

Unit 6: Eigenvectors (Characteristic Vectors)

1. Overview

Armed with the computational know-how of the previous unit, we now return to the discussion begun in Unit 4 and investigate the existence of vectors v such that for a given linear transformation f , $f(v)$ is a (non-zero) scalar multiple of v . Geometrically, this means that we seek vectors whose direction is preserved by the given linear transformation. Such a vector is called a characteristic vector (in German, an eigenvector), and the scalar c for which $f(v) = cv$ is called an eigenvalue or a characteristic value.

Study Guide
 Block 3: Selected Topics in Linear Algebra
 Unit 6: Eigenvectors (Characteristic Vectors)

2. Lecture 3.060

Eigenvectors
 (Characteristic Vectors)

$f(v) = cv$ ($v \neq 0$)

Example:
 $f(x,y) = \begin{pmatrix} x+4y \\ x+y \end{pmatrix}$
 $u = x+4y$
 $v = x+y$

$f(1,0) = (1,1) = c(1,0)$
 $f(0,1) = (4,1) = c(0,1)$
 $f(2,1) = (6,3) = 3(2,1)$

"Brute-Force" Technique
 $v = (x,y), cv = (cx,cy)$
 $f(v) = cv \rightarrow (x+4y, x+y) = (cx, cy) \rightarrow$
 $\begin{cases} x+4y = cx \\ x+y = cy \end{cases} \rightarrow \begin{cases} (1-c)x + 4y = 0 \\ x + (1-c)y = 0 \end{cases}$

$\therefore \begin{vmatrix} 1-c & 4 \\ 1 & 1-c \end{vmatrix} = 0$
 $\therefore 1-2c+c^2-4 = 0$
 $c^2-2c-3 = 0$
 $(c-3)(c+1) = 0$
 $c = 3 \text{ or } c = -1$

$c = 3 \rightarrow \begin{cases} 2x + 4y = 0 \\ x - 2y = 0 \end{cases} \rightarrow x = 2y$
 $(2,1) \rightarrow 3(2,1)$

$c = -1 \rightarrow \begin{cases} 2x + 4y = 0 \\ x + 2y = 0 \end{cases} \rightarrow x = -2y$
 $(-2,1) \rightarrow -1(-2,1) = (2,-1)$

Geometric Interpretation:
 $\begin{cases} u = x+4y \\ v = x+y \end{cases}$
 Preserves the directions of the two lines $x = \pm 2y$

a.

Relative to \vec{e}_1 and \vec{e}_2 , matrix of f is $\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$.

Relative to $\vec{v}_1 = 2\vec{e}_1 + \vec{e}_2$ and $\vec{v}_2 = -2\vec{e}_1 + \vec{e}_2$, the matrix of f is $\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$

$f(\vec{v}_1) = 3\vec{v}_1 = 3\vec{v}_1 + 0\vec{v}_2$
 $f(\vec{v}_2) = -\vec{v}_2 = 0\vec{v}_1 - 1\vec{v}_2$

Matrix Approach
 $v = [u_1, \dots, u_n]$
 $f(u_1) = a_{11}u_1 + \dots + a_{n1}u_n$
 $f(u_2) = a_{12}u_1 + \dots + a_{n2}u_n$

$v = x_1u_1 + \dots + x_nu_n$
 $X = [x_1 \dots x_n]$
 $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$
 $f(v) = cv \rightarrow XA = cX \rightarrow XA - cX = 0 \rightarrow X(A - cI) = 0$
 $(A - cI)^{-1}$ exists $\rightarrow X = 0$
 $\therefore \det(A - cI) = 0$

$A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}, cI = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$
 $A - cI = \begin{bmatrix} 1-c & 4 \\ 1 & 1-c \end{bmatrix}$
 $\bar{A} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow \bar{A} - cI = \begin{bmatrix} 3-c & 0 \\ 0 & -1-c \end{bmatrix}$

Note #1
 May use A^T rather than A
 $XA = cX \rightarrow (XA)^T = (cX)^T \rightarrow A^T X^T = cX^T \rightarrow A^T X^T - cX^T = 0 \rightarrow (A^T - cI) X^T = 0 \rightarrow \det(A^T - cI) = 0$
 iff $\det(A - cI) = 0$

b.

Note #2
 $\det(A - cI)$ is independent of A
 i.e., if A and \bar{A} are two matrices which represent f then $\det(A - cI) = \det(\bar{A} - cI)$
 Namely:
 $A = B^{-1}\bar{A}B$
 $|A - cI| = |B^{-1}\bar{A}B - cI|$

$= |B^{-1}\bar{A}B - cB^{-1}IB|$
 $= |B^{-1}(\bar{A} - cI)B|$
 $= |B^{-1}| |\bar{A} - cI| |B|$
 $= |B^{-1}| |\bar{A} - cI| |B|$
 $|A - cI| = |\bar{A} - cI|$

Note #3
 Every matrix satisfies its characteristic equation

$c^2 - 2c - 3 = 0$
 $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$
 $A^2 = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 2 & 5 \end{bmatrix}$
 $A^2 - 2A - 3I = \begin{bmatrix} 5 & 8 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
 $\bar{A} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow \bar{A}^2 = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$
 $\bar{A}^2 - 2\bar{A} - 3I = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -6 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

c.

