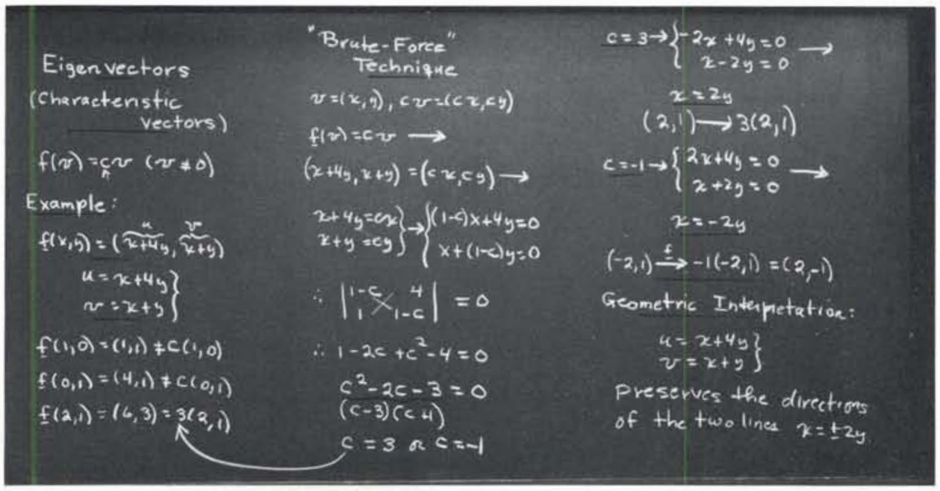
Unit 6: Eigenvectors (Characteristic Vectors)

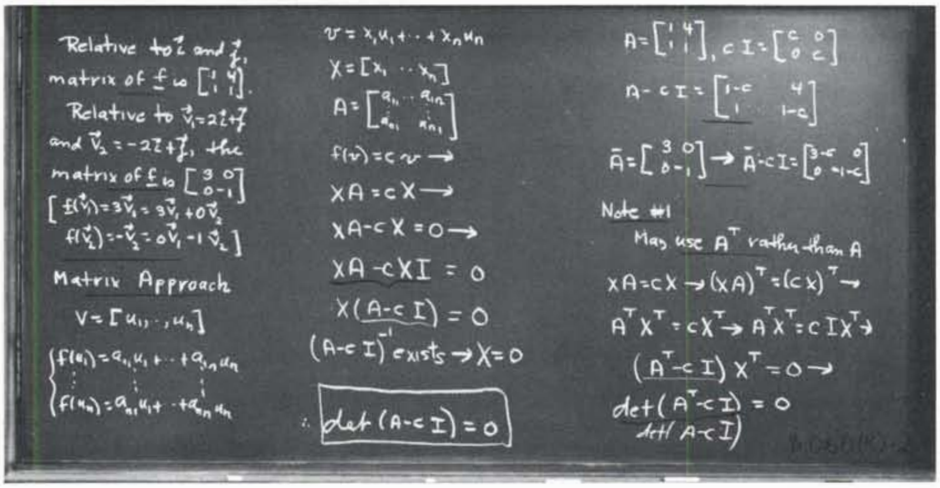
1. Overview

Armed with the computational know-how of the previous unit, we now return to the discussion begun in Unit 4 and investigate the existence of vectors v such that for a given linear transformation f, f(v) is a (non-zero) scalar multiple of v. Geometrically, this means that we seek vectors whose direction is preserved by the given linear transformation. Such a vector is called a characteristic vector (in German, an eigenvector), and the scalar c for which f(v) = cv is called an eigenvalue or a characteristic value.

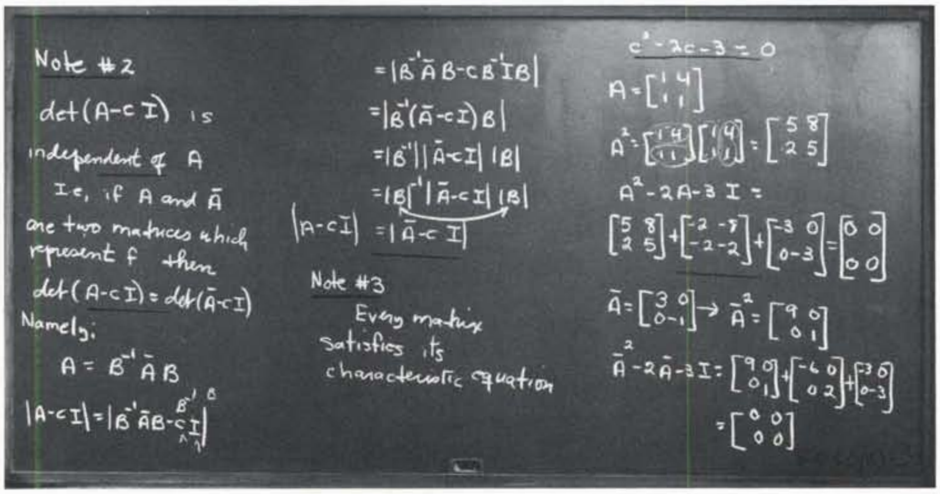
2. Lecture 3.060



a.



b.



c.

