

Unit 2: The Dimension of a Vector Space

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1. Overview

In this unit, we analyze the notion of what is meant by the dimension of a vector space. The basic idea involves finding the fewest number of vectors that span the given space, and this, in turn, involves some knowledge of the concept of linear independence.

Study Guide  
 Block 3: Selected Topics in Linear Algebra  
 Unit 2: The Dimension of a Vector Space

2. Lecture 3.020

**Spanning Vectors**  
 $\alpha_1, \dots, \alpha_n \in V$ . Then  
 $S(\alpha_1, \dots, \alpha_n) = \left\{ \sum_{i=1}^n c_i \alpha_i : c_i \in \mathbb{R} \right\}$   
 is a subspace of  $V$ , called the space spanned by  $\alpha_1, \dots, \alpha_n$ .

**Example 1 ( $n=1$ )**  
 Pick  $\alpha_1 \in V$   
 $S(\alpha_1) = \{c\alpha_1 : c \text{ real}\}$

$\beta, \gamma \in S(\alpha_1) \rightarrow$   
 $\beta = c_1 \alpha_1, \gamma = c_2 \alpha_1 \rightarrow$   
 $\beta + \gamma = (c_1 + c_2) \alpha_1 \in S(\alpha_1)$   
 $r\beta = r(c_1 \alpha_1) = (rc_1) \alpha_1 \in S(\alpha_1)$   
 [Note: Geometrically  $S(\alpha_1)$  may be viewed as a line]  
 Key Point  
 $S(\alpha_1, \dots, \alpha_n)$  need not be "larger than"  $S(\alpha_1, \dots, \alpha_{n-1})$

**Example 2**  
 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$   
 $S(\alpha_1, \alpha_2) = \{c_1 \alpha_1 + c_2 \alpha_2\}$   
 $S(\alpha_1, \alpha_2, \alpha_3) = \{c_1 \alpha_1 + c_2 \alpha_2 + c_3 \alpha_3\}$   
 $\therefore S(\alpha_1, \alpha_2) \subseteq S(\alpha_1, \alpha_2, \alpha_3)$   
 $\beta \in S(\alpha_1, \alpha_2, \alpha_3) \rightarrow$   
 $\beta = c_1 \alpha_1 + c_2 \alpha_2 + c_3 \alpha_3$   
 $= c_1 \alpha_1 + c_2 \alpha_2 + c_3 (\alpha_1 + \alpha_2)$   
 $= (c_1 + c_3) \alpha_1 + (c_2 + c_3) \alpha_2$   
 $\therefore \beta \in S(\alpha_1, \alpha_2)$ , or:  
 $S(\alpha_1, \alpha_2, \alpha_3) = S(\alpha_1, \alpha_2)$

a.

**Definition**  
 $\alpha_1, \dots, \alpha_n \in V$  are called linearly dependent  $\leftrightarrow$   
 $\alpha_k = \sum_{i=1}^k a_i \alpha_i$   
 otherwise they are linearly independent

**Example 3**  
 $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, 5\begin{pmatrix} 1 \\ 1 \end{pmatrix}$   
 and  $\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$  are lin. dep since  $5\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

**Equivalent Def**  
 $\alpha_1, \dots, \alpha_n$  are lin. indep  $\leftrightarrow$   
 $c_1 \alpha_1 + \dots + c_n \alpha_n = 0 \rightarrow$   
 $c_1 = \dots = c_n = 0$   
 Linear Dependence implies Redundancy

**Key Point**  
 If  $\beta \in S(\alpha_1, \dots, \alpha_n)$  and  $\alpha_1, \dots, \alpha_n$  are linearly independent then:  
 $\beta$  has unique representation in the form  $\beta = \sum_{i=1}^n a_i \alpha_i$ ; i.e.  
 $a_1 \alpha_1 + \dots + a_n \alpha_n = b_1 \alpha_1 + \dots + b_n \alpha_n \rightarrow$   
 $(a_1 - b_1) \alpha_1 + \dots + (a_n - b_n) \alpha_n = 0 \rightarrow$   
 $a_1 - b_1 = 0, \dots, a_n - b_n = 0 \rightarrow$   
 $a_1 = b_1, \dots, a_n = b_n$   
 [In terms of Example 2  
 $\alpha_3 = 0\alpha_1 + 0\alpha_2 + 1\alpha_3$  are lin. dep  
 $= 1\alpha_1 + 1\alpha_2 + 0\alpha_3$   
 $= c_1 \alpha_1 + c_2 \alpha_2$   
 $(1-c_1) \alpha_3$ ]

b.

