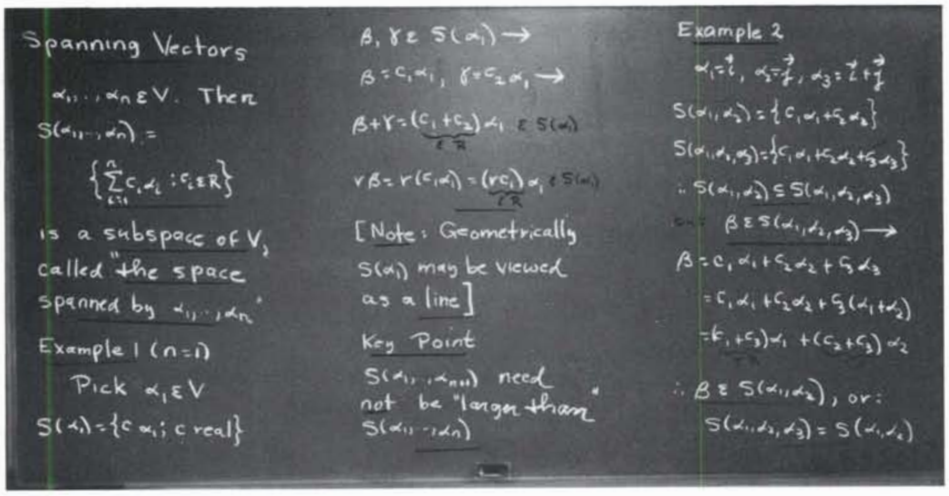
Unit 2: The Dimension of a Vector Space

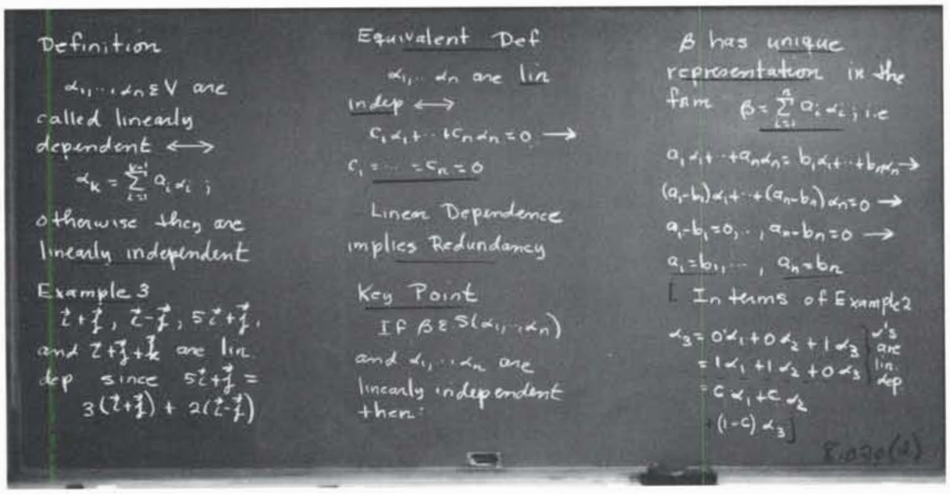
1. Overview

In this unit, we analyze the notion of what is meant by the dimension of a vector space. The basic idea involves finding the fewest number of vectors that span the given space, and this, in turn, involves some knowledge of the concept of linear independence.

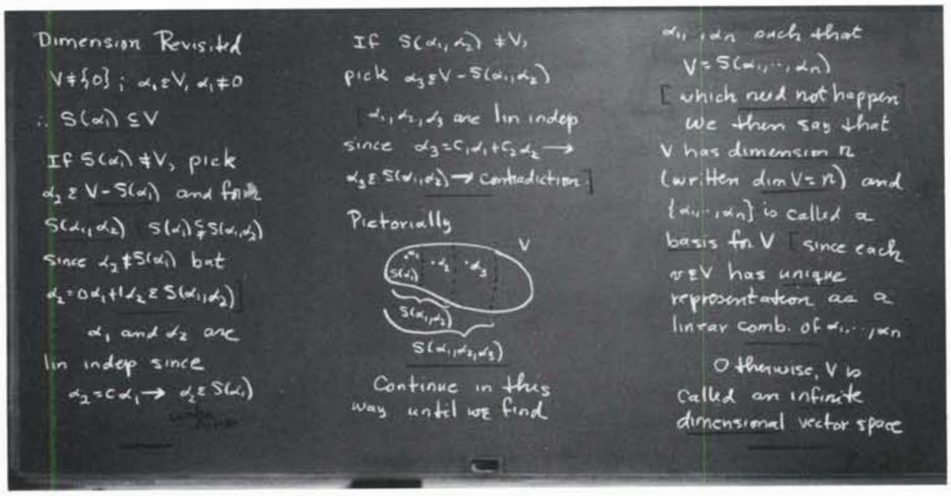
2. Lecture 3.020



a.



b.



c.

3. Exercises:

3.2.1(L)

Let V be the vector space of 4-tuples (relative to a particular set of four vectors) and let $\alpha_1 \varepsilon V$ be defined by $\alpha_1 = (1,2,3,4)$.

a. Describe the space $S(\alpha_1)$ [i.e. the space spanned by α_1] and show that $\alpha_2 = (2,5,7,7) \notin S(\alpha_1)$.

b. Describe the space $S(\alpha_1,\alpha_2)$.

