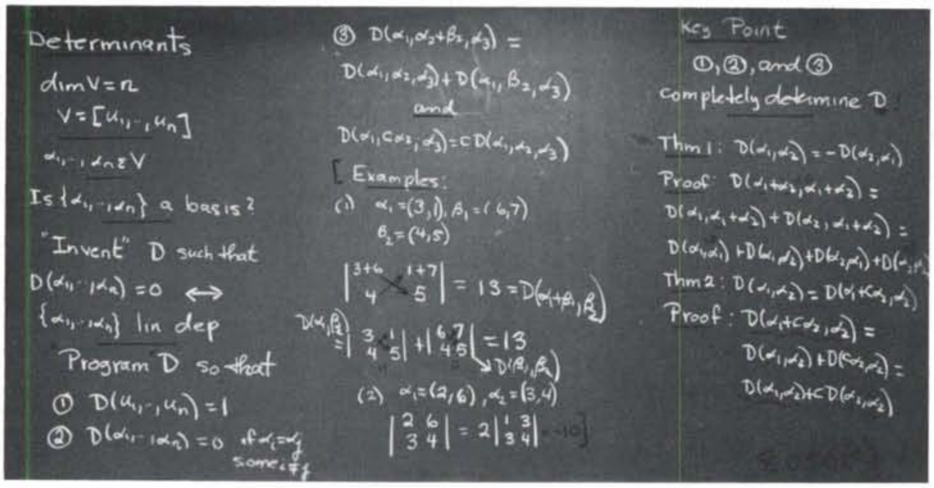
Unit 5: Determinants

1. Overview

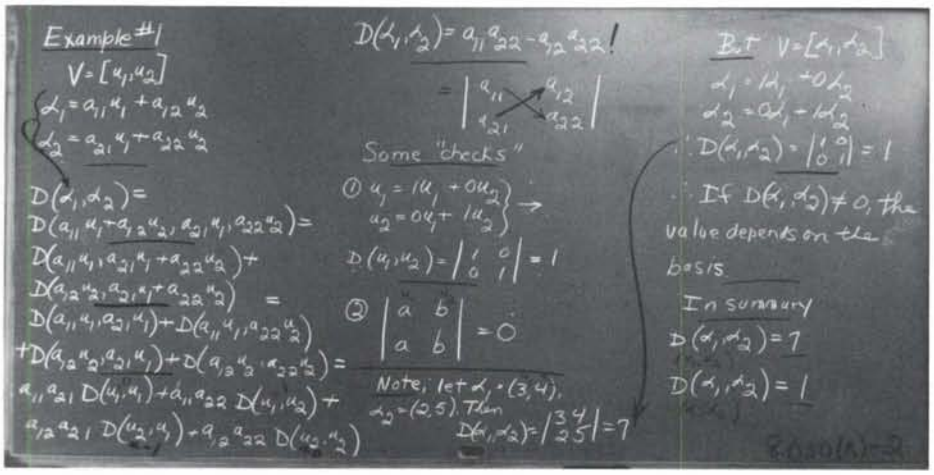
The usual "pre-new-math" traditional treatment of determinants is quite sterile and often subject to misinterpretation. It seems that the reasons for teaching determinants on the pre-calculus level are often better handled by the technique of row-reducing matrices. Our aim in this unit is to explain why determinants are important; what they are designed to do; and how one works with them efficiently.

In a sense, this treatment could have come at a different place in our course, but we prefer to interrupt the theme developed in the previous unit in order to introduce matrices now. After this unit, we shall continue with our study of linear transformations, but hopefully we shall be able to handle things much more elegantly because of our knowledge of determinants.