**Dimension Revisited**  
 $V \neq \{0\}; \alpha_i \in V, \alpha_i \neq 0$   
 $S(\alpha_i) \in V$   
 If  $S(\alpha_i) \neq V$ , pick  $\alpha_2 \in V - S(\alpha_1)$  and form  $S(\alpha_1, \alpha_2) \subseteq S(\alpha_1)$   
 since  $\alpha_2 \notin S(\alpha_1)$  but  $\alpha_2 = 0\alpha_1 + 1\alpha_2 \in S(\alpha_1, \alpha_2)$   
 $\alpha_1$  and  $\alpha_2$  are lin. indep since  $\alpha_2 = c\alpha_1 \rightarrow \alpha_2 \in S(\alpha_1)$

If  $S(\alpha_1, \alpha_2) \neq V$ , pick  $\alpha_3 \in V - S(\alpha_1, \alpha_2)$   
 $\alpha_1, \alpha_2, \alpha_3$  are lin. indep since  $\alpha_3 = c_1 \alpha_1 + c_2 \alpha_2 \rightarrow \alpha_3 \in S(\alpha_1, \alpha_2) \rightarrow$  contradiction

**Pictorially**

Continue in this way until we find  $\alpha_1, \dots, \alpha_n$  such that  $V = S(\alpha_1, \dots, \alpha_n)$  [which need not happen]  
 We then say that  $V$  has dimension  $n$  (written  $\dim V = n$ ) and  $\{\alpha_1, \dots, \alpha_n\}$  is called a **basis** for  $V$  since each  $v \in V$  has unique representation as a linear comb. of  $\alpha_1, \dots, \alpha_n$   
 Otherwise,  $V$  is called an infinite dimensional vector space

c.

3. Exercises:

3.2.1(L)

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Let  $V$  be the vector space of 4-tuples (relative to a particular set of four vectors) and let  $\alpha_1 \in V$  be defined by  $\alpha_1 = (1, 2, 3, 4)$ .

- Describe the space  $S(\alpha_1)$  [i.e. the space spanned by  $\alpha_1$ ] and show that  $\alpha_2 = (2, 5, 7, 7) \notin S(\alpha_1)$ .
- Describe the space  $S(\alpha_1, \alpha_2)$ .

[Note: At this time, Exercise 3.1.10 of the previous unit should no longer be viewed as optional. If you have not done this exercise before, you should do it now, especially if the notation  $S(\alpha_1, \alpha_2)$  is strange to you.]

- Show that  $\alpha_3 = (3, 7, 8, 9) \notin S(\alpha_1, \alpha_2)$ .
- For what value(s) of  $y$  and  $z$  does  $(3, 7, y, z)$  belong to  $S(\alpha_1, \alpha_2)$ ?

3.2.2(L)

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- Let  $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in V$ . Show that

$$S(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = S(\alpha_1, \alpha_3, \alpha_4, \alpha_2).$$

- Show that  $S(\alpha_1, \alpha_2, \alpha_3) = S(3\alpha_1, \alpha_2, \alpha_3)$ .
- Show that  $S(\alpha_1, \alpha_2, \alpha_3) = S(\alpha_1 + \alpha_2, \alpha_2, \alpha_3)$ .

3.2.3(L)

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Let  $\alpha_1 = (1, 2, 3, 4)$ ,  $\alpha_2 = (2, 5, 7, 7)$ , and  $\alpha_3 = (3, 7, 8, 9)$ .

- Use the row-reduced matrix technique to find  $\beta_1, \beta_2, \beta_3$  such that  $(x_1, x_2, x_3, x_4) \in S(\alpha_1, \alpha_2, \alpha_3)$  if and only if  $(x_1, x_2, x_3, x_4) = x_1\beta_1 + x_2\beta_2 + x_3\beta_3$ .
- Using part (a), show how  $x_4$  must be related to  $x_1, x_2$ , and  $x_3$  if  $(x_1, x_2, x_3, x_4) \in S(\alpha_1, \alpha_2, \alpha_3)$ . In particular, show that  $(4, 9, 13, 14) \notin S(\alpha_1, \alpha_2, \alpha_3)$ .
- Show that  $\{\beta_1, \beta_2, \beta_3\}$ , where the  $\beta$ 's are as in part (a), is a linearly independent set.