[Note: At this time, Exercise 3.1.10 of the previous unit should no longer be viewed as optional. If you have not done this exercise before, you should do it now, especially if the notation $S(\alpha_1,\alpha_2)$ is strange to you.]

c. Show that $\alpha_3 = (3,7,8,9) \notin S(\alpha_1,\alpha_2)$.

d. For what value(s) of y and z does $(3,7,y,z)$ belong to $S(\alpha_1,\alpha_2)$?

3.2.2(L)

a. Let $\alpha_1,\alpha_2,\alpha_3,\alpha_4 \varepsilon V$. Show that

$$S(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = S(\alpha_1,\alpha_3,\alpha_4,\alpha_2).$$

b. Show that $S(\alpha_1,\alpha_2,\alpha_3) = S(3\alpha_1,\alpha_2,\alpha_3)$.

c. Show that $S(\alpha_1,\alpha_2,\alpha_3) = S(\alpha_1 + \alpha_2,\alpha_2,\alpha_3)$.

3.2.3(L)

Let $\alpha_1 = (1,2,3,4)$, $\alpha_2 = (2,5,7,7)$, and $\alpha_3 = (3,7,8,9)$.

a. Use the row-reduced matrix technique to find β_1,β_2,β_3 such that $(x_1,x_2,x_3,x_4) \varepsilon S(\alpha_1,\alpha_2,\alpha_3)$ if and only if $(x_1,x_2,x_3,x_4) = x_1\beta_1 + x_2\beta_2 + x_3\beta_3$.

b. Using part (a), show how x_4 must be related to x_1, x_2, and x_3 if $(x_1,x_2,x_3,x_4) \varepsilon S(\alpha_1,\alpha_2,\alpha_3)$. In particular, show that $(4,9,13,14) \notin S(\alpha_1,\alpha_2,\alpha_3)$.

c. Show that $\{\beta_1,\beta_2,\beta_3\}$, where the β's are as in part (a), is a linearly independent set.

3.2.4(L)

Let $\alpha_1 = (1,2,3,4)$, $\alpha_2 = (2,3,5,5)$, $\alpha_3 = (2,4,7,6)$, and $\alpha_4 = (-1,2,3,4)$.

a. Use the row-reduced matrix technique to determine $S(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$. In particular, show that $(x_1,x_2,x_3,x_4) \in S(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ if and only if $x_4 = 5x_2 - 2x_3$.

b. Show that $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$ is linearly dependent by exhibiting α_4 as a linear combination of α_1, α_2, and α_3.

c. Express (4,7,12,11) as a linear combination of α_1, α_2, α_3, and α_4.

3.2.5(L)

Let $\alpha_1 = (1,2,3)$, $\alpha_2 = (2,4,6)$, $\alpha_3 = (3,7,8)$, $\alpha_4 = (1,3,2)$, and $\alpha_5 = (1,-2,7)$.

a. Show that $S(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) = S(\beta_1,\beta_2)$ where $\beta_1 = (1,0,5)$ and $\beta_2 = (0,1,-1)$. In particular, what is the dimension of $S(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)$?

b. Express β_1 and β_2 as linear combinations of α_1 and α_3. Also show how α_2, α_4, and α_5 may be expressed as linear combinations of α_1 and α_3.

3.2.6(L)

a. Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \in V$. Suppose that $\{\alpha_1, \ldots, \alpha_n\}$ span V and that $\{\beta_1, \ldots, \beta_m\}$ is linearly independent. By appropriately investigating the set $\{\beta_1, \ldots, \beta_m, \alpha_1, \ldots, \alpha_n\}$, <u>in the given order</u>, conclude that $n \geqslant m$.

b. Let $\alpha_1 = (1,1,1,1,1)$, $\alpha_2 = (1,2,2,3,3)$ and $\alpha_3 = (2,3,4,3,6)$. Augmenting $\alpha_1,\alpha_2,\alpha_3$ by $u_1 = (1,0,0,0,0)$, $u_2 = (0,1,0,0,0)$, $u_3 = (0,0,1,0,0)$, $u_4 = (0,0,0,1,0)$, and $u_5 = (0,0,0,0,1)$ <u>in the given order</u>, construct a basis for E^5 which includes α_1, α_2, and α_3 by using the row-reduced matrix technique.

3.2.7

Let $\alpha_1 = (1,3,-1,2)$, $\alpha_2 = (2,0,1,3)$, $\alpha_3 = (-1,1,0,0)$.

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3.2.7 continued

a. Show that $(x_1,x_2,x_3,x_4) \in S(\alpha_1,\alpha_2,\alpha_3) \leftrightarrow 5x_1 + 5x_2 + 8x_3 - 6x_4 = 0$.

b. What is the dimension of $S(\alpha_1,\alpha_2,\alpha_3)$ and what is a natural basis for $S(\alpha_1,\alpha_2,\alpha_3)$? [That is, find β_1,β_2,β_3 such that $(x_1,x_2,x_3,x_4) \in S(\alpha_1,\alpha_2,\alpha_3) \leftrightarrow (x_1,x_2,x_3,x_4) = x_1\beta_1 + x_2\beta_2 + x_3\beta_3$.]

3.2.8

Let $\alpha_1 = (1,2,3)$, $\alpha_2 = (2,5,4)$, $\alpha_3 = (3,8,9)$, and $\alpha_4 = (4,9,9)$.

a. Show that the dimension of $S(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = 3$. In particular, express (0,0,1), (0,1,0), and (1,0,0) as a linear combination of α_1, α_2, and α_4.

b. Express α_3 as a linear combination of α_1, α_2, and α_4.

c. Express (2,1,4) as a linear combination of α_1, α_2, and α_4.

3.2.9(L)

Show that the space of all polynomials cannot have finite dimension.