $\alpha = 3u_1 + 2u_2$ and $\beta = u_1 + 4u_2$.

3.7.3

a. Suppose we now impose the additional property on our bilinear function that $\alpha \cdot \beta = \beta \cdot \alpha$ for all $\alpha, \beta \varepsilon V$, where $V = [u_1, u_2]$.

If $u_1 \cdot u_1 = 1$, $u_1 \cdot u_2 = 2$, and $u_2 \cdot u_2 = 3$, compute

$(x_1u_1 + x_2u_2) \cdot (y_1u_1 + y_2u_2)$.

b. Let $V = E^3$ and suppose $v_1 = \vec{i} + 2\vec{j} + 3\vec{k}$ and $v_2 = 2\vec{i} + 5\vec{j} - 2\vec{k}$. Find a vector $v_2{}^* \varepsilon V$ such that $S(v_1, v_2) = S(v_1, v_2{}^*)$ and $v_1 \cdot v_2{}^* = 0$ where now the dot product is the usual one.

c. Returning to part (a), find $u_2{}^*$ such that $S(u_1, u_2) = S(u_1, u_2{}^*)$, where $u_1 \cdot u_2{}^* = 0$.

3.7.4

Consider the bilinear function defined by

$$A = \begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$$

on the space $V = [u_1, u_2]$.

a. Compute $(x_1u_1 + x_2u_2) \cdot (y_1u_1 + y_2u_2)$, and in particular, compute $v \cdot v$ for each $v \varepsilon V$ where $v = x_1u_1 + x_2u_2$.

b. Find a vector $u_1{}^* \varepsilon V$ such that $S(u_1, u_2) = S(u_1{}^*, u_2)$ and $u_1{}^* \cdot u_2 = 0$. Moreover, prove that relative to $[u_1{}^*, u_2]$ the matrix of the bilinear function is

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3.7.4 continued

$$\begin{bmatrix} -\frac{1}{5} & 0 \\ 0 & 5 \end{bmatrix}.$$

c. $v \varepsilon V$ is called a null vector if $v \neq 0$ but $v \cdot v = 0$. Find all null vectors of V.

3.7.5

Let $V = [u_1, u_2, u_3]$ and let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

define a symmetric bilinear function on V relative to $\{u_1, u_2, u_3\}$.

a. Find a new basis for V, $\{u_1, u_2^*, u_3^*\}$ where u_2^* and u_3^* are linear combinations of u_1, u_2, and u_3, but such that $u_1 \cdot u_2^* = u_1 \cdot u_3^* = u_2^* \cdot u_3^* = 0$, while $u_1 \cdot u_1 = u_2^* \cdot u_2^* = u_3^* \cdot u_3^* = 1$.

b. If $v = x_1 u_1 + x_2 u_2^* + x_3 u_3^*$, show that $v \cdot v = 0 \leftrightarrow$

$$x_3^2 = x_1^2 + x_2^2.$$

c. Check the result of (b) by converting $v = 3u_1 + 4u_2^* + 5u_3^*$ into u_1, u_2, u_3-components and showing that $v \cdot v = 0$.

d. With A as in part (a), find a matrix B such that

$$BAB^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

e. Explain how the equation

$$x_1^2 + 2x_2^2 + 4x_3^2 + 2x_1x_2 + 2x_1x_3 + 6x_2x_3 = m$$

is equivalent to solving the equation

$$y_1^2 + y_2^2 - y_3^2 = m.$$

3.7.6

Let $V = [u_1, u_2, u_3]$ and let a dot product be defined on V by

$$A = \begin{bmatrix} 3 & 4 & 4 \\ 4 & 6 & 5 \\ 4 & 5 & 6 \end{bmatrix}.$$

Find a basis $\{u_1^*, u_2^*, u_3^*\}$ for V such that $u_1^* \cdot u_1^* = u_2^* \cdot u_2^* = u_3^* \cdot u_3^* = 1$ and $u_i^* \cdot u_j^* = 0$ if $i \neq j$.

3.7.7

Verify the construction of the orthogonal basis developed in the lecture of this unit by showing geometrically that $-u_1 - \frac{1}{2} u_2 + u_3$ is perpendicular to the plane determined by u_1 and u_2 where $u_1 = \vec{i} + \vec{j} + \vec{k}$, $u_2 = 2\vec{i} + \vec{j} + \vec{k}$, and $u_3 = 2\vec{i} + \vec{j} + \vec{k}$.

3.7.8

Consider the vector space $V = \{f: \int_a^b f(x)dx$ exists and f continuous$\}$ where a and b are given constants. Show that if we define $f \cdot g = \int_a^b f(x)g(x)dx$, for all f and g in V, then $f \cdot g$ is an inner product.

3.7.9

a. Suppose $\{u_1, \ldots, u_n\}$ is a set of orthogonal non-zero vectors and that $c_1 u_1 + \ldots + c_n u_n = 0$. Prove that $c_1 = \ldots = c_n = 0$.

b. (Optional) Suppose dim V = n and W is any proper subspace of V. Define $W_p = \{v \varepsilon V: v \cdot w = 0$ for each $w \varepsilon W\}$. Show that W_p is a proper subspace of V and that

$$V = W \oplus W_p.$$

3.7.10 (Optional)

Use the Gram-Schmidt Orthogonalization Process to find an orthogonal basis for $V = [u_1, u_2, u_3, u_4]$ if the dot product on V is defined by the matrix

$$\begin{bmatrix} 4 & 1 & 2 & 1 \\ 1 & 7 & 3 & 2 \\ 2 & 3 & 2 & 1 \\ 1 & 2 & 1 & 9 \end{bmatrix}.$$

(This exercise is optional only because of the amount of computational detail, but it is worth doing if only to see how the method works for spaces of dimension greater than three.)