3. Exercises:

3.6.1(L)

a. Use the method shown in the lecture to find the eigenvectors of f if $f:V \to V$, where $V = [u_1, u_2]$, is the linear transformation defined by

$$f(u_1) = -3u_1 + 2u_2$$

$$f(u_2) = 4u_1 - u_2$$

b. Find a basis for V such that relative to this basis the matrix of f is diagonal.

c. Interpret the results of (a) and (b) geometrically in terms of f mapping the xy-plane onto the uv-plane.

3.6.2

Let $V = [u_1, u_2]$ and let the linear transformation $f:V \to V$ be defined by

$$f(u_1) = 8u_1 - 15u_2$$

$$f(u_2) = 2u_1 - 3u_2$$

Find the characteristic values of f and determine a basis for V which consists of eigenvectors. What is the matrix of f relative to this basis?

3.6.3(L)

If A is an n by n matrix and P is a non-singular n by n matrix, show that

$$|A - cI| = |PAP^{-1} - cI|.$$

3.6.4(L)

Let $V = [u_1, u_2]$ and let $f:V \to V$ be the linear transformation defined by

(continued on next page)

3.6.4(L) continued

$f(u_1) = (\cos\alpha)u_1 + (\sin\alpha)u_2$

$f(u_2) = (-\sin\alpha)u_1 + (\cos\alpha)u_2$

Show that f has no eigenvectors unless α is an integral multiple of π.

3.6.5

Let $V = [u_1,u_2,u_3,u_4]$ and let $f:V\to V$ be the linear transformation defined by

$f(u_1) = 8u_1 + 4u_3$

$f(u_2) = 9u_1 + 2u_2 + 6u_3$

$f(u_3) = -9u_1 - 4u_3$

$f(u_4) = 2u_2 + 3u_4$

a. Find the characteristic values of f.

b. Find a set of linear independent eigenvectors of f. In particular, describe the subspace V_2 of V where

$V_2 = \{v\varepsilon V: f(v) = cv\}$.

3.6.6(L)

a. Let $f:V\to V$ be a linear transformation. Suppose v_1 and v_2 are non-zero vectors in V, and that $f(v_1) = c_1v_1$ and $f(v_2) = c_2v_2$ where $c_1 \neq c_2$. Prove that $\{v_1,v_2\}$ is a linearly independent set.

b. Proceed inductively from (a) to show that if $v_3 \neq 0$ and if c_3 is unequal to 0, c_1, or c_2 and if $f(v_3) = c_3v_3$ then $\{v_1,v_2,v_3\}$ is a linearly independent set.

3.6.7(L)

Let $V = [u_1, u_2, u_3]$ and let $f:V \to V$ be the linear transformation defined by

$$f(u_1) = 2u_1 + u_3$$

$$f(u_2) = -u_1 + 2u_2 + 3u_3$$

$$f(u_3) = u_1 + 2u_3$$

a. Let A denote the transpose of the matrix of coefficients of f given above, and use A to find the characteristic values of f.

b. Find a basis for V which consists solely of eigenvectors of f.

c. What is the matrix of f relative to the basis found in part (b)?

d. Use the basis found in (b) to find a vector $\xi \varepsilon V$ such that $f(\xi) = v_o$, where v_o is a given vector in V.

e. Using the technique described in (d), find a vector $v \varepsilon V$ such that $f(v) = \alpha_1 + 4\alpha_2 + 12\alpha_3$ where $\{\alpha_1, \alpha_2, \alpha_3\}$ denotes the basis found in (b).

f. With A as in part (a), find a matrix P such that $P^{-1}AP = D$, where D is the diagonal matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

3.6.8 (Optional)

a. Let A be the same matrix as that given in part (a) of the previous exercise. Show that A satisfies the matrix equation

$$X^3 - 6X^2 + 11X - 6I = 0,$$

that is,

$$A^3 - 6A^2 + 11A - 6I = 0.$$

3.6.8 continued

How is this fact connected to the characteristic values of f where f is as given in the previous exercise?

b. In particular, use (a) to compute A^7 as a linear combination of I, A, and A^2.