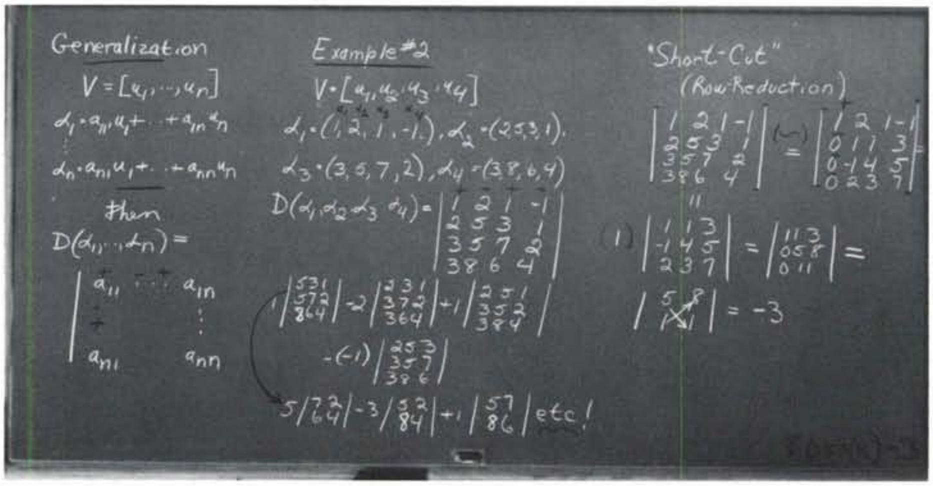
2. Lecture 3.050



a.



b.



c.

3. Read: Thomas, Appendix I. (optional). This may be read before or after doing the exercises, or it may be omitted entirely if you so desire. Our thought is that the treatment in the text is relatively compact and complete (at least in so far as the various recipes are concerned). Consequently, it may serve to tie together any loose ends that might still seem to be dangling to you after our own treatment of the subject.

4. Exercises:

3.5.1(L)

a. Compute

$$\begin{vmatrix} 3+2 & 1+5 \\ 4 & 7 \end{vmatrix}$$

by writing it as the sum of two "simpler" 2 by 2 determinants.

b. Use a similar technique to evaluate

$$\begin{vmatrix} 3+2 & 1+5 \\ 4+6 & 7+9 \end{vmatrix}$$

as the sum of four 2 by 2 determinants.

c. Let $A = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 5 \\ 6 & 9 \end{bmatrix}$. Show that $|A| + |B| \neq |A + B|$.

3.5.2(L)

a. Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

3.5.2(L) continued

Form the matrix AB and compute its determinant by using the method of part (b) in Exercise 3.5.1. In particular, show that $|AB| = |A|\ |B|$.

b. Show that if the n by n matrix is invertible, then

$$|A^{-1}| = |A|^{-1}.$$

c. Verify part (b) by direct computation in the specific case that

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix}.$$

3.5.3(L)

Show that if P is any invertible n by n matrix and A is any n by n matrix, then $|PAP^{-1}| = |A|$.

3.5.4 (optional)

Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

a. Show that $|A + I| = |A| + |I| \leftrightarrow a_{11} + a_{22} = 0$.

b. The trace of a square matrix is the sum of its diagonal elements. If B is invertible, it turns out that $B^{-1}AB$ and A have the same trace. Verify this in the special case where A is as above and

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}.$$

3.5.5(L)

Consider

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 6 \\ 2 & 5 & 8 & 9 \\ 3 & 4 & 9 & 8 \end{vmatrix}.$$

a. Evaluate this determinant by using the method of co-factors until it is reduced to a sum of 2 by 2 determinants.

b. Evaluate the same determinant but now by using row-reduction techniques that replace all but the first entry of the first column by 0.

3.5.6

Evaluate

$$\begin{vmatrix} 1 & 1 & 1 & 2 & 3 \\ 2 & 3 & 2 & 4 & 5 \\ 2 & 5 & 3 & 4 & 7 \\ 3 & 4 & 4 & 4 & 8 \\ 4 & 9 & 2 & 3 & 9 \end{vmatrix}$$

by the technique of Exercise 3.5.5, part (b).

3.5.7 (optional)

a. Prove directly from our axioms that

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & 0 \\ a_{21} & 0 \end{vmatrix} + \begin{vmatrix} a_{11} & 0 \\ 0 & a_{22} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} \\ a_{21} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} \\ 0 & a_{22} \end{vmatrix}$$

$$= a_{11}a_{22} - a_{21}a_{12}.$$

(continued on next page)

3.5.7 continued

b. Derive a similar result for

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

c. Count the inversions to determine the value of

$$\begin{vmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix}$$

3.5.8 (optional)

Use the various theorems at our disposal to show that

a. $$\begin{vmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{vmatrix} = (r_3 - r_2)(r_3 - r_1)(r_2 - r_1).$$

b. $$\begin{vmatrix} 1 & 1 & 1 & 1 \\ r_1 & r_2 & r_3 & r_4 \\ r_1^2 & r_2^2 & r_3^2 & r_4^2 \\ r_1^3 & r_2^3 & r_3^3 & r_4^3 \end{vmatrix} = (r_4 - r_3)(r_4 - r_2)(r_4 - r_1)r_3 - r_2)(r_3 - r_1)(r_2 - r_1).$$