3. Exercises:

3.6.1(L)

- a. Use the method shown in the lecture to find the eigenvectors of f if $f:V \rightarrow V$, where $V = [u_1, u_2]$, is the linear transformation defined by

$$f(u_1) = -3u_1 + 2u_2$$

$$f(u_2) = 4u_1 - u_2$$

- b. Find a basis for V such that relative to this basis the matrix of f is diagonal.
- c. Interpret the results of (a) and (b) geometrically in terms of f mapping the xy -plane onto the uv -plane.

3.6.2

Let $V = [u_1, u_2]$ and let the linear transformation $f:V \rightarrow V$ be defined by

$$f(u_1) = 8u_1 - 15u_2$$

$$f(u_2) = 2u_1 - 3u_2$$

Find the characteristic values of f and determine a basis for V which consists of eigenvectors. What is the matrix of f relative to this basis?

3.6.3(L)

If A is an n by n matrix and P is a non-singular n by n matrix, show that

$$|A - cI| = |PAP^{-1} - cI|.$$

3.6.4(L)

Let $V = [u_1, u_2]$ and let $f:V \rightarrow V$ be the linear transformation defined by

(continued on next page)

3.6.4(L) continued

$$f(u_1) = (\cos \alpha)u_1 + (\sin \alpha)u_2$$

$$f(u_2) = (-\sin \alpha)u_1 + (\cos \alpha)u_2$$

Show that f has no eigenvectors unless α is an integral multiple of π .

3.6.5

Let $V = [u_1, u_2, u_3, u_4]$ and let $f: V \rightarrow V$ be the linear transformation defined by

$$f(u_1) = 8u_1 + 4u_3$$

$$f(u_2) = 9u_1 + 2u_2 + 6u_3$$

$$f(u_3) = -9u_1 - 4u_3$$

$$f(u_4) = 2u_2 + 3u_4$$

- Find the characteristic values of f .
- Find a set of linear independent eigenvectors of f . In particular, describe the subspace V_2 of V where

$$V_2 = \{v \in V : f(v) = cv\}.$$

3.6.6(L)

- Let $f: V \rightarrow V$ be a linear transformation. Suppose v_1 and v_2 are non-zero vectors in V , and that $f(v_1) = c_1v_1$ and $f(v_2) = c_2v_2$ where $c_1 \neq c_2$. Prove that $\{v_1, v_2\}$ is a linearly independent set.
- Proceed inductively from (a) to show that if $v_3 \neq 0$ and if c_3 is unequal to 0, c_1 , or c_2 and if $f(v_3) = c_3v_3$ then $\{v_1, v_2, v_3\}$ is a linearly independent set.

3.6.7(L)

Let $V = [u_1, u_2, u_3]$ and let $f: V \rightarrow V$ be the linear transformation defined by

$$f(u_1) = 2u_1 + u_3$$

$$f(u_2) = -u_1 + 2u_2 + 3u_3$$

$$f(u_3) = u_1 + 2u_3$$

- Let A denote the transpose of the matrix of coefficients of f given above, and use A to find the characteristic values of f .
- Find a basis for V which consists solely of eigenvectors of f .
- What is the matrix of f relative to the basis found in part (b)?
- Use the basis found in (b) to find a vector $\xi \in V$ such that $f(\xi) = v_0$, where v_0 is a given vector in V .
- Using the technique described in (d), find a vector $v \in V$ such that $f(v) = \alpha_1 + 4\alpha_2 + 12\alpha_3$ where $\{\alpha_1, \alpha_2, \alpha_3\}$ denotes the basis found in (b).
- With A as in part (a), find a matrix P such that $P^{-1}AP = D$, where D is the diagonal matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} .$$

3.6.8 (Optional)

- Let A be the same matrix as that given in part (a) of the previous exercise. Show that A satisfies the matrix equation

$$X^3 - 6X^2 + 11X - 6I = 0,$$

that is,

$$A^3 - 6A^2 + 11A - 6I = 0.$$

3.6.8 continued

How is this fact connected to the characteristic values of f where f is as given in the previous exercise?

- b. In particular, use (a) to compute A^7 as a linear combination of I , A , and A^2 .

MIT OpenCourseWare
<http://ocw.mit.edu>

Resource: Calculus Revisited: Complex Variables, Differential Equations, and Linear Algebra
Prof. Herbert Gross

The following may not correspond to a particular course on MIT OpenCourseWare, but has been provided by the author as an individual learning resource.

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.