3.2.4(L)

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Let  $\alpha_1 = (1, 2, 3, 4)$ ,  $\alpha_2 = (2, 3, 5, 5)$ ,  $\alpha_3 = (2, 4, 7, 6)$ , and  $\alpha_4 = (-1, 2, 3, 4)$ .

- Use the row-reduced matrix technique to determine  $S(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ . In particular, show that  $(x_1, x_2, x_3, x_4) \in S(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$  if and only if  $x_4 = 5x_2 - 2x_3$ .
- Show that  $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$  is linearly dependent by exhibiting  $\alpha_4$  as a linear combination of  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_3$ .
- Express  $(4, 7, 12, 11)$  as a linear combination of  $\alpha_1$ ,  $\alpha_2$ ,  $\alpha_3$ , and  $\alpha_4$ .

3.2.5(L)

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Let  $\alpha_1 = (1, 2, 3)$ ,  $\alpha_2 = (2, 4, 6)$ ,  $\alpha_3 = (3, 7, 8)$ ,  $\alpha_4 = (1, 3, 2)$ , and  $\alpha_5 = (1, -2, 7)$ .

- Show that  $S(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = S(\beta_1, \beta_2)$  where  $\beta_1 = (1, 0, 5)$  and  $\beta_2 = (0, 1, -1)$ . In particular, what is the dimension of  $S(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ ?
- Express  $\beta_1$  and  $\beta_2$  as linear combinations of  $\alpha_1$  and  $\alpha_3$ . Also show how  $\alpha_2$ ,  $\alpha_4$ , and  $\alpha_5$  may be expressed as linear combinations of  $\alpha_1$  and  $\alpha_3$ .

3.2.6(L)

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- Let  $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m \in V$ . Suppose that  $\{\alpha_1, \dots, \alpha_n\}$  span  $V$  and that  $\{\beta_1, \dots, \beta_m\}$  is linearly independent. By appropriately investigating the set  $\{\beta_1, \dots, \beta_m, \alpha_1, \dots, \alpha_n\}$ , in the given order, conclude that  $n \geq m$ .
- Let  $\alpha_1 = (1, 1, 1, 1, 1)$ ,  $\alpha_2 = (1, 2, 2, 3, 3)$  and  $\alpha_3 = (2, 3, 4, 3, 6)$ . Augmenting  $\alpha_1, \alpha_2, \alpha_3$  by  $u_1 = (1, 0, 0, 0, 0)$ ,  $u_2 = (0, 1, 0, 0, 0)$ ,  $u_3 = (0, 0, 1, 0, 0)$ ,  $u_4 = (0, 0, 0, 1, 0)$ , and  $u_5 = (0, 0, 0, 0, 1)$  in the given order, construct a basis for  $E^5$  which includes  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_3$  by using the row-reduced matrix technique.

3.2.7

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Let  $\alpha_1 = (1, 3, -1, 2)$ ,  $\alpha_2 = (2, 0, 1, 3)$ ,  $\alpha_3 = (-1, 1, 0, 0)$ .

(continued on next page)

3.2.7 continued

- a. Show that  $(x_1, x_2, x_3, x_4) \in S(\alpha_1, \alpha_2, \alpha_3) \leftrightarrow 5x_1 + 5x_2 + 8x_3 - 6x_4 = 0$ .
- b. What is the dimension of  $S(\alpha_1, \alpha_2, \alpha_3)$  and what is a natural basis for  $S(\alpha_1, \alpha_2, \alpha_3)$ ? [That is, find  $\beta_1, \beta_2, \beta_3$  such that  $(x_1, x_2, x_3, x_4) \in S(\alpha_1, \alpha_2, \alpha_3) \leftrightarrow (x_1, x_2, x_3, x_4) = x_1\beta_1 + x_2\beta_2 + x_3\beta_3$ .]

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3.2.8

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Let  $\alpha_1 = (1, 2, 3)$ ,  $\alpha_2 = (2, 5, 4)$ ,  $\alpha_3 = (3, 8, 9)$ , and  $\alpha_4 = (4, 9, 9)$ .

- a. Show that the dimension of  $S(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 3$ . In particular, express  $(0, 0, 1)$ ,  $(0, 1, 0)$ , and  $(1, 0, 0)$  as a linear combination of  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_4$ .
- b. Express  $\alpha_3$  as a linear combination of  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_4$ .
- c. Express  $(2, 1, 4)$  as a linear combination of  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_4$ .

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3.2.9(L)

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Show that the space of all polynomials cannot have finite dimension.

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